Concerning a Quantum-like Uncertainty Relation for Pairs of Complementary Fuzzy Sets

MIECZYSLAW ALBERT KAAZ*

Veren Deutscher Ingenieur,
Duesseldorfs, Federal Republic of Germany

Submitted by L. Zadeh

It appears far more natural and rewarding to consider "fuzziness" spread over the category of closed sets and semicontinuous set-valued mappings than over a set of points and point-to-point functions considered hitherto; for, none will argue that we actually "see" points (and not sets and singletons) when we study fuzzy sets and systems. This change of attitude merely necessitates an extension of the membership function to set-valued mappings and—with regard to dynamics—a transcription of ordinary differential equations to differential inclusions (i.e., orientor field equations in the case of control problems). However, the proper perspective now requires topology in the spaces under consideration and a shift from Boolean algebra to Brouwerian algebra since fuzziness disobeys some of the fundamental laws, notably tertium-non-datur and transitivity. Also, in faithful adherence to intuitionistic thinking, we have to consider the cardinality of sets as being, at the most, potentially infinite, never actually infinite. This emphasizes the constructivity as well as the subjectivity aspects of any fuzzy set theory. Both hypotheses (concerning topology and logic), taken jointly, already program (in the limit) a fundamental uncertainty relation in fuzzy set theory, very much like the celebrated Heisenberg uncertainty relation in quantum mechanics, if we make the fuzzy quantum (the greatest negligible set) and the physical quantum correspond to one another. These ideas will be introduced into the formalism of orientor field control in a companion paper to follow. © 1987 Academic Press, Inc.

I. REVIEW OF RELEVANT QUANTUM MECHANICAL PRINCIPLES

In his critical study of fuzzy reasoning Lee [22] came to the conclusion that the problem of uncertainty in fuzzy logic is not well taken care of. As far as we know, this statement still holds today, over ten years later.

Now, the deepest probed and best understood concept of uncertainty is undoubtedly that associated with the well-known Heisenberg uncertainty relations. Because of their relevance to the results of this paper, we shall take a closer look at the foundations of quantum mechanics and the

* Present address: Am Golfplatz 50, D-4100 Duisburg 29 Federal Republic of Germany.
inherent uncertainty notion. Recent results in this discipline establish an overall unified and technically cogent development of the theory of quantum mechanical systems from a completely geometrical point of view. In tune with this trend, we say today that every irreducible complemented modular lattice \( L \) of finite rank is a geometry if and only if every line in \( L \) has at least three points lying on it. Thus, quantum mechanical systems exhibit logics which form some sort of projective geometries (cf. [14] and [38]), and which are consequently nondistributive, using the weaker law of modular identity instead. Moreover, quantum logic has to be an orthocomplementary lattice because in modular nondistributive lattices mere complementary operations fail to be one-to-one operations.

When quantum logic is represented as the lattice of subspaces of a Hilbert space, the physical quantities, or observables, are represented by selfadjoint linear operators. It is these operators which play the role of "random variables" in quantum theory, in contradistinction to classical probability theory, where the random variables are just real-valued functions on the space of possible outcomes which are measurable with respect to a Boolean \( \sigma \)-algebra of subsets which can—in an obvious way—be thought of as a Boolean \( \sigma \)-algebra of propositions. If the quantum logic of subspaces of a Hilbert space is represented as a logical \( \sigma \)-structure, certain equivalence classes of the set of random variables on the component Boolean \( \sigma \)-algebras determine, in a natural way, linear operators on a Hilbert space. Thus, physical quantities of the quantum theory are, in a sense, just equivalence classes of the classical theory, each one of them reflecting a physical quality of some sort.

Since there is no implication operation in quantum logic (comparable to the Boolean operation \( \neg a \cup b \), see [15]), use is made of a relation implication of partial order type. Thus, partial order and orthocomplementation play in quantum mechanics the part of implication and negation, respectively. Indeed, let \( L \) be a \( \sigma \)-lattice with elements \( \sqcap, \sqcup \) (first) and \( \sqcap, \sqcap \) (last) and an orthocomplementation \( \perp: a \mapsto a^\perp; a, a^\perp \in L \), which satisfies the following axioms:

1. \( (a^\perp)^\perp = a, a \in L \),
2. \( (a \leq b) \Rightarrow (b^\perp \leq a^\perp), a, b \in L \),
3. \( a \lor a^\perp = \sqcup, a \in L \).

**Definition 1.1.** A \( \sigma \)-lattice \( L \) is called a logic if, in addition to (i)–(iii), it also satisfies the implication

\[ (a \leq b; a, b \in L) \Rightarrow (b = a \lor (a^\perp \land b)) \].

**Definition 1.2.** An observable is said to be a map \( \mu \), carrying a Borel set \( E \in \mathcal{B}(\mathbb{R}) \) from \( \mathbb{R} \) into \( L \) and such that
(i) \( \mu_x(\emptyset) = 0 \),
(ii) \( \mu_x(E) = \mu_x(F) \) whenever \( E \cap F = \emptyset \).
(iii) \( \mu_x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} \mu_x(E_i) \) if \( E_i \cap E_j = \emptyset, \ i \neq j \), \( \{ E_i \} \subset B(\mathbb{R}) \).

**Definition 1.3.** A state is said to be a map \( p_\phi \) from \( L \) into \( \mathbb{R} \cup \{ +\infty \} \cup \{ -\infty \} \) such that

(i) \( p_\phi(\emptyset) = 0 \),
(ii) \( p_\phi(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} p_\phi(a_i), \ a_i \perp a_j, \ i \neq j \), \( \{ a_i \} \subset L \).

In general, \( p_\phi \) may attain the "values" \( +\infty \) or \( -\infty \), in which case \( p_\phi(a) \geq 0 \) is called a positive signed state and \( p_\phi(a) \leq 0 \) a negative signed state. In our restricted terminology, a state shall be a positive signed state characterized by \( p_\phi(\emptyset) = 1 \); the reason being that every \( \infty \)-dimensional Hilbert space is diffeomorphic with its unitsphere (see [5]).

We are now in a position to formulate the four axioms common to classical mechanics and quantum mechanics, using the following symbolism: \( A \) denotes the set of all observables \( \alpha \) of a physical system \( \Sigma \), \( \Phi \) stands for the set of all states \( \varphi \) of \( \Sigma \), and \( p: A \times \Phi \times B(\mathbb{R}) \to [0, 1] \) is a real-valued function.

If both the observable \( \alpha \in A \) and the state \( \varphi \in \Phi \) are fixed, then \( B(\mathbb{R}) \to [0, 1] \), i.e., \( E \mapsto p(\alpha, \varphi, E) \), is a probability measure on \( B(\mathbb{R}) \). The axioms, equally valid in classical mechanics and in quantum mechanics, run as follows:

**A1** \( p(\alpha, \varphi, \emptyset) = 0, \ p(\alpha, \varphi, \mathbb{R}) = 1 \),

\[
p(\alpha, \varphi, E_1, E_2, \ldots) = \sum_{i=1}^{\infty} p(\alpha, \varphi, E_i),
\]

\( \alpha \in A, \ \varphi \in \Phi, \ E_i \in B(\mathbb{R}), \ E_i \cap E_j = \emptyset, \ i \neq j \).

**A2** \( \bigwedge_{\varphi \in \Phi} (p(\alpha, \varphi, E) = p(\alpha', \varphi, E)) \Rightarrow (\alpha = \alpha') \),

\( \bigwedge_{\alpha \in A} (p(\alpha, \varphi, E) = p(\alpha, \varphi', E)) \Rightarrow (\varphi = \varphi') \).

**A3** \( (\alpha_1, \alpha_2, \ldots, \in A) \land (E_1, E_2, \ldots, \in B(\mathbb{R})) \)

\[
\bigwedge_{\alpha \in A} \bigvee_{i \neq j} \left[ (p(\alpha_i, \varphi, E_i) + p(\alpha_j, \varphi, E_j) \leq 1) \right]
\Rightarrow \bigwedge_{\varphi \in \Phi} \bigvee_{\beta \in A} \bigvee_{F \in B(\mathbb{R})} \left[ p(\beta, \varphi, F) \right]
\]

\[
= \sum_{i=1}^{\infty} p(\alpha_i, \varphi, E_i)
\]
(A4) Let $\mu: \mathcal{B}(\mathbb{R}) \to L$ be an $L$-measure (defined below). There exists an observable $x \in A$ such that $\mu = \mu_x$, i.e., a unique $L$-measure $\mu_x$ assigned to every observable $x \in A$.

The above axioms have the following interpretations:

(A1) $p(\alpha, \varphi, E)$ is the probability that a measurement of an operator (observable) $x$ for a system $\Sigma$ in state $\varphi$ yields a result belonging to $E$, a member of the smallest Boolean $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ of Borel sets on $\mathbb{R}$.

(A2) states that the only way to distinguish between any two observables (resp. any two states) is by experiment: two observables (resp. two states) yielding the same measurement results for all Borel sets and all states (resp. observables) are considered to be identical.

(A3) and (A4) will be appreciated as necessary postulates from the following arguments: Let us, first, form the cartesian product $\mathcal{E} = A \times \mathcal{B}(\mathbb{R})$ and call its elements $(\alpha, E), (\beta, F), \ldots$, experimental sentences. By the introduction in $\mathcal{E}$ of an equivalence relation $\approx$, defined by

$$(\alpha, E) \approx (\beta, F) \iff p(\alpha, \varphi, E) = p(\beta, \varphi, F) \text{ for each } \varphi \in \Phi,$$

we obtain the quotient space $L = \mathcal{E}/\approx$ with the (previously mentioned) equivalence classes $[(\alpha, E)], [(\beta, F)], \ldots$, as elements displaying various qualities of the system.

The ordering of $L$ is accomplished by the equivalence:

$$[(\alpha, E)] \leq [(\beta, F)] \iff (p(\alpha, \varphi, E) \leq p(\beta, \varphi, F) \text{ for each } \varphi \in \Phi),$$

$\leq$ being the relation of partial order in $L$. We obviously have then

$$[(\alpha, E)]^\perp = [(\alpha, \mathbb{R} - E)],$$

and since $p(\alpha, \varphi, \mathbb{R} - E) = 1 - p(\alpha, \varphi, E)$, the mapping $\perp : L \to L$, i.e., $[(\alpha, F)] \mapsto [(\alpha, F)]^\perp$, is a well-defined orthocomplementation in the partially ordered set $L$. Henceforth, we shall call $L$ the logic of the probability function $p$. Actually, $p$ induces in $\mathcal{E}$ a relation-implication (called $p$-implication by Maczynski [24]) which is defined by

$$(\alpha, E) \Rightarrow_p (\beta, F) \iff (p(\alpha, \varphi, E) \leq p(\beta, \varphi, F)),$$

expressing that the truth of $(\beta, F)$ is more probable than that of $(\alpha, E)$ in every state $\varphi \in \Phi$. The $p$-implication is, of course, reflexive and transitive; we may use it to replace the equivalence relation in $\mathcal{E}$ by the expression:

$$((\alpha, E) \approx (\beta, F))$$

iff

$$[(p(\alpha, \varphi, E) \Rightarrow p(\beta, \varphi, F)) \wedge (p(\beta, \varphi, F) \Rightarrow p(\alpha, \varphi, E))].$$
Now, let $\mu_\alpha : \mathcal{B}(\mathbb{R}) \to L$, i.e., $E \mapsto [(\alpha, E)]$, be an $L$-measure on $\mathcal{B}(\mathbb{R})$ for every $\alpha \in A$ and $p_\varphi : L \to [0, 1]$, $[(\alpha, E)] \mapsto p(\alpha, \varphi, E)$, be a probability measure on $L$. Then (A2) confirms that $\mu_\alpha \neq \mu_\beta$ and $\varphi \neq \psi$ iff $p_\varphi \neq p_\psi$. Hence

$$p(\alpha, \varphi, E) = p_\varphi \mu_\alpha(E),$$

and while the family $\{\mu_\alpha\}_{\alpha \in A}$ exhausts $L$, the family $\{p_\varphi\}_{\varphi \in \Phi}$ is full (i.e., $(p_\varphi(a) \leq p_\varphi(b)) \Rightarrow (a \leq b)$ for every $\varphi \in \Phi$ and $a, b \in L$). Mackey [23] has shown that these two families completely define $A$, $\Phi$, and $p$. This is due to the fact that the properties of $p(\alpha, \varphi, E)$ depend essentially on $L$. Moreover, it is the logic $L$ which decides whether $p(\alpha, \varphi, E)$ describes a system in classical mechanics or in quantum mechanics. This probability satisfies indeed the axioms (A1)–(A3). The elements of the cartesian product $\mathcal{E} = A \times \mathcal{B}(\mathbb{R})$ have appropriately been called experimental sentences, a typical sentence $(\alpha, E)$ stating that "a measurement of the observed quantity $\alpha$ yields a result in $E$." Sentences entering the same equivalence class represent jointly a physical quality of some sort. However, no certainty about the truth or falsity of $(\alpha, E)$ exists before the actual experiment is carried out; the theory merely states the probability of the truth of the sentence $(\alpha, E)$ in the system state $\varphi$. But if we wish to speak of a logic of experimental sentences, we must associate a logical value with each sentence. The only reasonable approach for this appears to be the interpretation of $p(\alpha, \varphi, E) \in [0, 1]$ as the logical value of $(\alpha, E)$ in state $\varphi \in \Phi$. If

$$p(\alpha, \varphi, E) + p(\beta, \varphi, F) \leq 1 \quad \text{for every } \varphi \in \Phi,$$

we shall hold $(\alpha, E)$ and $(\beta, F)$ as contradictory sentences, meaning that the sum of the two logical values is again a logical value within the unit interval. $(\alpha, E)$ and $(\beta, F)$ are certainly contradictory if $E \cap F = \emptyset$. We therefore comprehend (A3) as saying:

"For any sequence of pairwise contradictory experimental sentences $(\alpha_i, E_i)$, $i = 1, 2, ..., \text{there exists an experimental sentence } (\beta, F) \text{ whose logical value is the sum of the logical values of the sentences } (\alpha_i, E_i) \text{ for } j = 1, 2, ..."$

and we have the result: $(\beta, F) \equiv ((\alpha_1, E_1) \lor (\alpha_2, E_2) \lor \cdots)$, which reveals that without (A3) there would be no connecting relation between different observables of the same system, and it would be impossible to speak about relations between different physical quantities of the same system. This axiom is, therefore, indispensable.
Let \( \neg (\alpha, E) = (\alpha, \mathbb{R} - E) \). The function \( \neg \) obviously has the properties of negation in the set \( \mathcal{E} \) since:

\[ \neg (\alpha, E) \text{ is false whenever } (\alpha, E) \text{ is true, and} \]

\[ \neg \neg (\alpha, E) = (\alpha, E). \]

We now see that the set of experimental sentences \( \mathcal{E} \), endowed with this negation (\( \neg \)) and with the \( p \)-implication (\( p \) satisfying the axioms \( (A1)-(A3) \)), forms a kind of logical system. The result of a logical identification of equivalent sentences of a formalized theory \( T \) is a so-called Lindenbaum–Tarski algebra of \( T \). Thus, the logic \( (L, \leq, \bot) \) in \( \mathcal{E} \) may be conceived of as a Lindenbaum–Tarski algebra for the formalized system of experimental sentences. In this case, the sentence \( "(\alpha, E) \) or (\( \beta, F \))\) has sense (i.e., has a definable probability of truth) whenever \( (\alpha, E) \) and \( (\beta, F) \) are mutually contradictory (conforming to \( (A3) \)). In classical mechanics the sentences \( "(\alpha, E) \) or (\( \beta, F \))\) and \( "(\alpha, E) \) and (\( \beta, F \))\) always have physical sense; in this case \( L \) is a Boolean algebra. It is, on the other hand, a fundamental hypothesis of quantum mechanics that—due to the Heisenberg uncertainty relations—the above sentences do not always have physical sense in quantum mechanics; in this case \( L \) is an orthomodular \( \sigma \)-orthocomplemented poset (partially ordered set).

As for \( (A4) \), we should note that every observable \( \alpha \in A \) is uniquely determined by a corresponding \( L \)-measure \( \mu_\alpha \). However, so far we have not demanded that there be a certain observable to every \( \sigma \)-measure on \( \mathbb{B}(\mathbb{R}) \). This requirement is imposed on the system by \( (A4) \). Its physical significance is the following:

Given a set of experimental sentences \( Q_E \in \mathcal{E}, \) one for every Borel set in \( \mathbb{R} \) such that \( E \rightarrow [Q_E] \) is an \( L \)-measure, we are able to define an observable as a physical quantity corresponding to each of the experimental sentences.

This axiom makes it possible to study the properties of \( Q_E \) as well as the relations between observables. Moreover, we may identify the set of observables with the set of all \( L \)-measures. Now, let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a real Borel function on \( \mathbb{R} \), and let \( \alpha \in A \) be an observable. It is not hard to show that \( \mu_\alpha \circ f^{-1} \) is an \( L \)-measure. Also, \( (A4) \) implies that there corresponds an observable to this measure; let us denote it by \( f(\alpha) \). We obtain immediately the equality \( \mu_{f(\alpha)} = \mu_\alpha \circ f^{-1} \), and, by virtue of \( p(f(\alpha), \varphi, E) = p\mu_{f(\alpha)}(E) \), finally \( p(f(\alpha), \varphi, E) = p\mu_{f(\alpha)}(E) = p\mu_\alpha f^{-1}(E) = p(\alpha, \varphi, f^{-1}(E)) \). This result suggests the physical construction of the observable \( f(\alpha) \):

if the measurement of \( f(\alpha) \) yields a result in \( E \), then the measurement of \( \alpha \) yields a result in \( f^{-1}(E) \).

In classical mechanics \( L = \mathbb{B}(\mathbb{R}^6); \) if \( p_\varphi: \mathbb{B}(\mathbb{R}^6) \rightarrow [0, 1] \) is a state,
where \( \mu^\varphi : E \rightarrow p_\varphi f^{-1}(E) \) is the Lebesgue measure on \( \mathbb{R} \). In contrast to this, the logic of quantum mechanics is an orthocomplementary poset selected in such a way that the theory based on this logic generates results very close to the experimental results. This requirement appears to be optimally fulfilled by the

**Quantum Mechanical Postulate.** The logic \( L \) for a quantum mechanical system is isomorphic with the partially ordered set \( L(H) \) of all closed subspaces of a separable \( \infty \)-dimensional Hilbert space.

\( \perp \) is in this case the orthogonal complementation of a subspace, i.e., for \( N \in L(H) \) one has \( N^\perp = \{ x \in H : \langle x, a \rangle = 0 \text{ for every } a \in N \} \).

We now converge to the main points of our concern:

(a) commutativity and noncommutativity of operators in \( H \),

(b) the notion of measure on the subspaces of \( H \) (Lebesgue measure fails in such spaces), and

(c) the Heisenberg uncertainty relations.

Projection in Hilbert space \( H \) will always be understood as an orthogonal projection, i.e., a selfadjoint and idempotent operator \( P^* = P = P^2 \), acting in \( H \). According to Maczynski [25], two projections \( P \) and \( Q \) commute, i.e., \( PQ = QP \), iff the following inequality holds for all vectors \( u \in H \):

\[
\| Pu \|^2 + \| Qu \|^2 \leq \| u \|^2 + \lim_{n \to \infty} \| (PQ)^n u \|^2.
\]

Based on this result, we characterize the commutativity of projections in \( H \) by associating with every pair of projections \( P, Q \) a real number \( \delta(P, Q) \in [0, 1] \), such that

\[
\delta(P, Q) = 0 \quad \text{iff} \quad P \text{ and } Q \text{ commute.}
\]

For noncommuting projections, \( \delta(P, Q) \) is interpreted as a "commutativity gap," measuring the degree of noncommutativity of \( P \) and \( Q \). For 1-dimensional projections \( \delta(P, Q) \) coincides with the square root of the transition probability \( |\varphi, \psi|^2 \) between two states \( \varphi, \psi \) (unit vectors) in quantum mechanics. Indeed, for projections \( P_\varphi \) and \( P_\psi \) onto the 1-dimensional sub-
spaces generated by \( \varphi \) and \( \psi \), and assuming \( P_\varphi \neq P_\psi \), we have \( \delta(P_\varphi, P_\psi) = |(\varphi, \psi)| \). Thus,

\[
\delta(P_\varphi, P_\psi) = 0 \quad \text{whenever } P_\varphi \text{ and } P_\psi \text{ commute or else } P_\varphi \perp P_\psi.
\]

Generally, two nonvanishing projections \( P \) and \( Q \)

commute iff \[
\sup_{|u| = 1} \lim_{n \to \infty} \left( \|Pu\|^2 + \|Qu\|^2 - \|(PQ)^n u\|^2 \right) = 1,
\]

do not commute iff \[
1 < \sup_{|u| = 1} \lim_{n \to \infty} \left( \|Pu\|^2 + \|Qu\|^2 - \|(PQ)^n u\|^2 \right) \leq 2.
\]

We might add that the commutativity gap is equal to zero if one of the projectors vanishes; it is greatest when the transition probability approaches unity.

The definition of the commutativity gap may, of course, be extended to arbitrary selfadjoint operators in a Hilbert space. Indeed, using the spectral theorem, we may identify each selfadjoint operator \( A \) with the spectral measure \( P^A \) associating with every Borel set \( E \in \mathcal{B}(\mathbb{R}) \) a projection \( P_E^A \) in \( H \). Hence, a commutativity gap between two selfadjoint operators \( A \) and \( B \) acting in \( H \) can be defined as

\[
\delta_0(A, B) = \sup_{E,F \in \mathcal{B}(\mathbb{R})} (P_E^A, P_F^B).
\]

The commutativity gap \( \delta_0 \), defined by means of spectral measures, differs from the commutativity gap \( \delta \) defined earlier; namely, if \( P \) is a projection, then the spectral measure \( \mu \) corresponding to \( P \) is concentrated on the set \( \{0, 1\} \), and we have \( \mu(1) = P, \mu(0) = P^\perp \). Hence, for projection, the relation between \( \delta_0 \) and \( \delta \) is given by \( \delta_0(P, Q) = \max(\delta(P, Q), \delta(P^\perp, Q), \delta(P^\perp, Q^\perp), \delta(P, Q^\perp)) \). Since two selfadjoint operators commute iff their spectral projections commute, we also have

\[
(\delta_0(A, B) = 0) \equiv (A \text{ and } B \text{ commute}).
\]

We compute \( \delta_0(A, B) \) for selfadjoint operators with a pure point spectrum acting in a separable Hilbert space from the formula

\[
\delta_0(A, B) = \sup |(\varphi, \psi)|,
\]

over all unit eigenvectors \( \varphi \) of \( A \) that are not eigenvectors of \( B \) and over all unit eigenvectors \( \psi \) of \( B \) that are not eigenvectors of \( A \). If there are no such eigenvectors, \( A \) and \( B \) commute and we have \( \delta(A, B) = 0 \).

The failure of the Lebesgue measure has been more than adequately compensated by the so-called Gleason definition of measure on the subspaces of \( H \). Looking at quantum mechanics from the central limit point of
view (Cushen and Hudson [11]), i.e., in the light of a noncommutative analogue of axiomatic probability theory of the foundations of quantum mechanics, the field of events appears replaced by a non-Boolean lattice of propositions, in practice by the lattice $L$ of subspaces of a separable Hilbert space $H$. A countably additive function from $L$ to the closed unit interval constitutes a state (see Definition I.3), the noncommutative analogue of a probability measure.

The deep theorem of Gleason [13] asserts that every state on the lattice of subspaces of $H$ is of the form

$$\mathcal{M} \to \text{tr} \rho P_{\mathcal{M}},$$

$P_{\mathcal{M}}$ being the projection onto the subspace (manifold) $\mathcal{M}$, $\rho$ a density operator in $H$, i.e., a positive operator of unit trace (conversely, every density operator determines, by the above relation, a state). Our non-probabilistic formulation of this important theorem runs as follows.

**Theorem I.1 (Gleason).** A Gleason measure on $H$ is a real-valued functional on the complete complementary lattice of closed subspaces of $H$, which is $\sigma$-additive on the sequences of pairwise orthogonal closed subspaces. All measures on the closed subspaces $\mathcal{M} \subset H$ are of the kind

$$\mu(\mathcal{M}) = \text{tr} TP_{\mathcal{M}},$$

$T$ being a unique selfadjoint operator of the trace class, while $P_{\mathcal{M}}$ denotes the orthogonal projection on $\mathcal{M}$.

A lattice homomorphism from the field of Borel subsets of $\mathbb{R}$ to the lattice of propositions (i.e., $\mathcal{B}(\mathbb{R}) \to L$) determines an observable, analogous to the classical random variable identified with its set-function inverse. In case of a Hilbert space, an observable determines a projection-valued measure on $\mathbb{R}$ and thus, by the spectral theorem, a selfadjoint operator; conversely, every selfadjoint operator $\alpha$ with spectral resolution

$$\alpha = \int \lambda dP_\lambda(\lambda)$$

induces a lattice homomorphism sending each Borel set $E$ into the range of $P_\alpha(E)$. We may thus use the terms "state" and "density operator"—on the one hand—and "observable" and "selfadjoint operator"—on the other hand—interchangeably. Hence, a state $\varphi$ and an observable $\alpha$ induce a probability measure on $\mathbb{R}$ by $E \to \text{tr} \varphi P_\alpha(E)$, which is interpreted as the probability distribution for the measured value of the observable in state $\varphi$. The mean value is

$$\langle \alpha \rangle_\varphi = \int \lambda d(\text{tr} \varphi P_\lambda(\lambda)).$$
The noncommutativity theory manifests itself (in general) in the non-
existence of joint probability distributions for observables which, as
operations, do not commute. Such observables are assumed to be com-
patible in the sense that it is impossible to measure both simultaneously.
This is particularly the case for a pair of canonically conjugate position and
momentum operators \( q \) and \( p \). It is customary to state the Heisenberg
uncertainty relations in the two equivalent forms:

\[
\Delta p \cdot \Delta q \geq h = 2\pi \hbar, \tag{1.1}
\]

\[
\Delta E \cdot \Delta t \geq h. \tag{1.2}
\]

This equivalence is a consequence of the relativistic standpoint that energy
\( E \) and momentum \( p \) are quantities of the same kind; indeed,

\[
p_x = \frac{h\delta}{i\delta x}
\]

is the spacial part of the relativistic 4-vector,

\[
E = \frac{h\delta}{i\delta t}
\]

is its temporal component.

But then \( q \) and \( t \) must also be of the same kind; this follows independently
from the canonical equations of Hamilton, if \( p \) is kept constant (a case
known in physics as cyclic coordinate).

Epistemological Reflections

The decisive move of quantum mechanics has been to give-up all "objec-
tivity" of natural events. Indeed, the new lesson quantum mechanics has
tought us is that the same physical object may present itself in two
mutually excluding forms: as corpuscle and as wave. There is, however, no
contradiction between the actually observed properties of an electron, for
example; this appears only when we assume that the electron is entitled to
these properties even when we do not care to observe them. Both, cor-
puscle and wave views may be reconciled only if it is assumed that the
momentum of a particle at a known position cannot be known.

Every observation presumes a causal chain and yields a "visible" result. However, we may not assemble the single results and causal chains into a
model of a nature existing per se. Which of the mutually complementary
sides of nature we will be facing, depends largely on our freely chosen
experimental set-up, and the knowledge of one side excludes, of course, the
knowledge of the complementary one. This appears to be the current state
of our comprehension of quantum mechanics.

C. F. von Weizsäcker [40] has discussed the multi-valuedness of quan-
tum mechanical truth statements at length. It appears that the truth values
are true, false and undetermined; but, since we must mind complemen-

tarity, how are we to know the false or the undetermined to the true? We must look at von Weizsäcker's plea for dropping the tertium-non-datur with this inconsequence in mind. Nevertheless, a quantum physicist reporting on his experimental findings will say: "I know that A holds true." This phrase stresses the fact that the notion of the object bears a relation to the knowledgeable subject, both with regard to his knowledge and to his will. Whenever I want to know something definite, I must give-up all knowledge relating to the complementary. According to von Weizsäcker [40],

"there is a sharp boundary between awareness and non-awareness, which, however, is stripped of any objectiveness in nature, for I can place that boundary at will; only, I cannot make it vanish altogether."

Now, the observation of quantum mechanical events by means of some measuring apparatus obeying the laws of classical mechanics involves a cut. While the relations on the observer side as well as on the side of the observed object are sharply determined, this cut is both necessary and convenient to allow a perfect coupling of the two sides and to account for any interference due to measurement. This represents a restriction to the use of classical notions in the form of uncertainty relations (1.1) and (1.2). The position of this cut is of no importance for the formulation of laws of nature as long as its existence is acknowledged. With this understanding, the question referring to the exact position and the exact momentum of a particle must be considered as a virtual question only (the same as the simultaneity of events in the face of a finite speed of light) and rejected as ill posed in substance.

A cut of the type described above must also exist between technical cybernetics and biocybernetics, and indeed between any pair of exact natural sciences. It is the author's conviction that the class of all natural sciences constitutes a Stone space, i.e., a compact and totally disconnected Hausdorff space. The nonobservance of the respective boundaries and cuts is likely to deprive us of the desired knowledge and may well have been the cause of interdisciplinary failures in the past.

That which we call science is concerned with two fundamental objects:

1) To disseminate the knowledge about nature which would put man in a position to utilize the resources and forces of nature to his advantage.

2) To allocate to man the right position in nature on the basis of a real insight into the relations prevailing in nature.

These are also the principles of technical sciences with, perhaps, a greater emphasis on the first one.
II. SENTENCES IN BROUWERIAN ALGEBRAS

In this section we shall lay down the algebraic foundation for the treatment of systems defined on fuzzy sets. Since fuzzy sentences are stated in the language of a Brouwer algebra, it is advisable to be aware of the difference between classical and intuitionistic reasonings. To emphasize this concern, consider a propositional function $\omega(x)$ in the arithmetic of natural numbers.

To a mathematician, the problem of the truth of $\omega(x)$ is solved if there is a proof that the sentence $\delta = \bigcup \omega(x)$ is a theorem in arithmetic; he actually dispenses with the task of producing a construction of a number $n \in \mathbb{N}$ such that $\omega(n)$ is a true statement. To prove $\delta$, a mathematician would first prove $\gamma = \neg \bigcup \neg \omega(x)$ and then use the tautology $\gamma \Rightarrow \delta$. The truth of $\delta$ follows then from the modus ponens rule

$$\frac{\gamma \land (\gamma \Rightarrow \delta)}{\delta}$$

(if ..., then $\Rightarrow$).

This procedure is unacceptable to the intuitionist; he demands a construction of a number $n \in \mathbb{N}$ satisfying $\omega(n)$; failing this, both $\delta$ and $\gamma \Rightarrow \delta$ are rejected. Moreover, since constructivity is restricted to finite sets, intuitionistic logic will only admit a potential infinity, never an actual one. But many sets have an infinite number of elements and the axiom of infinity is one of seven axioms of the Zermelo–Fraenkel set theory; hence intuitionists reject the notion of a set as well as the whole set theory. Also, negation and disjunction are understood differently in intuitionistic logic; for, while (see [32])

$$\omega \Rightarrow \neg \neg \omega$$ is accepted, $\neg \neg \omega \Rightarrow \omega$ is rejected, and $\omega_1 \cup \omega_2$ is considered true if $\omega_1$ or $\omega_2$ is true and there is a method determining just which summand is true.

Consequently, the tautology $\omega \cup \neg \omega$ as well as the tertium-nondatur law have no validity in intuitionistic logic. Now, the metatheory of intuitionistic logic (the consequence operation) coincides with the theory of pseudo-Boolean algebras in the same sense as does the metatheory of classical logic with the theory of Boolean algebras. Hence follows that the theory of pseudo-Boolean algebras is the theory of lattices of open subsets, and every investigation of intuitionistic logic consists in an examination of lattices of open subsets of topological spaces. This in itself is a surprising fact.

First, we shall establish the required terminology.

**Definition II.1.** A relation $R$ on a nonvoid set $X$ and satisfying the conditions

...
COMPLEMENTARY FUZZY SETS

(i) \( xRx \) (reflexivity),
(ii) \((xRy \land yRx) \Rightarrow (x = y)\) (antisymmetry),
(iii) \((xRy \land yRz) \Rightarrow xRz\) (transitivity),

is said to be a relation of partial order, and the so ordered set a partially ordered set (poset).

A relation which is merely reflexive and transitive is called quasi-order. On the other hand, a relation that is connected and satisfies (i), (ii), and (iii) is said to be a linear order. Connectivity requires that every pair of elements of a linearly ordered set be comparable, i.e. (putting \( x \leq y \) instead of \( xRy \)) we must have either \( x \leq y \) or \( y \leq x \) for all pairs of the ordered set. A collection of sets ordered linearly is called a monotonic family.

Let \( A \) be an arbitrary nonvacuous set. Every mapping

\[ f' : A^n \to A, \quad n \in \mathbb{N}, \]

is called an \( n \)-argument operation defined on the set \( A \).

**Definition 11.2.** An abstract algebra (briefly, algebra) is the name given to any ordered pair \((A, 0)\) in which \( A \) is a nonvoid set and \( 0 = \{o_1, o_2, \ldots\} \) is the set of (usually finitely many) operations defined on \( A \).

Let \((L, \cup, \cap)\) be an abstract algebra with two binary operations: \( \cup \) and \( \cap \) and, possibly, a smallest (\( \sqcup \)) and a greatest (\( \sqcap \)) element. We shall call \( x \cup y \) the sum of the elements \( x \) and \( y \), and \( x \cap y \) the product of the elements \( x \) and \( y \). Consider now the following axioms:

1. \((L1)\) \( x \cup y = y \cup x, \ x \cap y = y \cap x; \)
2. \((L2)\) \( x \cup (y \cup z) = (x \cup y) \cup z, \ x \cap (y \cap z) = (x \cap y) \cap z; \)
3. \((L3)\) \( x \cup x = x, \ x \cap x = x; \)
4. \((L4)\) \( x \cup (x \cap y) = x, \ x \cap (x \cup y) = x; \)
5. \((L5)\) \( x \cap (y \cup z) = (x \cap y) \cup (x \cap z), \ x \cup (x \cap z) = (x \cup y) \cap (x \cup z); \)
6. \((L6)\) \( x \cup \bot = x, \ x \cap \bot = x; \)
7. \((L7)\) \( x \cup -x = \bot, \ x \cap -x = \bot. \)

**Definitions 11.3.** (1) \( L = (L, \cup, \cap) \) is called a lattice if it satisfies the axioms \((L1)-(L4)\). It admits partial ordering in such a way that for all two-element subsets \( \{x, y\} \subseteq L \) the sum coincides with the least upper bound (lub) and the product coincides with the greatest lower bound (glb); it is ordered if each pair of elements has a lub and a glb. A lattice is called complete if its every subset \( A \subseteq L \) possesses the least element \( \sqcup \) and the greatest element \( \sqcap \). (lub \( \equiv \) join; glb \( \equiv \) meet). The following duality principle applies:
If a sentence \( T_1 \) is a consequence of \((L1)-(L4)\), then the sentence \( T_2 \), obtained by interchanging \( \cup \) and \( \cap \), is also a consequence of \((L1)-(L4)\).

(2) \( \mathcal{L} \) is a distributive lattice if it satisfies axioms \((L1)-(L5)\).

(3) Lattice \( \mathcal{L} = (L, \cup, \cap, \neg, \sqcup, \sqcap) \), fulfilling all axioms \((L1)-(L7)\) is said to be a Boolean algebra.

The family \( 2^X \) of all subsets of a nonvoid set \( X \) is a Boolean algebra with set-theoretical operations.

**Definition II.4.** A nonvoid set \( V \) of elements of a lattice \( \mathcal{L} \) is called a filter whenever

\[
(a \cap b \in V) \equiv (a \in V \land b \in V) \quad \text{for all} \quad a, b \in L,
\]
equivalently,

\[
(a, b \in V) \Rightarrow (a \cap b \in V) \quad \text{and} \quad (a \in V \land a \leq b) \Rightarrow (b \in V).
\]

A nonvoid set \( A \) of elements of a lattice is called an ideal (the dual to a filter) whenever

\[
(a \cap b \in A) \equiv (a \in A \land b \in A) \quad \text{for all} \quad a, b \in L,
\]
equivalently,

\[
(a, b \in A) \Rightarrow (a \cup b \in A) \quad \text{and} \quad (a \leq b \land b \in A) \Rightarrow (a \in A).
\]

**Definitions II.5.**

(1) If the elements of a lattice \( \mathcal{L} \) are subsets of a space \( X \) and \( \cup \) and \( \cap \) coincide with the set-theoretical union and intersection, respectively, then \( L \) is said to be a set-lattice.

(2) There are two notions of "complement" to an element \( a \) in a lattice \( L \) corresponding to the set-theoretical complement of a set \( A \), being either the greatest subset \(-A\) of \( X \) disjoint from \( A \) or the least subset of \( X \) whose union with \( A \) yields the whole space \( X \). These subsets are, in general, not equivalent. If \( L \) has the least element \( \sqcup \), then \( c \in L \) is called a \( \cap \)-complement of \( L \) provided that \( c \) is the greatest element such that \( a \cap c = \sqcup \). If \( L \) has the greatest element \( \sqcap \), then \( c \in L \) is called a \( \cup \)-complement of \( L \) provided that \( c \) is the least element such that \( a \cup c = \sqcap \).

(3) If \( L \) is a distributive lattice and \( a \cup c = \sqcap \) and \( a \cap c = \sqcup \), then \( c \) is a complement of \( a \in L \). The only other complement of importance is the \( \cap \)-complement, which—if existing—is called a pseudo-complement.

(4) An element \( c \in L \) is called pseudo-complement of a relative to \( b \)
(or modulo $b$) or relative pseudo-complement if $c$ is the greatest element such that $a \cap c \leq b$. It is denoted by $a \sqsupset b$. By definition,

$$\{x \leq a \sqsupset b\} \equiv (a \cap x \leq b) \quad \text{for all } x \in L.$$  \hspace{1cm} (2.1)

Since $a \cap b \leq b$, we infer that $b \leq a \sqsupset b$, provided that $a \sqsupset b$ exists, and if $\sqcup$ and $\sqcap$ exist in $L$, we have

$$\{a \sqcap b\} \equiv (a \sqsupset b = \sqcap) \quad \text{and} \quad \sqcap \sqsupset b = b$$  \hspace{1cm} (2.2)

For (further) information on implicative lattices and semi-lattices see Nemitz [30].

(5) An ordered set $L \neq \emptyset$ in which every pair of elements has a meet and satisfies (2.1) and (2.3) is (following Curry [10]) called an implicative system, where

$$a \cap (a \sqsupset b) \leq b.$$  \hspace{1cm} (2.3)

(6) We say that $a \sqsupset b$ is a complement of a relative to $b$ (or a modulo $b$) if it is the least element $c \geq a \cap b$ such that $a \cup c = \sqcap$. If an element $a$ of a distributive lattice $L$ has the complement $\sqcap$, then, for every $b \in L$, the complement $a \sqcap b$ of $a$ relative to $b$ exists and $a \sqsupset b = \sqcap a \cup b$, where $\sqcap a \cup b$ is the pseudo-complement of $a$ relative to $b$ (it happens to be also the complement of $a$ relative to $b$).

(7) The notions dual to the relative pseudo-complement and to the relative complement are the pseudo-difference and the difference. An element $c$ is said to be the pseudo-difference of $b$ and $a$ if it is the least element such that $a \cup c \geq b$; it is denoted by $b \setminus a$. By definition:

$$\{x \geq b \setminus a\} \equiv (a \cup x \geq b) \quad \text{for } x \in L.$$  \hspace{1cm} (2.4)

Since $a \cup b \geq b$, we have $b \geq b \setminus a$ provided that $b \setminus a$ exists. Obviously $(a \setminus a) = (\sqcup \in L)$; and if $\sqcap \in L$ exists, then $(a \geq b) \equiv (b \setminus a = \sqcap$ and $b \setminus \sqcap = \sqcup)$. Thus, the pseudo-difference $b \setminus a$ is the least element $c \leq a \cup b$ such that $a \cup c = a \cup b$. Suppose that $\sqcup \in L$. We shall say that $b \setminus a$ is the difference of $b$ relative to $a$ if it is the greatest element $c \leq a \cup b$ such that $a \cap c \sqcup \sqcap$. If an element $a$ of a distributive lattice $L$ has a complement $\sqcap$, then the difference $b \setminus a$ exists and $b \setminus a = b \cap \sqcap = b$. 

**Definition II.6.** A relatively pseudo-complemented lattice has the unit element, but need not have the zero element $\sqcup$. If $\sqcup \in L$ exists, then $L$ is a pseudo-Boolean algebra $(L, \cup, \cap, \sqcup, \sqcap, \sqcup, \sqcap)$. An element of a pseudo-
Boolean algebra is said to be dense if \( \neg a = \uplus \); it is dense iff \( \neg \neg a = \uplus \).

Element \( a \) is called regular if \( a = 11\neg \).

As already stated, \((a \Rightarrow \neg \neg a)\) is and \((\neg \neg a \Rightarrow a)\) is not admissible in intuitionistic logic; rejected is also the tautology \( a \cup \neg a \) and, therefore, also tertium non datur. Every pseudo Boolean algebra may be conceived of as an algebra \((L, \cup, \cap, \equiv, \neg)\) with three binary operations and one unary operation. In the terminology of Wechler \([39]\), a pseudo-Boolean lattice is termed an \(L\)-fuzzy algebra; the corresponding dual lattice is known as Brouwer algebra.

**DEFINITION 11.7.** Lattices dual to pseudo-Boolean algebras are called Brouwerian algebras. Brouwer lattice is a notion between a lattice and a Boolean ring. Indeed, a lattice with unity is called Brouwer lattice if the pseudo-difference \(a \div b\) exists for all elements \(a, b \in L\) in the sense of \((2.4)\).

An abstract algebra \(\mathfrak{L} = (L, \cup, \cap, \div, \rhd)\) is said to be a Brouwerian algebra provided that

(i) \((L, \cup, \cap)\) is a lattice with the unit element,

(ii) \(\div\) is a binary operation satisfying equivalence \((2.4)\),

(iii) \(\rhd\) is a unary operation defined by \(\rhd a = \cap \div a\).

An important example of a Brouwerian algebra is order topology

\[
\mathcal{C}_\preceq = (O_{\preceq}, \cup, \cap, \div, \rhd),
\]

where \(O_{\preceq}\) is the collection of all open sets of the quasi-ordered (i.e., reflexive and transitive) set \((C, \preceq)\), \(\cup\) and \(\cap\) are the set-lattice operations, respectively, and \(\div\) and \(\rhd\) are defined thus

\[
R \div S = \left\{ c \in C : \left( \bigwedge_{c' \in C} (c \preceq c' \land c' \in S \Rightarrow c' \in R) \right) \right\},
\]

\[
\rhd R = \left\{ c \in C : \left( \bigwedge_{c' \in C} (c \preceq c' \Rightarrow c' \notin R) \right) \right\}.
\]

A set \(B \subseteq C\) is considered to be open in \(C\) if

whenever \(x \in B\) and \(x \preceq y\), then \(y \in B\).

The quasi-ordering relation is, in this case, the set-theoretical inclusion.

**THEOREM 11.1.** Let \(\mathfrak{A} = (A, \cup, \cap, \div, \rhd)\) be any distributive Brouwerian algebra. For every \(n \in \mathbb{N}\), let \(A_{2n}, B_{2n+1} \subseteq A\) be such that

(i) \(a_{2n} = \bigwedge_{a \in A_{2n}} a\) and \(b_{2n+1} = \bigwedge_{b \in B_{2n+1}} b\) exist,
(ii) for any \( c \in A \), \( \{ a \wedge c : a \in A_{2n} \} \in \{ A_{2k} : k \in \mathbb{N} \} \), \( \{ c \vee b : b \in B_{2n+1} \} \in \{ A_{2k} : k \in \mathbb{N} \} \),

(iii) for any \( c, d \in A \), \( \{ (d \wedge a) \vee c : a \in A_{2n} \} \in \{ A_{2k} : k \in \mathbb{N} \} \).

Then there exists an order topology \( C \leq \) and a monomorphism \( h \) from \( A \) to \( C \) preserving \( a_{2n} \) and \( b_{2n+1} \) for all \( n \in \mathbb{N} \).

The proof follows from the following two lemmas:

**Lemma II.2.** Let \( A \) be as in Theorem II.1 and let \( (Q) \) be a set of infinite joins and meets in \( A \):

\[
 a_{2n} = \bigvee_{a \in A_{2n}} a, \quad n \in \mathbb{N}, \\
 b_{2n+1} = \bigwedge_{b \in B_{2n+1}} b, \quad n \in \mathbb{N}.
\]

Finally, let \( x, y \) be elements of \( A \) for which the relation \( x \leq y \) does not hold. Then there exists a \( Q \)-ideal \( \Delta \) such that \( x \notin \Delta \) and \( y \in \Delta \).

**Lemma II.3.** Assuming that \( \Delta \in C \), where \( C \) is the set of all \( Q \)-ideals of \( A \), then \( a \cdot b \in \Delta \) if and only if, for each \( \Delta' \in C \) such that \( \Delta \subset \Delta' \), the following implication holds: if \( b \in \Delta' \), then \( a \in \Delta' \).

The condition:

\[
 a \cup \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (b_i \cup a) \quad (2.5)
\]

is fulfilled in every Brouwerian algebra, but

\[
 a \cap \bigwedge_{i \in I} a_i = \bigwedge_{i \in I} (a_i \cap a) \quad (2.6)
\]

is satisfied only in distributive Brouwerian algebras. Definitions II.6 and II.7 disclose the duality between the relative pseudo-complement \( a \supset b = \neg a \cup b \) and the pseudo-difference \( a \divideq b = \neg a \cap b \). This duality is also present in the Heyting–Brouwer logic (see \[33\]).

**Definition II.8** (Rauszer \[33\]). An abstract algebra \( (A, \cup, \cap, \supset, \divideq) \) is said to be a semi-Boolean algebra if \( (A, \cup, \cap, \supset) \) is a relatively pseudo-complemented lattice and \( \divideq \) is a pseudo-difference. The algebra \( (A, \cup, \cap, \supset, \divideq, \neg, \top, \bot, \oplus, \oslash) \) is occasionally called Heyting–Brouwer algebra whenever

(i) \( (A, \cup, \cap, \supset, \neg, \top) \) is a Heyting algebra with \( \neg a = a \supset \top \),

(ii) \( (A, \cup, \cap, \divideq, \bot, \oplus) \) is a Brouwer algebra with \( \neg a = \bot \divideq a \).
To every pseudo-Boolean algebra $A$ there exists a complete semi-Boolean algebra $A'$ and a monomorphism $g: A \to A'$. To every Brouwerian algebra $A$ there exists a complete semi-Boolean algebra $A'$ and a monomorphism $h: A \to A'$. Semi-Boolean algebras play the same role in Heyting–Brouwer logic as Boolean algebras do in classical logic and as Brouwerian algebras in fuzzy (set) logic. Now, a Brouwerian algebra may be regarded as a set-lattice $K(X)$ of all closed subsets of a topological space $X$ if a pseudo-Boolean algebra is taken to be a set lattice $G(X)$ of all open subsets of $X$. In this case, which will occur frequently in the sequel, we shall (following Kuratowski [19]) identify the pseudo-difference with the closed set-difference $\overline{A - B}$ for $A, B \subseteq 2^X$.

Sentences valid in any Brouwerian algebra

\[
(x \leq y) \Rightarrow (x \vdash z \leq y \vdash z) \cap (z \vdash y \leq z \vdash x),
\]

\[
(x \leq y) \equiv (x \vdash y = 0),
\]

\[
z \vdash (x \cap y) = (z \vdash x) \cup (z \vdash y),
\]

\[
(x \cup y) \vdash z = (x \vdash z) \cup (y \vdash z),
\]

\[
z \vdash \vdash x \leq x,
\]

\[
z \vdash \vdash x = \vdash x.
\]

It is easy to convert these formulas to sentences in a Brouwer set-algebra using the aforementioned equivalent pseudo-difference.

III. TOPOLOGICAL NEGLIGIBILITY IN HYPERSPACES AND LATTICES

In this section we redevelop some topological concepts and facts of interest in the general fuzzy set theory, of which a few will be of utmost importance for the main result of this paper and in the immediately following application papers. With regard to the latter, topological hyperspaces, Carathéodory (set-valued) mappings and differential inclusions deserve particular attention.

**Definitions III.1.** (1) A topological space is a set $1$ (whose elements are points) and a function (called closure: $A \to \overline{A}$) assigning to each set $A \subseteq 1$ a set $\overline{A} \subseteq 1$ in accordance with the following axioms (due to Kuratowski):

- **(A1)** $\overline{A \cup B} = \overline{A} \cup \overline{B},$
- **(A2)** $A \subseteq \overline{A},$
- **(A3)** $\emptyset = \emptyset,$
- **(A4)** $A = \overline{A}$
With the added axiom

$$(\text{A5}) \quad \{p\} = \{\bar{p}\}, \quad p \in I. \quad (1, \bar{I}) \text{ becomes a topological } T_1\text{-space.}$$

$T_2, T_3, T_3, T_3, T_4$ denote Hausdorff, regular, Tychonov, and normal spaces, respectively.

(2) A topological Boolean algebra is any Boolean algebra with an Int (interior) operation which assigns to each element of the Boolean algebra its interior in accordance with the following axioms (due to Hausdorff):

1. $\text{Int}(a \cap b) = \text{Int}(a) \cap \text{Int}(b)$, for all elements of the algebra,
2. $\text{Int}(a) \subseteq a$,
3. $\ast = \text{Int}(\ast)$,
4. $\text{Int}(a) = \text{Int}(\text{Int}(a))$.

(Capital letters are used in set-algebras).

(3) A topological Brouwer algebra makes use of the Kuratowski axioms. In this case the pseudo-difference is the closure of the difference set.

A topological Brouwer algebra is symbolized by $(2^Y, \cup, \cap, \setminus, \subseteq, \supseteq)$; its elements are closed subsets of $Y$, $\emptyset$ being the void set (which is an isolated element of $2^Y$), $\sqcap = Y$, and the three binary operations have the meanings defined earlier. $2^Y$ is ordered by inclusion and is endowed with an exponential (Vietoris) topology defined by the subbase of $2^Y$, i.e., by assuming that the collection of all sets is either of the form $\{K: K \subseteq G\} \in 2^G$ or of the form $\{K: K \cap G \neq \emptyset\} = 2^Y - 2^Y \setminus G$, $G$ being an open set. For a finite system of open sets $G_0, \ldots, G_n$, the family of sets

$$B(G_0, G_1, \ldots, G_n) = 2^{G_0} \cap (2^Y - 2^Y \setminus G_1) \cap \cdots \cap (2^Y - 2^Y \setminus G_n)$$

represents an open base of $2^Y$ with the sets $G_i$ ordered by inclusion: $G_0 \supseteq G_1 \supseteq \cdots$. The topology thus defined (and called exponential or Vietoris topology) is the coarsest one in which for $A$ open (resp. closed) the sets $2^A$ are open (resp. closed) in $2^Y$. It turns out that $Y$ is the greatest and $\emptyset$ the smallest (but isolated) set of $2^Y$.

**Definition III.2.** Let $X$ and $Y$ be two topological spaces and $F$ a set-valued mapping, $F: X \to 2^Y$, assigning to each $x \in X$ a set $F(x) \in 2^Y$. $F$ is said to be lower semicontinuous (lsc) in two cases:

(i) when the sets $\{x: F(x) \subseteq K\}$ are closed in $X$ for $K$ closed in $Y$, and

(ii) when the sets $\{x: F(x) \cap G \neq \emptyset\}$ are open in $X$ for $G$ open in $Y$. 
F is (dually) called upper semicontinuous (usc) in two cases:

(i) when the sets \( \{ x : F(x) \cap K \neq \emptyset \} \) are closed in \( X \) for \( K \) closed in \( Y \), and

(ii) when the sets \( \{ x : F(x) \subseteq G \} \) are open in \( X \) for \( G \) open in \( Y \).

F is called a continuous mapping if it is lsc and usc at all points of \( X \).

**Definition III.3.** Let there be given a transformation \( F: X \to \mathcal{P}(Y) \), where \( \mathcal{P}(Y) \) is the power set of \( Y \) (the existence of such a power set is guaranteed by an axiom of set theory). Selector of this transformation is the name given to any function \( f: X \to Y \) satisfying the relation

\[
f(x) \in F(x) \quad \text{for every } x \in X.
\]

The existence of a selector is, of course, a consequence of the axiom of choice (of the Zermelo-Fraenkel set theory).

In applicational problems involving selection, we shall mostly be looking for selectors exhibiting topologically interesting properties (under suitable assumptions on \( X, Y \) and \( F \)). Distinct from this type of selection is the definition of a set containing exactly one element from each set belonging to a given distribution of space \( X \). Such a set may be called distribution selector. A special case of the selection problem is the so-called choice function. This calls for the construction of a function \( f: R \to X \) sending the family \( R \) of nonvoid (not necessarily disjoint) subsets of space \( X \) into \( X \), such that \( f(E) \in E \) for every \( E \in R \). The power set is, however, too rich in subsets for most applications; sharper theorems may be obtained from a family of closed or compact sets. This rule will also be followed in our work. It is appropriate to quote now Chang's definition of topology as a family of fuzzy sets (to be defined later) in \( X = \{ x_i \} \) of individual variables with an associated membership function.

**Definition III.4 (Chang [9]).** A family of fuzzy sets in \( X \) is said to be a topology if it satisfies the following axioms:

1. \( \emptyset, X \in T \) (the coarsest topology axiom),
2. \( (A, B \in T) \Rightarrow (A \cap B \in T) \),
3. \( (A_i \in T, i \in I \text{ (an index set)}) \Rightarrow (\bigvee_{i \in I} A_i \in T) \).

**Negligible Sets**

As is known, linear topological spaces constitute a generalization of \( n \)-dimensional vector spaces; actually, under a linear topological space \( E \) we understand a linear space over the ring of coefficients from \( \mathbb{R} \), while both \( E \) and \( \mathbb{R} \) are endowed with topologies with respect to which the addition of
elements in \( E \) and their multiplication with coefficients from \( \mathbb{R} \) are continuous operations. It is possible to introduce in a linear topological space \( s \) the notion of distance between its elements \( x = \{\xi_1, ..., \xi_n, ...\} \) and \( y = \{\eta_1, ..., \eta_n, ...\} \) by means of the formula

\[
\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|}.
\]

The metric in \( E \) induces in this set a natural topology if, by assuming as neighbourhood \( V(x_0) (0 < \varepsilon < +\infty) \), of any point \( x_0 \) the set of all such elements \( y \in E \) that \( \rho(x_0, y) < \varepsilon \), for given \( \varepsilon > 0 \), the set \( E \) is transformed into a topological space. The topological space \( E \) is then called metrizable provided that the topology introduced by this metric is identical with the original topology of the topological space \( E \).

However, \( s \) is not a normable space by the Kolmogorov criterion of normability, requiring that there be a convex bounded zero neighbourhood in a normable space. Let \( s \) denote the linear metric space which is the countable infinite product of lines. It is known that \( s \) is homeomorphic to Hilbert space \( l^2 \) and to all separable infinite-dimensional Fréchet spaces (and thus, of course, to all such Banach spaces), see [1] and [7].

Any locally convex, complete, linear metric space is said to be a Fréchet space. A normed Fréchet space is, of course, a Banach space; but, the countable infinite product of lines \( R \times R \times ... = R^\infty \) is an example of a Fréchet space that is not a Banach space. It is known that all separable \( \infty \)-dimensional Fréchet spaces are homeomorphic to each other. In all these spaces, and in the Hilbert cube \( R^\infty \), closed sets of infinite deficiency play an important role.

**Definition III.4.** A closed set \( K \) in a Fréchet space \( X \) is said to have infinite deficiency (or \( \infty \) codimension) if \( X - K \) is \( \infty \)-dimensional, \( \bar{K} \) being the closure of the linear subspace spanned by elements of \( K \). A homeomorphism of a space \( X \) onto itself is said to be stable if it is a finite product of homeomorphisms, each of which is the identity on some open set.

It is convenient to define a Fréchet manifold (F-manifold) to be a separable metric space which admits an open cover by sets homeomorphic to open subsets of \( s \). Banach manifolds may be defined similarly. This leads to the study of homeomorphisms of F-manifolds onto dense subsets of themselves. We owe a first result of this type to Klee, who proved that, for any compact set \( K \) in \( l^2 \), \( l^2 \) is homeomorphic to \( l^2 - K \). It was subsequently shown that for various types of subsets \( K \) of certain linear metric spaces \( X \), \( X \) is homeomorphic to \( X - K \). Bessaga [6] introduced the term “negligible” for such sets \( K \).
DEFINITION III.5 (Bessaga [6]). A subset $A$ of a topological space $X$ is said to be negligible if $X$ is homeomorphic to the difference set $X - A$.

The topological classification of sets defines the class of smallest sets as comprising the so-called meager sets, i.e., nowhere dense sets, sets of first Baire category and sets of zero measure. The family of sets of the first category is a $\sigma$-ideal, hence hereditary and countably additive. The next class of still small sets is occupied by compact sets and negligible sets. It is our intention to provide simple criteria for the negligibility of sets in two cases of importance in the general control theory: in topological vector spaces admitting continuous norms (i.e., containing radially convex bodies) and in certain Banach spaces.

DEFINITION III.6. A set $A$ in a topological vector space is called narrow if there exists an incomplete continuous norm $v(\cdot)$ on $X$ such that $A$ is closed in the Banach space $Y = \text{compl} X$, the completion of $X$ with respect to the norm $v(\cdot)$. A subspace of a Banach space $X$ is said to be strongly $\sigma$-narrow (so-narrow) if there is a continuous norm $v(\cdot)$ on $X$ such that the original unit cell $u$ of $X$ is incomplete and $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n$ is closed in $Y = \text{compl} X$.

Completeness is, of course, not a topological property, i.e., the homeomorphic image of a complete space may be noncomplete. The concept of so-narrowness is reasonable provided that $v(\cdot)$ exists; this is, in particular, true in nonconjugate spaces with a base, while in separable reflexive Banach spaces the required norm $v(\cdot)$ does not exist. If $X$ is an $\infty$-dimensional topological vector space admitting a continuous norm (in particular a Banach space), every compact set $A$ of $X$ is narrow. For, under the assumptions of this statement, $X$ also admits a continuous incomplete norm, and $A$ is still compact under this new norm and thus closed in the completion with respect to it. Now, $A$ is negligible if $X$ is an $\infty$-dimensional topological vector space admitting a continuous norm, and $A$ is a compact set of $X$. If $X$ happens to be an $\infty$-dimensional Banach space and $A$ a countable union of compact sets of $X$, then $A$ is negligible. This assertion holds for the linear normed space $X = c_0$ (the linear manifold of all sequences converging to zero) since $A$ is then so-narrow. The proof for non-separable spaces $X$ follows from the fact that $X$ is homeomorphic to the Banach space $Y \times c_0$. As a result of the above considerations we obtain the two announced criteria.

THEOREM III.1. Every narrow set of any linear topological space is negligible.

THEOREM III.2. Every so-narrow set of any Banach space is negligible.
The proof of Theorem III.1 follows easily from certain arguments of the detailed proof of Theorem III.2 provided by Bessaga [6]. To this basic conception of negligible sets we shall add the notions of strongly negligible and metrically negligible sets due to Anderson [2].

Definition III.7. (1) A subset $K$ of a space $X$ is strongly negligible if, for any open cover $G$ of $X$, there is a homeomorphism $h: X \to X - K$ such that $h$ is limited by $G$, i.e., for any $p \in X$, there exists a $g \in G$ such that both $p$ and $h(p)$ are elements of $G$.

(2) A set $K$ in a metric space is referred to as metrically negligible in $X$ if, for each $\varepsilon > 0$, there exists a homeomorphism $h$ of $X$ onto $X - K$ such that $h$ moves no point by more than $\varepsilon$.

Clearly, in a metric space $X$, strong negligibility of a set $K$ implies metric negligibility since we may select an open cover of $X$ of mesh less than $\varepsilon$. It is nontrivial, but follows from the researches of Anderson that, in an $F$-manifold, metric negligibility of a set $K$ implies strong negligibility of $K$. It is of some importance, particularly for the analysis of dynamic systems, that studies of the conditions under which $X$ and $X - K$ are diffeomorphic are in progress.

Second Thoughts on Fuzzy Set Principles

The elements of fuzzy set theory are, by definition and nomenclature, sets endowed with a grading property, called fuzzy membership. A fuzzy set constitutes—so to speak—the primitive notion (Grundbegriff) in a formalized theory of fuzzy sets, comparable to the notion of set in Zermelo-Fraenkel set theory. In a less formal conception of fuzzy set methodology, we might give the fuzzy set a precise meaning (as some writers, notably Kwakernaak [21], have already done). This involves a generalization of the characteristic function of a set.

The characteristic function $\chi_A: X \to \{0, 1\}$ of a nonvoid set $A \subset X$ is defined by:

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \in X - A. \end{cases} \quad (3.1)$$

Mapping $\chi: 2^X \to \{0, 1\}^X$ with these values is clearly one-to-one and onto; thus $2^X \sim \{0, 1\}^X$ manifests agreement between the $Y^X$ and $2^X$ notations. The following formulas are well known.

$$\chi_X = 1, \quad \chi_{\emptyset} = 0,$$

$$\chi_{A \cap B} = \chi_A \cdot \chi_B,$$

$$\chi_{(A - B)} = \chi_A - \chi_{A \cap B} = \chi_A (1 - \chi_B).$$
\[ \chi(x - B) = \chi - B = 1 - \chi_B, \]
\[ (A = \bigcup_t A_t) \Rightarrow (\chi_A(x) = \max_t \chi_{A_t}(x)), \]
\[ (A = \bigcap_t A_t) \Rightarrow (\chi_A(x) = \min_t \chi_{A_t}(x)), \]
\[ (A = \lim_{n \to \infty} A_n) \equiv (\chi_A(x) = \lim_{n \to \infty} \chi_{A_n}(x)). \]

(3.2)

The concept of a characteristic function of a set may easily be extended to a sequence of sets and, more generally, to a set-valued function, as required for fuzzy control in orientor field notation. Thus, let \( F : T \to 2^Y \) be a set-valued mapping such that \( F(t) = F_t \subset Y \) for \( t \in T \). Then, the characteristic function \( \chi_F \) of \( F(t) \) associates with each element \( y \in Y \) a function \( \chi_F(y) \in \{0, 1\}^T \), defined by
\[ \chi_F(y) = \begin{cases} 1 & \text{for } y \in F_t, \\ 0 & \text{for } y \in Y - F_t. \end{cases} \]

(3.3)

On the other hand, the characteristic function of a sequence \( F_1, F_2, ..., F_n, ... \), assumes as values sequences of numbers
\[ y^{(1)}, y^{(2)}, ..., y^{(n)}, ..., \]
for which
\[ y^{(n)} = \begin{cases} 1 & \text{if } y \in F_n, \\ 0 & \text{if } y \in (-F_n). \end{cases} \]

(3.4)

The novelty of the membership function vis-à-vis the characteristic function is the multivaluedness of the former.

**Definition III.8.** (1) We choose to call a function \( \mu_A : X \to L \) mapping a nonvoid set \( A \) into a complete Brouwer lattice (algebra) \( L \) a grading function. It coincides with Zadeh's membership function if \( L \) reduces to the unit interval.

(2) The pair \((A, \mu)\) is called a fuzzy set in space \( X \).

If \( X \) happens to be a topological space, \( \mu \) becomes a mapping of a topological space into a Brouwerian algebra and we may avail ourselves of the excellent results of Kuratowski [18]. Kwakernaak [21] has suggested an interesting modification of Definition III.8 by projecting \( X \) onto a universe of discourse \( P \) and mapping the result into the unit interval, which amounts to a representation of the grading function \( \mu \) as the composition \( t \circ a \), where \( a : X \to P \) at \( t : P \to [0, 1] \). However, no proof of the com-
mutation of the diagram \((X \to P \to [0, 1] \leftarrow X)\) has so far been offered. In any case, the extended grading function corresponding to (3.3) would be of the form

\[
\chi'(y) = 1 - \frac{1}{1 + \max(0, \|y - a\|)}.
\] (3.5)

We shall now advance the

**FUNDAMENTAL ASSUMPTION.** The commutation gap \(\delta\) bears the same relation to quantum mechanics as the grading function \(\mu\) to the fuzzy set theory.

Both \(\delta\) and \(\mu\) are grading functions; they are necessary conditions for the existence of uncertainties and reflect the “subjectivity” of sentences in the respective theories. Let us review the applicability of some of the fundamental laws to fuzzy sets. Most important, the tertium-non-datur law fails in fuzzy set theory; and, since \(A \Rightarrow \neg \neg A\), but not conversely, the law of involution fails as well. We shall also reject transitivity. On the other hand, the law of contradiction, the law of syllogism, the law of contraposition and both DeMorgan laws remain in force. This was the argument of Moisil [27] for calling the collection of fuzzy subsets \(A\) of a space \(X\) a DeMorgan lattice. However, in his formalization involution is an admissible law. The grading function is characterized by isotonicity. This has, of course, a bearing on the properties of the fuzzy measure which will no longer be additive or \(\sigma\)-additive, but merely monotone (see Sugeno [15]).

Fuzzy logic may be constructed following the principles of the Łukasiewicz multivalued logic, which requires the introduction of conjunction, disjunction, meet \(x \land y = \min(x, y)\), join \(x \lor y = \max(x, y)\) and one of two possible implications (see Wechler [39]):

Łukasiewicz implication \(x \rightarrow^L y = \min(1, 1 - x + y)\), which exists only if \(x, y\) run through \([0, 1]\) (this is equivalent to Theorem 1 in Lee [22]);

remainder implication

\[
x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y, \end{cases}
\]

valid in the whole linearly ordered set \([0, 1]\) (residuation in Wechler's terminology).

We recall that every linearly ordered set is a residual lattice with respect to composition as well as to meet, join, and

\[
x \rightarrow^L y = \begin{cases} 1 & \text{if } x \geq y \\ x & \text{if } x < y, \end{cases}
\]

provided that the axiom \((x \to y) \cup (y \to x) = 1\) is satisfied.
It is apparent that the two implications exhibit different behaviour in case of multi-valuedness.

OBSERVATION. To the multiplicity of implications in fuzzy logic corresponds a multiplicity of six relational implications in the quantum logic of Birkhoff-von Neumann (see Kaaz [15]). This in itself is an indication of the existence of a basic uncertainty.

It is also appropriate and timely to mention Kuratowski's proof [17], that the nondefinability of a well ordering in a first order logic (a theorem of Tarski's) is a consequence of the existence of analytic non-Borelian sets. This may—sentence by sentence—be carried over to the realm of fuzzy sets. But, to do it rigorously, a formal definition of truth (assumed to exist in the Kwakernaak treatment of the membership function) for sentences in fuzzy vernacular is required. According to Tarski [37], a classical definition of truth is a special case of the definition of satisfaction. If this principle also holds for fuzzified sentences, then a specialization of the definition of fuzzy satisfaction already provided by Lee [22] would answer our need.

For the benefit of the reader, we shall add a remark on the subsets of the real line. Suslin space is the name given to every metric space that is a continuous image of the irrationals. If $X$ is a Polish space (i.e., a separable and topologically complete space), then the family of all Suslin subsets is called a family of analytical sets (or $A$-sets, in honour of Alexandroff). Now, non-Borelian sets are known to exist in the space of irrationals. Hence Borelian sets must be in minority in the set of reals, and this, in turn, entails the existence of nonmeasurable sets in $\mathbb{R}$. Deep investigations show that these facts are a consequence of the axiom of choice, and, as long as this axiom is a canon in the valid set theory, most of the interesting problems in game theory will remain undeterminate. This reveals precisely the dilemma of complementary systems in the face of non-measurable sets (consult also Banach's problem of measure [4]).

All of these difficulties vanish if we accept the Axiom of Determinateness (AD) proposed by Mycielski [29], in which case all sets on $\mathbb{R}$ become measurable. This axiom would take the place of the axiom of choice in the Zermelo-Fraenkel set theory; it would, however, introduce restrictions in the foundations of mathematics of another kind and perhaps more severe than the uncertainties due to the axiom of choice. We have mentioned this simply because the range of Zadeh's membership function happens to be a homeomorphic image of the real line.
We now proceed to formulate a rather fundamental relation reflecting
the irreducible amount of uncertainty in fuzzy systems using the concept of
negligible set, the rules governing Brouwerian algebra and logic and
some general results touched upon in the previous sections. Let us denote
by \( H \) the greatest negligible set and consider the topologized Brouwerian
algebra

\[
\mathcal{Y} = (2^Y, \cup, \cap, \setminus, \sqcup, \sqcap),
\]

in which \( Y \) is understood to be a topological space, \( 2^Y \) being endowed with
Vietoris topology; \( \cup \) and \( \cap \) are the lattice operations; \( \setminus \), the pseudo-difference, is identified with the operation \( A - B, A, B \in 2^Y \); \( \emptyset \), the isolated
void set, is set equal to \( \sqcup \), and \( Y \) to \( \sqcap \). We shall consider \( \mathcal{Y} \) as the range of
the set-valued function

\[
F: T \times X \to 2^Y,
\]

and the subsets \( F(t, x) \), closed in \( Y \), to be orientor fields satisfying the dif-
ferential inclusion

\[
\dot{x} \in F(t, x), \quad x(t_0) = x_0,
\]

which is easily recognized as a generalized differential equation for control
systems if we put

\[
\dot{x} = f(t, x, u) \in f(t, x, U(t, x)) = F(t, x).
\]

The hinted at generalization provides for the set of controls to be a
function of \( t \) and \( x \). Since \( u \) is then a hidden parameter, the solution for the
trajectory is thus easier to obtain. A number of selection methods and fixed
point theorems assist this task. But this will be the subject of a subsequent
paper. Remembering that the tertium-non-datur law fails in Brouwerian
logic, we are free to consider the set \( Y - A \cap \overline{A} \) to be in any way nonvoid.
By putting

\[
Y - A \cap \overline{A} \supset H \quad (4.4)
\]

we obtain a relation indicating the lowest limit of uncertainty for fuzzy sets.
This relation is fundamental in the same degree as \( H \). Its similarity to the
Heisenberg uncertainty relations, derived on a completely different basis, is
striking for the following reasons:

1. \( (4.4) \) is an inequality relation just like (1.1) and (1.2).
(2) It is a product relation of "complementary quantities" as relations (1.1) and (2.1).

(3) The logics underlying (1.1) and (1.2) and (4.4) are non-Boolean; conversion to Boolean logic is obtained in the limit case \( h = 0 \) and correspondingly \( H = \emptyset \).

(4) In the sense of property (3), the Gleason measure in quantum mechanics and the monotone measure in fuzzy set theory (see Sugeno [15]) reduce to Lebesgue measures.

(5) The uncertainties in quantum logic and in fuzzy set logic give rise to more than one implication relation (six in the former and at least two in the latter).

(6) One gap function measures the commutativity gap in quantum mechanics, the other measures the determinability gap due to fuzziness. Both represent necessary adjustments between the laws of quantum mechanics, respectively fuzzy set theory, on the one hand, and those of classical analysis (mechanics), on the other hand. Because of the limited information acquisition by means of the tools of one discipline in the domain of the other, only one of two complementary elements may be determined at a time; the freedom of preference introduces the element of subjectivity. Hence our statements apply only to the object of study and may not be generalized.

**Interpretation of Relation (4.4)**

To estimate the semantical content of the relation (4.4), it is quite instructive to ponder it from the mathematical (i) physical (ii), and intuitive (iii) points of view.

(i) The development of formal mathematical systems from Hilbert to Gödel has brought to the fore as the most important criteria facing mathematical theories their consistency (freedom from contradiction), completeness and decidability.

A formalized theory \( T \) is said to be consistent if it is impossible to obtain a formal proof in \( T \) of a formula \( \alpha \) and for its negation \( \neg \alpha \) simultaneously. For, if a theory admits a contradiction but for two sentences (i.e., formulas containing no free variables) \( \alpha \) and \( \beta \), then it is not hard to prove, using the laws of contradiction, tertium-non-datur and the logical implication \((\alpha \Rightarrow \beta) \equiv (\neg \alpha \lor \beta)\), that every conceivable statement would become a theorem of that theory. Needless to say that such a theory is worthless.

A theory \( T \) is said to be complete if there exists a proof for every sentence (more precisely: for the sentence or for its negation) in the formalized language of \( T \). Consequently, \( T \) is called incomplete if it contains an undecidable sentence. Finally, \( T \) is considered to be decidable if there is a
method by which may be ascertained in a finite number of steps whether an arbitrary formula of \( T \) is a theorem or not. If a decidability algorithm exists, the investigation reduces to mere mechanical operations.

Now, an important theorem due to Gödel states that the consistency proof of every formalized theory containing the arithmetic of natural numbers may only be carried out on the basis of a mathematical theory that is more extensive than the one whose consistency is to be proved. In particular, the consistency proof of formalized arithmetic may be obtained only on the basis of a mathematical theory containing all of arithmetic and, necessarily, additional theorems not belonging to arithmetic. Since every physical and technical theory involves the arithmetic of natural numbers, the proof of its consistency demands the existence of formulas from without itself; but, the consistency of the arithmetic of natural numbers itself cannot be proved in spite of the conviction that nothing is more consistent than the counting: 1, 2, 3,.... Worse still, we have to accept that the arithmetic of natural numbers is incomplete and undecidable, and, as shown under strengthened conditions, every consistent formalized theory embracing the arithmetic of natural numbers is incomplete.

In the face of these facts, mathematical ingenuity becomes invaluable.

(ii) Long before the discoveries of Gödel, physicists (very much concerned about the consistency of their science) have introduced the doctrine that a physical statement may become a law in the 3-dimensional space if and only if it exhibits invariance vis-à-vis the Lorentz transformation in the 4-dimensional Minkowski space. The consistency proof is here obtained by the theorem "the physical statement is Lorentz invariant in a higher dimensional space."

(iii) Consider now a hypothetical ideal vessel filled with an ideal liquid of some sort. Looking at this system "microscopically," we would say that the vessel is truly filled in the state of overflow, for otherwise—even if one drop only is missing—the assertion of an ideally filled vessel would be false. Consider the limit case when a single liquid drop decides on the ideal filling of the vessel. An observer of that drop is performing a perfect observation using an extremely accurate instrument, but he will be blind to the complementary quantity of liquid whose percentage change is unmeasurably small.

It is contemplation of this sort which leads us to the conviction that the classes of fuzzy sets are never disjoint and, therefore, transitivity \(^1\) would be ill applied in fuzzy set theory. Indeed, Dedekind's concepts of density and continuity would require suitable replacements. This implies, in turn, that actual fuzzy sets will exhibit indeterminacies very much greater than the uncertainty quantum \( H \) in (4.4).

\(^1\) Transitivity of order relation.
REFERENCES