Superpositions for Nonlinear Operators.*

I. Strong Superpositions and Linearizability

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A notion of superposition for nonlinear operators is defined and strong and weak superpositions are distinguished. The classification and representation of operators having a strong superposition is treated. The algebra induced by the superposition on the domain of operators with strong superpositions, as well as their relation to linearizable operators is delimited. The foundations presented here are intended as a model in the development of more general classes of nonlinear operators, to be treated in two subsequent papers.

1. INTRODUCTION

Does there exist an analogue of the superposition principle, which is valid for a substantial class of nonlinear operators? In recent years the increasing need to cope with nonlinear problems has triggered a renaissance of interest in this basic and intriguing question. The development and history of this field are not well known and much of the literature is relatively inaccessible. Natural as the question is, it must have been considered by several mathematicians. The earliest record was found by Temple [1] and it seems that Vessiot [2] first studied a generalization of the superposition principle in 1893.1 He posed the problem: for which class of ordinary differential equations

\[ \dot{y} = f(t, y) \quad \left[ \frac{dy}{dt} \right], \]  

1.1

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1 In an oral communication R. Bellman credited Abel with starting the field. Abel considered the general question of linearization of nonlinear operators. This work led to the Abel–Schröder functional equation.
does there exist a function $\phi$ such that the general solution, $y$, of (1.1) can be expressed as

$$y = \phi(y_1, \ldots, y_n; k),$$

where the $y_i$, $i = 1, \ldots, n$, are particular solutions of (1.1), $k$ is an arbitrary constant and $\phi$ is independent of the particular solutions used? Using the theory of transitive groups in three variables, Vessiot showed that the only nonlinear differential equation of type (1.1) having such a property is the Riccati equation, whose solutions have the well-known cross-ratio property corresponding to (1.2). This result, though disappointing, characterizes the special position that this remarkable equation occupies. It is the boundary of the linear and essentially nonlinear first-order differential equations.

In the same year Guldberg [3] provided a generalization of Vessiot's result for systems of first-order differential equations.

Still in 1893 Vessiot's efforts attracted a reprimand from Marius Sophus Lie [4] who pointed out that Lie's own theory of fundamental solutions of differential equations includes Vessiot's theory as a special case. Lie did not appreciate the importance of Vessiot's "special case" and his own more general theory does not readily apply to the problem which Vessiot considered. Hence, this criticism does not reflect the value of Vessiot's contribution.

In 1960 Temple [1] reconsidered the problem and refined Vessiot's approach. He concluded that even though Guldberg's theory can be used to extend the class of differential equations soluble by composition—relation (1.2)—still, only very few types of nonlinear equations can be so treated. Temple pointed out a way of classifying nonlinear equations soluble by composition by using and extending Lie's classification of transitive groups of a finite number of variables. He also provided a class of nonlinear differential equations whose general solution cannot be obtained by composition.

It is clear, therefore, that the constraint imposed by (1.2)—that the general solution be expressible in terms of a finite number of particular solutions—needs to be abandoned in favor of something more general. Hence, we make no requirements about the general solution. Instead, we investigate the class of nonlinear equations having the property, that any two solutions of such an equation may be combined, by means of a binary operation, to yield another solution of the same equation, the binary operation being independent of the particular solutions used. Independent studies based on similar notions are given in [5–7] and for some of the ensuing results applications have been found [5, 8–12]. This approach reveals, among other things, a rather natural classification for certain nonlinear operators, the resulting classes having properties, in some respects, analogous to the class of linear operators.

Briefly, the motivation for considering "nonlinear superpositions" is to provide a basis for a more systematic study of nonlinear problems. For some
interesting remarks pertaining to specific applications, the reader is referred to [1, 5, 6].

The superposition principle is formalized in the next section where weak and strong superpositions are distinguished. This principle suggests a classification of some nonlinear operators which is treated in Sections 3, 5, and 6. The algebra induced by a strong superposition on the domain and range of an operator is examined in Section 4. Furthermore, the relation between operators which may be "linearized" and those having a strong superposition is discussed.

In a subsequent paper, also dealing with strong superpositions, generalized affine transformations will be presented. It will be followed by a paper on operators having a weak superposition.

2. Formulation

At present, the most general definition of superposition [13] is

**DEFINITION 1.** Let $T$ be an operator $T : D_1 \to D_2$, $T$ has a superposition with respect to an element $e \in D_2$ iff $\exists$ a binary operation $\ast$ [e.g., an $F : D_1 \times D_1 \to$ with $F(x, y) = x \ast y$] on $D_1$ such that

$$T(x \ast y) = e \quad \text{when} \quad T(x) = T(y) = e.$$  \hspace{1cm} (2.1)

The term *superposition* is used in this definition with some propriety, for if $x$ and $y$ are solutions of the equation $T(u) = e$, then $x \ast y$ is also a solution. Trivial superpositions, in the sense of $F(x_1, x_2) = x_i$, $i = 1$ or $2$, or $F(x_1, x_2) = v$, $v$ fixed, exist but are excluded from the discussion.

Notice that the superposition is specific to an element $e$ in the range space $D_2$. The sets $D_i$, $i = 1, 2$, are usually taken to be linear spaces with the additive identity element of $D_2$ being the element $e$ cited in the definition. In that case, and when $T$ is a linear operator, the operation $\ast$ is the addition of the linear space $D_1$. Consequently, the class of operators having a superposition in the sense of Definition 1 includes the linear operators (more precisely additive operators). To see that this is a proper inclusion consider the following simple example. Let

$$T(u) = u\dot{u} - \dot{u}^2.$$ \hspace{1cm} (2.2)

An easy calculation shows that

$$T(xy) = x^2 T(y) + y^2 T(x).$$ \hspace{1cm} (2.3)

Hence if $x$ and $y$ are solutions of $T(u) = 0$, the product $xy$ is also a solution. $T$ is nonlinear but it does have a superposition.
For convenience we adopt the convention when we write \( T : D_1 \rightarrow D_2 \) to mean that \( T \) is defined on some nonempty subset \( D(T) \) of \( D_1 \); similarly the range of \( T \) is referred as \( R(T) \subset D_2 \).

Nonlinear operators exist which have a more restrictive superposition.

**Definition 2.** Let \( T : D_1 \rightarrow D_2 \), \( T \) has a **strong superposition** with respect to an element \( e \in D_2 \) iff \( \exists \) binary operations \( *_i \), \( i = 1, 2 \) on \( D_i \) (e.g., \( *_i \leftrightarrow F_i : D_i \times D_i \rightarrow D_i \), \( F_i(x, y) = x *_i y \)) such that

\[
T(x *_1 y) = T(x) *_2 T(y) \tag{2.4}
\]

\( \forall x, y \in D(T) \subset D_1 \) with \( *_2 \) having the property

\[
e *_2 e = e. \tag{2.5}
\]

To distinguish the different roles played by \( *_i (F_i) \) we refer to \( *_i \) as a **composition** for \( T \) and \( *_2 \) as the corresponding **separation**. The pair \( (*_1, *_2) \), or alternatively \( (F_1, F_2) \) with \( *_2 \) obeying (2.5) is called a strong superposition for \( T \) with respect to \( e \).

The linear operators, when \( D_i \) \( i = 1, 2 \) are linear spaces, have a strong superposition with respect to 0 \( \in D_2 \). Addition in \( D_1 \) corresponds to \( *_1 \), addition in \( D_2 \) to \( *_2 \) and 0 + 0 = 0 which corresponds to (2.5).

In functional form (2.4) is

\[
T[F_i(x, y)] = F_d[T(x), T(y)] \quad \forall x, y \in D(T) \tag{2.6}
\]

and (2.5) is

\[
F_d(e, e) = e. \tag{2.7}
\]

It is advantageous to retain both the operational, \( *_i \), and functional, \( F_i \), notations.

We refer to superpositions in the sense of Definition 1 as **weak superpositions** to contrast them to those of Definition 2. Operators having weak superpositions will be discussed in a subsequent paper.

**3. \( C_{F_1,F_2} \) Classes**

We consider now the following classification of operators having a strong superposition:

**Definition 3.** By the class of operators \( C_{F_1,F_2} \) we mean the operators \( T : D_1 \rightarrow D_2 \) which for a given \( (F_1, F_2) \) satisfy (2.6).
There exist such classes which are nonempty, the class of linear operators being one example. Another example is $C_{F_1,F_2}$ for

$$F_1(x, y) = (x^{c+1} + y^{c+1})/(c+1), \quad c \neq -1,$$

$$F_2(x, y) = x + y,$$

$$T(u) = L(u'u'), \quad u' = \frac{du}{ds}, \quad u = u(x).$$

$T : D \to D$, $D$ is the space of analytic functions of a complex variable, $L$ is a linear differential operator on $D$ and the principal branches of the functions in the indicated operations are taken.

It will be shown that many other nonempty $C_{F_1,F_2}$ classes exist.

**Theorem 1.** (a) If $T \in C_{F_1,F_2}$ and $T^{-1}$ exists then $T^{-1} \in C_{F_2,F_1}$

(b) If $T_1 \in C_{F_1,F_2}$, $T_2 \in C_{F_2,F_3}$ and the composition $T_2T_1$ is defined then $T_2T_1 \in C_{F_1,F_3}$.

**Proof.** (a) Let $x, y \in D(T^{-1}) \Rightarrow T^{-1}(x), T^{-1}(y) \in D(T)$. Since $T \in C_{F_1,F_2}$, applying (2.6) on $T^{-1}(x)$ and $T^{-1}(y)$ we have

$$T[F_1(T^{-1}(x)), T^{-1}(y)] = F_3[T(T^{-1}(x)), T(T^{-1}(y))] = F_3(x, y).$$

Applying $T^{-1}$ to both sides,

$$T^{-1}[F_3(x, y)] = F_3[T^{-1}(x), T^{-1}(y)] \Rightarrow T^{-1} \in C_{F_2,F_1}.$$

(b) Let the range of $T_1, R(T_1) \subset D(T_2)$ and let $x, y \in D(T_1) = D(T_2T_1)$.

$$(T_2T_1)[F_2(x, y)] = T_2[F_1(T_1(x), T_1(y))] = T_2[F_3(T_1(x), T_1(y))]$$

since $T_1 \in C_{F_1,F_2}$. Now $T_1(x), T_1(y) \in D(T_2)$ and $T_2 \in C_{F_2,F_3}$,

$$\therefore \quad (T_2T_1)[F_1(x_1, y)] = F_3[(T_2T_1)(x), (T_2T_1)(y)] \Rightarrow T_2T_1 \in C_{F_1,F_3}.$$

Consider the specialization of this theorem for the classes

$C_F \equiv C_{F,F} \quad (D_1 = D_2 = D, F_1 = F_2 = F)$. 

**Corollary 1.** If $T \in C_F$ and $T^{-1}$ exists then $T^{-1} \in C_F$. Furthermore if $T_1, T_2 \in C_F$ and $T_1T_2$, $(T_2T_1)$ is defined then $T_1T_2, (T_2T_1) \in C_F$.

By the space of analytic functions $D = H(\Omega)$ here and subsequently, we mean the linear space of analytic functions on some nonempty domain $\Omega$ of the complex plane. The domain $\Omega$ as well as the topology of $D$ are inconsequential in this discussion and are not specified.
This is a generalization of the well-known properties of linear operators, 
\([C, \text{ for } F(x, y) = x + y]\), that the inverse of a linear operator when it exists
is linear, and the composition of two linear operators when defined is linear. 
The theorem gives the corresponding conclusions when the composition and
the separation of an operator are not the same.

Another ramification of this corollary is obtained by letting

\[
G = \{T : D \to D \mid D(T) = D, T \text{ one-to-one and onto}\}
\]

and

\[
C_f' = \{T \in G \mid T[F(x, y)] = F[T(x), T(y)] \forall x, y \in D\}.
\]

G is a group under composition and \(C_f'\) is a proper subgroup of G.

To see that the class of operators having the same composition and separation
properly contains the linear operators consider the following example: 
Let \(D, u, u', c\) and \(L\) be as in the example in Section 2 and

\[
T(u) = [L(u' u')]^{1/(c+1)}. \tag{3.1}
\]

Notice that \(T\) may also be written as \(T(u) = [L_1(u^{c+1})]^{1/(c+1)}\), where

\[
L_1 = L \left(\frac{1}{c + \frac{1}{dz}}\right).
\]

Now

\[
T[(x^{c+1} + y^{c+1})^{1/(c+1)}] = [L_1(x^{c+1} + y^{c+1})]^{1/(c+1)}
\]

\[
= [L_1(x^{c+1}) + L_1(y^{c+1})]^{1/(c+1)}
\]

\[
= [(T(x))^{c+1} + T(y)^{c+1}]^{1/(c+1)},
\]

where the indicated operations are taken with respect to the principal branches
of the functions occurring.

The operator (3.1) is nonlinear and it belongs to the class \(C_f\) for

\[F(x, y) = (x^{c+1} + y^{c+1})^{1/(c+1)}.\]

It is interesting to note the case \(c = 0\). Then

\[T(u) = L(u'),\]

which is a linear operator and as expected \(F(x, y) = x + y\).

This particular example was constructed by applying the conclusions of
Theorem 1 to

\[F_1(x, y) = (x^{c+1} + y^{c+1})^{1/(c+1)}, \quad F_2(x, y) = x + y,\]

\[L(u' u') \in C_{F_1, F_2}, F_3 = F_1 \quad \text{and} \quad T(u) = u^{1/(c+1)} \in C_{F_2, F_1}.\]
In Sections 5 and 6 a specialization of the classes $C_F$ and $C_{F_1,F_2}$ is examined in greater detail.

4. COMMUTATIVITY AND STAR SPACES

As of yet no algebraic structure was imposed on the sets $D_i$. Assume that the $D_i$, $i = 1, 2$, are linear spaces over the same field $F$. A new algebraic structure, in terms of the operations $*_i$, is now constructed on $D_i$.

In addition to the operations $*_i$ on $D_i$ it is desirable to have something analogous to scalar multiplication in linear spaces. To this end we consider mappings of the form

$$s_i : F \times D \rightarrow D \quad i = 1, 2$$ \hspace{1cm} (4.1)

and use the notation $s_i(a, x) = aV_i x$. The mapping $s_i$ is called a scalar operation on $D_i$ over the field $F$. Furthermore for an operator $T \in C_{F_1,F_2}$ we search for scalar operations such that

$$T[(aV_1 x) *_1 (bV_1 y)] = (aV_2 T(x)) *_2 (bV_2 T(y)), \hspace{1cm} (4.2)$$

which is evidently a generalization of the linearity relation

$$L(ax + by) = aL(x) + bL(y).$$

It is (4.2) which necessitates that the linear spaces $D_i$ be taken over the same field.

Scalar operations having property (4.2) for certain operators $T \in C_{F_1,F_2}$ exist and can be given explicitly for certain commutative $*_i$ operations. This will be shown below. Before restricting the development to commutative operations, we proceed with an example of an operator having a non-commutative strong superposition. Note that operators having noncommutative weak superpositions are known [13]. Consider the linear space $\mathcal{F}$ over the field of reals, $R$, where

$$\mathcal{F} = \{ h \mid h : R \rightarrow R \}$$

with

$$(ah)(t) = ah(t) \quad a \in R$$

and

$$(h_1 + h_2)(t) = h_1(t) + h_2(t).$$

Define the operator $L : \mathcal{F} \rightarrow \mathcal{F}$ by

$$L(x(t)) = \begin{cases} x(t) & t \geq c \\ 0 & t < c \end{cases}$$

for some $c \in R$. It is easily seen that $L$ is linear.
Now choose \( f, g \in \mathcal{F} \) and \( f \) and \( g \) are onto, \( f \) is one-to-one, \( g(0) = 0 \), \( f \) and \( g \) are not linear functions and \( fg = gf \) [e.g., \( g(t) = t^n, f(t) = t^m \), with \( m \) being a positive odd integer, could be such a pair]. Notice that
\[
L(g(x(t))) = gL(x(t)) \quad \forall x \in \mathcal{F}.
\]
Define the nonlinear operator \( T: \mathcal{F} \to \mathcal{F} \) by
\[
T(x(t)) = L(f(x(t))).
\]
The operations
\[
x *_1 y = f^{-1}[f(g(x)) + f(y)] \quad \text{and} \quad x *_2 y = g(x) + y
\]
are a noncommutative superposition, with respect to \( 0 \in \mathcal{F} \), for \( T \) since
\[
T(x *_1 y) = L(f(g(x)) + f(y)) = L(g(x)) + L(f(y))
\]
\[
= g(L(f(x))) + L(f(y)) = g(T(x)) + T(y)
\]
\[
= T(x) *_2 T(y).
\]
Further, if
\[
T(x) = T(y) = 0
\]
then
\[
T(x *_1 y) = g(0) + 0 = 0.
\]
Now the operations given by
\[
x *'_2 y = f^{-1}(f(x) + f(y)) \quad \text{and} \quad x *'_2 y = x + y
\]
form another superposition, with respect to \( 0 \), for \( T \). Hence this example also shows that an operator need not have a unique superposition.

For scalar operations let
\[
a \nabla_1 x = f^{-1}(af(x))
\]
and
\[
a \nabla_2 x = ax \quad \text{for} \ a \in \mathbb{R} \ \text{and} \ x \in \mathcal{F}.
\]
\( T \) satisfies (4.2) with \( \nabla_1 \) and \( *'_1 \). Surprisingly \( T \) also satisfies (4.2) with \( \nabla_i \) and \( *_i \) as it is shown now. Let \( a, b \in \mathbb{R} \) and \( x, y \in \mathcal{F} \), then
\[
T[(a \nabla_1 x) *_1 (b \nabla_1 y)] = [T[f^{-1}(f(g(a \nabla_1 x)) + f(b \nabla_1 y))]]
\]
\[
= T[f^{-1}[g(f(a \nabla_1 x)) + f(b \nabla_1 y))]]
\]
\[
= T[f^{-1}[g(f(a \nabla_1 x)) + b f(y))]
\]
\[
= L(g(a \nabla_1 x)) + L(b f(y))
\]
\[
= g[L(a \nabla_1 x)) + L(b f(y))
\]
\[
= g[aL(f(x))] + bL(f(y))
\]
\[
= g(aT(x)) + bT(y)
\]
\[
= g(a \nabla_2 T(x)) + b \nabla_2 T(y)
\]
\[
= (a \nabla_2 T(x)) *_2 (b \nabla_2 T(y)).
\]
In the remainder of this paper we will treat only commutative superpositions.
Let $T : D_1 \rightarrow D_2$ and $T \in C_{F_1, F_2}$ for some $F_i$ $i = 1, 2$, and assume further that $D_1$ under $\ast_1$ is an abelian group.

**Theorem 2.** Let $T : D_1 \rightarrow D_2$, $T \in C_{F_i, F_2}$, $D(T) = D_1$ and $R(T) = D_2$ ($T$ is onto). If $(D_1, \ast_1)$ is an abelian group then $(D_2, \ast_2)$ is an abelian group.

**Proof.** It is easily seen that $T$ is a homomorphism from $(D_1, \ast_1)$ onto $(D_2, \ast_2)$ and the homomorphic image of an abelian group is an abelian group.

If $T$ is also one-to-one then it is an isomorphism and the condition of the theorem is necessary and sufficient.

The condition that $\ast_1$ is an abelian group operation enables us to represent the function $F_1$, and in view of Theorem 2 also the function $F_2$, in a very convenient form. This representation is used extensively later on. Abel in 1826 [14] was the first to show that if $F(x, y)$ is an abelian group operation on the reals and satisfies certain other conditions, then there exist a one-to-one function, $f$, of one variable such that

$$F(x, y) = f^{-1}(f(x) + f(y)).$$

This result has since been generalized and it may be extended to ordered linear compacta (see [15] for extensive literature). All that we shall require here is that the $F_i$ are expressible in "abelian form" [relation (4.3)]. Explicitly

$$F_i(x, y) = x \ast_i y = f_i^{-1}(f_i(x) + f_i(y)) \quad i = 1, 2,$$

which relates $\ast_i$ with the $+$ of the linear space $D_i$.

**Definition 4.** Let $(D, \ast)$ be an abelian group, $\Gamma$ a field, $s : \Gamma \times D \rightarrow D$ 

$s(a, x) = a \triangledown x$ such that

1. $a \triangledown (x \ast y) = (a \triangledown x) \ast (a \triangledown y),$
2. $(a + b) \triangledown x = (a \triangledown x) \ast (b \triangledown x)$, $+$ is the field addition,
3. $a \triangledown (b \triangledown x) = (ab) \triangledown x$, $ab$ denotes the field multiplication,
4. $1 \triangledown x = x$,

where 1 is the multiplicative identity of $\Gamma$.

We call $(D, \ast, \triangledown)$ with properties (1) $\rightarrow$ (4) a star space over the field $\Gamma$.

Clearly $(D, \ast, \triangledown)$ is a linear space over $\Gamma$ where $\ast$ plays the role of addition and $\triangledown$ is equivalent to scalar multiplication. It is preferable to call $(D, \ast, \triangledown)$ a star space for the following reason: $D$ may already be a linear space but this does not prevent the definition of new operations $\ast, \triangledown$ on the set $D$ with $(D, \ast, \triangledown)$ from being a linear space also. Therefore, the two situations should be distinguished. Some operators in the classes $C_{F_i, F_2}$ can be considered linear by appropriately changing the algebraic operations on $D_i$, while the same...
operators are not linear with the original algebraic operations on \( D \). This, of course, is the motivation for introducing the change.

Assume now that we are given a linear space \( D \) and an operation * on \( D \) in abelian form, namely,

\[
x * y = f^{-1}(f(x) + f(y)).
\]

We obtain an explicit representation of a scalar operation \( \nabla \) in terms of \( f \) and such that \((D, *, \nabla)\) is a star space.

**Theorem 3.** Let \( D \) be a linear space over a field \( \Gamma \) and * an operation on \( D \) expressible in abelian form. Let

\[
a \nabla x = f^{-1}(af(x)) \quad \forall a \in \Gamma, \quad x \in D,
\]

then \((D, *, \nabla)\) is a star space.

**Proof.** \((D, *)\) is already an abelian group.

1. \[
a \nabla (x * y) = f^{-1}[af(f^{-1}(f(x) + f(y)))]
\]
   \[
   = f^{-1}[af(x) + af(y)]
\]
   \[
   = f^{-1}[f(f^{-1}(af(x))) + f(f^{-1}(af(y)))]
\]
   \[
   = f^{-1}(f(a \nabla x) + f(a \nabla y)) = (a \nabla x) * (a \nabla y).
\]

2. \[
(a + b) \nabla x = f^{-1}[(a + bf(a)] = f^{-1}(af(x) + bf(x))
\]
   \[
   = f^{-1}[f(f^{-1}(af(x))) + f(f^{-1}(af(x)))]
\]
   \[
   = f^{-1}(f(a \nabla x) + f(b \nabla x)) = (a \nabla x) * (b \nabla x).
\]

3. \[
a \nabla (b \nabla x) = f^{-1}(af(b \nabla x))
\]
   \[
   = f^{-1}[af(f^{-1}(bf(x)))] = f^{-1}(ab(x)) = (ab) \nabla x.
\]

4. \[
(1 \nabla x) = f^{-1}(1f(x)) = f^{-1}(f(x)) = x.
\]

We can obtain immediately

**Corollary 2.** (a) The identity \( e \) of \((D, *)\) is \( e = f^{-1}(0) \).

(b) The inverse of an element \( x \in (D, *) \) is \( x^{-1} = f^{-1}(-f(x)) \).

Let \( D \) be the linear space of analytic functions of a complex variable over the field of complex numbers \( \mathbb{C} \). Let

\[
x * y = (xe + ye)^{1/c}, \quad a \nabla x = a^{1/c}x,
\]

where \( c \neq 0 \) is a complex number and the principal branch of the function \( g(z) = z^{1/c} \) is chosen for all the indicated operations. Then \((D, *, \nabla)\) is a star space and on this space the operator defined by (3.1) can be considered as a linear operator.
The star space \((D, *, \nabla)\) may be made into a linear topological space in the standard way if we wish to carry the analogy with linearity further.

5. Representations

We are now in position to examine the relation between operators having superposition functions expressible in abelian form and linear operators. For convenience, if the \(F_i\) are given in abelian form by (4.4) we write

\[
C_{F_1, F_2} = C_{f_1, f_2}.
\]

**Theorem 4.** Let \(D_i, i = 1, 2\) be linear spaces and \(T : D_1 \to D_2. T \in C_{f_1, f_2}\) iff there exists an additive operator \(L : D_1 \to D_2, D(L) = D(T)\) such that

\[
T(u) = f_2^{-1}[L(f_1(u))] \quad \forall u \in D(T). \tag{5.1}
\]

**Proof.** (i) Let \(T \in C_{f_1, f_2}\) and define

\[
L(u) = f_2[T(f_1^{-1}(u))].
\]

If \(u, v \in D_1\) and \(x = f_2^{-1}(u), y = f_2^{-1}(v)\) then

\[
L(u + v) = L(f_1(x) + f_1(y)) = f_2[T(f_1^{-1}(f_1(x) + f_1(y)))]
\]

\[
= f_2[f_2^{-1}(f_2(T(x)) + f_2(T(y)))] = f_2(T(x)) + f_2(T(y))
\]

\[
= L(u) + L(v).
\]

Therefore \(L : D_1 \to D_2\) is an additive operator and by definition

\[
f_2^{-1}[L(u)] = T[f_1^{-1}(u)].
\]

Replacing \(u\) by \(f_1(u)\) we obtain

\[
T(u) = f_2^{-1}[L(f_1(u))].
\]

(ii) To show the converse let \(T(u) = f_2^{-1}[L(f_1(u))]\) where \(L\) is additive.

\[
T[f_1^{-1}(f_1(x) + f_1(y))] = f_2^{-1}[L(f_1(x)) + f_1(y)]
\]

\[
= f_2^{-1}[L(f_1(x)) + L(f_1(y))]
\]

\[
= f_2^{-1}(f_2(T(x)) + f_2(T(y)) \Rightarrow T \in C_{f_1, f_2}.\]
Recall the motivation for considering scalar operations relation (4.2). In view of Theorems 3 and 4 when $T \in \mathcal{C}_{f_1,f_2}$ and $D_1$ are linear spaces over the field of real numbers, then

$$T(a\nabla_1 x) = a\nabla_2 T(x) \quad \forall x \in D(T)$$

and all rational numbers $a$.

Equation (5.2) stems directly from the analogous relation for additive operators, namely, that additivity implies homogeneity over the rationals.

We formalize now the concept of homogeneity for operators in the $\mathcal{C}_{f_1,f_2}$ classes.

**Definition 5.** Let $T : D_1 \to D_2$ and $D_i \ i = 1, 2$ be star spaces over the same field $\Gamma$. $T$ is *star homogeneous* iff

$$T(a\nabla_1 x) = a\nabla_2 T(x) \quad \forall x \in D(T), \ a \in \Gamma.$$  

**Theorem 5.** Let $T : D_1 \to D_2$, $\ast_i$ be binary operations on $D_i \ i = 1, 2$ expressible in abelian form (4.4) and $\nabla_i$ constructed as in Theorem 3. Then $T$ is a star-homogeneous operator in $\mathcal{C}_{f_1,f_2}$ iff $\exists$ a linear operator $L : D_1 \to D_2$, $D(L) = D(T)$ such that

$$T(u) = f_2^{-1}[L(f_1(u))] \quad \forall u \in D(T).$$

**Proof.** The additivity of $L$ and the representation (5.4) follow from Theorem 4. The homogeneity for both $L$ and $T$ is easily obtained as in Theorem 4.

In view of this result star-homogeneous operators in a $\mathcal{C}_{f_1,f_2}$ are linearizable in the sense that the domain and range spaces—which are linear spaces—may be transformed into star spaces—which are also linear spaces—as shown in Section 4. On the star spaces such operators are linear.

**6. Classification**

In this section the classification of operators, which was discussed in Section 3, is pursued for the classes $\mathcal{C}_{f_1,f_2}$. Observe that if $T : D_1 \to D_2$, $f_i : D_1 \to D_i$, $L_i : D_i \to D_i$ where $L_i$ is one-to-one and additive $i = 1, 2$, then in view of Theorem 4 $\mathcal{C}_{f_1,f_2} = \mathcal{C}_{L_1 f_1, L_2 f_2}$. That is, $(f_1, f_2)$ and $(L_1 f_1, L_2 f_2)$ define the same class of operators.

**Theorem 6.** If $\mathcal{C}_{f_1,f_2} \cap \mathcal{C}_{g_1,g_2} \neq \emptyset$ then either

$$\mathcal{C}_{f_1,f_2} = \mathcal{C}_{g_1,g_2}.$$
or
\[ C_{f_1, f_2} \cap C_{g_1, g_2} = \{I\}, \]

I is the identity operator
\[ C_{f_1, f_2} = C_{f_1} \quad \text{and} \quad C_{g_1, g_2} = C_{g_1}. \]

Proof. Let \( T : D_1 \rightarrow D_2 \), \( T \neq I \) the identity operator \((D_1 \neq D_2)\) and \( T \in C_{f_1, f_2} \cap C_{g_1, g_2} \).

By Theorem 4
\[ T(u) = f_2^{-1}[L(f_1(u))] = g_2^{-1}[L'(g_1(u))], \]
where \( L \) and \( L' \) are additive operators.

Letting \( x = g_1(u) \) and rearranging we obtain
\[ (g_2 f_2^{-1})[L(f_1 g_1^{-1}(x))] = L'(x) \]
or
\[ h_2^{-1}[L(h_1(x))] = L'(x), \]
where
\[ h_i = f_i g_i^{-1} \quad i = 1, 2. \]

\( L' \) is an additive operator so invoking Theorem 4 again for \( L' \) we get
\[ h_i^{-1}(h_i(x) + h_i(y)) = x + y \quad i = 1, 2 \]
or
\[ h_i(x + y) = h_i(x) + h_i(y), \]
the additivity condition on \( D_i \).

Since \( h_i \) is additive \( h_i^{-1} \) is also additive. Let \( h_i^{-1} = I_i \ i = 1, 2 \), then
\[ g_i = L_i f_i \quad i = 1, 2. \]

Therefore \( C_{g_1, g_2} = C_{L_1 f_1, L_2 f_2} = C_{f_1, f_2} \) by the remark preceding the theorem.

Now if \( I \in C_{f_1, f_2} \cap C_{g_1, g_2} \) we must have \( D_1 = D_2 \) for otherwise the identity map would not be defined. Furthermore
\[ I(u) = u = f_2^{-1}[L_1(f_1(u))] = g_2^{-1}[L_2(g_1(u))] \]
and hence \( f_2 = L_1 f_1, g_2 = L_2 g_1 \).

It is easy to see that
\[ C_{f_1, L_1 f_1} = C_{f_1} \quad \text{and} \quad C_{g_1, L_2 g_2} = C_{g_1}. \]
Now if $\exists T \in C_{f_1} \cap C_{g_1}$ and $T \neq I$ then

$$T(u) = g_2^{-1}[L_1(f_1(u))] = g_2^{-1}[L_2(g_1(u))].$$

Therefore, $g_1 = Lf_1 \Rightarrow C_{f_1} = C_{g_1}$, which is the case already considered. Hence, the remaining case is

$$g_1 \neq Lf_1 \Rightarrow C_{f_1, f_2} \cap C_{g_1, g_2} = C_{f_1} \cap C_{g_1} = \{I\}.$$

This theorem together with Theorem 1 show that the $C_{f_1, f_2}$ classification of operators yields a separation of the operators in addition to providing some information about their inverses and compositions. Furthermore, this classification is consistent with considering the linear operators as one such class.

The formulation established here will serve as a model in the investigation of more general classes of nonlinear operators. This will be considered in the subsequent papers.

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