Topology and its Applications 157 (2010) 2572–2579

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Tree maps having chain movable fixed points

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ARTICLE INFO

Article history:
Received 9 February 2010
Received in revised form 21 July 2010
Accepted 21 July 2010

Keywords:
Chain recurrent point
Chain reachable set
Chain equivalent set
Chain movable fixed point
Turbulence
Topological entropy

ABSTRACT

In this paper we discuss some basic properties of chain reachable sets and chain equivalent sets of continuous maps. It is proved that if $f: T \to T$ is a tree map which has a chain movable fixed point v, and the chain equivalent set $CE(v, f)$ is not contained in the set $P(f)$ of periodic points of f, then there exists a positive integer p not greater than the number of points in the set $\mathrm{End}([CE(v, f)]) - P_v(f)$ such that f^p is turbulent, and the topological entropy $h(f) \geqslant (\log 2)/p$. This result generalizes the corresponding results given in Block and Coven (1986) [2], Guo et al. (2003) [6], Sun and Liu (2003) [10], Ye (2000) [11], Zhang and Zeng (2004) [12]. In addition, in this paper we also consider metric spaces which may not be trees but have open subsets U such that the closures $\overline{U}$ are trees. Maps of such metric spaces which have chain movable fixed points are discussed.

1. Introduction

Let (X, d) be a metric space. Denote by $C^0(X)$ the set of all continuous maps from X to itself. For any $f \in C^0(X)$ and $x \in X$, the *orbit* of x under f, denoted by $O(x, f)$, is the set $\{f^n(x)\colon\ n = 0, 1, 2, \ldots\}$, where $f^0 = id_X$, $f^1 = f$, and $f^n = f \circ f^{n-1}$ $(n \geqslant 2)$ is the n-fold composition of f. Let $\mathbb{N}$ be the set of all positive integers. For any $n \in \mathbb{N}$, write $\mathbb{N}_n = \{1, \ldots, n\}$. A point $x \in X$ is called a *periodic point* of f with *period* n if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leqslant i < n$. $x \in X$ is called a *fixed point* of f if $f(x) = x$. Let $\mathrm{Fix}(f)$ be the set of all fixed points of f, let $P_n(f)$ be the set of all periodic points of f with period n, and let $P(f) = \bigcup_{n=1}^{\infty} P_n(f)$. A map $f \in C^0(X)$ is called a *periodic homeomorphism* of *period* n if $\mathrm{Fix}(f^n) = X$ and $\mathrm{Fix}(f^i) \neq X$ for $1 \leqslant i < n$.

For any $x, y \in X$ and $\varepsilon > 0$, a sequence $(x_0, x_1, \ldots, x_n)$ of points in X with $n \geqslant 1$ is called an ε*-pseudo orbit* or an ε*-chain* of f from x to y if $x_0 = x$, $x_n = y$ and $d(x_i, f(x_{i-1})) < \varepsilon$ for each $i \in \mathbb{N}_n$. Write

$$W_\varepsilon(x, f) = \{w \in X\colon \text{there is an } \varepsilon\text{-chain of } f \text{ from } x \text{ to } w\}, \tag{1.1}$$

and

$$W(x, f) = \bigcap \bigl\{W_\varepsilon(x, f)\colon\ \varepsilon > 0\bigr\}. \tag{1.2}$$

$W(x, f)$ is called the *chain reachable set* of x under f, and every point in $W(x, f)$ is called a *chain reachable point* of x. A point $x \in X$ is called a *chain recurrent point* of f if $x \in W(x, f)$. Denote by $CR(f)$ the set of all chain recurrent points of f. A map $f \in C^0(X)$ is said to be *pointwise chain recurrent* if $CR(f) = X$. Two points x and y are said to be *chain equivalent* under f if $y \in W(x, f)$ and $x \in W(y, f)$. Let

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2010.07.029

$$CE(x,f)=\begin{cases}\{y\in X\colon\ y \text{ is chain equivalent to } x \text{ under } f\}, & \text{if } x\in CR(f);\\ \emptyset, & \text{if } x\in X-CR(f).\end{cases} \tag{1.3}$$

$CE(x, f)$ is called the *chain equivalent set* of x under f. A fixed point x of f is said to be *chain movable* if $CE(x, f) - \{x\} \neq \emptyset$. Evidently, we have

$$CE(x,f)\subset W(x,f),\quad \text{for any } x\in X. \tag{1.4}$$

A metric space A is called an *arc* if there is a homeomorphism $h : [0, 1] \to A$. In this case, the points $h(0)$ and $h(1)$ are called the *endpoints* of A, and we write $\partial A = \mathrm{End}(A) = \{h(0), h(1)\}$, $\mathrm{Int}(A) = A - \partial A$. A continuous map f from a metric space X to itself is said to be *turbulent* in a subspace Y of X if there exist two arcs A_1 and A_2 in Y such that $A_1 \cap A_2 = \partial A_1 \cap \partial A_2$ and $f(A_1) \cap f(A_2) \supset A_1 \cup A_2$.

A connected metric space G is called a *graph* if there exist finitely many arcs $A_1, A_2, \dots, A_n$ in G such that $G = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \partial A_i \cap \partial A_j$ for $1 \leqslant i < j \leqslant n$. Let S^1 be the unit circle in the complex plane $\mathbb{C}$. A graph C is called a *circle* if it is homeomorphic to S^1. A graph T is called a *tree* if it contains no circle. Let G be a graph, $x \in G$, and U be a neighborhood of x in G such that the closure $\overline{U}$ is a tree. The number of connected components of $\overline{U} - \{x\}$, denoted by $val_G(x)$ or $val(x)$, is called the *valence* of x in G. The point x is called an *endpoint* of G if $val(x) = 1$. Denote by $\mathrm{End}(G)$ the set of all endpoints of G, and $\mathrm{Int}(G) = G - \mathrm{End}(G)$. A continuous map from a graph (resp. a tree, resp. an interval) to itself is called a *graph* (resp. *tree*, resp. *interval*) *map*.

The study of chain recurrence and turbulence of maps has attracted a lot of attention, and many interesting results were given, for instance, see [1–9] and [10–12]. In [2], Block and Coven proved that, if an interval map f is pointwise chain recurrent, then either f^2 is the identity map, or f^2 is turbulent. Guo, Zeng and Hu [6] proved that, if f is a pointwise chain recurrent 3-star map, then either f^{12} is the identity map or f^{12} is turbulent. Sun and Liu [10] studied chain equivalent sets of tree maps. Ye [11] studied topological entropies of transitive tree maps. In [12], pointwise chain recurrent tree maps were discussed.

In this paper we will study tree maps having chain movable fixed points. For any finite set V, denote by $N(V)$ the number of elements in V. For any non-empty subset Y of a tree T, denote by $[Y]$ or $[Y]_T$ the least connected closed subset of T containing Y. Our main result is the following theorem, which is a generalization of the corresponding results given in [2,6,10–12].

Theorem 4.2. *Let $f : T \to T$ be a tree map which has a chain movable fixed point v, let $P_v(f)$ be the connected component of $P(f)$ containing v, and let $CE(v, f)$ be defined as in* (1.3). *Write $\mu = N(\mathrm{End}(P_v(f)) - \{v\})$ and $m = N(\mathrm{End}([CE(v, f)]) - P_v(f))$.*

(1) *If the chain equivalent set $CE(v, f) \subset P_v(f)$, then $CE(v, f) = P_v(f) \subset \bigcup_{i=1}^{\mu} P_i(f)$, and $f|CE(v, f)$ is a periodic homeomorphism.*
(2) *If the chain equivalent set $CE(v, f) \not\subset P_v(f)$, then there exists $p \in \mathbb{N}_m$ such that f^p is turbulent, and the topological entropy $h(f) \geqslant (\log 2)/p$.*

In addition, in this paper we also consider compact connected metric spaces linked by stars. Maps of such spaces which have chain movable fixed points are discussed.

2. Chain reachable sets and chain equivalent sets

In this section we discuss some basic properties of chain reachable sets and chain equivalent sets. Let (X, d) be a metric space. For any non-empty subset Y of X, any $x \in X$ and any $\varepsilon > 0$, write $B(x, \varepsilon) = \{y \in X\colon\ d(y, x) < \varepsilon\}$, and $B(Y, \varepsilon) = \{y \in X\colon\ d(y, Y) < \varepsilon\}$. Denote by $\overline{Y}$ and $\mathring{Y}$ the closure and the interior of Y in X, respectively. For any $f \in C^0(X)$, the subset Y of X is said to be *f-invariant* if $f(Y) \subset Y$. Write

$$V_\varepsilon(x,f,1)=B\big(f(x),\varepsilon\big), \tag{2.1}$$

$$V_\varepsilon(x,f,n+1)=B\big(f\big(V_\varepsilon(x,f,n)\big),\varepsilon\big), \tag{2.2}$$

$$W_\varepsilon(x,f,n)=\bigcup_{i=1}^{n}V_\varepsilon(x,f,i). \tag{2.3}$$

Then $\bigcup_{n=1}^{\infty} W_\varepsilon(x, f, n) = \bigcup_{n=1}^{\infty} V_\varepsilon(x, f, n)$, and

$$W_\varepsilon(x,f,n+1)=W_\varepsilon(x,f,n)\cup B\big(f\big(W_\varepsilon(x,f,n)\big),\varepsilon\big)\quad \text{for all } n\geqslant 1. \tag{2.4}$$

Comparing (1.1) with (2.1) and (2.2), we have

$$W_\varepsilon(x,f)=\bigcup_{n=1}^{\infty}V_\varepsilon(x,f,n)=B\big(f(x),\varepsilon\big)\cup B\big(f\big(W_\varepsilon(x,f)\big),\varepsilon\big). \tag{2.5}$$

The following lemma is obvious, and the proof is omitted.

Lemma 2.1. *For any $f \in C^0(X)$ and any $x \in X$, both the chain reachable set $W(x, f)$ and the chain equivalent set $CE(x, f)$ are f-invariant closed subsets of X. Furthermore, if X is compact, then $W(x, f) = f(W(x, f)) \cup \{f(x)\}$ and $CE(x, f) = f(CE(x, f))$.*

Proposition 2.2. *Let (X, d) be a compact metric space, $f \in C^0(X)$, $x \in X$, $X_0 = W(x, f) \cup \{x\}$, and let $f_0 = f|X_0 : X_0 \to X_0$. Then $W(x, f) = W(x, f_0)$, that is, the chain reachable sets of x under f and under f_0 are the same.*

Proof. It suffices to show $W(x, f) \subset W(x, f_0)$ since $W(x, f_0) \subset W(x, f)$ is obvious. By (1.1), we have

$$W_\varepsilon(x, f) \supset \overline{W_{\varepsilon'}(x, f)} \quad \text{for any } \varepsilon > \varepsilon' > 0. \tag{2.6}$$

It follows from (2.6) and (1.2) that

$$W(x, f) = \bigcap_{\varepsilon>0} \overline{W_\varepsilon(x, f)}. \tag{2.7}$$

For any given $\varepsilon > 0$, since X is compact, there exists $\mu = \mu(\varepsilon) \in (0, \varepsilon/3]$ such that

$$f\big(B(y, \mu)\big) \subset B\big(f(y), \varepsilon/3\big) \quad \text{for all } y \in X.$$

By (2.6) and (2.7), there exists $\delta = \delta(\mu(\varepsilon)) \in (0, \mu] \subset (0, \varepsilon/3]$ such that

$$W_\delta(x, f) \subset B\big(W(x, f), \mu\big). \tag{2.8}$$

For any $w \in W(x, f)$, take a δ-chain $(x_0, x_1, \ldots, x_n)$ of f from x to w. Note that $\{x_1, \ldots, x_n\} \subset W_\delta(x, f)$. By (2.8), for $i = 1, \ldots, n-1$, there is a point $y_i \in W(x, f)$ such that $d(y_i, x_i) < \mu$. Put $y_0 = x_0 (= x)$ and $y_n = x_n (= w)$. It is easy to check that $(y_0, y_1, \ldots, y_n)$ is an ε-chain in $W(x, f) \cup \{x\}$. Thus we have $w \in W(x, f_0)$, and hence $W(x, f) \subset W(x, f_0)$. □

Proposition 2.3. *Let (X, d) be a compact metric space, and $f \in C^0(X)$. Then*

(1) *$CR(f) = CR(f|CR(f))$;*
(2) *$CE(x, f) = CE(x, f|CE(x, f))$ for any $x \in CR(f)$.*

A proof of (1) of Proposition 2.3 can be found in [1, p. 117], and (2) of Proposition 2.3 can be proved by an argument analogous to the proof of Proposition 2.2.

Proposition 2.4. *Let X be a compact metric space, and $f \in C^0(X)$. If v is a chain movable fixed point of f, then v is not isolated in the chain equivalent set $CE(v, f)$.*

Proof. Take a point $y \in CE(v, f) - \{v\}$. For any $\varepsilon \in (0, d(y, v)]$, there exists $\delta \in (0, \varepsilon/2]$ such that $f(B(v, \delta)) \subset B(v, \varepsilon/2)$. For any $n \in \mathbb{N}$, take a (δ/n)-chain $(x_{0n}, x_{1n}, \ldots, x_{k_n n})$ of f from v to y. Then there is a point $w_n \in \{x_{0n}, x_{1n}, \ldots, x_{k_n n}\} \cap (\overline{B(v, \varepsilon)} - B(v, \delta))$. Let w be a cluster point of the sequence $(w_1, w_2, \ldots)$. Then $w \in CE(v, f) \cap \overline{B(v, \varepsilon)} - B(v, \delta)$. Thus v is not isolated in $CE(v, f)$. □

Proposition 2.5. *Let (X, d) be a compact metric space, Y be a non-empty closed subset of X, and $f \in C^0(X)$. If $f(Y) \subset \mathring{Y}$, then, for any $x \in Y$, the chain reachable set $W(x, f) \subset \mathring{Y}$.*

Proof. The proposition is trivial if $Y = X$. We now assume that Y is a proper subset of X. Then $f(Y)$ and $X - \mathring{Y}$ are disjoint non-empty closed sets. Let $\varepsilon_0 = d(f(Y), X - \mathring{Y})$. Then $\varepsilon_0 > 0$. For any $x \in Y$ and $y \in W(x, f)$, let $(x_0, x_1, \ldots, x_n)$ be an ε_0-chain of f from x to y. Then $x_0 = x \in Y$. For $i \in \mathbb{N}_n$, if $x_{i-1} \in Y$, then $d(x_i, f(Y)) \leqslant d(x_i, f(x_{i-1})) < \varepsilon_0$, which implies that $x_i \notin X - \mathring{Y}$, i.e. $x_i \in \mathring{Y}$. Thus we have $y = x_n \in \mathring{Y}$, and hence $W(x, f) \subset \mathring{Y}$. □

Lemma 2.6. *Let (X, d) be a metric space, and $f : X \to X$ be a uniformly continuous map. Then, for any $n \in \mathbb{N}$, there exists a function $\delta_n : (0, \infty) \to (0, \infty)$ such that*

$$d\big(x_n, f^n(x_0)\big) < \varepsilon \quad \textit{for any } \varepsilon > 0 \textit{ and any } \delta_n(\varepsilon)\textit{-chain } (x_0, x_1, \ldots, x_n) \textit{ of } f.$$

Proof. If $n = 1$ then the lemma holds by taking δ_1 to be the identity map of $(0, \infty)$. We now assume that the lemma holds for some $n \in \mathbb{N}$, and the function δ_n has been chosen. For any $\varepsilon > 0$, since f is uniformly continuous, there exists $\mu(\varepsilon) > 0$ such that $d(f(x), f(y)) < \varepsilon/2$ for any $x, y \in X$ with $d(x, y) < \mu(\varepsilon)$. Let $\delta_{n+1}(\varepsilon) = \min\{\delta_n(\mu(\varepsilon)), \varepsilon/2\}$. Then, for any $\delta_{n+1}(\varepsilon)$-chain $(x_0, x_1, \ldots, x_n, x_{n+1})$, one has $d(x_n, f^n(x_0)) < \mu(\varepsilon)$, and $d(x_{n+1}, f^{n+1}(x_0)) \leqslant d(x_{n+1}, f(x_n)) + d(f(x_n), f^{n+1}(x_0)) < \delta_{n+1}(\varepsilon) + \varepsilon/2 \leqslant \varepsilon$. □

Lemma 2.7. *Let (X, d) be a metric space, and $f : X \to X$ be a uniformly continuous map. Then, for any $v \in \mathrm{Fix}(f)$ and any $n \in \mathbb{N}$, we have*

$$W(v, f^n) = W(v, f), \qquad CE(v, f^n) = CE(v, f), \quad \text{and} \quad CR(f) = CR(f^n).$$

Proof. Let $\delta_n : (0, \infty) \to (0, \infty)$ be as in Lemma 2.6.

(1) For any $y \in W(v, f)$ and any $\varepsilon > 0$, let $(x_0, x_1, \dots, x_m)$ be a $\delta_n(\varepsilon)$-chain of f from v to y. If $m/n \notin \mathbb{N}$, we can replace $(x_0, x_1, \dots, x_m)$ by $(v, \dots, v, x_0, x_1, \dots, x_m)$. Thus we may assume that $m/n \in \mathbb{N}$. By Lemma 2.6, $(x_0, x_n, x_{2n}, \dots, x_m)$ is an ε-chain of f^n from v to y. This means that $y \in W(v, f^n)$, and we have $W(v, f) \subset W(v, f^n)$. Conversely, $W(v, f^n) \subset W(v, f)$ is obvious. Thus $W(v, f^n) = W(v, f)$.

(2) For any $y \in X$ and $v \in \mathrm{Fix}(f)$, it is analogous to show that $v \in W(y, f^n)$ if and only if $v \in W(y, f)$. This with $W(v, f^n) = W(v, f)$ implies $CE(v, f^n) =: CE(v, f)$.

(3) Consider any $x \in CR(f)$, any $\varepsilon > 0$ and any $\delta_n(\varepsilon)$-chain $(x_0, x_1, \dots, x_k)$ of f from x to x. Let $x_{jk+i} = x_i$ for every $j \in \mathbb{N}$ and $i \in \mathbb{N}_k$. Then $(x_0, x_n, x_{2n}, \dots, x_{kn})$ is an ε-chain of f^n from x to x. Thus $CR(f) \subset CR(f^n)$. Conversely, $CR(f^n) \subset CR(f)$ is obvious. Hence we have $CR(f) = CR(f^n)$. □

3. Maps of spaces linked by stars

In this section we consider such a metric space X that may not be a tree but has an open subset U of which the closure $\overline{U}$ is a tree.

Definition 3.1. Let (X, d) be a metric space which contains an arc A with a homeomorphism $\lambda : [0, 1] \to A$. Let $v = \lambda(0)$ and $J = A - \lambda(1)$. The semi-open arc J is called a *twig* in X and v is called the *endpoint* of the twig J if $d(\lambda([0, t]), X - J) > 0$ for any $t \in (0, 1)$.

Proposition 3.2. *Let (X, d) be a compact metric space which has a twig J with endpoint v, and $f \in C^0(X)$. Suppose that v is a chain movable fixed point of f, $v \notin f(X - J)$, and $B(v, \varepsilon) - \mathrm{Fix}(f) \neq \emptyset$ for any $\varepsilon > 0$. Then f is turbulent in J.*

Proof. For the convenience of statement, we may assume that $J = [0, 1)$, $\overline{J} = [0, 1]$ and $v = 0$. Since $0 \notin f(X - J)$, there exists $x_1 \in (0, 1)$ such that $f(X - J) \cap [0, x_1] = \emptyset$. Since 0 is a chain movable fixed point of f, by Proposition 2.4, there is $v_2 \in (0, x_1] \cap CE(0, f)$ such that $f([0, v_2]) \subset [0, x_1]$. Noting that $v_2 \in W(0, f)$ and $0 \in W(v_2, f)$, from Proposition 2.5 we get

$$f([0, c]) \not\subset [0, c) \quad \text{for any } c \in (0, v_2] \tag{3.1}$$

and

$$f(X - [0, c)) \not\subset X - [0, c] \quad \text{for any } c \in (0, v_2]. \tag{3.2}$$

Noting that $[0, \varepsilon] - \mathrm{Fix}(f) \neq \emptyset$ for any $\varepsilon \in (0, v_2]$, from (3.1) we obtain

Claim 1. For any $\varepsilon \in (0, v_2]$, there exists $x \in (0, \varepsilon]$ such that $f(x) > x$.

By Claim 1, there exist $0 \leqslant y_3 < y_4 \leqslant v_2$ such that $f(y_3) = y_3$,

$$f(x) > x \quad \text{for all } x \in (y_3, y_4],$$

and $\max(f([y_3, y_4])) > \max(f([0, y_3]))$. Write $y_5 = \min(f([y_4, 1]) \cap J)$. If $y_5 > y_3$, then, for any $c \in (y_3, \min\{y_4, y_5\})$, we have $f(X - [0, c)) \subset X - [0, c]$. But this contradicts (3.2). Thus $y_5 \leqslant y_3$, and there exists $y_6 \in (y_4, 1)$ such that $f(y_6) = y_3$ and $f((y_3, y_6)) \cap [0, y_3] = \emptyset$. Write $y_7 = \max(f([y_3, y_6]) \cap [0, 1])$. Then $y_7 \geqslant f(y_4) > y_4$.

If $y_7 < y_6$, take a point $c_0 \in (y_7, y_6)$. Then we have $f([0, c_0]) \subset [0, y_7] \subset [0, c_0)$, which with (3.1) implies $v_2 \in [y_4, c_0)$. Write $c_1 = \min(f([y_4, c_0]))$. Then $c_1 > y_3$. Take a point $c_2 \in (y_3, \min\{c_1, y_4\})$. Then $v_2 \in (c_2, c_0)$, $f([c_2, c_0]) \subset (c_2, c_0)$, and it follows from Proposition 2.5 that $W(v_2, f) \subset (c_2, c_0)$. But this contradicts that $0 \in W(v_2, f)$. Thus $y_7 \geqslant y_6$, and there exists $y_8 \in (y_3, y_6)$ such that $f(y_8) = y_6$. Let $A_1 = [y_3, y_8]$, and $A_2 = [y_8, y_6]$. Then A_1 and A_2 are arcs in J, $A_1 \cap A_2 = \partial A_1 \cap \partial A_2$, and $f(A_1) \cap f(A_2) \supset A_1 \cup A_2$. Hence, f is turbulent in J. Proposition 3.2 is proven. □

For $n \in \mathbb{N}$, write $S_n = \{z \in \mathbb{C}\colon\ z^n \in [0, 1]\}$, and $\mathrm{Ent}_S(S_n) = \{z \in \mathbb{C}\colon\ z^n = 1\}$. The subspace S_n of the complex plane $\mathbb{C}$ is called the *unit n-star*. Let T be a tree. If there exists a homeomorphism $h : S_n \to T$, then T is called an *n-star*, the point $h(0)$ is called the *center* of T, and we write

$$\mathrm{Cent}(T) = h(0), \qquad \mathrm{Ent}_S(T) = h(\mathrm{Ent}_S(S_n)), \qquad \mathrm{Int}_S(T) = T - \mathrm{Ent}_S(T).$$

Note that an arc A is also a star, but the center of A is dependent on the choice of h. If an endpoint (resp. an interior point) of A is chosen to be the center, then A is a 1-star (resp. 2-star). If T is an n-star with $n \geqslant 2$ (resp. $n = 1$), then $\mathrm{Ent}(T) = \mathrm{Ent}_S(T)$ (resp. $\mathrm{Ent}(T) = \mathrm{Ent}_S(T) \cup \{\mathrm{Cent}(T)\}$).

Definition 3.3. Let X be a compact connected metric space, and $T \subset X$ be an n-star with $n \in \mathbb{N}$. X is called a *space linked by the n-star T* if there exists a continuous map $\lambda : X \to T$ such that $\lambda^{-1}(x) = \{x\}$ for all $x \in T - \mathrm{Ent}_S(T)$.

Note that each tree can be regarded as a space linked by a star. Thus the following proposition also holds for trees.

Proposition 3.4. *Let X be a compact connected metric space linked by an n-star T, $n \in \mathbb{N}$, v be the center of T, and $f \in C^0(X)$. Suppose that v is a chain movable fixed point of f, and $P_v(f)$ is the connected component of $\bigcup_{i=1}^{n} P_i(f)$ containing v. If $f^{-1}(v) = P_v(f) = \{v\}$, then there exist an arc $A \subset T$ and $p \in \mathbb{N}_n$ such that $v \notin \mathrm{Int}(A)$ and f^p is turbulent in A.*

Proof. Let the retraction $\lambda : X \to T$ be as in Definition 3.3. Suppose that $\mathrm{Ent}_S(T) = \{y_1, \ldots, y_n\}$. For $i \in \mathbb{N}_n$, let A_i be the arc in T with endpoints v and y_i, $J_i = A_i - \{y_i\}$, and let $Y_i = \lambda^{-1}(A_i)$. Then Y_i is a compact connected subspace of X, Y_i has a twig J_i with endpoint v, and $Y_i \cap Y_j = \{v\}$ for $1 \leqslant i < j \leqslant n$. Since v is a chain movable fixed point of f, it follows from Proposition 2.4 that there exists $k \in \mathbb{N}_n$ such that $CE(v, f) \cap (J_k - \{v\}) \neq \emptyset$. Because $f^{-1}(v) = \{v\}$, there exists the smallest $p \in \mathbb{N}_n$ such that $f^p(Y_k) \subset Y_k$ and $v \notin f^p(Y_k - J_k)$. By Lemma 2.7, we have $CE(v, f) = CE(v, f^p)$. It is easy to show that $CE(v, f^p) \cap Y_k = CE(v, f^p|Y_k)$. Hence, v is also a chain movable fixed point of the map $f^p|Y_k : Y_k \to Y_k$. Since $P_v(f) = \{v\}$, for any $\varepsilon > 0$, we have $B(v, \varepsilon) \cap Y_k - \mathrm{Fix}(f^p|Y_k) \neq \emptyset$. Thus, by Proposition 3.2, f^p is turbulent in $J_k \subset A_k$. Let $A = A_k$. Then $v \notin \mathrm{Int}(A)$. □

4. Tree maps having chain movable fixed points

Let T be a tree. For any non-empty subset X of T, denote by $[X]$ or $[X]_T$ the least connected closed subset of T containing X. Then $[X]$ is a subtree of T if X contains more than one point. For any $x, y \in T$, write $[x, y] = [\{x, y\}]$, $[x, y) = (y, x] = [x, y] - \{y\}$, and $(x, y) = (x, y] - \{y\}$. Since T is compact, for any $f \in C^0(T)$ and $x \in T$, the set $CR(f)$ of chain recurrent points, the chain equivalent set $CE(x, f)$ and the chain reachable set $W(x, f)$ are independent of the choice of metric on T. Thus we may assume that the metric d on T satisfies

$$d(x, y) = d(x, z) + d(y, z) \quad \text{for any } x, y \in T \text{ and any } z \in [x, y]. \tag{4.1}$$

For any finite subset V of T, denote by $N(V)$ the number of points in V.

Lemma 4.1. *Let $f : T \to T$ be a tree map, v be a fixed point of f, and $P_v(f)$ be the connected component of $P(f)$ containing v. If $P_v(f) - \{v\} \neq \emptyset$, then $P_v(f)$ is a subtree of T, $f(P_v(f)) = P_v(f) \subset \bigcup_{i=1}^{\mu} P_i(f)$ with $\mu = N(\mathrm{End}(P_v(f)) - \{v\})$, and $f|P_v(f) : P_v(f) \to P_v(f)$ is a periodic homeomorphism.*

Proof. Noting that $\bigcup_{i=0}^{\infty} f^i(P_v(f))$ is also a connected subset of $P(f)$, we have $P_v(f) = \bigcup_{i=0}^{\infty} f^i(P_v(f))$, and hence $f(P_v(f)) = P_v(f)$. Write $\mu_0 = N(\mathrm{End}(\overline{P_v(f)}) - \{v\})$. If $P_v(f) \not\subset \bigcup_{i=1}^{\mu_0} P_i(f)$, then there exist $y \in P_v(f)$ with period $p > \mu_0$ and $j \in \mathbb{N}_{p-1}$ such that $O(y, f) \cap [v, y) = \emptyset$ and $y \in (v, f^j(y))$. Therefore, there exists $w_0 \in (v, y)$ such that $f^j(w_0) = y$. This means that $w_0 \notin P(f)$, and hence v and y are not in the same connected component of $P(f)$. But this contradicts that $y \in P_v(f)$. Thus $P_v(f) \subset \bigcup_{i=1}^{\mu_0} P_i(f)$. It follows that $P_v(f)$ is a closed subset of T (and hence, is a subtree of T), $\mu_0 = \mu$, and $f|P_v(f)$ is a periodic homeomorphism. □

The main result of this section is the following theorem.

Theorem 4.2. *Let $f : T \to T$ be a tree map which has a chain movable fixed point v, let $P_v(f)$ be the connected component of $P(f)$ containing v, and let $CE(v, f)$ be defined as in (1.3). Write $\mu = N(\mathrm{End}(P_v(f)) - \{v\})$ and $m = N(\mathrm{End}([CE(v, f)]) - P_v(f))$.*

(1) *If the chain equivalent set $CE(v, f) \subset P_v(f)$, then $CE(v, f) = P_v(f) \subset \bigcup_{i=1}^{\mu} P_i(f)$, and $f|CE(v, f)$ is a periodic homeomorphism.*
(2) *If the chain equivalent set $CE(v, f) \not\subset P_v(f)$, then there exists $p \in \mathbb{N}_m$ such that f^p is turbulent, and the topological entropy $h(f) \geqslant (\log 2)/p$.*

(For the definition of topological entropy, see [1, p. 191].)

Proof. (1) Note that $P_v(f) \subset CE(v, f)$ is always true. Thus, if $CE(v, f) \subset P_v(f)$, then $CE(v, f) = P_v(f)$. Since v is a chain movable fixed point of f, we have $P_v(f) - \{v\} = CE(v, f) - \{v\} \neq \emptyset$. By Lemma 4.1, $f|CE(v, f)$ is a periodic homeomorphism, and $CE(v, f) \subset \bigcup_{i=1}^{\mu} P_i(f)$.

(2) Let the chain reachable set $W(v, f)$ be defined as in (1.2). If $CE(v, f) \not\subset P_v(f)$, then

$$W(v, f) - P_v(f) \supset CE(v, f) - P_v(f) \supset \mathrm{End}\big([CE(v, f)]\big) - P_v(f) \neq \emptyset.$$

For any $\varepsilon > 0$ and $n \in \mathbb{N}$, let $V_\varepsilon(v, f, n)$, $W_\varepsilon(v, f, n)$ and $W_\varepsilon(v, f)$ be defined as in (2.1), (2.2), (2.3) and (2.5). Since $v \in \mathrm{Fix}(f)$, we have $v \in W_\varepsilon(v, f, n) = V_\varepsilon(v, f, n) \subset V_\varepsilon(v, f, n+1)$. Since the metric d on T satisfies (4.1), $W_\varepsilon(v, f, n)$ and $W_\varepsilon(v, f)$ are connected open subsets of T. Hence, by Lemma 2.1, $W(v, f) = \bigcap_{\varepsilon>0} W_\varepsilon(v, f)$ is a subtree of T, and $f(W(v, f)) = W(v, f)$. There are two cases.

Case 1. There is a point $y_m \in W(v, f) - P_v(f)$ such that $f(y_m) \in P_v(f)$.

In this case, we have $y_m \in CE(v, f) - P(f)$. Take a sufficiently small $\varepsilon_0 > 0$ such that $y_m \notin W_{\varepsilon_0}(v, f, m+1)$. For any $k \in \mathbb{N}$, write $\varepsilon_k = \varepsilon_0/2^k$. Then there exists $n_k \geqslant 2$ such that $y_m \in W_{\varepsilon_k}(v, f, n_k + m)$ but $y_m \notin W_{\varepsilon_k}(v, f, n_k + m - 1)$. Let $(x_{k,0}, x_{k,1}, \ldots, x_{k,n_k+m})$ be an ε_k-chain of f from v to y_m. Then

$$x_{k,n_k+i} \in W_{\varepsilon_k}(v, f, n_k + i) - W_{\varepsilon_k}(v, f, n_k + i - 1) \quad \text{for } i = 0, 1, \ldots, m. \tag{4.2}$$

Since T is compact, there exist integers $1 \leqslant k(1) < k(2) < \cdots$ and points $y_0, y_1, \ldots, y_{m-1}$ in $W(v, f)$ such that, for each $i \in \{0, 1, \ldots, m-1\}$, the sequence $(x_{k(j),n_{k(j)}+i})_{j=1}^{\infty}$ converges to y_i. Obviously, we have

$$f(y_{i-1}) = y_i \quad \text{for } i = 1, \ldots, m$$

and

$$\{y_0, y_1, \ldots, y_m\} \subset CE(v, f) - P(f).$$

If there exist integers $0 \leqslant i < q \leqslant m$ such that $y_q \in [v, y_i)$, then, for large enough j, we have

$$y_q \in [v, x_{k(j),n_{k(j)}+i}] \subset W_{\varepsilon_{k(j)}}(v, f, n_{k(j)} + i),$$

which implies that $x_{k(j),n_{k(j)}+m} = y_m = f^{m-q}(y_q) \in W_{\varepsilon_{k(j)}}(v, f, n_{k(j)} + i + m - q) \subset W_{\varepsilon_{k(j)}}(v, f, n_{k(j)} + m - 1)$. But this contradicts (4.2). Therefore, we have

$$y_q \notin [v, y_i) \quad \text{for any integers } 0 \leqslant i < q \leqslant m,$$

and hence, since $\mathrm{End}([CE(v, f)]) - P_v(f)$ has only m points, there exist integers $0 \leqslant i_0 < q_0 \leqslant m$ such that $y_{i_0} \in [v, y_{q_0})$. Note that there is a unique point $w_0 \in P_v(f)$ such that $y_{i_0} \in (w_0, y_{q_0})$ and $[w_0, y_{q_0}] \cap P_v(f) = \{w_0\}$. Let $p = \max(\{k(q_0 - i_0)\colon k \in \mathbb{N}\} \cap \mathbb{N}_{m-i_0})$, and let $y_{m+\beta} = f^\beta(y_m)$, for all $\beta \in \mathbb{N}$. Then

$$p \in \mathbb{N}_{m-i_0} \subset \mathbb{N}_m, \quad q_0 + p > m, \quad \text{and} \quad f^p(y_{q_0}) = y_{q_0+p} \in P_v(f).$$

Since $f^{q_0-i_0}(y_{i_0}) = y_{q_0}$, we have $f^{q_0-i_0}([w_0, y_{i_0}]) \supset [w_0, y_{q_0}] \supset [w_0, y_{i_0}]$, which implies $f^{k(q_0-i_0)}([w_0, y_{i_0}]) \supset [w_0, y_{q_0}]$ for all $k \in \mathbb{N}$. Specially, we have $f^p([w_0, y_{i_0}]) \supset [w_0, y_{q_0}]$. Hence, there exists a point $x_0 \in (w_0, y_{i_0}]$ such that $f^p(x_0) = y_{q_0}$. Let $A_1 = [w_0, x_0]$, $A_2 = [x_0, y_{q_0}]$. Then $A_1 \cap A_2 = \partial A_1 \cap \partial A_2 = \{x_0\}$, and

$$f^p(A_1) \cap f^p(A_2) \supset \left[f^p(w_0), y_{q_0}\right] \cap [y_{q_0}, y_{q_0+p}] \supset [w_0, y_{q_0}] \supset A_1 \cap A_2.$$

This means that f^p is turbulent.

Case 2. There is no point $y_m \in W(v, f) - P_v(f)$ such that $f(y_m) \in P_v(f)$.

In this case, we have $f(W(v, f) - P_v(f)) \subset W(v, f) - P_v(f)$. Let $T_0 = W(v, f)$, and let $f_0 = f|T_0 : T_0 \to T_0$. By Propositions 2.2 and 2.3, we have $W(v, f) = W(v, f_0) \supset CE(v, f) = CE(v, f_0) \supset P_v(f) = P_v(f_0)$. Hence, for the convenience of statement, in the following we may assume that $W(v, f) = T$ and $f_0 = f$.

We now construct an identification space T^* of T as follows. Let $v^* = P_v(f)$, and let $T^* = (T - P_v(f)) \cup \{v^*\}$. Then T^* is a partition of T. Define a metric d^* on T^* by, for any $x, y \in T^* - \{v^*\}$,

(i) $d^*(x, y) = d(x, y)$, if x and y are in the same connected component of $T - P_v(f)$;
(ii) $d^*(x, y) = d(x, P_v(f)) + d(y, P_v(f))$, if x and y are in different connected components of $T - P_v(f)$;
(iii) $d^*(x, v^*) = d(x, P_v(f))$.

Evidently, under this metric d^*, T^* is still a tree. Define a map $f^* : T^* \to T^*$ by $f^*(v^*) = v^*$ and $f^*(x) = f(x)$ for all $x \in T^* - \{v^*\}$. It is easy to check that f^* is continuous. Let $\pi : T \to T^*$ be the natural projection, defined by $\pi(x) = x$ for any $x \in T - P_v(f)$ and $\pi(w) = v^*$ for any $w \in P_v(f)$. Then

$$d^*\big(\pi(x), \pi(y)\big) \leqslant d(x, y) \quad \text{for any } x, y \in T, \tag{4.3}$$

and $f^*\pi = \pi f$, that is, the following diagram is commutative.

$$\begin{array}{ccc} T & \xrightarrow{\quad f \quad} & T \\ \pi \downarrow & & \downarrow \pi \\ T^* & \xrightarrow[\quad f^* \quad]{} & T^* \end{array}$$

Consider any sequence $(x_0, x_1, \dots, x_n)$ of points in T. If $d(f(x_{i-1}), x_i) < \varepsilon$ for some $i \in \mathbb{N}_n$ and some $\varepsilon > 0$, then it follows from (4.3) that $d^*(f^*\pi(x_{i-1}), \pi(x_i)) < \varepsilon$. Conversely, if $d^*(f^*\pi(x_{i-1}), \pi(x_i)) < \varepsilon$ and $v^* \notin [f^*\pi(x_{i-1}), \pi(x_i)]_{T^*}$, then we have $d(f(x_{i-1}), x_i) = d^*(f^*\pi(x_{i-1}), \pi(x_i)) < \varepsilon$. If $d^*(f^*\pi(x_{i-1}), \pi(x_i)) < \varepsilon$ and $v^* \in [f^*\pi(x_{i-1}), \pi(x_i)]_{T^*}$, then there exists an ε-chain $(y_0, y_1, \dots, y_{k_i})$ of f from x_{i-1} to x_i such that $\{y_1, \dots, y_{k_i-1}\} \subset P_v(f)$. Therefore, for any $x, y \in T$ and any $\varepsilon > 0$, there exists an ε-chain of f^* from $\pi(x)$ to $\pi(y)$ if and only if there exists an ε-chain of f from x to y. Thus we have $CR(f^*) = \pi(CR(f))$, and

$$CE\big(\pi(x), f^*\big) = \pi\big(CE(x, f)\big) \quad \text{for any } x \in CR(f).$$

Let $P_{v^*}(f^*)$ be the connected component of $P(f^*)$ containing v^*. Noting that $P(f^*) = \pi(P(f))$ also holds, we have $P_{v^*}(f^*) = \{v^*\}$. Since $f(W(v, f) - P_v(f)) \subset W(v, f) - P_v(f)$, it follows that $(f^*)^{-1}(v^*) = \{v^*\}$. Since $CE(v, f) - P_v(f) \neq \emptyset$, we have

$$CE\big(v^*, f^*\big) - \big\{v^*\big\} = \pi\big(CE(v, f)\big) - \pi\big(P_v(f)\big) = \pi\big(CE(v, f) - P_v(f)\big) \neq \emptyset.$$

Thus v^* is a chain movable fixed point of f^*. By Proposition 3.4, there exist $p \in \mathbb{N}_m$ and arcs A_1^*, A_2^* and A_0^* in T^* such that $A_1^* \cup A_2^* \subset A_0^*$, $v^* \notin \mathrm{Int}(A_0^*)$, and

$$A_1^* \cap A_2^* = \partial A_1^* \cap \partial A_2^*, \qquad \big(f^*\big)^p\big(A_1^*\big) \cap \big(f^*\big)^p\big(A_2^*\big) \supset A_1^* \cup A_2^*.$$

For $i = 0, 1, 2$, let $A_i = \overline{\pi^{-1}(A_i^*) - P_v(f)}$. Then A_i is an arc, $v \notin \mathrm{Int}(A_0)$, $A_1 \cup A_2 \subset A_0$, $A_1 \cap A_2 = \partial A_1 \cap \partial A_2$, and $f^p(A_1) \cap f^p(A_2) \supset A_1 \cup A_2$. Thus f^p is turbulent.

In addition, it is well known that if φ is a turbulent tree map then the topological entropy $h(\varphi) \geqslant \log 2$. Hence we have $h(f^p) \geqslant \log 2$, and by [1, Proposition VIII. 2], we obtain $h(f) \geqslant (\log 2)/p$. Theorem 4.2 is proven. □

From Theorem 4.2 we obtain the following corollary, which was first given by Sun and Liu [10].

Corollary 4.3. *([10, Theorems 2.2 and 2.1]) Let T be a tree with n endpoints, and $f \in C^0(T)$. Suppose that f has a chain movable fixed point v.*

(1) *If $[CE(v, f)] - \bigcup_{i=1}^{n} P_n(f) \neq \emptyset$, then there exist $p \in \mathbb{N}_n$ such that f^p is turbulent.*
(2) *If $[CE(v, f)] - \bigcup_{i=1}^{n-1} P_n(f) \neq \emptyset$ and $v \in \mathrm{End}([CE(v, f)])$, then there exists $p \in \mathbb{N}_{n-1}$ such that f^p is turbulent.*

Proof. Let $P_v(f)$, m and μ be the same as in Theorem 4.2 and its proof. Let $m' = N(\mathrm{End}([CE(v, f)]))$. Then $\max\{m, \mu\} \leqslant m' \leqslant n$, and $\max\{m, \mu\} \leqslant m' - 1 \leqslant n - 1$ if $v \in \mathrm{End}([CE(v, f)])$. By Lemma 4.1, we have $P_v(f) \subset \bigcup_{i=1}^{\mu} P_i(f)$, which leads to $[CE(v, f)] - P_v(f) \supset [CE(v, f)] - \bigcup_{i=1}^{n} P_i(f) \neq \emptyset$, and to $[CE(v, f)] - P_v(f) \supset [CE(v, f)] - \bigcup_{i=1}^{n-1} P_i(f) \neq \emptyset$ if $v \in \mathrm{End}([CE(v, f)])$. Hence, by (2) of Theorem 4.2, there exists $p \in \mathbb{N}_m \subset \mathbb{N}_n$ (and $p \in \mathbb{N}_m \subset \mathbb{N}_{n-1}$ if $v \in \mathrm{End}([CE(v, f)])$) such that f^p is turbulent. □

Let $n = N(\mathrm{End}(T))$. In Theorem 4.2, if $P_v(f) = T$, then $T \subset \bigcup_{i=1}^{n} P_i(f)$. If $P_v(f) \neq T$ but $CR(f) = T$, then $CE(v, f) = T$ and $1 \leqslant m = N(\mathrm{End}(CE(v, f)) - P_v(f)) \leqslant n$. Thus, from Theorem 4.2 we obtain the following corollary, which was given by Zhang and Zeng [12].

Corollary 4.4. *([12, Main Theorem]) Let T be a tree with n endpoints, β_n be the least common multiple of $1, 2, \dots, n$, and let $f \in C^0(T)$. If f is pointwise chain recurrent, then one of the following two statements holds*:

(1) *f^r is the identity map of T, for some $r \in \mathbb{N}_{\beta_n}$;*
(2) *f^r is turbulent, for some $r \in \mathbb{N}_{\beta_n}$.*

It is well known that every tree map $f : T \to T$ has at least a fixed point. If f is transitive (that is, f has an orbit dense in T), then $CE(v, f) = T$ and $P_v(f) = \{v\}$ for any $v \in \mathrm{Fix}(f)$. Thus, from (2) of Theorem 4.2 we obtain the following corollary, which was first given by Ye [11].

Corollary 4.5. *([11, Theorem 2.4]) Let $f : T \to T$ be a transitive tree map, and $n = N(\mathrm{End}(T))$. Let $m = n$ if $\mathrm{Fix}(f) \cap \mathrm{End}(T) = \emptyset$, and $m = n - 1$ if $\mathrm{Fix}(f) \cap \mathrm{End}(T) \neq \emptyset$. Then there exists $p \in \mathbb{N}_m$ such that f^p is turbulent, and hence, $h(f) \geqslant (\log 2)/m$.*

In addition, we have the following proposition, which is also a direct corollary of Theorem 4.2.

Proposition 4.6. *Let $f : T \to T$ be a tree map, v be a fixed point of f, and $m = N(\mathrm{End}(T) - \{v\})$. If the chain equivalent set $CE(v, f)$ is not connected, then there exists $p \in \mathbb{N}_m$ such that f^p is turbulent, and the topological entropy $h(f) \geqslant (\log 2)/p$.*

Example 4.7. Let $f : T \to T$ be a tree map, v be a fixed point of f, and $m = N(\mathrm{End}(T) - \{v\})$. If $\mathrm{End}(T) \subset CE(v, f)$, and there exist a subtree T_0 of T and $n \in \mathbb{N}$ such that $f^n(T_0) \subset \mathrm{Int}(T_0)$, then, by Proposition 4.6, there exists $p \in \mathbb{N}_m$ such that f is turbulent, and the topological entropy $h(f) \geqslant (\log 2)/p$.

Specially, if $f : [0, 1] \to [0, 1]$ is a continuous map, $f(0) = 0$, $1 \in CE(0, f)$, and there exist $0 < r < s < 1$ such that $f([r, s]) \subset (r, s)$, then f is turbulent, and the topological entropy $h(f) \geqslant \log 2$.

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