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ULTRAFILTERS ON A COUNTABLE SET

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An ultrafilter on a set is a proper collection of subsets of that set which is maximal among such collections having the finite intersection property. Ultrafilters were popularized by N.Bourbaki for their use in describing topological convergence, but for some time there was little discussion of the possible structural properties that an individual ultrafilter might possess. This paper is concerned only with ultrafilters on a countable set and a method of describing them by building them from certain minimal ultrafilters; most of the work here has come from Chapters 1, 2, and 4 of [1]. The first part of this paper describes a certain tree of ultrafilters and the third part describes an ordering in which this tree is embedded; the remaining two parts deal with minimal ultrafilters and with products of ultrafilters.

Theorems and lemmas are always thought of here as being proved in Zermelo-Fraenkel set theory with the axiom of choice, ZFC. If a theorem T is to be proved in some other set theory, say the theory Γ, it is stated in ZFC in the following manner:

Theorem. $\Gamma \vdash$ T.

An ordinal is always the set of all inferior ordinals and we shall always regard a cardinal as an initial ordinal – the first two infinite initial ordinals are ω and ω_1. The cardinal $2^{\aleph_0}$ will often be written 'c'; 'κ' and 'λ' are reserved for cardinals. The proposition $\aleph_1 = c$ will often be called 'CH'. The symbol '$[A]^{\lambda}$' stands for the set of all subsets of A of power λ and '$[A]^{<\lambda}$' consists of those subsets of power less than λ. The set of all functions from A to B will be written '${}^{A}B$'.

Some concepts from model theory and topology will occasionally appear; βX is, of course, the Stone-Čech compactification of X, N is the discrete countable space. The cardinal number of a set A is written $\overline{\overline{A}}$.

There is also some notation in this paper which is less widely used. We shall let St(ω) be the set of all ultrafilters on ω and $St^{\omega}(\omega)$ the set of non-principal ultrafilters; ultrafilters on ω will usually be lower case Roman letters, especially 'p' and 'q'. The ultraproduct of the structures $\{\mathfrak{A}_i : i \in I\}$ by the ultrafilter D on I shall be written 'Prod(D,λi. $\mathfrak{A}_i$)'.

§ 1. The Rudin-Frolik ordering

The Rudin-Frolik ordering of ultrafilters was defined by Z.Frolik [3] – who called it the 'producing relation' – and nearly defined by M.E. Rudin [6]. M.E.Rudin, in unpublished work, first showed that it was an ordering; Frolik used it to prove that $\beta\mathrm{N} - \mathrm{N}$ is not homogeneous.

1.1. Definition. A sequence $X \in {}^{\omega}\mathrm{St}(\omega)$ is discrete if there is a sequence $\langle a_n : n \in \omega \rangle$ of subsets of ω such that $a_n \in X(n)$ and $a_n \cap a_m = 0$ when $n \neq m$. If X is any such sequence, discrete or not, $\overline{X}$ is the set of those ultrafilters $p \in \mathrm{St}(\omega)$ such that for each $a \in p$ there is an $n \in \omega$ with $a \in X(n)$.

1.2. Definition. (i) If $X \in {}^{\omega}\mathrm{St}^{\omega}(\omega)$ and $p \in \mathrm{St}(\omega)$, then $\Sigma(X, p) = \{ a \subseteq \omega : \{ n : a \in X(n)\} \in p \}$.

(ii) If $X \in {}^{\omega}\mathrm{St}(\omega)$ is discrete and $p \in \overline{X}$, then $\Omega(X, p) = \{ a \subseteq \omega : \forall b \in p \; \exists n \in a \; (b \in X(n)) \}$.

The operations Σ and Ω are inverse to each other; this fact is the point of the following lemma.

1.3. Lemma. *If* $X \in \mathrm{St}(\omega)$ *is discrete and* $p \in \mathrm{St}(\omega)$, *then*

(i) $\Sigma(X, p) \in \overline{X}$;

(ii) *if* $p \in \overline{X}$, *then* $\Omega(X, p) \in \mathrm{St}(\omega)$;

(iii) *if* $p \in \overline{X}$, $\Sigma(\Omega(X, p)) = p$;

(iv) $\Omega(X, \Sigma(X, p)) = p$;

(v) $\Sigma(X, p) \in X$ *if and only if* p *is principal;*

(vi) *if* $p \in \overline{X}$, *then* $\Omega(X, p)$ *is principal if and only if* $p \in X$.

The following definition provides an equivalence relation among ultrafilters so that two equivalent ones have exactly the same set theoretical properties.

1.4. Definition. If f is a permutation of ω such that $q = \{ f(a) : a \in p \}$ where $p, q \in \mathrm{St}(\omega)$, then one says that $f(p) = q$. If for some permutation f, $f(p) = q$, then $p \equiv q$. The equivalence class, under $\equiv$, containing p is $\underset{\sim}{p}$.

1.5. Lemma. *If $X, Y \in {}^{\omega}\mathrm{St}(\omega)$ are discrete and φ is a bijection from* $\mathrm{ran}(X)$ *to* $\mathrm{ran}(Y)$ *then there is a unique homeomorphism Ψ extending φ mapping $\overline{X}$ onto $\overline{Y}$.*

Proof: Let $\Psi(p) = \Sigma(X, \Omega(X, p))$; Ψ extends φ because of 1.3 (v) and (vi); it is onto because $\Psi(\Sigma(X, \Omega(Y, p))) = p$. To see that Ψ is well defined one uses the fact that $\Psi(\Sigma(X, \Omega(X, q))) = \Sigma(Y, \Omega(X, q))$. The rest of the lemma follows by checking the effects of Ψ in the topology of $\beta\mathrm{N}$.

1.6. Lemma. *If $p \in \mathrm{St}(\omega)$, $a \in p$ and a is infinite, then $p_a = \{b \cap a : b \in p\}$ is an ultrafilter on a and $p_a \equiv p$.*

Proof: This is obvious if p is principal or if $\omega - a$ is finite. Suppose that $a_0 = \omega - a$ is infinite, choose $a_\omega \subseteq a$ such that $a_\omega \in p$ and $a_0 - a_\omega$ is infinite. Let $\{a_n : n \geq 1\}$ be a partition of $a_0 - a_\omega$ into mutually disjoint infinite pieces. Take f_n to be a bijection of a_{n+1} onto a, and f_ω to be the identity on a_ω, then $f(p_a) = p$ where $f = \cup\{f_\alpha : \alpha \in \omega + 1\}$.

The next lemma is due to M.E.Rudin.

1.7. Lemma. *The following are equivalent:*
(1) *There is a discrete $X \in {}^{\omega}\mathrm{St}^{\omega}(\omega)$, such that $q = \Sigma(X, p)$;*
(2) *There are discrete sequences $X, Y \in {}^{\omega}\mathrm{St}^{\omega}(\omega)$ such that* $\mathrm{ran}(X) \cap \mathrm{ran}(Y) = 0$, $\mathrm{ran}(X) \subseteq \overline{Y}$, *and there is an $r \in \mathrm{St}^{\omega}(\omega)$ with* $r = \Sigma(X, p) = \Sigma(Y, q)$.

Proof: Assume (1) in the form $q = \Sigma(Z, p)$ and let $Y \in {}^{\omega}\mathrm{St}^{\omega}(\omega)$ with $\mathrm{ran}(Y)$ discrete. Using 1.5, one may extend the natural homeomorphism of $\mathrm{ran}(Y)$ onto N to a φ mapping $\overline{Y}$ onto $\beta\mathrm{N}$. For this φ, $\varphi^{-1}(q) = \Sigma(Y, q) = \Sigma(\varphi^{-1}(Z), p)$ since $\Psi(\Sigma(Y, q)) = \Sigma(\mathrm{N}, q) = q$. To establish (2), one sets $r = \varphi^{-1}(q)$ and $X = \varphi^{-1}(Z)$.

Conversely, if X, Y, p, q and r play the roles required by (2), one selects a homeomorphism φ of $\mathrm{ran}(Y)$ onto N; by 1.5 this can be extended to a map of $\overline{Y}$ onto $\beta\mathrm{N}$. One obtains (1) by putting $q = \Sigma(\Psi(X), p)$.

1.8. Definition. When the state of affairs of 1.7 takes place one says that $\underset{\sim}{p} < \underset{\sim}{q}$.

The ordering of 1.8 is what we shall call the 'Rudin-Frolik' ordering. In [3], Frolik showed that each element has 2^c successors and at most c predecessors. Since any two ultrafilters which can be mapped to each other by a homeomorphism of $\beta N - N$ have exactly the same predecessors he was able to conclude that $\beta N - N$ is not homogeneous. The next results are due to M.E.Rudin: 1.9 is essentially in her paper [6], 1.10 is an unpublished result due to her, and 1.11 uses 1.10 to conclude that the Rudin-Frolik ordering is indeed an ordering.

1.9. Lemma. *Let $X, Y \in {}^{\omega}\mathrm{St}^{\omega}(\omega)$ be discrete and let $p = \overline{X} \cap \overline{Y}$, then there exist subsequences X' and Y', of X and Y respectively, such that $\Omega(X,p) = \Omega(X',p)$, $\Omega(Y,p) = \Omega(Y',p)$, and such that one of the following three holds*:

(i) $\operatorname{ran} X' \cap \operatorname{ran} Y' = 0$ *and* $\operatorname{ran} X' \subseteq \overline{Y'}$;

(ii) $\operatorname{ran} X' \cap \operatorname{ran} Y' = 0$ *and* $\operatorname{ran} Y' \subseteq \overline{X'}$;

(iii) $\operatorname{ran} X' = \operatorname{ran} Y'$.

1.10. Theorem. *If $\underset{\sim}{p} < \underset{\sim}{q}$, then $\underset{\sim}{p} \neq \underset{\sim}{q}$.*

Proof: Suppose that $\underset{\sim}{p} < \underset{\sim}{q}$, then there exist an r, X, and Y with $\operatorname{ran}(X) \cap \operatorname{ran}(Y) = 0$, $\operatorname{ran}(X) \subseteq \overline{Y}$, and $r = \Sigma(X, p_1) = \Sigma(Y, p_2)$ where $p_1 \equiv p_2 \equiv p$. One may permute the elements of the sequences X and Y to obtain new sequences X' and Y' such that $r = \Sigma(X', p) = \Sigma(Y', p)$. Let us hereafter call these new sequences 'X' and 'Y' again. Suppose that $\langle a_n \subseteq \omega : n \in \omega \rangle$ renders X discrete as in 1.1; let $C(n) =$ $= \{k \in \omega : a_n \in Y(k) \text{ and } n < k\}$. One now defines two sets R and B by stages:

$$R_0 = \{k : k \notin \cup \{C(n) : n \in \omega\}\} ;$$

let l_0 be the least integer not in R_0, then

$$B_0 = \bigcup \{C(n) : n \in R_0\} \cup C(l_0) ,$$

$$R_{m+1} = \bigcup \{C(n) : n \in \bigcup \{B_i : i \leq m\}\} ;$$

Let l_0 be the least integer not in R_0, then

$$B_0 = \bigcup \{ C(n) : n \in R_0 \} \cup C(l_0) ,$$

$$R_{m+1} = \bigcup \{ C(n) : n \in \bigcup \{ B_i : i \leq m \} \} .$$

Let l_{m+1} be the least integer not contained in $\bigcup \{ R_i : i \leq m + 1 \} \cup \cup \bigcup \{ B_i : i \leq m \}$. Now, take

$$B_{m+1} = \bigcup \{ C(n) : n \in \bigcup \{ R_i : i \leq m + 1 \} \} \cup C(l_{m+1}) .$$

To finish the theorem one observes that putting $R = \bigcup \{ R_i . i \in \omega \}$, $B = \bigcup \{ B_i : i \in \omega \}$ provides two sets such that $\omega = R \cup B, R \cap B = 0$, and $R \in \Omega(X, r)$ exactly when $B \in \Omega(Y, r)$. This contradicts the assumption that $\Omega(X, r) = \Omega(Y, r) = p$.

1.11. Theorem. (i) *$\underset{\sim}{r} < \underset{\sim}{q} < \underset{\sim}{p}$ implies that $\underset{\sim}{r} < \underset{\sim}{p}$.*
(ii) *$\{ \underset{\sim}{p} : \underset{\sim}{p} < \underset{\sim}{q} \}$ is a linear ordering.*

Proof: (i) If $\underset{\sim}{r} < \underset{\sim}{q} < \underset{\sim}{p}$, then there are sequences X and Y for which $p = \Sigma(X, q) = \Sigma(Y, r)$. If case (ii) of 1.9 holds then $r < p$. Case (iii) would mean that $\underset{\sim}{r} = \underset{\sim}{q}$ in violation of 1.10. It only remains to show that case (i) is impossible. If (i) held, one would have $\underset{\sim}{r} < \underset{\sim}{q} < \underset{\sim}{p} < \underset{\sim}{r}$. By the method of 1.7 one can obtain sequences X, Y, and Z with $r = \Sigma(X, r) = \Sigma(Y, q) = \Sigma(Z, p)$, but $r = \Sigma(X, r)$ means that $\underset{\sim}{r} < \underset{\sim}{r}$, contradicting 1.10 again.

Theorems 1.10 and 1.11 serve to show that the Rudin-Frolik ordering is a tree; we shall see in the next theorem that it is not well-founded – part 2 contains stronger versions of this result.

1.12. Theorem. *The Rudin-Frolik ordering is not well-founded.*

Proof: Let X_0 be a discrete sequence, and choose X_{n+1} to be a discrete sequence of elements of $\overline{X}_n - \mathrm{ran}(X_n)$. Since βN is compact, there is a $p \in \bigcup \{ \overline{X}_n : n \in \omega \}$. It follows that $\underset{\sim}{q}_{n+1} < \underset{\sim}{q}_n$ where $q_i = \Omega(X_i, p)$.

§ 2. The semigroup of ultrafilters

The following definition of the product of ultrafilters, similar to the definition of product measure, turns βN into a semigroup.

2.1. Definition. If p and q are ultrafilters on ω then $p \cdot q =$ $= \{a \subseteq \omega \times \omega : \{m : \{n : \langle n, m\rangle \in a\} \in p\} \in q\}$. One writes $p \cdot q$ for the r such that $r = p \cdot q$, a convention justified by the next lemma.

2.2. Lemma. (i) *$p \cdot q$ is an ultrafilter.*
(ii) $p \cdot (q \cdot r) \equiv (p \cdot q) \cdot r$.
(iii) *If $p \equiv r$, then $p \cdot q \equiv r \cdot q$.*
(iv) *If $p \equiv r$, then $q \cdot p \equiv q \cdot r$.*

Proof: Each part can be demonstrated by an easy computation.

2.3. Definition. Let St be the set $\{ \underset{\sim}{p} : p \in \mathrm{St}(\omega)\}$. If p is principal one writes '$\underset{\sim}{1}$' for '$\underset{\sim}{p}$'.

The definitions above make $\langle \mathrm{St}, \cdot, \underset{\sim}{1}\rangle$ into a semigroup with unity. Soon we shall see that it is non-commutative, has a trivial center, is right cancellable, and has no left or right identities apart from $\underset{\sim}{1}$.

2.4. Theorem. (i) *If X is discrete, $X \in \mathrm{St}^\omega(\omega)$, and $\Sigma(X, p) = q$, then $\Sigma(r \cdot X, p) \equiv r \cdot q$ when $(r \cdot X)(n) \equiv r \cdot X(n)$.*

(ii) *If X is discrete, $X \in \mathrm{St}^\omega(\omega)$, and for each n, $X(n) \equiv p$, then $\Sigma(X, q) \equiv p \cdot q$.*

(iii) *If X is discrete, $X \in \mathrm{St}^\omega(\omega)$, $t \in \mathrm{St}(\omega)$, and $p = \Sigma(X, q)$, then there is a discrete sequence Y such that* $\mathrm{ran}(Y) \cap \mathrm{ran}(X) = 0$, $p \cdot t =$ $= \Sigma(Y, q \cdot t)$ *and* $\mathrm{ran}(X) \subseteq \overline{Y}$.

Proof: (i) Suppose that $a \in r \cdot q$, let $a_m = \{n : \langle n, m\rangle \in a\}$. Since $\Sigma(X, p) = q$, one finds that $\{k : \{m : a_m \in r\} \in X(k)\} = p$. This means that $\{k : a \in r \cdot X(k)\} \in p$ so that $a \in \Sigma(r \cdot X, p)$, thus $r \cdot q \subseteq \Sigma(r \cdot X, q)$. Since both $r \cdot q$ and $\Sigma(r \cdot X, q)$ are ultrafilters, they must be equal.

(ii) Let $\{C_n : n \in \omega\}$ be a partition of ω with $C_n \in X(n)$, set $v_m =$ $= \{\langle n, m\rangle : n \in \omega\}$. Let g_m be a bijection from v_m onto c_m such that if h is the bijection from v_m onto ω sending $\langle n, m\rangle$ to n, then $g_m h^{-1}(p) =$

$= p_m = \{ a \cap c_m : a \in t \}$ where $t \equiv p$. Such a g_m exists by 1.6. Letting $g = \bigcup \{ g_m : m \in \omega \}$, one has that $g(a) \in \Sigma(X, q)$ for $a \in p \cdot q$. This means that $g(p \cdot q) \subseteq \Sigma(X, q)$ so that $g(p \cdot q) = \Sigma(X, q)$.

(iii) By (ii), there is a discrete sequence P with $p \cdot t = \Sigma(P, t)$ and $P(n) \equiv p$ for each n. Suppose that $\{ c_k : k \in \omega \}$ renders P discrete. Using 1.6 we may choose a discrete sequence Y_k with $c_k \in Y_k(n)$, for each n, and $P(n) = \Sigma(Y_k, q)$. The 'Y' of the theorem can be any sequence whose range is $\bigcup \{ \operatorname{ran}(Y_k) : k \in \omega \}$. To see this one maps ω in a one-to-one manner onto $\omega \times \omega$ sending c_n onto $\omega \times \{n\}$.

2.5. Lemma. *If X is a discrete sequence of ultrafilters such that each $X(n)$ is minimal above $\underset{\sim}{1}$ in the Rudin-Frolik ordering then $\underset{\sim}{q}$ is minimal above $\underset{\sim}{p}$ whenever $q \equiv \Sigma(X, p)$.*

Proof: Suppose $\underset{\sim}{r} < \underset{\sim}{q}$, then there is a discrete Y such that $\Sigma(X, p) = \Sigma(Y, r)$. By our assumption case (i) of 1.9 is impossible, but either case (ii) or case (iii) implies that $r \leq p$.

2.6. Definition. If S is a function from ω into $\operatorname{St}(\omega)$ then $F[0] = F(0)$ and $F[n+1] = F[n] \cdot F(n+1)$.

2.7. Theorem. *For each $k \in \omega$, there are discrete sequences of ultrafilters $X_0, ..., X_{k-1}$ such that $F[k+1] = \Sigma(X_i, F(k-i) \cdots F(k+1))$ where, for each n, $X_i(n) \equiv F[k-i-1]$ and $\operatorname{ran}(X_i) \subseteq \overline{\operatorname{ran}(X_{i+1})}$, while $\operatorname{ran}(X_i) \cap \operatorname{ran}(X_j) = 0$ when $i \neq j$.*

Proof: One proceeds by induction on k; when $k = 0$, $F[1] = \Sigma(Y, F(0))$ and one may take Y as the 'X_0' of the theorem.

By definition $F[k+1] = F[k] \cdot F(k+1)$, so that $F[k+1] = \Sigma(Y, F(k+1))$ where for each n, $Y(n) \equiv F[k]$. Suppose that Y is rendered discrete by the disjointed set $\{ c_n : n \in \omega \}$. Using the induction hypothesis one can obtain discrete sequences Z_n^i, $i < n-1$, such that $c_n \in \bigcap \{ \operatorname{ran}(Z_n^i) : i \in \omega \}$, $Y(n) \equiv \Sigma(Z_n^i, F(n-1-i) \cdots F(n))$ and for each m, $Z_n^i(m) \equiv F[n-2-i]$.

Let $X_0 = Y$, and let X_{l+1} be a discrete sequence whose range is $\bigcup [\operatorname{ran} Z_n^l : n \in \omega]$. It is clear that $\operatorname{ran}(X_l) \subseteq \overline{\operatorname{ran}(X_{l+1})}$ and $\operatorname{ran}(X_l) \cap \operatorname{ran}(X_s) = 0$ when $s \neq l$ because of the corresponding facts concerning the sequences Z_n^i. The rest of the theorem follows from 1.9 and 2.4 (iii).

Frolik spoke of the operation Σ as an infinite sum of ultrafilters, we shall now define something akin to an infinite product. The theorem immediately above was introduced for its application to this notion of a product.

2.8. Definition. Let F be a sequence of ultrafilters. Fix a well ordering of sufficiently many sets to permit one to define $\Pi(F, p) = \Sigma(X, p)$ where X is the first discrete sequence such that $X(n) \equiv F[n]$. If for each n, $F(n) = p$, then $\Pi(F, p) = p^{\omega}$.

Surely $\underset{\sim}{\Pi(F, p)}$ is well defined. This equivalence class of ultrafilters appears in the next theorem which serves to show that many different orderings can be embedded into the Rudin-Frolik tree.

2.9. Theorem. *For each function F from ω to* $\mathrm{St}^{\omega}(\omega)$ *and each non-principal ultrafilter p, there is an isomorphism φ of* $\mathrm{Prod}(p, \lambda n \cdot \langle n, \epsilon\rangle)$ *into the Rudin-Frolik predecessors of* $\underset{\sim}{\Pi(F, p)}$.

Proof: From the definition of $\Pi(F, p)$ we know that $\Pi(F, p) = \Sigma(X, p)$ where $X(n) \equiv F[n]$. We may suppose that the sequence X is rendered discrete by the disjointed sequence $\langle c_n \subseteq \omega : n \in \omega\rangle$. One can now use 2.7 and 1.6 to obtain discrete sequences $X_0^{n+1}, \ldots, X_{n-1}^{n+1}$ which have the properties of 2.7 (except that 'k' of 2.7 is 'n' here) but such that for each i and m, $c_n \in X_i^n(m)$. Given a pressing down function on ω (one with $f(n) < n$, for positive n), one can define a discrete sequence X_f such that $\mathrm{ran}(X_f) = \bigcup \{\mathrm{ran}(X_{f(n+1)}^{n+1}) : n \in \omega\}$. The embedding φ is defined by $\varphi([f]) = \underset{\sim}{\Omega(X_f, \Pi(F, p))}$.

It remains to be shown that $\underset{\sim}{\Omega(X_f, \Pi(F, p))} < \underset{\sim}{\Omega(X_g, \Pi(F, p))}$ whenever

$$\mathrm{Prod}(p, \lambda n. \langle n, \epsilon\rangle) \models [f] < [g] \ .$$

Whenever the latter holds we may suppose there is an $a \in p$ such that if $n + 1 \in a$, then $\underset{\sim}{\Omega(X_{f(n+1)}^{n+1}, F[n+1])} < \underset{\sim}{\Omega(X_{g(n+1)}^{n+1}, F(n+1))}$. The desired result then follows from 1.6.

2.10. Corollary. *The Rudin-Frolik ordering contains a chain of type* ω_1^*.

Proof: Such a chain can be found in $\mathrm{Prod}(p, \lambda n.\langle n, \epsilon\rangle)$.

2.11. Corollary. *The Rudin-Frolik ordering contains a chain ordered like the reals.*

Proof: The structure $\mathrm{Prod}(p, \lambda n.\langle n, \epsilon\rangle)$ has type $\omega + \theta + \omega^*$ where θ is the type of a dense linear ordering. Since $\omega + \theta + \omega^*$ is the type of an $\aleph_1$ saturated ordering, θ must be an η_1 ordering and therefore contains a chain similar to the reals.

Let us now find the theory of the ultraproduct which appears in 2.9 above. The next theorem gives axioms for this theory; if one were to assume the continuum hypothesis the theorem would follow immediately from basic results concerning saturated models. Here '$\equiv$' denotes elementary equivalence.

2.12. Theorem. *If p and q are non-principal ultrafilters then* $\mathrm{Prod}(p, \lambda n.\langle n, \epsilon\rangle) \equiv \mathrm{Prod}(q, \lambda n.\langle n, \epsilon\rangle)$. *In fact, both are equivalent to* $\langle \omega + \omega^*, <\rangle$.

Proof: We expand the language appropriate for these structures to a new language having two new countably infinite sets of constants, $\{c_n, d_n : n \in \omega\}$, and a unary function parameter, S. The theory Γ consists of the consequences of that set of axioms which specify a strict ordering with a first element, c_0, and a last element, d_0, such that element other than d_0 has an immediate successor which is given by S. Furthermore, we require that $Sd_0 = d_0$, $Sd_{n+1} = d_n$, $Sc_n = c_{n+1}$, and that $c_n < d_m$ for each n and m.

The structure $\langle \omega + \omega^*, <\rangle$ is a reduct of a model of Γ. By eliminating quantifiers in Γ one can see that Γ is complete; it follows that the appropriate expansions of $\langle \omega + \omega^*, <\rangle$, $\mathrm{Prod}(p, \lambda n.\langle n, \epsilon\rangle)$ are all elementarily equivalent and are therefore themselves equivalent.

2.13. Corollary. (i) ZFC & CH $\vdash$ *If p and q are non-principal then* $\mathrm{Prod}(p, \lambda n.\langle n, \epsilon\rangle)$ *and* $\mathrm{Prod}(q, \lambda n.\langle n, \epsilon\rangle)$ *are isomorphic.*

(ii) *If $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are Rudin-Frolik minimal then the orderings of their predecessors are mutually isomorphic.*

The process of exponentiation can be continued through the countable ordinals.

2.14. Definition. First, fix a well ordering of enough sets to carry out the rest of this definition. Already we have defined p^ω and p^n, $n \in \omega$; let $p^{\alpha+1}$ be the equivalence class of the first ultrafilter q such that $q \equiv p \cdot p^\alpha$, and if δ is a positive limit ordinal then $p^\delta = \Sigma(X, p)$ where $\langle \delta_n : n \in \omega \rangle$ is the first sequence of countable ordinals whose union is δ and X is the first discrete sequence with $X(n) \equiv p^{\alpha_n}$.

2.15. Theorem. *For each $\alpha \in \omega_1$, there is a discrete sequence X such that $X(n) = p^{\alpha_n}$ for some $\alpha_n \in \alpha$, $p^\alpha = \Sigma(X, p)$, and, in addition, these discrete sequences can be chosen so that if $\beta \in \alpha$, $p^\alpha = \Sigma(X, p)$, $p^\beta = \Sigma(Y, p)$, $X(n) = p^{\alpha_n}$ and $Y(n) = p^{\beta_n}$, then $\beta_n \in \alpha_n$ for all but finitely many $n \in \omega$.*

Proof: If α is a limit ordinal the definition of p^α insures that it is of the form $\Sigma(X, p)$. If $\alpha = \beta + 1$, then, by induction, $p^\beta = \Sigma(X, p)$ where $X(n) \equiv p^{\beta_n}$. By 2.4 (i), $\Sigma(p \cdot X, p) = p \cdot p^\beta \equiv p^{\beta+1} \equiv p^\alpha$, where $(p \cdot X)(n) = p \cdot X(n) = p^{\beta_n+1}$. It is easily shown by induction that the sequences defined in this way satisfy the final clause of the theorem.

2.16. Corollary. *If $\beta \in \alpha$, then $\underset{\sim\sim}{p^\beta} < \underset{\sim\sim}{p^\alpha}$.*

2.17. Theorem. (i) *There is an order preserving monomorphism of the ultraproduct* $\mathrm{Prod}(p, \langle \omega_1, \epsilon \rangle)$ *into* $\{ \underset{\sim}{q} : \exists \alpha \in \omega_1 (\underset{\sim}{q} < \underset{\sim\sim}{p^\alpha})\}$.

(ii) ZFC & CH $\vdash$ *If p is Rudin-Frolik minimal then the set of predecessors of $\underset{\sim\sim}{p^\alpha}$, $\omega \in \alpha \in \omega_1$, is isomorphic to* $\mathrm{Prod}(p, \lambda n.\langle n, \epsilon \rangle)$.

Proof: (i) If $f \in {}^\omega\omega_1$ let $\Psi([f]) = \underset{\sim\sim}{\Sigma(X, p)}$, where X is the first discrete sequence for which $X(n) \equiv p^{f(n)}$. By the methods that have already been exploited above, this Ψ can be seen to be an embedding.

(ii) Both of these structures are models of the theory Γ of 2.12.

In the remainder of this section we shall list some additional properties of the semigroup of ultrafilters; first, 2.19 shall provide a cancellation law, then we will consider some infinite distributive laws.

2.18. Lemma. (i) *If* $p \cdot q \equiv r \cdot q$ *then* $p \equiv r$.

(ii) *If* $p \cdot q \equiv r \cdot s$ *and* q *is Rudin-Frolik minimal then either* $s \equiv q$ *and* $p \equiv r$, *or there is an ultrafilter* t *such that* $p \equiv r \cdot t$ *and* $s \equiv t \cdot q$.

(iii) *If* $p \cdot q \equiv r \cdot s$ *and* p *is Rudin-Frolik minimal then either* $s \equiv q$ *and* $p \equiv r$, *or there is an ultrafilter* t *such that* $r \equiv p \cdot t$ *and* $q \equiv t \cdot s$.

Proof: Each of the parts of this lemma follows easily from 1.11 and 2.4 (ii).

2.19. Corollary. *If* $p \cdot q \equiv p \cdot s$ *and* p *is Rudin-Frolik minimal then* $q \equiv s$.

Proof: By 2.18 (iii), either $q \equiv s$ or there is a t with $p \equiv p \cdot t$. Thus $p > t$, so that either $p \equiv 1$, in which case $q \equiv s$, or $t \equiv 1$, in which case $q \equiv s$ since $q \equiv t \cdot s$.

There are several ways in which one might try and extend the distributive law for ultrafilter products to cover the case of an infinite sequence of ultrafilters; some of these extend rules hold, others do not. In 2.21 one can find a distributive law and an application for it and 2.22 is devoted to a useful counterexample.

2.20. Lemma. $\Sigma(X,p) < \Sigma(Y,p)$ *if and only if* $\{n : X(n) < Y(n)\} \in p$.

Theorem 2.21 below is due to K.Kunen.

2.21. Theorem. (i) *If* Z *is a discrete sequence with* $Z(n) \equiv \Sigma(X, Y(n))$ *then* $\Sigma(X, \Sigma(Y, q)) \equiv \Sigma(Z, q)$.

(ii) *Let* RF *be the Rudin-Frolik ordering. The ordering of the Rudin-Frolik successors of* p *is isomorphic to* $\mathrm{Prod}(p, RF)$.

Proof: Part (i) follows directly from 2.20. For part (ii), one sets $\Psi([X]) = \Sigma(Y, p)$ where Y is the first discrete sequence, in some fixed well ordering, such that $X(n) \equiv Y(n)$. Using 2.20 one sees that Ψ is a monomorphism of the desired sort.

2.22. Lemma. *If* Y *is a discrete sequence of ultrafilters with* $Y(n) \equiv p^{n+2}$, *then* $\Sigma(Y, p) \equiv p^{\omega+1}$.

Proof: Let X be a discrete sequence with $X(n) \equiv p^{n+1}$ and $\Sigma(X, p) = p^{\omega}$.

Using 2.20, one finds that $p^{\omega} < \Sigma(Y, p)$ so that there is a discrete sequence Z such that $\Sigma(Y, p) = \Sigma(Z, p^{\omega})$. Now by 1.11, there is a set $a = \{n : Y(n) \in \overline{\operatorname{ran}(X)}\} \in p$. Choose a set of subsequences X_n of X such that $Y(n) \in \overline{\operatorname{ran}(X_n)}$ when $n \in a$, and such that there are mutually disjoint neighborhoods in $\beta\mathbf{N}$ each containing the range of exactly one sequence X_n. Let T be a discrete sequence with $T(n) \equiv \Omega(X_n, Y(n))$ for $n \in a$. It is easily seen that $\Sigma(T, p) \equiv p^{\omega}$ so that $\{n : T(n) \equiv p^{n+1}\} \in p$. This means that $\{n : \{m : X_n(m) \equiv p\} \in p^{n+1}\} \in p$ so that one can define a subsequence X_n' of X_n such that for each n, $X_n'(m) \equiv p$ and $\{n : \Sigma(X_m', p^{m+1}) = Y(m)\} \in p$. Let X' be a discrete sequence having each X_n' as a subsequence. It is clear that $\Sigma(X', p^{\omega}) = \Sigma(Y, p)$, but $\Sigma(X', p^{\omega}) \equiv p \cdot p^{\omega} \equiv p^{\omega+1}$.

2.23. Corollary. *If p is non-principal then $p^{\omega+1} = p \cdot p^{\omega} < p^{\omega} \cdot p$.*

This corollary shows that the semigroup of ultrafilters has a trivial center. It also provides a counterexample to the following erroneous distributive law: $\Sigma(X \cdot p, q) \equiv \Sigma(X, p \cdot q)$. If we choose X so that $X(n) \equiv p^{n+1}$ then $\Sigma(X \cdot p, q) \equiv p^{\omega+1}$ while $\Sigma(X, p^2) \equiv \Sigma(X, p) \cdot p$, by 2.21 (i). In this instance, $\Sigma(X, p^2) \equiv p^{\omega} \cdot p$ which is not equal to $p^{\omega+1}$ by 2.23.

§ 3. The Rudin-Keisler ordering

There is a natural ordering of the ultrafilters which includes the Rudin-Frolik tree as a suborder; one says that p is less than q if it is a quotient of q under some mapping of the natural numbers. This idea commonly appears in work on ultrafilters – and on filters and measures as well – but the first uses, which are known to me, of the properties of this ordering are those of M.E.Rudin and H.J.Keisler. M.E.Rudin noticed that this ordering could be used to establish Frolik's theorem on the non-homogeneity of $\beta N - N$ in place of the Rudin-Frolik ordering. H.J.Keisler used it to provide some counterexamples to statements in model theory obtained by abandoning a requirement, in certain theorems of model theory, that the language involved by countable – some of this appears in [4].

3.1. Definition. (i) If $f \in {}^{\omega}\omega$, p and q are ultrafilters, then $f(p) = q$ whenever $q = \{ f^{-1}(a) : a \in p \}$.

(ii) One says that $p \leq_0 q$ if there is an f such that $f(q) = p$, and $p <_0 q$ if $p \leq_0 q$ while $p \not\equiv q$.

(iii) If $p \leq_0 q$ then one also says that $\underset{\sim}{p} \leq_0 \underset{\sim}{q}$, similarly $\underset{\sim}{p} <_0 \underset{\sim}{q}$ whenever $p <_0 q$.

3.2. Lemma. (i) *If* $\underset{\sim}{p} < \underset{\sim}{q}$, *then* $\underset{\sim}{p} <_0 \underset{\sim}{q}$.

(ii) $\{ \underset{\sim}{q} : \underset{\sim}{q} <_0 \underset{\sim}{p} \}$ *has power at most* c.

(iii) $\{ \underset{\sim}{q} : \underset{\sim}{p} <_0 \underset{\sim}{q} \}$ *has power* 2^c.

(iv) *If* $\underset{\sim}{p} \leq_0 \underset{\sim}{q} \leq_0 \underset{\sim}{r}$, *then* $\underset{\sim}{p} \leq_0 \underset{\sim}{r}$.

(v) *The ordering* $<_0$ *is well defined on the* $\equiv$ *equivalence classes of ultrafilters.*

Let $\mathfrak{S} = \langle \omega, b \rangle_{b \subseteq \omega}$. Keisler had considered $<_0$ in the following form (which is equivalent to $f(p) = q$): $q = \{ a \subseteq \omega : \mathrm{Prod}(p, \mathfrak{S}) \models [f] \in a \}$. This says, roughly, that q is a non-standard principal ultrafilter in the model determined by p. The next theorem has been independently proved by many people including K.Kunen, H.J.Keisler, and M.E.Rudin. It is stated here for ultrafilters on a countable set but is really quite general.

3.3. Theorem. *If $f(p) = p$, then $\{n : f(n) = n\} \in p$.*

Proof: Let $S = \{n : f(n) = n\}$, $R = \{n : f(n) > n\}$, and $T = \{n : f(n) < n\}$; one can prove the theorem by showing that $R \cup T \notin p$.

Suppose that $T \in p$, let $T_n = \{m : n$ is the least integer such that $f^n(m) \notin T\}$. Here 'f^n' means the n-fold iterate of f, so that $\bigcup\{T_n : n > 0\} = T$. Each of the disjoint sets $\bigcup\{T_{2n} : n \in \omega\}$ and $\bigcup\{T_{2n+1} : n \in \omega\}$ can be in p only if the other is as well, this gives a contradiction.

Suppose $R_n = \{m : n$ is the least integer such that $f^n(m) \notin R\}$. Just as in the previous case one can see that neither $\bigcup\{R_{2n} : n \in \omega\}$ nor $\bigcup\{R_{2n+1} : n \in \omega\}$ can be in p. The set $\omega - \bigcup\{R_n : n \in \omega\}$ can be partitioned into two pieces, in the same manner, such that when one is in p the other is in $f(p) = p$, thus $R \notin p$.

3.4. Corollary. (i) *If $p \leq_0 q \leq_0 p$, then $p \equiv q$.*

(ii) *The relation $<_0$ partially orders the $\equiv$ equivalence classes of ultrafilters.*

The next theorem is due to H.J.Keisler; it shows that the present ordering is not a tree.

3.5. Theorem. *Both of the following hold $p \leq_0 p \cdot q$ and $q \leq_0 p \cdot q$; if q is non-principal, then $p <_0 p \cdot q$, while if p is non-principal, $q <_0 p \cdot q$.*

Proof: That $q \leq_0 p \cdot q$ and that strict less than obtains when q is non-principal follows from 3.2 (i). The function $f = \lambda\langle m, n\rangle n$ sends $p \cdot q$ onto p, so that $p \leq_0 p \cdot q$. If $f(p \cdot q) \equiv p$ then there is a permutation g of ω such that $gf(p \cdot q) = p$ and thus, by 3.3, f is identical with g on an element of p. This clearly can never happen when p is non-principal.

3.6. Theorem. *If no non-principal ultrafilter has less than c generators then the $<_0$ ordering has at least c mutually incomparable elements.*

Proof: We shall define two incomparable ultrafilters by stages; it shall be clear how to obtain c of them. For each $\alpha \in c$ there are filters Q_α and P_α such that if p and q are ultrafilters which extend them, they are the desired ultrafilters.

Let $\langle f_\alpha : \alpha \in c \rangle$ be a well ordering of ${}^\omega\omega$, suppose that P_β and Q_β have been defined for $\beta \in \alpha \in c$, and that each P_β and Q_β are proper filters having less than c generators and containing all complements of finite sets. Suppose, in addition, that if p extends P_β and q extends Q_β then $f_\beta(p) \neq q$ and $f_\beta(q) \neq p$. If $\alpha = \beta + 1$ let $P = P_\beta$ and $Q = Q_\beta$, and if α is a limit ordinal let P be the filter generated by the complements of finite sets along with $\cup \{P_\beta : \beta \in \alpha\}$. Define Q in the corresponding manner. Both P and Q have less than c generators and hence are not ultrafilters. Consider a set $a \subseteq \omega$ such that $a \in P$ and $\omega - a \in P$; if $f_\alpha(a) \in Q$ then let P' be the filter generated by P and $\{\omega - a\}$ If $f(\omega - a) \in Q$ let P' be the filter generated by $\{a\}$ and P. If neither a nor $\omega - a$ are in Q let Q' be generated by $[a]$ along with Q and P' by $\{\omega - a\}$ along with P.

Finally, consider a set $b \subseteq \omega$ such that $b \notin Q'$ and $\omega - b \notin Q'$. One carries out the above construction again beginning with P' and Q' and reversing the roles of P and Q. In this way one obtains Q'' and P''; finally let $Q'' = Q_\alpha$ and $P'' = P_\alpha$.

It follows from the above theorem that assuming its hypothesis there must be Rudin-Frolik incomparable ultrafilters. The hypothesis is often true; it holds in any model of set theory obtained by adding enough Cohen reals or Solovay reals (random reals) so that the CH fails in the resulting extension; it is also a consequence of Martin's axiom. We shall see below that it holds if c is real-valued measurable.

Let h be a correspondence between $s(\omega)$ and ${}^\omega 2$ which assigns to each set its characteristic function, and let d be the map from ${}^\omega 2$ to $[0, 1]$ which assigns to each function the real for which it represents a binary expansion. In this way we obtain a subset, $dh(p)$, of the interval starting with an ultrafilter p. If $a \subseteq \omega$ let $F(a)$ be the filter generated by a and $I(a)$ the ideal which is generated by a. It is readily seen that for any infinite-coinfinite set a, $m(d \cdot h(I(a))) = m(d \cdot h(F(a))) = 0$, where m is Lebesgue measure. The following theorem is an immediate consequence of this fact.

3.7. Theorem. *If there is an extension μ of Lebesgue mwasure which is finitely additive and such that $\mu(\cup \{ A_\alpha : \alpha \in \kappa \}) = 0$ whenever each $A_\alpha \subseteq [0, 1]$ and $\mu(A_\alpha) = 0$, then no ultrafilter on ω has less than κ^+ generators.*

§ 4. Minimal ultrafilters

Z.Frolik introduced the Rudin-Frolik ordering – the 'producing relation' of [3] -- to show that $\beta N - N$ is not homogeneous; W.Rudin has previously obtained the same result, assuming the continuum hypothesis, by showing that $\beta N - N$ contains both p-points and points which are not p-points and that the property of being a p-point is a topological invariant. The p-points, which we shall soon define, are minimal in the Rudin-Frolik ordering; they are again assuming the continuum hypothesis, mutually interchangeable by a homeomorphism of $\beta N - N$ onto itself – a fact which appeared in the paper [7] of W.Rudin.

It is convenient here to use the notion of a generic set in a partial ordering which has been developed by R.Solovay. The definition of a generic set is a direct descendant of the 'generic' sets of P.Cohen [2] which were intended to be ideal sets of natural numbers whose semantical properties made them quite different from the common real.

4.1. Definition. A dense subset D of a partial ordering $\mathcal{P} = \langle P, \leq \rangle$ is a subset of P such that any element of P is greater than an element of D. If $\mathcal{D}$ is a collection of dense subsets of $\mathcal{P}$ and $G \subseteq P$, then G is said to be $\mathcal{D}$ generic for $\mathcal{P}$ if the following hold:

(i) if $x, y \in G$, there is a $z \in G$ such that $z \leq x$ and $z \leq y$;
(ii) if $x \in G$ and $x \leq y$, then $y \in G$;
(iii) for each $D \in \mathcal{D}$, $D \cap G \neq 0$.

4.2. Definition. $MA(\aleph_\alpha)$ is the proposition: for each partial ordering $\mathcal{P}$ having no uncountable set of mutually incomparable elements and for each collection $\mathcal{D}$ of less than $\aleph_\alpha = c$ dense subsets of $\mathcal{P}$, there is a set which is $\mathcal{D}$ generic for $\mathcal{P}$. MA is the porposition: $\exists \alpha \geq 1 \; MA(\aleph_\alpha)$.

The proposition MA is known as 'Martin's axiom'; it was formulated, as $MA(\aleph_2)$, by A.Martin and shown consistent with ZF by him. The next lemma is essentially a theorem of Rasiowa and Sikorski – their result can be found in [5], page 87 – but stated here for κ rather than ω.

4.3. Lemma. *If $\mathcal{P} = \langle P, \leq \rangle$ is a partial ordering such that each descending* *chain in $\mathcal{P}$ of length less than κ has a lower bound, where κ is regular, and if $\mathcal{D}$ is a set of at most κ dense subsets of $\mathcal{P}$, then there is a $\mathcal{D}$ generic set for $\mathcal{P}$.*

4.4. Corollary. *The theories ZFC & CH and ZFC & MA$(\aleph_1)$ are identical.*

4.5. Theorem. *ZFC & MA $\vdash$ Let F be the set of all filters on ω generated by less than c elements and which contain no finite sets, let $\mathcal{F} = \langle F, \supseteq \rangle$, then any decreasing chain of elements of F of length less than c has a lower bound in $\mathcal{F}$.*

Proof: Suppose that $\langle F_\alpha : \alpha \in \kappa \in c\rangle$ is a descending chain in $\mathcal{F}$ and that $\{a_\alpha \subseteq \omega : \alpha \in \kappa\}$ constitutes a set of generators of all the F_α, $\alpha \in \kappa$. Form the partial ordering $\langle P, \leq\rangle$ consisting of elements

$$\langle s, t\rangle \in \cup \{{}^n 2 : n \in \omega\} \times [\kappa]^{<\omega} ;$$

two such elements bear the relation $\langle s_0, t_0\rangle \leq \langle s_1, t_1\rangle$ when $s_0 \supseteq s_1$, $t_0 \supseteq t_1$, and $s_0^{-1}(1) - a_\alpha = s_1^{-1}(1)$ whenever $\alpha \in t_1$. Each of the sets $A_\alpha = \{\langle s, t\rangle : \alpha \in t\}$ and $B_n = \{\langle s, t\rangle : n \in \operatorname{dom}(s)\}$ is dense in P. It is easily seen that P has no uncountable set of mutually incompatible elements, so, using MA, one may assume that there is a set G which is generic for these dense sets. The set $b = \{s^{-1}(1) : s \in G\}$ generates a filter which is a lower bound for the chain $\langle F_\alpha\rangle$.

4.6. Definition. An ultrafilter p is a p-point if for each partition $\{c_n : n \in \omega\}$ of ω with $c_n \notin p$, there is an $a \in p$ such that for each n, $a \cap c_n$ is finite.

4.7. Theorem. *The following are equivalent properties of a non-principal ultrafilter p:*

(i) *p is a p-point;*
(ii) *if $\{c_n \subseteq \omega : n \in \omega\}$ are elements of p, then there is an $a \in p$ such that for each n, $a \cap (\omega - c_n)$ is finite;*
(iii) *if U_n are open sets in $\beta\mathrm{N}$ with $p \in \bigcap \{U_n : n \in \omega\}$, then $p \in \operatorname{Int}(\bigcap \{U_n : n \in \omega\})$;*

(iv) *if $\langle\omega, R\rangle$ is a linear ordering then there is an $a \in p$ such that $\langle a, R \cap a^2\rangle$ has order type ω or ω^*;*

(v) *each sequence $\langle x_n\rangle$ in $D^{\aleph_0}$ (D is the Hausdorff space of two points) has a convergent subsequence $\langle x_n : n \in a\rangle$ where $a \in p$.*

Proof: Properties (i) and (ii) are clearly equivalent and they are equivalent to (iii) by translating them into the language of the topology of $\beta\mathbf{N}$.

To show that (i) implies (iv) one observes that any subinterval I of $\langle\omega, R\rangle$ with $I \in p$ has a cofinal subset and this set determines a decomposition of I into subintervals. If none of these subintervals were an element of p, then there would be infinitely many of them; so, by (i), there would be an increasing sequence in R of type ω. On the other hand, suppose $\langle\omega, R\rangle$ were decomposed so that some subinterval $\langle a_0, R \cap a_0^2\rangle$ were an element of p, one would then decompose a_0 into intervals. If no such subinterval were in p, then we would be finished, because R would have an increasing sequence, otherwise there would be a subinterval $\langle a_1, R \cap a_1^2\rangle$, $a_1 \subseteq a_0$, and $a_1 \in p$. Either we would finish the argument in finitely many steps or would obtain a descending sequence, $a_0 \supseteq a_1 \supseteq a_2 \supseteq \dots$, of elements of p. In this case there would be a partition $\{\omega - a_0, a_0 - a_1, a_1 - a_2, \dots\}$ which using (i), would provide an element of p of type ω, ω^*, or $\omega + \omega^*$, and therefore of type ω or ω^*.

For (iv) implies (v) one observes that $D^{\aleph_0}$ is the Cantor set and that, under its lexicographic ordering, any ω or ω^* sequence will converge.

To show that (v) implies (i) one lets $\{a_n : n \in \omega\}$ be a partition of ω and sets $x_n(m) = 1$ if $n \notin a_0 \cup \dots \cup a_n$ and $x_n(m) = 0$ otherwise. If $\langle x_n : n \in b\rangle$ converges, then either $b \cap (a_0 \cup \dots \cup a_n)$ is finite or $b \cap (\omega - (a_0 \cup \dots \cup a_n))$ is finite, the latter contradicts the supposition that $b \in p$ while $a_i \notin p$.

4.8. Definition. An ultrafilter p on ω is Ramsey if for each n and $m \in \omega$, and each $f : [\omega]^n \to m$ there is an $a \subseteq \omega$ such that f is constant on $[a]^n$; in the circumstances one says that f is homogeneous on a.

Most of the next theorem is due to K.Kunen; he noticed that (i), (ii), and (v) are equivalent and his proof that (iv) implies (v) is used here to show that they are equivalent to (ii) and (iii).

4.9. Theorem. *The following are equivalent properties of a non-principal ultrafilter p:*

(i) *p is Ramsey;*

(ii) *if $R \subseteq \omega^2$ such that for each n, $\{m : nRm\} \in p$ then there is an $a \in p$ such that $a = \{k_n : n \in \omega\}$, $k_n < k_{n+1}$, and kRk_{n+1} for each n;*

(iii) *if $\langle \omega, T \rangle$ is a tree then there is an $a \in p$ which is either a chain or an antichain;*

(iv) *if $\{c_n : n \in \omega\}$ is a partition of ω, $c_n \notin p$, then there is an $a \in p$ such that for each n, $a \cap c_n$ has at most one element;*

(v) *if for each n, $a_n \in p$, then there is a function g such that $g(n+1) \in a_{g(n)}$, $g(n) < g(n+1)$, and* $\mathrm{ran}(g) \in p$.

Proof: We shall show that (i) → (iv) → (v) → (i); having done this one can show that (ii) and (iii) are equivalent to the others by showing that (i) → (ii) → (iv) and (i) → (iii) → (iv).

To see that (i) → (ii) one defines a function $f: [\omega]^2 \to 2$ by $f(\{n, m\}) = 1$ exactly when nRm, where $n < m$. By hypothesis f must be homogeneously equal to one on a set in p, such a set satisfies the conclusion of (ii). To see that (ii) → (iv) one begins with a partition $\{c_n\}$ of ω, where $c_n \notin p$, and then one defines a relation kRl which holds exactly when $k \in c_n$, $l \in c_m$, and $n < m$.

The implications (i) → (iii) → (iv) are easily established. One proves (i) → (iii) by defining $f(\{m, n\}) = 1$ exactly when mTn; a homogeneous set for f gives either a chain or an antichain. To show that (iii) → (iv) one considers a tree having countably many branches joined at their base and each consisting of the elements of one piece of the partition $\{c_n\}$. There can be no chains lying in p and any antichain will serve to obtain the conclusion of (iv).

The implications (iv) → (v) → (i) still remain. To show that (iv) → (v), let $a_n \in p$ and suppose that $a_{n+1} \subseteq a_n$. Since p is a p-point it contains a set b such that $b - a_n$ is finite for each n. Say that $b = \{k_n : n \in \omega\}$ with $k_n < k_{n+1}$. Let $f(m)$ be the first n such $b - a_m \subseteq k_n$. Now let $A_0 = \omega - b$, $A_{k+1} = \{l \in b : f^k(0) < l \leq f^{k+1}(0)\}$ where f^0 is the function which is constantly zero, f^1 is f, and f^{k+1} is $f \cdot f^k$. By (iv) there is an $a \in p$ such that $a \cap A_i$ has exactly one element for each $i \in \omega$. If $\{l_i : i \in \omega\}$ is an enumeration of the distinct elements of

$\{a \cap A_i : i \in \omega\}$ in increasing order and $g(n) = l_{2n}$, then g has the properties required in (v).

The proof that (v) → (i) can be closely patterned upon a popular proof of Ramsey's theorem.

4.10. Theorem. *ZFC & MA* $\vdash$ *Let* $\{c_n : n \in \omega\}$ *be a partition of* ω, *and* $c'_n = \bigcup \{c_m : m > n\}$; *suppose that* $F = \{\omega - c : \forall n(\overline{\overline{c \cap c_n}} \leq 1)\} \cup \bigcup \{e : \omega - e \text{ is finite}\}$. *If G is a filter having less than c generators* $[a_\alpha]$, *and* $F \cup G \cup \{c'_n : n \in \omega\} \cup \{b\}$ *has the finite intersection property, then there is a* $d \subseteq \omega$ *such that for each* α, $d - a_\alpha$ *is finite, and* $\{c'_n : n \in \omega\} \cup F \cup \{b, d\}$ *has the finite intersection property.*

Proof: We first define a partial ordering $\langle P, \leq \rangle$ in which a typical element has the form

$$t = \langle\langle\langle s_0, l(0)\rangle, \ldots, \langle s_n, l(n)\rangle\rangle, \{\alpha_0, \ldots, \alpha_k\}\rangle,$$

where $n, k, l(i) \in \omega$, $s_i \in [\omega]^{<\omega}$, and $\alpha_i \in \kappa$ where G has κ generators $[a_\alpha : \alpha \in \kappa]$. Furthermore, we require that $s_i \subseteq c_{l(i)}$, $\bar{\bar{s}}_{i+1} > l(i)$ and $l(i+1) > l(i)$. If t' is another element of P having the same form as t except written with primes one sets $t' \leq t$ if $n' \geq n$, $k' \geq k$; each $\langle s_i, l(i)\rangle$ appearing in t appears in t'; each α_i appearing in t appears in t'; and for each s'_i appearing in t' but not in t and each α in t, $s'_i \subseteq a_\alpha$.

We shall now define some dense subsets of P, $A_\alpha = \{t \in P : \alpha \text{ appears in } t\}$ and $B_l = \{t \in P : \text{for some } \langle s_i, l(i)\rangle \text{ appearing in } t, l(i) > l\}$. It is obvious that A_α is dense because one can simply add an α to any element of P in which it does not appear and thus obtain a lesser element of P. To see that B_l is dense one first establishes the following fact: for each k, $m \in \omega$ and $\alpha \in \kappa$ there is an $l \geq k$ such that $\overline{\overline{b \cap c_l \cap a_\alpha}} > m$. If this were not the case one could choose $\{r_0^l, \ldots, r_{m-1}^l\}$ to be a set of m distinct elements of c_l which includes $b \cap c_l \cap a_\alpha$. Taking $r_i = \{r_i^l : l \geq k\}$ we should have that $b \cap c_l \cap a_\alpha \subseteq r_0 \cup \cdots \cup r_{m-1}$. This means that $b \cap c'_l \cap a_\alpha \cap (\omega - r_{m-1}) = 0$ which would contradict our hypothesis concerning the finite intersection property. It is easily seen that P has no uncountable set of mutually incompatable elements.

Using MA we may now select a G which is generic for P and intersects each A_α and each B_l; the 'd' of the theorem is $d = \bigcup \{s_i : \text{for some } t \in P$

and some l, $\langle s_i, l\rangle$ appears in $t\}$. The fact that d intersects A_α insures that $d - a_\alpha$ is finite; the fact d intersects the sets B_l insures that $\{c'_n : n \in \omega\} \cup F \cup \{b, d\}$ has the finite intersection property.

If G were only countably generated then 4.10 would be provable without any axioms other than those of ZFC.

4.11. Definition. (i) A partition $\{c_n : n \in \omega\}$ of ω is unbounded if $\{\bar{c}_n : n \in \omega\}$ is unbounded.

(ii) An ultrafilter p is selective for the partition $\{c_n : n \in \omega\}$ of ω if there is an $a \in p$ such that for each n, $\overline{\overline{a \cap c_n}} \leq 1$.

It has been shown above that the ultrafilters which are selective for every partition are exactly the Ramsey ultrafilters, thus the next theorem implies that there are p-points which are not Ramsey.

4.12. Theorem. ZFC & $MA \vdash$ *If* $\{c_n : n \in \omega\}$ *is an unbounded partition of* ω, *then there exists a p-point which is not selective for it.*

Proof: Let $F = \{\omega - c : \forall n(\overline{\overline{c \cap c_n}} \leq 1)\} \cup \{e : \omega - e \text{ is finite}\}$ as in 4.10; let $c'_n = \bigcup \{c_m : m > n\}$. Give a well ordering $\langle A_\alpha : \alpha \in c\rangle$ of all partitions of α we shall define a sequence $\langle F_\alpha : \alpha \in c\rangle$ of filters having the following properties:

(i) F_0 is the filter generated by F;

(ii) each F_α can be generated by $F \cup G_\alpha$, where G_α has power less than c;

(iii) if δ is a positive limit ordinal, then $F_\delta = \bigcup \{F_\alpha : \alpha \in \delta\}$;

(iv) if $A_\beta = [a_n^\beta : n \in \omega]$, either there is an n with $a_n^\beta \in F_{\beta+1}$ or for each n, $d \cap a_n^\beta$ is finite.

Once a sequence $\langle F_\alpha\rangle$ having these properties has been defined one can extend F_c to an ultrafilter p which would have to be a p-point, on account of (iv), and could not be selective for $\{c_n\}$ because it includes F.

Condition (iii) describes the manner in which F_δ is defined for limit ordinals; we will now define $F_{\beta+1}$. Call the partition $A_{\beta+1}$ '$\{a_n\}$', rather than '$\{a_n^\beta\}$', as before $a'_n = \bigcup\{a_m : m > n\}$. If $F_\beta \cup \{a'_n : n \in \omega\}$ lacks the finite intersection property, then there is an i such that $F_\beta \cup \{a_i\}$

has the finite intersection property; let $F_{\beta+1}$, in this case, be the filter generated by $F_\beta \cup \{a_i\}$.

We may suppose, then, that $F_\beta \cup \{a'_n : n \in \omega\}$ has the finite intersection property. One may, by (ii), choose a set G of $\kappa < \mathfrak{c}$ generators for F_β beyond those of F_0. By 4.10 there is a set d such that $d - a$ is finite for each $a \in G$, such that the filter generated by $F_0 \cup \{d\}$ includes F_β, and such that $F_0 \cup \{d\} \cup \{a'_n : n \in \omega\}$ has the finite intersection property. Again by 4.10 we can extend this set to $F_0 \cup \{d\} \cup$ $\cup \{a'_n : n \in \omega\} \cup \{y\}$ which also has the finite intersection property and such that $y \cap a_n$ is finite for each n. Let F_β be the filter generated by $F_0 \cup \{d\} \cup \{a'_n : n \in \omega\} \cup \{y\}$; this shows how to complete the construction of the filters F_α

4.13. Corollary. *ZFC & MA* $\vdash$ *There exists a dense set in* βN *of p-points which are not Ramsey.*

The existence of Ramsey ultrafilters was first shown, assuming the *CH*, by F.Galvin.

4.14. Theorem. *ZFC & MA* $\vdash$ *There exists in* βN *a dense set of Ramsey ultrafilters.*

Proof: Let $f : [\omega]^n \to m$ and let $A_f = \{F \in \mathcal{F} : \exists a \in F$ (f is homogeneous on a)$\}$ – here we are using the notation of 4.5. Let $B_a =$ $= \{F \in \mathcal{F} : a \in F$ or $(\omega - a) \in F\}$; the sets B_a are dense in $\mathcal{F}$ and, using Ramsey's theorem, one can see that the sets A_f are dense too. By 4.3 and 4.5 there is a C which is generic for these dense sets; $\cup G$ is a Ramsey ultrafilter. To show that they are dense one just relativizes the construction to an open set.

The next theorem is an obvious consequence of 4.7 and is stated here just to show the relationship between this section and part 2 of this paper.

4.15. Theorem. *If p is a p-point, then* $\underset{\sim}{p}$ *is Rudin-Frolik minimal.*

4.16. Definition. (i) The $P(\kappa)$ ultrafilters are those ultrafilters p such that for every sequence $\langle a_\alpha : \alpha \in \lambda \in \kappa \rangle$ there is a $b \in p$ such that for each $\alpha \in \lambda$, $b - a_\alpha$ is finite.

The $P(\aleph_1)$ ultrafilters are exactly the p-points; MA implies that there exist $P(c)$ ultrafilters. This last fact can be shown by adapting the construction of [7] to the combinatorical principle of 4.5.

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