Conjugacy of Cartan subalgebras of complex finite-dimensional Leibniz algebras ☆

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Abstract

In the present work the properties of Cartan subalgebras and their connection with regular elements in finite-dimensional Lie algebras are extended to the case of Leibniz algebras. It is shown that Cartan subalgebras and regular elements of a Leibniz algebra correspond to Cartan subalgebras and regular elements of a Lie algebra by a natural homomorphism. Conjugacy of Cartan subalgebras of Leibniz algebras is shown.

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1. Introduction

This work is devoted to the study of Leibniz algebras, which were introduced by French mathematician J.-L. Loday in [1] and considered further in works [2–6].

It is well known that Leibniz algebras are "noncommutative" generalization of Lie algebras. Investigations of nilpotent Leibniz algebras in [7–10] show that many nilpotent properties of Lie algebras can be extended to the case of nilpotent Leibniz algebras.

A characteristic of non-Lie Leibniz algebras is a nontriviality of the ideal generated by squares of the elements of the algebra (moreover, it is abelian).

Cartan subalgebras, their relations with regular elements and decomposition to weight spaces by Cartan subalgebras play the basic role in the structure theory of Lie algebras. Although some properties of regular elements and Cartan subalgebras were studied in [10], the relation between regular elements and Cartan subalgebras of the Leibniz algebra, on the one side, and regular elements and Cartan subalgebras of its factor algebra by the ideal generated by squares of elements (which evidently is a Lie algebra), on the other side, has been not clarified. Therefore, one of the aim of this work is to investigate the mentioned relation. Namely, it is proved that the images of regular elements and Cartan subalgebras by a natural homomorphism are regular elements and Cartan subalgebras, respectively.

The classical result of the structure theory of finite-dimensional Lie algebras on conjugacy of Cartan subalgebras is proven in the case of Leibniz algebras.

2. Preliminaries

Definition 2.1. An algebra L over a field F is called a Leibniz algebra if the Leibniz identity

$$\big[x, [y, z]\big] = \big[[x, y], z\big] - \big[[x, z], y\big]$$

holds for any $x, y, z \in L$, where $[\, , \,]$ is the multiplication in L.

Note that in the case of fulfilling the identity $[x, x] = 0$ the Leibniz identity can be easily reduced to the Jacobi identity. Thus, Leibniz algebras are "noncommutative" analogues of Lie algebras.

For an arbitrary algebra L, we define the following sequence:

$$L^1 = L, \qquad L^{n+1} = \big[L^n, L^1\big].$$

Definition 2.2. An algebra L is called nilpotent if there exists $s \in N$ such that $L^s = 0$.

For an arbitrary element $x \in L$, we consider the operator of right multiplication $R_x : L \to L$ where $R_x(z) = [z, x]$. The set $R(L) = \{R_x \mid x \in L\}$ is a Lie algebra with respect to operation of commutation, and the following identity holds:

$$R_x R_y - R_y R_x = R_{[y, x]}.$$

It is easy to see from this identity that the solvability of the Lie algebra $R(L)$ is equivalent to the solvability of the Leibniz algebra L. Moreover, if L is nilpotent then the Lie algebra $R(L)$ is also nilpotent.

The following lemma gives a decomposition of a vector space into the direct sum of two invariant subspaces with respect to a linear transformation.

Lemma 2.1 *(Fitting's Lemma). Let V be a vector space and $A : V \to V$ be a linear transformation. Then $V = V_{0A} \oplus V_{1A}$, where $A(V_{0A}) \subseteq V_{0A}$, $A(V_{1A}) \subseteq V_{1A}$ and $V_{0A} = \{v \in V \mid A^i(v) = 0 \text{ for some } i\}$ and $V_{1A} = \bigcap_{i=1}^{\infty} A^i(V)$. Moreover, $A|_{V_{0A}}$ is a nilpotent transformation and $A_{V_{1A}}$ is an automorphism.*

Proof. See [11, Chapter II, §4]. $\quad\square$

Definition 2.3. The spaces V_{0A} and V_{1A} are called the Fitting's null-component and the Fitting's one-component (respectively) of the space V with respect to the transformation A.

Definition 2.4. An element h of the Leibniz algebra L is said to be regular if the dimension of the Fitting's null-component of the space L with respect to R_h is minimal. In addition, its dimension is called a rank of the algebra L.

It is easy to see that the dimension of the Fitting's null-component of a linear transformation A equals to the order of zero root of characteristic polynomial of this transformation. Hence an element h is regular if and only if the order of zero characteristic root is minimal for R_h.

Note that in the case of Lie algebras the linear transformation R_h is degenerated (since $[h, h] = 0$ for any h) and therefore the rank of the Lie algebra is greater than zero.

The following lemma shows that for Leibniz algebras the rank is also greater than zero.

Lemma 2.2. *Let L be a finite-dimensional Leibniz algebra. Then the operator R_x is degenerated for any $x \in L$.*

Proof. See [10]. $\quad\square$

We also have the following generalization of Fitting's Lemma for Lie algebras of nilpotent transformations of a vector space.

Theorem 2.1. *Let G be a nilpotent Lie algebra of linear transformations of a vector space V and $V_0 = \bigcap_{A \in G} V_{0A}$, $V_1 = \bigcap_{i=1}^{\infty} G^i(V)$. Then the subspaces V_0 and V_1 are invariant with respect to G (i.e., they are invariant with respect to every transformation B of G) and $V = V_0 \oplus V_1$. Moreover, $V_1 = \sum_{A \in G} V_{1A}$.*

Proof. See [11, Chapter II, §4]. $\quad\square$

Remark 1. From [11] in case of vector space V over an infinite field and fulfillment of conditions of Theorem 2.1, we have the existence of the element $B \in G$ such that $V_0 = V_{0B}$ and $V_1 = V_{1B}$.

3. Conjugacy of the Cartan subalgebras of finite-dimensional Leibniz algebras

Let $\Im$ be a nilpotent subalgebra of a Leibniz algebra L and $L = L_0 \oplus L_1$ be the Fitting's decomposition of the algebra L with respect to the nilpotent Lie algebra $R(\Im) = \{R_x \mid x \in \Im\}$ of transformations of the vector space as in Theorem 2.1.

The set $l(\Im) = \{x \in L \mid [x, \Im] \subseteq \Im\}$ is said to be a left normalizer of the subalgebra $\Im$ in the algebra L.

The set $r(\Im) = \{x \in L \mid [\Im, x] \subseteq \Im\}$ is said to be a right normalizer of the subalgebra $\Im$ in the algebra L.

Definition 3.1. A subalgebra $\Im$ of a Leibniz algebra L is called a Cartan subalgebra if the following two conditions are satisfied:

(a) $\Im$ is nilpotent;
(b) $\Im$ coincides with the left normalizer of $\Im$ in the algebra L.

Note that the definition of a Cartan subalgebra of a Leibniz algebra is agree with the definition of a Cartan subalgebra for the Lie algebra.

In view of the antisymmetric identity in Lie algebras, the sets $l(\Im)$ and $r(\Im)$ coincide.

The following example show that in general these sets do not coincide for Leibniz algebras.

Example 3.1. Let L be a Leibniz algebra determined by the following multiplication:

$$[x, z] = x, \qquad [z, y] = y, \qquad [y, z] = -y, \qquad [z, z] = x,$$

where $\{x, y, z\}$ is the basis of the algebra L and omitted products are equal to zero.

Then $\Im = \{x - z\}$ is a Cartan subalgebra, but $r(\Im) = \{x, z\}$.

For Cartan subalgebras of Leibniz algebras similar to the case of Lie algebras, there is a characterization in terms of the Fitting's null-component, namely, the following proposition is true.

Proposition 3.1. *A nilpotent subalgebra $\Im$ of a Leibniz algebra L is a Cartan subalgebra if and only if $\Im$ coincides with L_0 in the Fitting's decomposition of the algebra L with respect to $R(\Im)$.*

Proof. See [10]. □

The following theorem establishes properties of the Fitting's null-component of the regular element of a Leibniz algebra.

Theorem 3.1. *Let L be a Leibniz algebra over an infinite field F and a be a regular element of the algebra L. Then the Fitting's null-component $\Im$ of the algebra L with respect to R_a is a Cartan subalgebra.*

Proof. See [10]. □

Another useful remark on regular elements and Cartan subalgebras of Leibniz algebras is the fact that if the Cartan subalgebra contains a regular element a, then $\mathfrak{I}$ is uniquely determined by the element a as the Fitting's null-component of the algebra L with respect to R_a, i.e., $\mathfrak{I} = L_{0R_a}$.

For the Leibniz algebra L, we consider the natural homomorphism φ into the factor algebra L/I, where $I = \text{ideal}\langle [x, x] \mid x \in L \rangle$.

Proposition 3.2. *Let L be a complex finite-dimensional Leibniz algebra. Then the image of a regular element of the algebra L by a homomorphism φ is a regular element of the Lie algebra L/I.*

Proof. Let a be a regular element of the algebra L. We will prove that the element $\bar{a} = a + I$ will be a regular element of the Lie algebra L/I. Suppose the opposite, i.e., $\bar{a} = a + I$ is not a regular element. Let $\bar{b} = b + I$ be any regular element of the Lie algebra L/I and $a - b \notin I$.

Since I is an ideal, then for any $x \in L$ we have $R_x(I) \subseteq I$. It means that the matrix of the transformation R_x has the following block form

$$R_x = \begin{pmatrix} X, & 0 \\ Z_x, & I_x \end{pmatrix}$$

in the basis $\{e_1, e_2, \ldots, e_m, i_1, i_2, \ldots, i_n\}$ of L, where $\{i_1, i_2, \ldots, i_n\}$ is the basis of I. Here X is the matrix of the transformation $R_x|_{\{e_1, e_2, \ldots, e_m\}}$ and I_X is the matrix of the transformation $R_x|_I$.

Let

$$R_a = \begin{pmatrix} A, & 0 \\ Z_a, & I_a \end{pmatrix}, \qquad R_b = \begin{pmatrix} B, & 0 \\ Z_b, & I_b \end{pmatrix}$$

be the matrices of the transformations R_a and R_b, respectively.

Let k (respectively k') be the order of the characteristic zero root of the matrix A (respectively B) and s and s' be the orders of the characteristic zero root of the matrices I_a and I_b, respectively. Then we have $k' < k$, $s < s'$.

Put $U = \{y \in L \setminus I \mid R_y = \begin{pmatrix} Y & 0 \\ Z_y & I_y \end{pmatrix}$ and Y has the order of the characteristic zero root less than $k\}$, $V = \{y \in L \setminus I \mid R_y = \begin{pmatrix} Y & 0 \\ Z_y & I_y \end{pmatrix}$ and I_y has the order of the characteristic zero root less than $s + 1\}$.

Since $b \in U$ and $a \in V$ these sets are nonempty.

Let us show that the set U is an open subset of the set $L \setminus I$ in the Zariski topology.

Let Y have the order of the characteristic zero root less than k. Then Y^k has the rank greater than $n - k$. It means that there exists a nonzero minor of the order $n - k + 1$. In the other words, there exists a nonzero polynomial of structural constants of the algebra L, hence the set U is open in the Zariski topology in the subset of the set $L \setminus I$.

One can analogously prove that the set V is open in $L \setminus I$. It is not difficult to check that the sets U and V are dense in $L \setminus I$. Therefore, there exists an element $y \in U \cap V$ such that Y has the order of characteristic zero root less than k and I_y has the order of the characteristic zero root less than $s + 1$. Thus, for this element y the order of characteristic zero root is not greater than $k + s - 1$, i.e., the rank of the algebra L is less than $k + s$ and we obtain the contradiction to the assumption that $\bar{a}$ is not a regular element of the Lie algebra $L \setminus I$. $\square$

Remark 2. For the Cartan subalgebra $\Im$ of the Leibniz algebra L, we consider the Lie algebra $R(\Im)$ of linear transformations L (which evidently is nilpotent) and the decomposition of the algebra L with respect to $R(\Im)$. Remark 1 implies existence of an element $R_b \in R(\Im)$ such that the Fitting's null-component with respect to the nilpotent Lie algebra of linear transformations $R(\Im)$ coincides with the Fitting's null-component with respect to the transformation R_b, i.e., $L_0 = L_{0R_b}$. Using Proposition 3.1 we obtain $\Im = L_{0R_b}$.

Let $\Im$ be a Cartan subalgebra of the Leibniz algebra L and $L = L_0 \oplus L_\alpha \oplus L_\beta \oplus \cdots \oplus L_\gamma$ be a decomposition of the algebra L into characteristic subspaces with respect to operator R_b possessing the property $\Im = L_{0R_b}$.

Lemma 3.1. *Let the element* $x = x_0 + x_\alpha + x_\beta + \cdots + x_\gamma$, *where* $x_\sigma \in L_\sigma$, $\sigma \in \{0, \alpha, \beta, \ldots, \gamma\}$ *satisfy the following conditions*:

(a) *there exists* $k \in N$ *such that* $[\cdots[[x, \underbrace{b], b], \ldots, b}_{k \text{ times}}] \in I$;

(b) $x \neq x_0$.

Then $\bar{x} = \bar{x}_0$.

Proof. Let k is the minimal of numbers having the property:

$$\left[\cdots\left[[x, \underbrace{b], b\right], \ldots, b}_{k \text{ times}}\right] \in I.$$

It is not difficult to see that the element

$$\left[\cdots\left[[x, \underbrace{b], b\right], \ldots, b}_{k-1 \text{ times}}\right] \in I$$

possesses the properties of the element x. That is why we can suppose $[x, b] \in I$.

To prove the lemma, it is sufficient to show that $x' = x_\alpha + x_\beta + \cdots + x_\gamma \in I$. Without any loss of generality we can also suppose that $[x', b] \in I$.

If $x_\sigma \in I$ for any $\sigma \in \{\alpha, \beta, \ldots, \gamma\}$ then the assertion of the lemma is evident. Suppose there exists $\sigma \in \{\alpha, \beta, \ldots, \gamma\}$, such that $x_\sigma \notin I$. For convenience, we put $\sigma = \alpha$.

Let $x_\alpha = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_t e_t$ be the decomposition with respect to Jordan basis of L_α. Then $[x_\alpha, b] = \alpha x_\alpha + \alpha_1' e_2 + \alpha_2' e_3 + \cdots + \alpha_{t-1}' e_t$ for some α_i', where $i = \{1, 2, \ldots, t-1\}$.

Consider multiplication:

$$[x', b] = \alpha x' + \alpha_1' e_2 + \alpha_2' e_3 + \cdots + \alpha_{t-1}' e_t + ([x_\beta, b] - \alpha x_\beta) + \cdots + ([x_\gamma, b] - \alpha x_\gamma).$$

For the element $x'' = [x', b] - \alpha x'$ (which has not the basis element e_1 in its decomposition), we have $[x'', b] \in I$ and $\overline{x''} = \alpha \overline{x'}$, and also if $x'' \in L_0$ then $x'' = 0$. Hence, $x' = x_\alpha$ and $[x', b] = \alpha x'$ which implies that $x_\alpha \in I$. This contradicts the assumption that $x_\alpha \notin I$.

Thus, conditions $[x'', b] \in I$; $x'' \neq x_0$ and $\overline{x''} = \alpha \overline{x'}$ hold for the element x''.

Let $x'' = x_\alpha' + x_\beta' + \cdots + x_\gamma'$, where the basis element e_1 is absent in the decomposition of x_α'. If $x_\alpha' \in I$ than we can suppose that x'' equals to $x_\beta' + \cdots + x_\gamma'$.

If $x_\alpha' \notin I$ then we analogously obtain the element $x''' = x_\alpha'' + x_\beta'' + \cdots + x_\gamma''$ satisfying the following conditions: $[x''', b] \in I$, $\overline{x'''} = \alpha^2 \overline{x''}$ and there are no basis elements e_1, e_2 in the decomposition of x_α''.

If we continue similar operations finite times and take into account the property of the Jordan basis, we obtain existence of the element $z = z_\alpha + z_\beta + \cdots + z_\gamma$ such that $[z_\alpha, b] = \alpha z_\alpha$ and conditions $[z, b] \in I$, $z \neq x_0$ and $\overline{z} = \alpha^q \overline{x'}$ are satisfied.

Consider the element $z' = [z, b] - \alpha z = ([z_\beta, b] - \alpha z_\beta) + \cdots + ([z_\gamma, b] - \alpha z_\gamma)$. It is evident that $[z', b] \in I$ and $\overline{z'} = \alpha^{q+1} \overline{x'}$. In the case of $z' = x_0$, we have $z_\beta = \cdots = z_\gamma = 0$, i.e., $z' = 0$ and $x' \in I$.

Suppose that $z' \neq x_0$. In this case we have shown the existence of the element $z' = z_\beta + \cdots + z_\gamma$ (such that there is not a component lying in L_α in the decomposition of z') for which the following conditions hold: $[z', b] \in I$; $z \neq x_0$ and $\overline{z'} = \alpha^{q+1} \overline{x'}$. If we continue analogously, we obtain the existence of an element d such that $[d, b] = \delta d$ and $d \in L_\delta$, where $\delta \in \{\alpha, \beta, \ldots, \gamma\}$, satisfying the following conditions: $[d, b] \in I$, $\overline{d} = q\overline{x'}$ for some nonzero number q. Hence, $d \in I$ and therefore $\overline{x'} = \overline{0}$, i.e., $x' \in I$. $\quad\square$

The following theorem establishes a relation for Cartan subalgebras as in Proposition 3.2.

Theorem 3.2. *Let L be a Leibniz algebra and $\Im$ is its Cartan subalgebra. Then the image of the subalgebra $\Im$ by homomorphism φ is the Cartan subalgebra of the Lie algebra L/I.*

Proof. Denote $\varphi(\Im) = \bar{\Im}$. Nilpotence of $\bar{\Im}$ follows from property of homomorphic image of nilpotent subalgebra.

Let $L_{\bar{0}}$ be the Fitting's null-component in the Fitting's decomposition of the algebra L/I with respect to $R(\bar{\Im})$.

Suppose that $\bar{\Im}$ is not a Cartan subalgebra, i.e., $\bar{\Im} \subset l(\bar{\Im})$. By Proposition 3.1, we have $\bar{\Im} \subset L_{\bar{0}}$, where

$$L_{\bar{0}} = \left\{ \bar{x} \in L/I \mid \text{for any } \bar{h} \in \bar{\Im} \text{ there exists } k \in N \text{ such that } \left[\cdots \left[[\bar{x}, \bar{h}], \bar{h} \right], \ldots, \bar{h} \right] = \bar{0} \right\}.$$

$$\underbrace{\qquad\qquad}_{k \text{ times}}$$

Hence, there exists an element $\bar{x} = x + I$ such that $\bar{x} \in L_{\bar{0}}$ and $\bar{x} \notin \bar{\Im}$ (it means that $x \notin I$, $x \notin \Im$), i.e.,

$$\left[\cdots \left[[\bar{x}, \bar{h}], \bar{h} \right], \ldots, \bar{h} \right] = \bar{0}$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{k\ \text{times}}$$

for any $\bar{h} \in \bar{\Im}$ and some $k \in N$. So, there exists x such that $x \notin \Im$, $x \notin I$ and for any

$$h \in \Im \quad \text{we have} \quad \left[\cdots \left[[x, h], h \right], \ldots, h \right] \in I$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{k\ \text{times}}$$

for some $k \in N$.

Thus, Lemma 3.1 is applicable to the element x, i.e., $x \neq x_0$, where $x_0 \in L_{0R_b}$, and

$$\left[\cdots \left[[x, b], b \right], \ldots, b \right] \in I$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{n\ \text{times}}$$

by some $k \in N$. Hence $\bar{x} = \bar{x}_0$, i.e., $\bar{x} \in \bar{\Im}$ which contradicts to the condition that $\bar{\Im}$ is not a Cartan subalgebra. $\square$

We present the example which demonstrates that the preimage by a natural homomorphism of a regular element (Cartan subalgebra) is not regular (Cartan subalgebra).

Example 3.2. Let the Leibniz algebra L with the basis $\{e_1, e_2, \ldots, e_5\}$ be defined by the following multiplication:

$$[e_2, e_1] = -e_3, \qquad [e_1, e_2] = e_3, \qquad [e_1, e_3] = -2e_1,$$
$$[e_3, e_1] = 2e_1, \qquad [e_3, e_2] = -2e_2, \qquad [e_2, e_3] = 2e_2,$$
$$[e_5, e_1] = e_4, \qquad [e_4, e_2] = e_5, \qquad [e_4, e_3] = -e_4,$$
$$[e_5, e_3] = e_5,$$

where omitted products are equal to zero.

It is not difficult to see that $I = \{e_4, e_5\}$ and L/I is isomorphic to the Lie algebra sl_2. Note that $\bar{e}_1 = e_1 + I$ is a regular element of the algebra L/I, but e_1 is not regular in L. Moreover, $\bar{\Im} = \{\bar{e}_1\}$ is a Cartan subalgebra of the algebra L/I, but $\Im = \{e_1, e_4, e_5\}$ is not a Cartan subalgebra of the algebra L.

Similar to the case of Lie algebras case, one can easily prove that if a nilpotent endomorphism R_x corresponds to an element x of the Leibniz algebra L over the field of zero characteristic then $\exp(R_x)$ is an automorphism of the algebra L. All kinds of products of such automorphisms are said to be *invariant automorphisms*. If τ is an automorphism then $\tau(\exp(R_x))\tau^{-1} = \exp(R_{\tau(z)})$.

Theorem 3.3. *Let $\Im_1$ and $\Im_2$ be Cartan subalgebras of the Leibniz algebra L. Then there exists an invariant automorphism δ such that $\delta(\Im_1) = \Im_2$.*

Proof. Let $\bar\Im$ be the image of the Cartan subalgebra $\Im$ by homomorphism φ. Then from the theory of Lie algebras we have the existence of a regular element $\bar a = a + I \in \bar\Im$ such that $\bar\Im = L_{\bar 0 R_{\bar a}}$. Take $b \in \Im$ such that $\Im = L_{0 R_b}$ (existence of the element b follows from Remark 2). As $a, b \in \Im \setminus I$ then $L_{0 R_b} \subseteq L_{0 R_a}$. Regularity of the element $\bar a$ implies $L_{\bar 0 R_{\bar a}} \subseteq L_{\bar 0 R_{\bar b}}$. Suppose $L_{\bar 0 R_{\bar a}} \neq L_{\bar 0 R_{\bar b}}$. Then there exists a nonzero element $\bar x = x + I$ such that $\bar x \in L_{\bar 0 R_{\bar b}}$ and $\bar x \notin R_{\bar 0 R_{\bar a}}$. Hence, for the element x we have

$$\big[\cdots\big[[x, \underbrace{b], b\big], \ldots, b}_{k \text{ times}}\big] \in I$$

for some $k \in N$ and

$$\big[\cdots\big[[x, \underbrace{a], a\big], \ldots, a}_{s \text{ times}}\big] \notin I$$

for any $s \in N$.

Note that

$$\big[\cdots\big[[x, \underbrace{b], b\big], \ldots, b}_{t \text{ times}}\big] \neq 0$$

for any $t \in N$. In fact, otherwise $x_0 \in L_{0 R_b}$ and $x_0 \in L_{0 R_a}$. But it contradicts the condition

$$\big[\cdots\big[[x, \underbrace{a], a\big], \ldots, a}_{s \text{ times}}\big] \notin I$$

for any $s \in N$. So,

$$\big[\cdots\big[[x, \underbrace{b], b\big], \ldots, b}_{t \text{ times}}\big] \neq 0$$

for any $t \in N$, i.e., $x \notin \Im$. Thus, we have obtained that for the element x, the condition

$$\big[\cdots\big[[x, \underbrace{b], b\big], \ldots, b}_{k \text{ times}}\big] \in I$$

holds for some $k \in N$ and $x \neq x_0$, where x_0 is the component in the decomposition of the element x as in Lemma 3.1. Using this lemma, we obtain $\bar x = \bar x_0 \in \bar\Im = L_{\bar 0 R_{\bar a}}$, i.e., we have the contradiction to the assumption $L_{\bar 0 R_{\bar a}} \neq L_{\bar 0 R_{\bar b}}$.

Thus, $\bar\Im = L_{\bar 0 R_{\bar b}}$ and $\Im = L_{0 R_b}$.

Let $\Im_1$ and $\Im_2$ be Cartan subalgebras of the Leibniz algebra L. Then by Theorem 3.2 $\bar{\Im}_1$ and $\bar{\Im}_2$ are Cartan subalgebras of the Lie algebra $\bar{L}$. Now using conjugacy of Cartan subalgebras $\bar{\Im}_1$ and $\bar{\Im}_2$ we have that there exists an invariant automorphism $\bar{\delta}$ such that $\bar{\delta}(\bar{\Im}_1) = \bar{\Im}_2$, i.e., $\delta(\bar{b}_1) = \bar{b}_2$. Let $\bar{\delta} = \exp R_{\bar{z}}$, where $\bar{z} = z + I$ and $z \notin I$. Then $(\exp R_z)(b_1) + I = b_2 + I$.

Let c be an element of the Leibniz algebra L such that $\exp R_z(c) = b_2$. Then $\exp R_z(b_1) + I = \exp R_z(c) + I$. It means that $\exp R_z(b_1 - c) \in I$. Hence $b_1 - c \in I$, where b_1 and c do not belong to the ideal I, i.e., $L_{0R_c} = \Im_1$, which immediately implies $\exp R_z(\Im_1) = \Im_2$. $\quad\square$

Theorems 3.1 and 3.3 imply that any Cartan subalgebra of the Leibniz algebra contains regular elements and all Cartan algebras have the same dimension coinciding with the rank of the algebra.

Acknowledgment

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