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High order difference schemes for the system of two space second order nonlinear hyperbolic equations with variable coefficients

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Abstract

In this paper, we develop implicit difference schemes of $O(k^4 + k^2h^2 + h^4)$, where $k > 0$, $h > 0$ are grid sizes in time and space coordinates, respectively, for solving the system of two space dimensional second order nonlinear hyperbolic partial differential equations with variable coefficients having mixed derivatives subject to appropriate initial and boundary conditions. The proposed difference method for the scalar equation is applied for the solution of wave equation in polar coordinates to obtain three level conditionally stable ADI method of $O(k^4 + k^2h^2 + h^4)$. Some physical nonlinear problems are provided to demonstrate the accuracy of the implementation.

Keywords: Difference method; Hyperbolic equation; ADI method; Polar coordinates; Nonlinear wave equation

1. Introduction

The second order nonlinear hyperbolic equations with variable coefficients are of common occurrence in mathematical physics and several other areas of science and engineering. Both explicit and implicit difference schemes have been discussed for the numerical integration of the two-dimensional wave equation by many authors. The wave equation with nonlinear first derivative terms have been studied by Lees [11] and Greenspan [4]. Using operator compact implicit method, fourth order difference schemes have also been developed for linear two-dimensional wave equation by Ciment and Leventhal [1, 2]. McKee [12] has studied conditionally stable Alternating Direction Implicit (ADI) method of $O(k^4 + h^4)$ for a two space hyperbolic equation with variable coefficients. Unconditionally stable ADI schemes of $O(k^2 + h^2)$ and $O(k^2 + h^4)$ for the wave equation with constant coefficients and variable coefficients were proposed by Jain et al. [7] and Iyengar and Mittal [6]. Jain et al. [8] have developed $O(k^4 + h^4)$ difference method for solving the

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system of one-dimensional nonlinear hyperbolic equations. Jain et al. [9] have also discussed difference methods of $O(h^4)$ for solving the system of three space nonlinear elliptic equations. Fourth order difference methods for two-dimensional linear elliptic equations with variable coefficients were proposed by Krishnaiah et al. [10]. Recently, Mohanty [13] has proposed fourth order difference methods for solving the system of two-dimensional nonlinear elliptic equations with variable coefficients and obtained fourth order convergent results for two-dimensional Navier Stokes equations for high Reynolds number. Mohanty and Jain [14] have developed two level implicit difference methods of $O(k^2 + h^4)$ for solving the system of general three space nonlinear parabolic equations with variable coefficients and obtained fourth order convergent results for three-dimensional unsteady Navier–Stokes equations. High order unconditionally stable difference methods for multidimensional heat equation in polar coordinates were discussed by Iyengar and Manohar [5]. But to the authors' knowledge no fourth order finite difference method for solving the system of two space nonlinear hyperbolic equations with variable coefficients has been developed so far. In this paper, we propose three level implicit difference methods of $O(k^4 + k^2h^2 + h^4)$ for the system of two-dimensional nonlinear hyperbolic equations with variable coefficients having mixed derivative terms. For the derivation of the difference schemes we follow the ideas given by Krishnaiah et al. [10] and Mohanty [13]. We obtain two sets of fourth order difference methods; one in the absence of mixed derivatives and we require 19-grid points and 7M evaluations of the function f; second when the coefficients of u_{xy} are not equal to zero and the coefficients of u_{xx} and u_{yy} are equal and we require 27-grid points and $11M$ evaluations of the function f. The proposed method for scalar equation has tested the wave equation in polar coordinates. We refine our procedure in such a way that the solutions retain the order and accuracy even in the vicinity of the singularity. A further refinement allows us to obtain conditionally stable ADI method of $O(k^4 + k^2h^2 + h^4)$. It is shown that for a fixed $p = k/h$, the proposed methods are of $O(h^4)$. This ADI method requires the solution of only tridiagonal system of equations parallel to coordinate axes, at each time step, independent of the order of the method. Some physical nonlinear problems like Vander Pole nonlinear wave equation, Dissipative nonlinear wave equation, etc. are solved to illustrate the utility of the methods derived. We also compare the numerical results obtained by using the present methods with those obtained by using the Central Difference Schemes (CDS) based on Central difference approximations.

2. Formulation of the methods

Consider the system of two space nonlinear hyperbolic equations of the form:

$$\begin{aligned}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} &= \boldsymbol{A}(x,y,t)\frac{\partial^2 \boldsymbol{u}}{\partial x^2} + 2\boldsymbol{B}(x,y,t)\frac{\partial^2 \boldsymbol{u}}{\partial x \partial y} + \boldsymbol{C}(x,y,t)\frac{\partial^2 \boldsymbol{u}}{\partial y^2}\\ &\quad + \boldsymbol{f}(x,y,t,u^{(1)},u^{(2)},\ldots,u^{(M)},u_x^{(1)},u_x^{(2)},\ldots,u_x^{(M)},\\ &\qquad u_y^{(1)},u_y^{(2)},\ldots,u_y^{(M)},u_t^{(1)},u_t^{(2)},\ldots,u_t^{(M)}) \end{aligned} \tag{1}$$

which is defined in a semi-infinite region $\Omega = \{(x,y,t)|0 < x,y < 1, t > 0\}$. For $\boldsymbol{S} = \boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ we assume the coefficients are $M \times M$ diagonal matrices

$$\boldsymbol{S}(x,y,t) = \begin{bmatrix} S^{(1)}(x,y,t) & & & 0 \\ & S^{(2)}(x,y,t) & & \\ & & \ddots & \\ 0 & & & S^{(M)}(x,y,t) \end{bmatrix}_{M \times M}$$

and $u = [u^{(1)}, u^{(2)}, \dots, u^{(M)}]^{\mathrm{T}}$ and $\boldsymbol{f} = [f^{(1)}, f^{(2)}, \dots f^{(M)}]^{\mathrm{T}}$ are vectors having M components each, T being the transpose of the matrices.

Throughout this article, we consider $i = 1(1)M$, M being a positive integer. The initial and boundary conditions associated with (1) are given by

$$u^{(i)}(x,y,0) = u_0^{(i)}(x,y), \quad u_t^{(i)}(x,y,0) = u_1^{(i)}(x,y), \quad 0 \leqslant x, \quad y \leqslant 1; \tag{2.1}$$

$$u^{(i)}(0,y,t) = g_0^{(i)}(y,t), \quad u^{(i)}(1,y,t) = g_1^{(i)}(y,t), \quad 0 \leqslant y \leqslant 1, \quad t \geqslant 0; \tag{2.2}$$

$$u^{(i)}(x,0,t) = h_0^{(i)}(x,t), \quad u^{(i)}(x,1,t) = h_1^{(i)}(x,t), \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0. \tag{2.3}$$

As usual, let $h > 0$ and $k > 0$ be the grid sizes in space and time directions, respectively. We replace the region Ω by a set of interior grid points $(x_l, y_m, t_j) \equiv (l,m,j)$ where $x_l = lh$, $y_m = mh$, $l,m = 0(1)N+1$ and $t_j = jk$, $0 < j < J$. The values of the exact solutions $u^{(i)}(x,y,t)$ and the coefficients $A^{(i)}(x,y,t), \dots$, etc. at the grid point (l,m,j) are denoted by $U_{l,m}^{(i)j}$ and $A_{l,m}^{(i)j}, \dots$, etc., respectively. Let $u_{l,m}^{(i)j}$ are approximate solutions of $U_{l,m}^{(i)j}$. We assume that $u^{(i)} \in C^4$ and $A^{(i)}$, $B^{(i)}$, $C^{(i)} \in C^2$, where C^m denotes the set of all functions of (x,y,t) continuous up to order m. The system of equations (1) are assumed to satisfy the conditions $[B^{(i)}]^2 < A^{(i)}.C^{(i)}$.

We now simply follow the approaches given by Krishnaiah [10] and Mohanty [13]. We consider two sets of fourth order difference methods described as follows.

For $S = A, B, C$ and G; let

$$S_{abc}^{(i)} = \frac{\partial^{a+b+c} S^{(i)}\ (x_l, y_m, t_j)}{(\partial x_l)^a\ (\partial y_m)^b\ (\partial t_j)^c}. \tag{3}$$

We need the following approximations. For $a,b,c = 0, \pm 1$, let

$$\bar{U}_{xl,m+b}^{(i)j+c} = (U_{l+1,m+b}^{(i)j+c} - U_{l-1,m+b}^{(i)j+c})/(2h), \tag{4.1}$$

$$\bar{U}_{yl+a,m}^{(i)j+c} = (U_{l+a,m+1}^{(i)j+c} - U_{l+a,m-1}^{(i)j+c})/(2h), \tag{4.2}$$

$$\bar{U}_{tl+a,m+b}^{(i)j} = (U_{l+a,m+b}^{(i)j+1} - U_{l+a,m+b}^{(i)j-1})/(2k), \tag{4.3}$$

$$\bar{U}_{xl\pm 1,m+b}^{(i)j+c} = (\pm 3\ U_{l\pm 1,m+b}^{(i)j+c} \mp 4U_{l,m+b}^{(i)j+c} \pm U_{l\mp 1,m+b}^{(i)j+c})/(2h), \tag{4.4}$$

$$\bar{U}_{yl+a,m\pm 1}^{(i)j+c} = (\pm 3\ U_{l+a,m\pm 1}^{(i)j+c} \mp 4U_{l+a,m}^{(i)j+c} \pm U_{l+a,m\mp 1}^{(i)j+c})/(2h), \tag{4.5}$$

$$\bar{U}_{tl+a,m+b}^{(i)j\pm 1} = (\pm 3\ U_{l+a,m+b}^{(i)j\pm 1} \mp 4U_{l+a,m+b}^{(i)j} \pm U_{l+a,m+b}^{(i)j\mp 1})/(2k), \tag{4.6}$$

$$\bar{U}_{xxl,m+b}^{(i)j+c} = (U_{l+1,m+b}^{(i)j+c} - 2U_{l,m+b}^{(i)j+c} + U_{l-1,m+b}^{(i)j+c})/(h^2), \tag{4.7}$$

$$\bar{U}^{(i)j+c}_{yyl+a,m} = (U^{(i)j+c}_{l+a,m+1} - 2U^{(i)j+c}_{l+a,m} + U^{(i)j+c}_{l+a,m-1})/(h^2), \tag{4.8}$$

$$\bar{U}^{(i)j}_{ttl+a,m+b} = (U^{(i)j+1}_{l+a,m+b} - 2U^{(i)j}_{l+a,m+b} + U^{(i)j-1}_{l+a,m+b})/(k^2), \tag{4.9}$$

$$\bar{U}^{(i)j+c}_{xyl,m} = (U^{(i)j+c}_{l+1,m+1} - U^{(i)j+c}_{l+1,m-1} - U^{(i)j+c}_{l-1,m+1} + U^{(i)j+c}_{l-1,m-1})/(4h^2). \tag{4.10}$$

The given system of differential equations (1) at the grid point (l, m, j) may be discretized as:

$$\begin{aligned}
&[6\delta_t^2 - L_1^{(i)}\delta_x^2 - L_2^{(i)}\delta_y^2 - L_3^{(i)}(4\mu_x\delta_x\mu_y\delta_y) - L_4^{(i)}\delta_x^2(2\mu_y\delta_y) \\
&\quad - L_5^{(i)}\delta_y^2(2\mu_x\delta_x) - L_6^{(i)}(8\mu_x\delta_x\mu_y\delta_y\mu_t\delta_t) - L_7^{(i)}\delta_x^2(2\mu_t\delta_t) \\
&\quad - L_8^{(i)}\delta_y^2(2\mu_t\delta_t) - L_9^{(i)}\delta_t^2(2\mu_x\delta_x) - L_{10}^{(i)}\delta_t^2(2\mu_y\delta_y) \\
&\quad - L_{11}^{(i)}\delta_t^2(4\mu_x\delta_x\mu_y\delta_y) - L_{12}^{(i)}\delta_x^2\delta_y^2 - L_{13}^{(i)}\delta_x^2\delta_t^2 - L_{14}^{(i)}\delta_y^2\delta_t^2]U^{(i)j}_{l,m} \\
&\quad = \frac{k^2}{2}[6\bar{\bar{F}}^{(i)j}_{l,m} + R_1^{(i)}\bar{F}^{(i)j}_{l+1,m} + R_2^{(i)}\bar{F}^{(i)j}_{l-1,m} + R_3^{(i)}\bar{F}^{(i)j}_{l,m+1} + R_4^{(i)}\bar{F}^{(i)j}_{l,m-1} \\
&\qquad + \bar{F}^{(i)j+1}_{l,m} + \bar{F}^{(i)j-1}_{l,m} + R_5^{(i)}(\bar{F}^{(i)j}_{l+1,m+1} - \bar{F}^{(i)j}_{l+1,m-1} \\
&\qquad - \bar{F}^{(i)j}_{l-1,m+1} + \bar{F}^{(i)j}_{l-1,m-1})] + \bar{T}^{(i)j}_{l,m},
\end{aligned} \tag{5}$$

where $\delta_x U_l = (U_{l+1/2} - U_{l-1/2})$ and $\mu_x U_l = \frac{1}{2}(U_{l+1/2} + U_{l-1/2})$ are central and average difference operators with respect to x-direction, etc. and $\bar{T}^{(i)j}_{l,m} = O(k^6 + k^4h^2 + k^2h^4)$ and

$$P_0^{(i)} = 2B_{000}^{(i)}/A_{000}^{(i)},$$

$$P_1^{(i)} = (2A_{100}^{(i)} + P_0^{(i)}A_{010}^{(i)})/A_{000}^{(i)}, \qquad P_2^{(i)} = (2C_{010}^{(i)} + P_0^{(i)}C_{100}^{(i)})/C_{000}^{(i)},$$

$$L_1^{(i)} = p^2[6A_{000}^{(i)} + \tfrac{1}{2}h^2(A_{200}^{(i)} + A_{020}^{(i)} + p^2A_{002}^{(i)} + P_0^{(i)}A_{110}^{(i)} - P_1^{(i)}A_{100}^{(i)} - P_2^{(i)}A_{010}^{(i)})],$$

$$L_2^{(i)} = p^2[6C_{000}^{(i)} + \tfrac{1}{2}h^2(C_{200}^{(i)} + C_{020}^{(i)} + p^2C_{002}^{(i)} + P_0^{(i)}C_{110}^{(i)} - P_1^{(i)}C_{100}^{(i)} - P_2^{(i)}C_{010}^{(i)})],$$

$$L_3^{(i)} = \frac{p^2}{2}[6B_{000}^{(i)} + \tfrac{1}{2}h^2(B_{200}^{(i)} + B_{020}^{(i)} + p^2B_{002}^{(i)} + P_0^{(i)}B_{110}^{(i)} - P_1^{(i)}B_{100}^{(i)} - P_2^{(i)}B_{010}^{(i)})],$$

$$L_4^{(i)} = \tfrac{1}{4}hp^2[P_0^{(i)}(A_{100}^{(i)} + 2B_{010}^{(i)}) - 2P_1^{(i)}B_{000}^{(i)} - P_2^{(i)}A_{000}^{(i)} + 2A_{010}^{(i)} + 4B_{100}^{(i)}],$$

$$L_5^{(i)} = \tfrac{1}{4}hp^2[P_0^{(i)}(C_{010}^{(i)} + 2B_{100}^{(i)}) - 2P_2^{(i)}B_{000}^{(i)} - P_1^{(i)}C_{000}^{(i)} + 2C_{100}^{(i)} + 4B_{010}^{(i)}],$$

$$L_6^{(i)} = \tfrac{1}{4}(kp^2B_{001}^{(i)}), \qquad L_7^{(i)} = \tfrac{1}{2}(kp^2A_{001}^{(i)}), \qquad L_8^{(i)} = \tfrac{1}{2}(kp^2C_{001}^{(i)}),$$

$$L_9^{(i)} = \tfrac{1}{4}(hP_1^{(i)}), \qquad L_{10}^{(i)} = \tfrac{1}{4}(hP_2^{(i)}), \qquad L_{11}^{(i)} = \tfrac{1}{8}(2p^2B_{000}^{(i)} - P_0^{(i)}),$$

$$L_{12}^{(i)} = \tfrac{1}{2}p^2[C_{000}^{(i)} + A_{000}^{(i)} + 2P_0^{(i)}B_{000}^{(i)}], \qquad L_{13}^{(i)} = \tfrac{1}{2}(p^2A_{000}^{(i)} - 1),$$

$$L_{14}^{(i)} = \tfrac{1}{2}(p^2C_{000}^{(i)} - 1),$$

$$R_1^{(i)} = 1 - \tfrac{1}{2}hP_1^{(i)}, \qquad R_2^{(i)} = 1 + \tfrac{1}{2}hP_1^{(i)}, \qquad R_3^{(i)} = 1 - \tfrac{1}{2}hP_2^{(i)},$$

$$R_4^{(i)} = 1 + \tfrac{1}{2}hP_2^{(i)}, \qquad R_5^{(i)} = \tfrac{1}{4}P_0^{(i)},$$

$$\begin{aligned}\bar{F}^{(i)j+c}_{l+a,m+b} = f^{(i)}(&x_{l+a}, y_{m+b}, t_{j+c}, U^{(1)j+c}_{l+a,m+b}, U^{(2)j+c}_{l+a,m+b}, \ldots, U^{(M)j+c}_{l+a,m+b},\\
&\bar{U}^{(1)j+c}_{xl+a,m+b}, \bar{U}^{(2)j+c}_{xl+a,m+b}, \ldots, \bar{U}^{(M)j+c}_{xl+a,m+b},\\
&\bar{U}^{(1)j+c}_{yl+a,m+b}, \bar{U}^{(2)j+c}_{yl+a,m+b}, \ldots, \bar{U}^{(M)j+c}_{yl+a,m+b},\\
&\bar{U}^{(1)j+c}_{tl+a,m+b}, \bar{U}^{(2)j+c}_{tl+a,m+b}, \ldots, \bar{U}^{(M)j+c}_{tl+a,m+b}),\end{aligned}$$

$$\begin{aligned}\bar{\bar{U}}^{(i)j}_{xl,m} = \bar{U}^{(i)j}_{xl,m} + \frac{h}{12A^{(i)}_{000}}&[(\bar{F}^{(i)j}_{l+1,m} - \bar{F}^{(i)j}_{l-1,m}) - (\bar{U}^{(i)j}_{ttl+1,m} - \bar{U}^{(i)j}_{ttl-1,m})\\
&+ 2B^{(i)}_{000}(\bar{U}^{(i)j}_{xxl,m+1} - \bar{U}^{(i)j}_{xxl,m-1}) + C^{(i)}_{000}(\bar{U}^{(i)j}_{yyl+1,m} - \bar{U}^{(i)j}_{yyl-1,m})]\\
+ \frac{h^2}{6A^{(i)}_{000}}&[A^{(i)}_{100}\bar{U}^{(i)j}_{xxl,m} + 2B^{(i)}_{100}\bar{U}^{(i)j}_{xyl,m} + C^{(i)}_{100}\bar{U}^{(i)j}_{yyl,m}],\end{aligned}$$

$$\begin{aligned}\bar{\bar{U}}^{(i)j}_{yl,m} = \bar{U}^{(i)j}_{yl,m} + \frac{h}{12C^{(i)}_{000}}&[(\bar{F}^{(i)j}_{l,m+1} - \bar{F}^{(i)j}_{l,m-1}) - (\bar{U}^{(i)j}_{ttl,m+1} - \bar{U}^{(i)j}_{ttl,m-1})\\
&+ A^{(i)}_{000}(\bar{U}^{(i)j}_{xxl,m+1} - \bar{U}^{(i)j}_{xxl,m-1}) + 2B^{(i)}_{000}(\bar{U}^{(i)j}_{yyl+1,m} - \bar{U}^{(i)j}_{yyl-1,m})]\\
+ \frac{h^2}{6C^{(i)}_{000}}&[A^{(i)}_{010}\bar{U}^{(i)j}_{xxl,m} + 2B^{(i)}_{010}\bar{U}^{(i)j}_{xyl,m} + C^{(i)}_{010}\bar{U}^{(i)j}_{yyl,m}],\end{aligned}$$

$$\begin{aligned}\bar{\bar{U}}^{(i)j}_{tl,m} = \bar{U}^{(i)j}_{tl,m} - \tfrac{1}{12}k\,&[(\bar{F}^{(i)j+1}_{l,m} - \bar{F}^{(i)j-1}_{l,m}) + A^{(i)}_{000}(\bar{U}^{(i)j+1}_{xxl,m} - \bar{U}^{(i)j-1}_{xxl,m})\\
&+ 2B^{(i)}_{000}(\bar{U}^{(i)j+1}_{xyl,m} - \bar{U}^{(i)j-1}_{xyl,m}) + C^{(i)}_{000}(\bar{U}^{(i)j+1}_{yyl,m} - \bar{U}^{(i)j-1}_{yyl,m})]\\
- \tfrac{1}{6}k^2\,&[A^{(i)}_{001}\bar{U}^{(i)j}_{xxl,m} + 2B^{(i)}_{001}\bar{U}^{(i)j}_{xyl,m} + C^{(i)}_{001}\bar{U}^{(i)j}_{yyl,m}],\end{aligned}$$

$$\begin{aligned}\bar{\bar{F}}^{(i)j}_{l,m} = f^{(i)}\,(&x_l, y_m, t_j, U^{(1)j}_{l,m}, U^{(2)j}_{l,m}, \ldots, U^{(M)j}_{l,m}, \bar{\bar{U}}^{(1)j}_{xl,m}, \bar{\bar{U}}^{(2)j}_{xl,m}, \ldots, \bar{\bar{U}}^{(M)j}_{xl,m},\\
&\bar{\bar{U}}^{(i)j}_{yl,m}, \bar{\bar{U}}^{(2)j}_{yl,m}, \ldots, \bar{\bar{U}}^{(M)j}_{yl,m}, \bar{\bar{U}}^{(1)j}_{tl,m}, \bar{\bar{U}}^{(2)j}_{tl,m}, \ldots, \bar{\bar{U}}^{(M)j}_{tl,m}).\end{aligned}$$

Note that, the system of difference equations (5) are valid for both the cases (I) $B^{(i)} = 0$ and (II) $A^{(i)} = C^{(i)}$, $B^{(i)} \neq 0$ (see [10]).

3. ADI schemes and stability analysis

In this section, we aim to discuss three level implicit ADI difference schemes for the solution of the two space wave equation in polar coordinates and ensure that the numerical methods discussed here retain their order and accuracy everywhere including the region in the vicinity of the singularity $r = 0$.

We consider the equation of the form

$$u_{tt} = u_{xx} + C(x)u_{yy} + \frac{\alpha}{x}u_x + g(x,y,t), \tag{6}$$

where α is a constant. The proposed method (5) for the linear equation (6) requires solution of a system of equations with a large band width at each time level. It is also difficult to study the stability of such an equation. Therefore, we modify the linear difference equation using the approximations of type

$$\frac{1}{x_{l\pm1}} = \frac{1}{x_l} \mp \frac{h}{(x_l)^2} + \frac{h^2}{(x_l)^3} + O(\mp h^3 + h^4),$$
$$G^j_{l\pm1,m} = G_{000} \pm hG_{100} + \tfrac{1}{2}h^2\, G_{200} + O(\pm h^3 + h^4),$$
$$G^j_{l,m\pm1} = G_{000} \pm hG_{010} + \tfrac{1}{2}h^2\, G_{020} + O(\pm h^3 + h^4),$$
$$G^{j\pm1}_{l,m} = G_{000} \pm kG_{001} + \tfrac{1}{2}k^2\, G_{002} + O(\pm k^3 + k^4)$$

in such a way that the order of the scheme is unchanged while the resulting difference equations require the solution of only tridiagonal matrices at each time level. This difference scheme for (6) in product form is given by:

$$[M_1]\,[M_2]\,\delta_t^2 u^j_{l,m} = p^2[S_1\delta_x^2 + S_2\delta_y^2 + S_3(2\mu_x\delta_x) + S_4(2\delta_y^2\mu_x\delta_x) + S_5\delta_x^2\delta_y^2]u^j_{l,m} + \Sigma G \equiv [R_u], \quad l,m = 1(1)N, \tag{7}$$

where

$$M_1 = 1 + \frac{\alpha}{24l}(1-p^2)\,(2\mu_x\delta_x) + \tfrac{1}{12}(1-p^2)\delta_x^2,$$

$$M_2 = 1 + \tfrac{1}{12}(1 - p^2C_{000})\delta_y^2,$$

$$S_1 = 1 + \frac{\alpha(\alpha-2)}{12l^2}, \qquad S_2 = C_{000} + \frac{\alpha h}{12l}C_{100} + \tfrac{1}{12}h^2\, C_{200},$$

$$S_3 = \frac{\alpha}{2l} + \frac{\alpha(2-\alpha)}{24l^3}, \qquad S_4 = \frac{\alpha}{24l}(1 + C_{000}) + \tfrac{1}{12}h\, C_{100}, \tag{8}$$

$$S_5 = \tfrac{1}{12}(1 + C_{000})$$

and $\Sigma G = \tfrac{1}{12}k^2\left[12G_{000} + \frac{\alpha h}{l}G_{100} + h^2\,(G_{200} + G_{020} + p^2G_{002})\right].$

The scheme (7) is of $O(k^4 + k^2h^2 + h^4)$. The additional terms are of high orders and do not affect the accuracy of the scheme but enables a factorization of the operators of the left-hand side of (7). In order to facilitate the computation we may write the scheme (7) in two step ADI form as:

$$[M_2]u^{*j+1}_{l,m} = [R_u], \tag{9.1}$$

$$[M_1]\delta_t^2 u^j_{l,m} = u^{*j+1}_{l,m}, \tag{9.2}$$

where $u_{l,m}^{*j+1}$ is any intermediate value and the intermediate boundary conditions required for the solution of $u_{l,m}^{*j+1}$ are obtained from (9.2). Note that, the left-hand side matrices represented by (9.1) and (9.2) are tridiagonal and are free from the terms $1/(l-1)$ and $1/(l+1)$, thus very easily solved for $l, m = 1(1)N$ in the region Ω and no fictitious points are needed to calculate the intermediate boundary conditions.

Now we examine the Von Neumann linear stability of the difference scheme (7). We seek a solution of the homogeneous part of the (7) of the form $u_{l,m}^{j} = \xi^j e^{i\theta_1 lh} e^{i\theta_2 mh}$, where $i = \sqrt{-1}$ and $\theta_1 h$, $\theta_2 h$ are real; then the characteristic equation for (7) becomes

$$(a + ia')\,\xi^2 - 2(b + ib')\,\xi + (a + ia') = 0 \tag{10}$$

where $a = [1 + \frac{1}{6}(1-p^2)(\cos\theta_1 h - 1)]\,[1 + \frac{1}{6}(1 - p^2 C_{000})(\cos\theta_2 h - 1)]$,

$$a' = \frac{\alpha}{12l}(1-p^2)\sin\theta_1 h[1 + \tfrac{1}{6}(1 - p^2 C_{000})(\cos\theta_2 h - 1)],$$

$$b = a + p^2\,[S_1(\cos\theta_1 h - 1) + S_2(\cos\theta_2 h - 1) + 2S_5(\cos\theta_1 h - 1)(\cos\theta_2 h - 1)],$$

$$b' = a' + p^2 \sin\theta_1 h[S_3 + 2S_4(\cos\theta_2 h - 1)].$$

For stability it is required that $B \leqslant A$ (see [15]), where $A = |a + ia'|$ and $B = |b + ib'|$. Since $\max \sin^2(\theta_r h/2) = 1$ (when $\theta_r h = \pi$, $r = 1$ and 2), we find that the scheme (7) is stable if

$$0 < \frac{9p^2(S_1 + S_2 - 4S_5)}{(2+p^2)\;(2 + p^2 C_{000})} \leqslant 1. \tag{11}$$

Now consider, the two-dimensional problem

$$u_{tt} = u_{rr} + \frac{\alpha}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + g(r,\theta,t). \tag{12}$$

For $\alpha = 1$ and 2 this equation represents two-dimensional wave equation in cylindrical and spherical coordinates, respectively. Replacing the variables x, y by r, θ, respectively, and setting $C = 1/r^2$ in (7), we get the ADI method in split form as (9.1) and (9.2), where

$$M_2 = 1 + \tfrac{1}{12}\left(1 - \frac{p^2}{l^2h^2}\right)\delta_y^2, \qquad S_2 = \frac{1}{l^2h^2} + \frac{(3-\alpha)}{6h^2l^4},$$

$$S_4 = \frac{\alpha}{24l} + \frac{(\alpha - 4)}{24h^2l^3}, \qquad S_5 = \tfrac{1}{12}\left(1 + \frac{1}{l^2h^2}\right)$$

and the other coefficients are listed in (8).

Similarly, for the two space problem

$$u_{tt} = u_{rr} + \frac{\alpha}{r}u_r + u_{zz} + g(r,z,t) \tag{13}$$

we have the ADI method (9.1), (9.2) with $C = 1$ and the variables x, y are replaced by r, z, respectively. For $\alpha = 1, 2$, Eq. (13) represents the two-dimensional wave equation in cylindrical and spherical symmetry. The coefficients are given by

$$M_2 = 1 + \tfrac{1}{12}(1 - p^2)\,\delta_y^2, \qquad S_2 = 1.0, \qquad S_4 = \frac{\alpha}{12l}, \qquad S_5 = \tfrac{1}{6}$$

and the other coefficients are given by (8).

If the solution at $r = 0$ is also to be determined, where $r = 0$ is a part of the boundary, then we need a difference equation valid at $r = 0$. We illustrate the procedure for one of the equations, say (13).

Let at $r = 0$, we have $u_r = 0$, then (13) at $r = 0$ becomes

$$u_{tt} = (1 + \alpha)u_{rr} + u_{zz} + g(0, z, t). \tag{14}$$

A suitable $O(k^4 + k^2h^2 + h^4)$ approximation is

$$\begin{aligned}[M_3]\,[M_4]\,\delta_t^2 u_{0,m}^j &= p^2\,[(1 + \alpha)\delta_r^2 + \delta_z^2 + \tfrac{1}{12}(2 + \alpha)\,\delta_r^2\delta_z^2]\,u_{0,m}^j \\ &\quad + \tfrac{1}{12}k^2\,[12G_{0,m}^j + h^2(G_{rr0,m}^j + G_{zz0,m}^j + p^2 G_{tt0,m}^j)], \\ &\equiv [S_u]\end{aligned} \tag{15}$$

where

$$M_3 = 1 + \tfrac{1}{12}(1 - (1 + \alpha)p^2)\,\delta_r^2$$

and

$$M_4 = 1 + \tfrac{1}{12}(1 - p^2)\,\delta_z^2.$$

An ADI form is

$$[M_4]\,u_{0,m}^{*j+1} = [S_u], \tag{16.1}$$

$$[M_3]\,\delta_t^2 u_{0,m}^j = u_{0,m}^{*j+1}, \tag{16.2}$$

where in (15) we are to use the condition $u_r = 0$ at $r = 0$.

This implies $u_{-1,m}^{*j+1} = u_{1,m}^{*j+1}$. Note that the equations (9.2) and (16.2) are the same, valid along lines parallel to r-axis. The intermediate boundary conditions are obtained from (16.2).

4. Numerical illustrations

In this section, we have solved two linear and five nonlinear problems whose exact solutions are known to us in the prescribed region Ω. We have also compared the proposed difference methods with those obtained by using the corresponding CDS. The initial and boundary conditions and right-hand side functions may be obtained using the exact solution. In all the above ADI methods, the integration is first carried along lines parallel to the θ-axis or z-axis, and then along the r-axis.

These methods produce tridiagonal systems for solution along lines parallel to the axes. They are all three level formulas. The starting values at the zeroth and first time level may be obtained by using the initial conditions. For the purpose of comparison, we have taken the exact solution for starting values at the first time level. The linear equations have been solved using the Gauss-elimination method whereas the nonlinear equations have been solved using the Newton–Raphson method. For a fixed $p = k/h$ (mesh ratio parameter), all the above methods behave like fourth order methods. For numerical simplicity, we have restricted ourselves to the values $M = 1$ and 2. All computations were carried out using double precision arithmetic at the Computer Service Centre, University of Delhi.

Problem 1. (a) Eq. (13) is to solve with the exact solution $u = e^{-2t}\cosh(r)\cosh(z)$. At $r = 0$, we have $u_r = 0$ and all other conditions are provided from the exact solution. The Root Mean Square (RMS) errors are tabulated in Table 1 at $t = 2.0$ for $\alpha = 1$ and 2 for a fixed $p = 0.4$.

(b) Eq. (12) is to solve with the exact solution $u = r^2\cos(\pi\theta)\sin t$. The RMS errors are tabulated in Table 1 at $t = 0.5$ for $\alpha = 1$ and 2 for a fixed $p = 0.025$.

Problem 2 (*Vander pole type nonlinear wave equation*).

$$u_{tt} = u_{xx} + u_{yy} + \varepsilon(u^2 - 1)u_t + f(x, y, t)$$

with the next solution $u = e^{-t}\sin(\pi x)\sin(\pi y)$. The maximum absolute errors are tabulated in Table 2 at $t = 0.5$ and 1.0 for $\varepsilon = 0.01$ and 0.001 for a fixed $p = 0.4$.

Problem 3 (*Dissipative nonlinear wave equation*).

$$u_{tt} = u_{xx} + u_{yy} - \frac{\partial}{\partial t}(u^2) + f(x, y, t)$$

with the exact solution $u = \sin(\pi x)\sin(\pi y)\cos t$.

The maximum absolute errors are tabulated in Table 3 at $t = 0.5$ and 1.0 for a fixed $p = 0.4$.

Problem 4.

$$u_{tt} = (1 + e^t\cos x\sin y)u_{xx} + (1 + e^t\sin x\cos y)u_{yy} + \varepsilon u(u_x + u_y + u_t) + f(x, y, t).$$

with the exact solution $u = \sin x\cos y\cos t$.

The maximum absolute errors are tabulated in Table 4 at $t = 0.5$ for a fixed $p = 0.2$.

Problem 5.

$$u_{tt} = (1 + e^{x+y+t})(u_{xx} + u_{yy}) + 2(1 + e^t\cos x\sin y)u_{xy} + \varepsilon u(u_x + u_y + u_t) + f(x, y, t)$$

with the exact solution $u = \cos x\sin y\sin t$.

The maximum absolute errors are tabulated in Table 5 at $t = 0.5$ for a fixed $p = 0.2$.

Table 1
Problem 1: The RMS errors

	Scheme (9)				CDS			
	Problem 1(a)		Problem 1(b)		Problem 1(a)		Problem 1(b)	
h	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
$\frac{1}{4}$	0.1530(− 04)	0.3840(− 04)	0.1174(− 05)	0.1515(− 04)	0.3559(− 03)	0.1290(− 02)	0.1435(− 02)	0.1430(− 02)
$\frac{1}{8}$	0.8669(− 06)	0.2270(− 05)	0.9172(− 07)	0.8738(− 06)	0.8843(− 04)	0.2980(− 03)	0.3332(− 03)	0.3333(− 03)
$\frac{1}{16}$	0.5855(− 07)	0.1397(− 06)	0.5589(− 08)	0.5117(− 07)	0.2183(− 04)	0.7281(− 04)	0.7848(− 04)	0.7865(− 04)

Table 2
Problem 2: The maximum absolute errors

	Scheme (5) when $M = 1$				CDS			
	$\varepsilon = 0.01$		$\varepsilon = 0.001$		$\varepsilon = 0.01$		$\varepsilon = 0.001$	
h	$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$
$\frac{1}{4}$	0.1105(− 02)	0.5833(− 03)	0.1106(− 02)	0.5835(− 03)	0.5946(− 01)	0.3547(− 01)	0.5951(− 01)	0.3550(− 01)
$\frac{1}{8}$	0.7714(− 04)	0.3139(− 04)	0.7720(− 04)	0.3138(− 04)	0.1591(− 01)	0.6722(− 02)	0.1592(− 01)	0.6723(− 02)
$\frac{1}{16}$	0.5017(− 05)	0.1851(− 05)	0.5021(− 05)	0.1850(− 05)	0.4108(− 02)	0.1426(− 02)	0.4113(− 02)	0.1462(− 02)

Table 3
Problem 3: The maximum absolute errors

	Scheme (5) when $M = 1$		CDS	
h	$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$
$\frac{1}{4}$	0.1208(− 02)	0.1223(− 02)	0.5763(− 01)	0.6217(− 01)
$\frac{1}{8}$	0.8255(− 04)	0.7346(− 04)	0.1542(− 01)	0.1373(− 01)
$\frac{1}{16}$	0.5327(− 05)	0.4561(− 05)	0.3972(− 02)	0.3269(− 02)

Table 4
Problem 4: The maximum absolute errors

	Scheme (5) when $M = 1$			CDS		
h	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 10$	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 10$
$\frac{1}{4}$	0.6347(− 05)	0.6414(− 05)	0.5103(− 03)	0.5282(− 03)	0.5400(− 03)	0.3069(− 02)
$\frac{1}{8}$	0.4649(− 06)	0.4664(− 06)	0.3832(− 04)	0.1545(− 03)	0.1574(− 03)	0.7136(− 03)
$\frac{1}{16}$	0.1373(− 07)	0.1440(− 07)	0.2418(− 05)	0.4015(− 04)	0.4091(− 04)	0.1829(− 03)

Table 5
Problem 5: The maximum absolute errors

	Scheme (5) when $M = 1$			CDS		
h	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 10$	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 10$
$\frac{1}{4}$	0.1073(− 04)	0.1065(− 04)	0.2060(− 04)	0.3982(− 03)	0.3983(− 03)	0.3846(− 03)
$\frac{1}{8}$	0.6987(− 06)	0.6933(− 06)	0.1314(− 05)	0.1015(− 03)	0.6235(− 04)	0.9725(− 04)
$\frac{1}{16}$	0.1583(− 07)	0.1938(− 07)	0.2986(− 07)	0.2554(− 04)	0.2555(− 04)	0.2446(− 04)

Problem 6.

$$\begin{aligned} u_{tt} &= (1 + \mathrm{e}^{t+x} \cos y)(u_{xx} + u_{yy}) + 2(1 + \mathrm{e}^{t} \cos x \sin y) u_{xy} \\ &\quad + \alpha uv(u_x + v_x + u_y + v_y + u_t + v_t) + f(x, y, t) \\ v_{tt} &= (1 + \mathrm{e}^{t+x+y})\,(v_{xx} + v_{yy}) + 2(1 + \mathrm{e}^{t} \sin x \cos y) v_{xy} \\ &\quad + \beta uv(u_x + v_x + u_y + v_y + u_t + v_t) + g(x, y, t) \end{aligned}$$

with the exact solution $u = \sin x \sin y \sin t$ and $v = \cos x \cos y \cos t$.

The maximum absolute errors are tabulated in Table 6 at $t = 0.5$ for a fixed $p = 0.2$.

Table 6
Problem 6: The maximum absolute errors

		Scheme (5) when $M = 2$			CDS		
h		$(\alpha,\beta) = (0.1, 0.01)$	$(\alpha,\beta) = (0.01, 0.1)$	$(\alpha,\beta) = (5, 10)$	$(\alpha,\beta) = (0.1, 0.01)$	$(\alpha,\beta) = (0.01, 0.1)$	$(\alpha,\beta) = (5, 10)$
$\frac{1}{4}$	u	0.1005(− 04)	0.9949(− 05)	0.1589(− 04)	0.5738(− 03)	0.5749(− 03)	0.5108(− 03)
	v	0.2883(− 04)	0.2877(− 04)	0.2399(− 04)	0.4753(− 03)	0.4758(− 03)	0.5300(− 03)
$\frac{1}{8}$	u	0.6255(− 06)	0.6204(− 06)	0.9284(− 06)	0.1621(− 03)	0.1624(− 03)	0.1463(− 03)
	v	0.1708(− 05)	0.1705(− 05)	0.1367(− 05)	0.1052(− 03)	0.1054(− 03)	0.1205(− 03)
$\frac{1}{16}$	u	0.9557(− 08)	0.9521(− 08)	0.1158(− 07)	0.4103(− 04)	0.4109(− 04)	0.3723(− 04)
	v	0.5865(− 07)	0.5864(− 07)	0.5789(− 07)	0.2581(− 04)	0.2584(− 04)	0.2946(− 04)

5. Conclusions

In this paper, we have proposed three level implicit difference methods of $O(k^4 + k^2h^2 + h^4)$ for solving the system of two space nonlinear second order hyperbolic equations. The method has been successfully applied to the wave equation in polar coordinates. These methods give accurate results everywhere including the region in the vicinity of $r = 0$. A special treatment is required if $r = 0$ is a part of the boundary and the solution is to be determined at $r = 0$. This is illustrated for Eq. (13) in the r–z plane. The solutions remain accurate at all points on $r = 0$ and the order is preserved. The proposed ADI schemes are conditionally stable and the range of stability of the mesh ratio being determined by the coefficients associated with the particular problem. From the numerical results we found that the proposed methods are of $O(h^4)$ and very close to the accurate one for a fixed p.

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