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Short Communication

Symmetrical formulas for rational interpolants

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Abstract: Symmetrical determinantal formulas for the numerator and denominator of an ordinary rational interpolant are presented and discussed. Degenerate cases are analysed.

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1. Introduction

Determinantal formulas for the numerator and denominator of a rational interpolant have been known for a long time [1,4] and there is an excellent review of their present status [5]. The fundamental problem consists of interpolating data values f_i at points x_i (f_i, $x_i \in \mathbb{C}$), $i = 0, 1, 2,\ldots, n$, and integers l, m are given for which $l + m = n$ and which specify the type of the interpolant.

Define

$$p^{[l/m]}(x) = \prod_{i=0}^{n}(x - x_1)\begin{vmatrix} f_0/(x-x_0) & f_0 & x_0 f_0 & \cdots & x_0^{m-1} f_0 & 1 & x_0 & \cdots & x_0^{l-1} \\ f_1/(x-x_1) & f_1 & x_1 f_1 & \cdots & x_1^{m-1} f_1 & 1 & x_1 & \cdots & x_1^{l-1} \\ \vdots & & & & & & & & \\ f_n/(x-x_n) & f_n & x_n f_n & \cdots & x_n^{m-1} f_n & 1 & x_n & \cdots & x_n^{l-1} \end{vmatrix} \tag{1}$$

and

$$q^{[l/m]}(x) = \prod_{i=0}^{n}(x - x_1)\begin{vmatrix} 1/(x-x_0) & f_0 & x_0 f_0 & \cdots & x_0^{m-1} f_0 & 1 & x_0 & \cdots & x_0^{l-1} \\ 1/(x-x_1) & f_1 & x_1 f_1 & \cdots & x_1^{m-1} f_1 & 1 & x_1 & \cdots & x_1^{l-1} \\ \vdots & & & & & & & & \\ 1/(x-x_n) & f_n & x_n f_n & \cdots & x_n^{m-1} f_n & 1 & x_n & \cdots & x_n^{l-1} \end{vmatrix} \tag{2}$$

Theorem 1 (A modified rational interpolation theorem). *With definitions* (1) *and* (2), $p^{[l/m]}(x)$ *and* $q^{[l/m]}(x)$ *are polynomials satisfying*

$$\partial\{p^{[l/m]}\} \leqslant l, \qquad \partial\{q^{[l/m]}\} \leqslant m, \tag{3}$$

$$p^{[l/m]}(x_i) = f_i q^{[l/m]}(x_i), \quad i = 0, 1, 2,\ldots, n. \tag{4}$$

Corollary. *For ordinary rational interpolation, define*

$$r^{[l/m]}(x)=p^{[l/m]}(x)/q^{[l/m]}(x).$$

Then, provided $q^{[l/m]}(x_i)\neq 0$, $i=0,1,2,\ldots,n$,

$$r^{[l/m]}(x_i)=f_i,\quad i=0,1,2,\ldots,n.$$

Proof. From (1) and (2), we see that $p^{[l/m]}(x)$ and $q^{[l/m]}(x)$ are both polynomials in x of degrees n at most. Using the identity

$$\frac{1}{x-x_i}=\frac{1}{x}+\frac{x_i}{x^2}+\frac{x_i^2}{x^3}+\cdots+\frac{x_i^{k-1}}{x^k}+\frac{x_i^k}{x^k(x-x_i)},\tag{5}$$

we see that

$$p^{[l/m]}(x)=\mathrm{O}(x^l),\qquad q^{[l/m]}(x)=\mathrm{O}(x^m)\quad\text{as } x\to\infty\tag{6}$$

and so (3) is proved. The determinants in (1) and (2) may be combined to show that

$$\left[p^{[l/m]}(x)-f_iq^{[l/m]}(x)\right]_{x=x_i}=0,\quad i=0,1,2,\ldots,n$$

and so (4) follows too. The corollary is obvious. □

2. Remarks

(1) *Relationships.* The determinants in (1) and (2) are $(n+1)\times(n+1)$ determinants, whereas the equivalent Cauchy determinants are $(n+2)\times(n+2)$; undoubtedly they are closely related. Stoer's first identities [6] may also be derived from (1) and (2) using Sylvester's rule.

(2) *Special cases.* If $m=0$, there are no 'f-type' columns in (1) and (2), $q^{[l/m]}(x)$ is in fact a constant and $p^{[l/m]}(x)/q^{[l/m]}(x)$ reduces to the Lagrange interpolating polynomial.

If $l=0$, there are no 'x-type' columns in (1) and (2). Provided $f_i\neq 0$, $i=0,1,\ldots,n$, $p^{[l/m]}(x)/q^{[l/m]}(x)$ reduces to the reciprocal of the Lagrange polynomial interpolating $f_0^{-1}, f_1^{-1},\ldots,f_n^{-1}$.

(3) *Construction.* Prof. J. Meinguet contributes the following ingenious constructive proof of (1)–(4). Define polynomials $p(x)$, $q(x)$, $w(x)$ by

$$\partial\{p(x)\}\leqslant l,\qquad \partial\{q(x)\}\leqslant m,\qquad w(x)=\prod_{i=0}^{n}(x-x_i),$$

$$p(x_i)=f_iq(x_i),\quad i=0,1,2,\ldots,n.$$

From Lagrange's interpolation formula (or by partial fraction decomposition), we have

$$\frac{p(x)}{w(x)}=\sum_{i=0}^{n}\frac{p(x_i)}{w'(x_i)(x-x_i)}=\sum_{i=0}^{n}\left(\frac{f_i}{x-x_i}\right)\frac{q(x_i)}{w'(x_i)},\tag{7}$$

$$\frac{q(x)}{w(x)}=\sum_{i=0}^{n}\left(\frac{1}{x-x_i}\right)\frac{q(x_i)}{w'(x_i)},\tag{8}$$

where we have made use of the fact that $p(x_i)=f_iq(x_i)$ for all i. By using Jacobi's device for obtaining identities from Lagrange's formula, we find

$$\sum_{i=0}^{n}\frac{x_i^kq(x_i)}{w'(x_i)}=0\quad\text{for } k=0,1,\ldots,l-1,$$

$$\sum_{i=0}^{n}\frac{x_i^jp(x_i)}{w'(x_i)}=\sum_{i=0}^{n}\frac{(x_i^j/f_i)q(x_i)}{w'(x_i)}=0\quad\text{for } j=0,1,\ldots,m-1,$$

because the degrees of the polynomials $q(x)$ and $p(x)$ are not greater than m and l, respectively; hence it follows that

$$\left(\frac{q(x_0)}{w'(x_0)},\dots,\frac{q(x_n)}{w'(x_n)}\right)\begin{pmatrix} f_0 & x_0 f_0 & \cdots & x_0^{m-1} f_0 & 1 & x_0 & \cdots & x_0^{l-1} \\ f_1 & x_1 f_1 & \cdots & x_1^{m-1} f_1 & 1 & x_1 & \cdots & x_1^{l-1} \\ \vdots & & & & & & & \\ f_n & x_n f_n & \cdots & x_n^{m-1} f_n & 1 & x_n & \cdots & x_n^{l-1} \end{pmatrix} = 0. \tag{9}$$

Provided only that this $(n+1)\times n$ matrix is of maximum rank, the determinantal formulas (1) (resp. (2)) immediately follow by elimination of the $(n+1)$ unknown quantities $q(x_i)/w'(x_i)$ between (7) and (9) (resp. (8) and (9)).

(4) *Confluence*. Suppose, for example, that derivative values $f_0, f_0', f_0^{(2)},\dots,f_0^{(k)}$ are the data supplied at the point x_0. This corresponds to the Hermite problem of confluence of $x_0, x_1,\dots,x_k$. The first $k+1$ rows of each of the determinants in (1) and (2) become a $(k+1)\times(n+1)$ block. From (1), we find that the first column of the block has entries

$$\frac{f_0}{x-x_0}, \frac{f_0+(x-x_0)f_0'}{(x-x_0)^2},\dots,\frac{f_0+(x-x_0)f_0'+\cdots+(x-x_0)^k f_0^{(k)}/k!}{(x-x_0)^{k+1}}. \tag{10}$$

The numerators of (10) are the first $k+1$ truncations of the Taylor series expansion of the implicit function about x_0. Similarly, the first column of (2) contains the entries

$$\frac{1}{x-x_0}, \frac{1}{(x-x_0)^2},\dots,\frac{1}{(x-x_0)^{k+1}}, \tag{11}$$

after confluence. The entries in the remaining columns, numbered $2, 3,\dots,n+1$, which are common to both the numerator and denominator polynomials retain their property of order reduction (9), which may be based on the rule

$$\frac{1}{(x-x_0)^r} = \frac{1}{x^r} + \frac{r x_0}{x^{r+1}} + \frac{r(r+1)x_0^2}{x^{r+2}2!} + \cdots + \binom{r+k-1}{r-1}\frac{x_0^k}{x^{k+r}} + \cdots,$$

valid for $|x| > |x_0|$, and where as usual

$$\binom{\alpha}{\beta} = {}^{\alpha}C_{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}.$$

The effect of column $m+s+1$ may be thought of as dealing with $O(x^{-s})$ in (11), and the entry in row r of this column is

$$M_{r,m+s+1} = \binom{s-1}{r-1} x_0^{s-r}, \quad r=1, 2,\dots,k+1;\ s=1, 2,\dots,l. \tag{12}$$

The effect of column $s+1$ may be thought of as dealing with $O(x^{-s})$ in (10), and the entry in row r of this column is

$$M_{r,s+1} = \sum_{j=1}^{r} \binom{s-1}{j-1} \frac{f_0^{(r-j)}}{(r-j)!} x_0^{s-j}, \quad r=1, 2,\dots,k+1;\ s=1, 2,\dots,m. \tag{13}$$

Note that $M_{r,m+s+1} = M_{r,s+1} = 0$ if $r > s$.

A full explicit solution similar to (1) and (2) for the Hermite interpolation problem now follows using blocks composed of the entries of the types in (10)–(13). However, much of the elegance of (1) and (2) has been lost.

(5) *Calculations*. Equations (1) and (2) are unsuitable for numerical work, for which the algorithms of Werner [7] or Graves–Morris and Hopkins [3] are recommended, see [2].

3. Degeneracies

It is well known that not all rational interpolation problems are soluble. Wuytack [8] gave a comprehensive analysis of these problems by correlating and ordering common factors of the solutions of the modified rational interpolation problem. His treatment is the closest possible approximation to a representation independent analysis. We now consider the question of unattained points and related issues in the context of the specific representations (1) and (2). The alternatives are characterised by the results of Theorems 2 and 4.

Define F_j to be cofactor of the element $(x-x_j)^{-1}$ in (2) and let F denote the matrix occurring in (9). For conciseness, we also define

$$N(x) \equiv p^{[l/m]}(x), \qquad D(x) \equiv q^{[l/m]}(x),$$

to be polynomials expressed by (1) and (2).

Theorem 2. *A necessary and sufficient condition that $D(x) \equiv 0$ is that*

$$F_j = 0, \quad j = 0, 1, 2, \dots, n. \tag{14}$$

Proof. Suppose that $D(x) \equiv 0$. Then

$$D(x_j) = 0, \quad j = 0, 1, 2, \dots, n$$

and (14) follows. For the converse, expand the determinant representation of $D(x)$ by its first column. □

Theorem 3. *For $l \geqslant 1$ and $m \geqslant 1$, a necessary and sufficient condition for $D(x) \equiv 0$ is that polynomials $n(x)$ and $d(x)$ exist, at least one of which is non-null, for which*

$$\partial\{n(x)\} \leqslant l-1, \qquad \partial\{d(x)\} \leqslant m-1 \tag{15}$$

and

$$n(x_i) = f_i d(x_i), \quad i = 0, 1, \dots, n. \tag{16}$$

Proof. First, suppose that $D(x) \equiv 0$, so that

$$F_k = 0, \quad k = 0, 1, 2, \dots, n, \tag{17}$$

by Theorem 2. Because F is an $(n+1) \times n$ matrix, its rank does not exceed n. Suppose that rank $F = n$, so that F has n linearly independent rows. This is contradicted by (17), so that the row rank, and therefore the column rank, of F is at most $n-1$. Hence numbers $\{\alpha_i, \beta_i\}$, not all zero, exist such that the following linear combination of the columns of F vanishes:

$$\sum_{i=0}^{m-1} \alpha_i x_j^i f_j + \sum_{i=0}^{l-1} \beta_i x_j^i = 0, \quad j = 0, 1, 2, \dots, n, \tag{18}$$

i.e.

$$\left(\sum_{i=0}^{m-1} \alpha_i x^i \right) f(x) + \sum_{i=0}^{l-1} \beta_i x^i = 0, \quad x = x_0, x_1, \dots, x_n, \tag{19}$$

and the necessity result is proved. For the converse, suppose that a result of the form (19) holds. Then (18) also holds, and represents a vanishing linear combination of columns of F. Hence $D(x) \equiv 0$. □

Theorem 4. *Let $D(x) \not\equiv 0$ and $m \geqslant 1$. A necessary and sufficient condition that $F_j = 0$ is that x_j is an unattained point, i.e.*

$$\lim_{x \to x_j} N(x)/D(x) \neq f_j. \tag{20}$$

Proof. If x_j is an unattained point, then Theorem 1 shows that $F_j = 0$ and the sufficiency condition is proved. To prove necessity, we consider a value of j for which $F_j = 0$.

Consider first the case $l = 0$. Then

$$f_i D(x_i) = N, \quad i = 0, 1, 2, \ldots, n,$$

where N is a constant, independent of i. Since $F_j = 0$, $D(x_j) = 0$ and therefore $N = 0$. If $f_j = 0$ also, by inspection of (2) we see that $D(x) \equiv 0$, contrary to hypothesis. Therefore $f_j \neq 0$ and so x_j is an unattained point.

Next, consider the case of $l \geqslant 1$. Since $F_j = 0$,

$$N(x_j) = D(x_j) = 0,$$

and we may define polynomials $n(x)$ and $d(x)$ by

$$n(x) = N(x)/(x - x_j) \quad \text{and} \quad d(x) = D(x)/(x - x_j), \tag{21}$$

which satisfy (15). From (4) and (21) it follows that

$$n(x_i) = f_i d(x_i) \quad \text{for all } i, \text{ except } i = j.$$

However, if x_j is an attained point,

$$n(x_j) = f_j d(x_j)$$

and therefore the conditions of Theorem 3 are satisfied. Therefore, $D(x) \equiv 0$, contrary to hypothesis. Hence x_j is necessarily an unattained point. □

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