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Spectrally arbitrary ray patterns

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Abstract

An $n \times n$ ray pattern A is said to be spectrally arbitrary if for every monic nth degree polynomial $f(x)$ with coefficients from $\mathbb{C}$, there is a matrix in the pattern class of A such that its characteristic polynomial is $f(x)$. In this article the authors extend the nilpotent-Jacobi method for sign patterns to ray patterns, establishing a means to show that an irreducible ray pattern and all its superpatterns are spectrally arbitrary. They use this method to establish that a particular family of $n \times n$ irreducible ray patterns with exactly $3n$ nonzeros is spectrally arbitrary. They then show that every $n \times n$ irreducible, spectrally arbitrary ray pattern has at least $3n - 1$ nonzeros.

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1. Introduction

The problem of classifying the spectrally arbitrary sign patterns, that is, sign patterns that allow the realization of every self-conjugate spectrum, was introduced in [6] by Drew et al. In their article, they developed the nilpotent-Jacobi method for showing that a sign pattern and all its superpatterns are spectrally arbitrary, and they conjectured that a particular family of tridiagonal patterns is spectrally arbitrary. Work on spectrally arbitrary sign patterns has continued in several articles including [1–3,8,9], where families of spectrally arbitrary patterns have been presented and where general properties of spectrally arbitrary patterns have been studied. In particular, in

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[1], Britz et al. showed that every $n \times n$ irreducible, spectrally arbitrary sign pattern must have at least $2n - 1$ nonzeros and they provided families of patterns that have exactly $2n$ nonzeros. This result is easily extended to zero–nonzero patterns over $\mathbb{R}$ and $\mathbb{C}$. In [4], Corpuz and McDonald studied the problem of classifying the spectrally arbitrary zero–nonzero patterns over $\mathbb{R}$, and they classified all $n \times n$ spectrally arbitrary zero–nonzero patterns when $n \leqslant 4$. They provided some initial ideas on the maximum number of nonzeros a zero–nonzero pattern must have before it is forced to be spectrally arbitrary. In [5], properties of reducible, spectrally arbitrary zero–nonzero and sign patterns over $\mathbb{R}$ were investigated.

In this paper we initiate the study of spectrally arbitrary ray patterns. We show how to use the nilpotent-Jacobi method in the complex case, provide a family of $n \times n$ spectrally arbitrary ray patterns with exactly $3n$ nonzeros, and show that every irreducible $n \times n$ spectrally arbitrary ray pattern has at least $3n - 1$ nonzeros.

2. The nilpotent-Jacobi method

Recall that an $n \times n$ ray pattern P is a matrix with entries $p_{ij} \in \{e^{i\theta} : 0 \leqslant \theta < 2\pi\} \cup \{0\}$. Its *ray pattern class* is

$$Q(P) = \{A = [a_{ij}] \in M_n(\mathbb{C}) : a_{ij} = r_{ij}\, p_{ij}, \quad \text{where } r_{ij} \in \mathbb{R}^+ \quad \text{for } 1 \leqslant i, j \leqslant n\}.$$

A ray pattern P is said to be *spectrally arbitrary* if each monic nth degree polynomial with coefficients from $\mathbb{C}$ is the characteristic polynomial of some matrix in $Q(P)$.

In the paper entitled *Spectrally Arbitrary Sign Patterns* [6], Drew et al. describe a method for establishing that a sign pattern and all its superpatterns are spectrally arbitrary. Their method worked for those sign patterns in whose classes certain types of nilpotent matrices could be found. We extend their method to the ray pattern case in the following manner:

1. Find a nilpotent matrix in the given ray pattern class.
2. Change $2n$ of the positive coefficients (denoted $r_1, r_2, \ldots, r_{2n}$) of the $e^{\theta_{ij}}$ in this nilpotent matrix to variables $t_1, t_2, \ldots, t_{2n}$.
3. Express the characteristic polynomial of the resulting matrix as:
$$x^n + (f_1(t_1, \ldots, t_{2n}) + ig_1(t_1, \ldots, t_{2n}))x^{n-1} + \cdots + (f_{n-1}(t_1, \ldots, t_n) \\ + ig_{n-1}(t_1, \ldots, t_n))x + (f_n(t_1, \ldots, t_{2n}) + ig_n(t_1, \ldots, t_{2n})).$$
4. Find the Jacobi matrix
$$J = \frac{\partial(f_1, g_1, \ldots, f_n, g_n)}{\partial(t_1, \ldots, t_{2n})}.$$
5. If the determinant of J, evaluated at $(t_1, t_2, \ldots, t_{2n}) = (r_1, r_2, \ldots, r_{2n})$ is nonzero, then by continuity of the determinant in the entries of a matrix, there is a neighborhood U of $(r_1, r_2, \ldots, r_{2n})$ such that all the vectors in U are strictly positive and the determinant of J evaluated at any of these vectors is nonzero. Moreover, by the Implicit Function Theorem there is a neighborhood $V \subseteq U$ of $(r_1, r_2, \ldots, r_{2n}) \subseteq \mathbb{R}^{2n}$, a neighborhood W of $(0, 0, \ldots, 0) \subseteq \mathbb{R}^{2n}$, and a function $(h_1, \ldots, h_{2n})$ from W into V such that for any $(a_1, b_1, \ldots, a_n, b_n) \in W$ there exists a strictly positive vector $(s_1, s_2, \ldots, s_{2n}) = (h_1, h_2, \ldots, h_{2n})(a_1, b_1, \ldots, a_n, b_n) \in V$ where $f_j(s_1, s_2, \ldots, s_{2n}) = a_j$ and $g_j(s_1, s_2, \ldots, s_{2n}) = b_j$. Taking positive scalar multiples of the corresponding matrices, we see that each monic nth degree polynomial over $\mathbb{C}$ is the characteristic polynomial of some matrix in this ray pattern class.

Next consider a superpattern of our pattern. Represent the new nonzero entries by $c_1 e^{\theta_{i_1 j_1}}, \ldots, c_k e^{\theta_{i_k j_k}}$. Let $\widehat{f_i}(t_1, t_2, \ldots, t_{2n}, c_1, c_2, \ldots, c_k)$ and $\hat{g}_i(t_1, t_2, \ldots, t_{2n}, c_1, c_2, \ldots, c_k)$ represent the new functions in the characteristic polynomial, and $\hat{J} = \frac{\partial(\widehat{f_1}, \hat{g}_1, \ldots, \widehat{f_n}, \hat{g}_n)}{\partial(t_1, \ldots, t_{2n})}$ the new Jacobi matrix. As above, let $(a_1, b_1, \ldots, a_n, b_n) \in W$ and $(s_1, s_2, \ldots, s_{2n}) = (h_1, h_2, \ldots, h_{2n})(a_1, b_1, \ldots, a_n, b_n)$. Then $a_j = f_j(s_1, s_2, \ldots, s_{2n}) = \widehat{f_j}(s_1, s_2, \ldots, s_{2n}, 0, 0, \ldots, 0)$, $b_j = g_j(s_1, s_2, \ldots, s_{2n}) = \widehat{g_j}((s_1, s_2, \ldots, s_{2n}, 0, 0, \ldots, 0)$ and the determinant of $\widehat{J}$ evaluated at $(t_1, \ldots, t_{2n}, c_1, c_2, \ldots, c_k) = (s_1, s_2, \ldots, s_{2n}, 0, 0, \ldots, 0)$ is equal to the determinant of J evaluated at $(t_1, \ldots, t_{2n}) = (s_1, s_2, \ldots, s_{2n})$ and hence is nonzero. By the implicit function theorem, there exists a neighborhood $\widehat{V} \subseteq V$ of $(s_1, s_2, \ldots, s_{2n})$, a neighborhood T of $(0, 0, \ldots, 0) \subseteq \mathbb{R}^k$ and a function $(q_1, q_2, \ldots, q_{2n})$ from T into $\widehat{V}$ such that for any vector $(d_1, \ldots, d_k) \in T$, there exists a strictly positive vector $(e_1, e_2, \ldots, e_{2n}) = (q_1, q_2, \ldots, q_{2n})(d_1, d_2, \ldots, d_k) \in \widehat{V}$ where $\widehat{f_j}(e_1, e_2, \ldots, e_{2n}, d_1, d_2, \ldots, d_k) = a_j$ and $\widehat{g_j}(e_1, e_2, \ldots, e_{2n}, d_1, d_2, \ldots, d_k) = b_j$. Choosing $(d_1, d_2, \ldots, d_k) \in T$ strictly positive we see that there are also matrices in the superpattern's class with every characteristic polynomial corresponding to a vector in W. Taking positive scalar multiples of the corresponding matrices, we see that each monic nth degree polynomial over $\mathbb{C}$ is the characteristic polynomial of some matrix in this superpattern's class.

3. A family of spectrally arbitrary ray patterns

In this section we show that for all $n \geqslant 4$, there are $n \times n$ spectrally arbitrary ray patterns of the following form:

$$A_n = \begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 1 & e^{\theta i} & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \\ -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & -i & -i & \cdots & \cdots & \cdots & -i & -i \end{bmatrix},$$

where θ can take on infinitely many values between 0 and 2π. Note that each of these ray patterns is in unreduced lower Hessenberg form and has exactly $3n$ nonzeros.

Theorem 1. *For each $n \geqslant 4$, there exist infinitely many choices for θ with $0 \leqslant \theta \leqslant 2\pi$ so that A_n, and all of its superpatterns, are spectrally arbitrary ray patterns.*

Proof. For convenience, we restrict θ to $0 \leqslant \theta \leqslant \frac{\pi}{2}$. Let $q = \cos\theta$. Consider,

$$B = \begin{bmatrix} a_1 & 1 & 0 \ldots & \ldots & \ldots & 0 \\ a_2 & q + i\sqrt{1-q^2} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & 0 & \ldots & 0 & 1 & 0 \\ a_{n-1} & b_n & \ldots & \ldots & 0 & 1 \\ a_n & ib_{n-1} & \ldots & \ldots & ib_2 & ib_1 \end{bmatrix}.$$

Then $B \in Q(A_n)$ whenever $a_2, a_3, \ldots a_{n-3}, a_n, b_n$ are positive, and $a_1, a_{n-2}, a_{n-1}, b_1, b_2, \ldots, b_{n-1}$ are negative. We note that there are additional choices for θ and the signings of these constants that result in spectrally arbitrary patterns.

Recall that the coefficient of x^{n-j} in the characteristic polynomial for B_n consists of the sum of the signed weighted products of disjoint cycles whose total length is j. The reader can verify that the characteristic polynomial of B_n is as follows:

Term	Coefficient
x^{n-1}	$(-a_1 - q)$
	$+\mathrm{i}\left(-b_1 - \sqrt{1-q^2}\right)$
x^{n-2}	$\left(-a_2 - b_1\sqrt{1-q^2} + a_1 q\right)$
	$+\mathrm{i}\left(-b_2 + a_1 b_1 + a_1\sqrt{1-q^2} + b_1 q\right)$
x^{n-j}	$\left(-a_j + a_1 b_{j-2}\sqrt{1-q^2} - b_{j-1}\sqrt{1-q^2}\right)$
$3 \leqslant j \leqslant n-3$	$+\mathrm{i}\left(-b_j - a_1 b_{j-2} q + b_{j-1} q + \sum_{k=1}^{j-1} a_k b_{j-k}\right)$
x^2	$\left(-a_{n-2} - b_n + a_1 b_{n-4}\sqrt{1-q^2} - b_{n-3}\sqrt{1-q^2}\right)$
	$+\mathrm{i}\left(-b_{n-2} - a_1 b_{n-4} q + b_{n-3} q + \sum_{k=1}^{n-3} a_k b_{n-k-2}\right)$
x^1	$\left(-a_{n-1} + a_1 b_n + a_1 b_{n-3}\sqrt{1-q^2} - b_{n-2}\sqrt{1-q^2}\right)$
	$+\mathrm{i}\left(-b_{n-1} - a_1 b_{n-3} q + b_1 b_n + b_{n-2} q + \sum_{k=1}^{n-2} a_k b_{n-k-1}\right)$
x^0	$\left(-a_n + a_1 b_{n-2}\sqrt{1-q^2}\right)$
	$+\mathrm{i}\left(-b_n a_1 b_1 - a_1 b_{n-2} q + \sum_{k=1}^{n-1} a_k b_{n-k}\right)$

Let f_j and g_j represent the real and complex parts of the coefficient of x^{n-j}, respectively, viewed as functions of $a_1, b_1, \ldots, a_n, b_n$.

Consider the Jacobi matrix C given by

$$C = \frac{\partial(f_1, g_1, \ldots, f_n, g_n)}{\partial(a_1, b_1, \ldots, a_n, b_n)}.$$

Notice that

$$c_{jk} = \begin{cases} -1 & \text{if } j = k \text{ and } j \leqslant 2n-6, \\ 0 & \text{if } j < k \text{ and } j \leqslant 2n-6. \end{cases}$$

Hence

$$\begin{aligned} \det(C) &= \det(C[\{1, \ldots, 2n-6\}])\det(C[\{2n-5, 2n-4, \ldots, 2n\}] \\ &= \det(C[\{2n-5, 2n-4, \ldots, 2n\}]). \end{aligned}$$

Notice

$$C[\{2n-5,2n-4,\dots,2n\}]=\begin{bmatrix}-1 & 0 & 0 & 0 & 0 & -1\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & -\sqrt{1-q^2} & -1 & 0 & 0 & a_1\\ b_1 & q+a_1 & 0 & -1 & 0 & b_1\\ 0 & a_1\sqrt{1-q^2} & 0 & 0 & -1 & 0\\ b_2 & -a_1q+a_2 & b_1 & a_1 & 0 & -a_1b_1\end{bmatrix},$$

which has determinant equal to b_2. Thus the Jacobian of this system of equations is nonzero at any nilpotent realization we choose.

We finish our proof by showing that there are infinitely many choices of q with $0<q<1$, and nonzero real numbers $a_1, b_1, \dots a_n, b_n$ with the correct signs for which B is nilpotent. Starting with $j=1$ and working up to $j=n-3$, set the real and complex parts of each coefficient of x^{n-j} equal to zero, and then solve for a_j and b_j, respectively. Notice that

$$\begin{aligned}a_1 &= -q,\\ b_1 &= \left(\sqrt{1-q^2}\right)(-1),\\ a_2 &= -2q^2+1,\\ b_2 &= \left(\sqrt{1-q^2}\right)(-q).\end{aligned}$$

Proceeding in this manner, observe that for $3 \leqslant j \leqslant n-3$, $a_j = h_j$ for some polynomial $h_j = h_j(q)$, whose lowest degree term is a positive constant when j is even and a positive multiple of q when j is odd; and that $b_j = \left(\sqrt{1-q^2}\right) l_j$ for some polynomial $l_j = l_j(q)$, whose lowest degree term is a negative constant when j is odd and a negative multiple of q when j is even. This leaves us with six equations to solve, the first three of which are:

$$\begin{aligned}a_{n-2} &= -b_n + h_{n-2},\\ b_{n-2} &= \left(\sqrt{1-q^2}\right) l_{n-2},\\ a_{n-1} &= a_1 b_n + h_{n-1}.\end{aligned}$$

The fourth equation is

$$b_{n-1} = b_1 b_n - a_1 b_{n-3} q + b_{n-2} q + a_1 b_{n-2} + \sum_{k=2}^{n-3} a_k b_{n-k-1} + a_{n-2} b_1,$$

which yields, via substitution

$$b_{n-1} = -q a_1 b_{n-3} + b_1 h_{n-2} + \sum_{k=2}^{n-3} a_k b_{n-k-1} = \left(\sqrt{1-q^2}\right) l_{n-1}.$$

The fifth equation is

$$a_n = h_n.$$

Here the $h_j = h_j(q)$ and $l_j = l_j(q)$ satisfy the same properties as above with respect to the lowest degree term when $n-2 \leqslant j \leqslant n-1$. Moreover $h_n = h_n(q) \neq 0$, which has a positive lowest degree term. Substituting into the final equation we get:

$$
\begin{aligned}
0 &= -b_n a_1 b_1 - a_1 b_{n-2} q + \sum_{k=1}^{n-3} a_k b_{n-k} + a_{n-2} b_2 + a_{n-1} b_1 \\
&= -b_2 b_n + \left(\sqrt{1-q^2}\right)(q^2 l_{n-2} - q h_{n-2} - h_{n-1}) + \sum_{k=1}^{n-3} a_k b_{n-k}.
\end{aligned}
$$

And hence

$$
b_n = h_{n-2} - q l_{n-2} + \frac{h_{n-1}}{q} - \sum_{k=1}^{n-3} \frac{h_k l_{n-k}}{q} = -\frac{l_n(q)}{q},
$$

where $l_n = l_n(q)$ satisfies the same properties as above with respect to the lowest degree term. This implies that b_n is in fact positive for values of q sufficiently close to zero. Substituting back into the equations for a_{n-2} and a_{n-1} we see that they are also nonzero polynomials in q, however their lowest degree terms have negative coefficients. Since there are finitely many choices for q that make at least one of the $h_j(q) = 0$ or $l_j(q) = 0$, by choosing values for q sufficiently close to zero, we can find infinitely many values of q (and hence θ) which make B_n nilpotent, while $a_2, \ldots, a_{n-3}, a_n, b_n$ are positive and $a_1, a_{n-2}, a_{n-1}, b_1, b_2, \ldots, b_{n-1}$ are negative.

Given that A_n is a spectrally arbitrary ray pattern with a nilpotent realization whose Jacobian is nonvanishing, it follows from the nilpotent-Jacobi method outlined in Section 2 that all superpatterns of A_n are also spectrally arbitrary. □

Lemma 2. *The* 3×3 *ray pattern*

$$
A_3 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & \frac{1-\mathrm{i}}{\sqrt{2}} & 1 \\ -1 & \mathrm{i} & \mathrm{i} \end{bmatrix}
$$

is spectrally arbitrary.

Proof. The matrix

$$
B = \begin{bmatrix} -a_1 & 1 & b_3 \\ a_2 & 1-\mathrm{i} & 1 \\ -a_3 & \mathrm{i}b_2 & \mathrm{i}b_1 \end{bmatrix}
$$

is in the pattern class $Q(A_3)$ whenever $a_1, b_1, \ldots, a_3, b_3$ are positive. The characteristic polynomial of B is

$$
\begin{aligned}
&x^3 + ((a_1 - 1) + i(1 - b_1))x^2 + ((b_1 - a_2 - a_1 + a_3 b_3) + i(a_1 - a_1 b_1 - b_2 + b_1))x \\
&\quad + (a_3 - a_3 b_3 + a_1 b_1) + i(a_2 b_1 - a_1 b_2 - a_2 b_3 b_2 + a_3 b_3 + a_1 b_1).
\end{aligned}
$$

The Jacobian of this system of equations is

$$
-a_3 + a_2 b_2 - a_2 b_3 b_2 - a_3 b_1 + a_3 b_3 b_2,
$$

which is nonzero at the nilpotent realization achieved by setting

$$
a_1 = a_3 = b_1 = b_2 = 1 \quad \text{and} \quad a_2 = b_3 = 2. \quad \square
$$

4. The minimum number of nonzeros in a spectrally arbitrary ray pattern

In the previous section we have shown that for every $n \geqslant 3$ there exist $n \times n$ irreducible, spectrally arbitrary ray patterns with exactly $3n$ nonzeros. The main theorem in this section establishes that the minimum number of nonzeros in an $n \times n$ irreducible, spectrally arbitrary ray pattern is at least $3n - 1$, and hence must be either $3n$ or $3n - 1$.

The following lemma is well known in the case of sign patterns. It extends easily to ray patterns.

Lemma 3. *Let A be an $n \times n$ irreducible, complex matrix and let $G = G(A)$ with edge set E. Then there exists $S \subseteq E$ such that $|S| = n - 1$ and the induced subgraph G_S is a tree. Moreover, for each such S, there exists $n \times n$ diagonal matrix D with positive real numbers on the diagonal such that $B = DAD^{-1}$ has the property that $b_{jk} = 1$ for all $(j, k) \in S$.*

For $S \subseteq \mathbb{C}$, let $d(S)$ denote the transcendence dimension of the extension field $(\mathbb{Q}(i))(S)$ over $\mathbb{Q}(i)$, the field of complex rational numbers. That is, $d(S)$ is the cardinality of a maximum cardinality set of algebraically independent transcendentals over $\mathbb{Q}(i)$ that lie in S. Note that if $S, T \subseteq \mathbb{C}$, then $d(S \cup T) \leqslant d(S) + d(T)$; that if $S \subseteq T \subseteq \mathbb{C}$, then $d(S) \leqslant d(T)$; and that if K is a finite algebraic extension of $(\mathbb{Q}(i))(S)$, then $d(K) = d(\mathbb{Q}(i)(S))$ (see [7, Section 14.6] for further details).

In [1] it was shown that every irreducible, spectrally arbitrary sign pattern contains at least $2n - 1$ nonzero entries. We adapt that proof to the ray pattern case to obtain:

Theorem 4. *An $n \times n$ irreducible, spectrally arbitrary ray pattern must have at least $3n - 1$ nonzero entries.*

Proof. Let P be an $n \times n$ spectrally arbitrary ray pattern. For each j and k such that $p_{jk} \neq 0$, express p_{jk} as $p_{jk} = u_{jk} + \mathrm{i}\epsilon_{jk}\sqrt{1 - u_{jk}^2}$ for some u_{jk} with $-1 \leqslant u_{jk} \leqslant 1$ and $\epsilon_{jk} \in \{-1, 1\}$. For each j and k such that $p_{jk} = 0$, let $u_{jk} = 0$. Let $U = \{u_{jk} \mid 1 \leqslant j, k \leqslant n\}$. Since U is a finite subset of $\mathbb{R}$, we can choose $V = \{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\} \subseteq \mathbb{R}$ so that $d(V \cup U) = 2n + d(U)$.

Since P is a spectrally arbitrary ray pattern, we can find $A \in Q(P)$ with characteristic polynomial $p_A(x) = x^n + \sum_{j=1}^{n}(\alpha_j + \mathrm{i}\beta_j)x^{n-j}$. Write each nonzero a_{jk} as $a_{jk} = r_{jk}\Big(u_{jk} + \mathrm{i}\epsilon_{jk}\sqrt{1 - u_{jk}^2}\Big)$ where r_{jk} is a positive real number. If $a_{jk} = 0$, then set $r_{jk} = 0$. (By Lemma 3, we can assume that at least $n - 1$ of the r_{jk} are equal to 1.) Since the coefficients of $p_A(x)$ are polynomials in the entries of A, it follows that the real and complex parts of the coefficients of $p_A(x)$ are polynomials in the real and complex parts of the entries of A. Thus

$$(\mathbb{Q}(i))(V) \subseteq (\mathbb{Q}(i))\left(\left\{r_{jk}u_{jk}, r_{jk}\sqrt{1 - u_{jk}^2} : 1 \leqslant j, k \leqslant n\right\}\right).$$

Then

$$(\mathbb{Q}(i))(V \cup U) \subseteq (\mathbb{Q}(i))\left(\left\{r_{jk}u_{jk}, r_{jk}\sqrt{1 - u_{jk}^2} : 1 \leqslant j, k \leqslant n\right\} \cup U\right).$$

Also note that

$$\begin{aligned}&(\mathbb{Q}(i))\left(\left\{r_{jk}u_{jk}, r_{jk}\sqrt{1 - u_{jk}^2} : 1 \leqslant j, k \leqslant n\right\} \cup U\right)\\ &\quad\subseteq (\mathbb{Q}(i))\left(\left\{r_{jk} : 1 \leqslant j, k \leqslant n\right\} \cup \left\{\sqrt{1 - u_{jk}^2} : 1 \leqslant j, k \leqslant n\right\} \cup U\right),\end{aligned}$$

and that $(\mathbb{Q}(i))(\{r_{jk} : 1 \leqslant j, k \leqslant n\} \cup \{\sqrt{1-u_{jk}^2} : 1 \leqslant j, k \leqslant n\} \cup U)$ is a finite algebraic extension of $(\mathbb{Q}(i))(\{r_{jk} : 1 \leqslant j, k \leqslant n\} \cup U)$ since

$$1 - \left(\sqrt{1-u_{jk}^2}\right)^2 = u_{jk}^2 \in (\mathbb{Q}(i))\left(\left\{r_{jk} : 1 \leqslant j, k \leqslant n\right\} \cup U\right)$$

for all j and k. Thus

$$d\left(\left\{r_{jk} : 1 \leqslant j, k \leqslant n\right\} \cup \left\{\sqrt{1-u_{jk}^2} : 1 \leqslant j, k \leqslant n\right\} \cup U\right) = d(\{r_{jk} : 1 \leqslant j, k \leqslant n\} \cup U).$$

Let m be the number of nonzero entries in A, and hence of nonzero r_{jk}. Since at least $n-1$ of the r_{ij} are known to be 1,

$$d(\{r_{jk} : 1 \leqslant j, k \leqslant n\}) \leqslant m - (n-1).$$

Hence

$$\begin{aligned} 2n + d(U) &= d(V \cup U) \leqslant d\left(\left\{r_{jk} : 1 \leqslant j, k \leqslant n\right\} \cup U\right) \\ &\leqslant d(\{r_{jk} | \ 1 \leqslant j, k \leqslant n\}) + d(U) \\ &\leqslant m - (n-1) + d(U). \end{aligned}$$

and so

$$m \geqslant 3n - 1. \qquad \square$$

Corollary 5. *There are no spectrally arbitrary* 2×2 *ray patterns.*

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