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Monodromic vs geodesic computation of Virasoro classical conformal blocks

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Abstract

We compute 5-point classical conformal blocks with two heavy, two light, and one superlight operator using the monodromy approach up to third order in the superlight expansion. By virtue of the AdS/CFT correspondence we show the equivalence of the resulting expressions to those obtained in the bulk computation for the corresponding geodesic configuration.

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1. Introduction

Conformal blocks represent important elements of any conformal field theory [1]. In general, they are certain functions of conformal dimensions $\{\Delta_i\}$ and Virasoro central charge c which are completely defined by the conformal symmetry. In the limit where c goes to infinity they are approximated by the so-called classical conformal blocks. Recently, a remarkable interpretation of the classical conformal blocks in the context of the AdS₃/CFT₂ correspondence has been inves-

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tigated [2–8]. It was shown that some class of classical CFT_2 conformal block can be described by means of a particular classical mechanics in AdS_3 .

There exist different types of the classical conformal blocks depending on the behavior of the conformal dimensions as $c \rightarrow \infty$ [9]. One can distinguish between two limiting cases: conformal blocks with only heavy and only light operators. In this note we focus on the case of two heavy operators of equal conformal dimensions producing in the bulk either a conical defect or BTZ black hole [2,3,5,6]. The bulk geodesic configuration corresponding to the n -point block with only two heavy fields consists of $2n - 5$ massive scalar particles propagating in the background geometry produced by the heavy operators [7]. It was pointed out in [10,8] that these configurations are nothing but the ordinary Witten exchange diagrams with the difference that there is no integration over positions of the vertices of the geodesic graph.

While the 4-point conformal block in the heavy-light approximation¹ was explicitly computed [5,6] an exact consideration of the n -point case is hampered by technical difficulties related to solving associated higher order algebraic equations. To analyze multi-particle configurations we proposed to use an additional approximation procedure with respect to a small parameter which can be chosen to be a mass of one of the particles whose worldline ends on the boundary (*i.e.*, it is a conformal dimension of one of external fields) [7]. This allows to iteratively reconstruct n -point heavy-light classical conformal block starting from the $(n - 1)$ -point one. Such a *super-light* approximation applies when there is a known expression for the n -point heavy-light conformal block. This is the case with the 5-point heavy-light block considered as a deformation of the exactly known heavy-light 4-point block with respect to a small classical conformal dimension of the third light operator. In this paper we use this procedure to analyze the bulk/boundary correspondence of the conformal block/geodesic Witten diagram computation of the 5-point heavy-light block in the sub-leading orders of the super-light expansion.

Our computation of the 5-point classical conformal block on the boundary relies on the study of the monodromy properties of the auxiliary Fuchsian differential equation. It is reduced to the computation of the so-called accessory parameters which are partial derivatives of the conformal block function. In the bulk we use the geodesic approach involving a different but related set of quantities – angular momenta of external and intermediate geodesic segments, and the mechanical action of the geodesic configuration or, equivalently, the geodesic length. Analogously to the accessory parameters, the external angular momenta are defined as derivatives of the mechanical action which in turn is related to the classical conformal block [6]. We expect that the geodesic description can be successively derived from the monodromy approach by finding counterparts of the monodromy approach constituents on the bulk side. In this paper we compare the systems of algebraic equations describing both the accessory and angular parameters and find out that they are generally different but have a common physically relevant root which leads to the conformal block function.

The paper is organized as follows. In Section 2 we apply the monodromy approach to find equations on the accessory parameters for the classical 5-point conformal block. In Section 3 we formulate the super-light approximation procedure and compute the accessory parameters up to the third order in the dimension of the super-light operator. In Section 4 we perform the corresponding bulk computation. In Section 5 we discuss explicit relations between the classical

¹ For the general 4-point block the basic computation tool is Zamolodchikov recursion relations [11,12], for the recent development see [13]. Alternative recursion in the important case of the vacuum conformal blocks was proposed recently in [14]. For multi-point blocks the AGT combinatorial representation is applicable (at finite value of the central charge in rational and non-rational CFTs with Virasoro symmetry see [15–17].

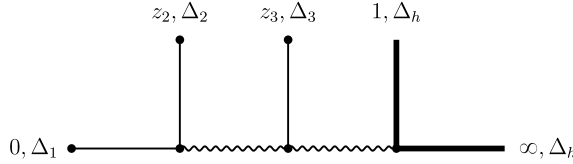


Fig. 1. The 5-point classical heavy-light conformal block. Two bold lines on the right represent heavy operators. As usual the projective invariance is used to fix three insertion positions as $z_1 = 0, z_4 = 1, z_5 = \infty$.

conformal block and the geodesic length, and between the accessory parameters and external angular momenta. Then we discuss a relation between two systems of algebraic equations on accessory/angular parameters. Using the super-light expansion method we find a perturbative solution to the bulk equations on angular parameters up to the third order. In Section 6 we compute the 5-point classical conformal block and the geodesic length up to the third order in the superlight dimension and find out exact match of two answers. Section 7 contains our conclusions.

2. Classical conformal block and monodromy problem

An n -point conformal block function $\mathcal{F}(z_i, \Delta_i, \tilde{\Delta}_j)$ is a holomorphic contribution to the correlation function of n primary fields in points z_i coming from a given set of Virasoro representations in the intermediate channels. In addition to the central charge c and conformal weights Δ_i of the external operators the conformal block depends on conformal weights $\tilde{\Delta}_j$ in the intermediate channels. For the spherical topology the fusion channel is represented by the corresponding fishbone diagram.

In the classical limit $c \rightarrow \infty$ the conformal blocks are exponentiated (see, e.g., [19,20]) as

$$\mathcal{F}(z_i, \Delta_i, \tilde{\Delta}_j) = \exp \left[-\frac{c}{6} f(z_i, \epsilon_i, \tilde{\epsilon}_j) \right], \tag{2.1}$$

where $\epsilon_i = \Delta_i/c$ and $\tilde{\epsilon}_j = \tilde{\Delta}_j/c$ are called respectively external and intermediate classical conformal dimensions and $f(z_i, \epsilon_i, \tilde{\epsilon}_j)$ is the main object of our study – classical conformal block. To compute the classical conformal block in the 5-point case we apply the monodromy method (see, e.g., [21,22] for a general discussion). From now on, a function $f(z_i)$ denotes the 5-point classical block related to the quantum block with the diagram presented on the Fig. 1 (we omit conformal dimensions as they always remain the same).

The 5-point conformal block can be obtained by considering an auxiliary 6-point correlation function $\langle V_{12}(z)V_1(z_1)\cdots V_5(z_5) \rangle$, where $V_i(z_i)$ are primary operators with dimensions $\Delta_i, i = 1, \dots, 5$ and $V_{12}(z)$ is the second level degenerate operator. Below we use the standard Liouville parametrization $c = 1 + 6(b + b^{-1})^2$ [9]. Because of the decoupling condition $(b^{-2}L_{-1}^2 + L_{-2})V_{12}(z) = 0$ the 6-point correlation function obeys the following second order differential equation [1]

$$\left[\frac{1}{b^2} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^5 \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right) \right] \langle V_{12}(z)V_1(z_1)\cdots V_5(z_5) \rangle = 0. \tag{2.2}$$

In the classical limit $b \rightarrow 0$ the 6-point auxiliary correlation function behaves as

$$\langle V_{12}(z)V_1(z_1)\cdots V_5(z_5) \rangle \Big|_{b \rightarrow 0} \rightarrow \psi(z) \exp(b^{-2} f(z_i)), \tag{2.3}$$

where the exponential factor $f(z_i)$ is the sought 5-point classical conformal block and the auxiliary holomorphic function $\psi(z)$ is governed by a Fuchsian equation arising from the above decoupling condition

$$\frac{d^2\psi(z)}{dz^2} + T(z)\psi(z) = 0. \tag{2.4}$$

Here $T(z)$ is the holomorphic component of the classical stress–energy tensor

$$T(z) = \sum_{i=1}^5 \left(\frac{\epsilon_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right), \tag{2.5}$$

where ϵ_i are the classical conformal dimensions and c_i are the accessory parameters related to the classical conformal block as follows

$$c_i(z) = \frac{\partial f(z)}{\partial z_i}, \quad i = 1, \dots, 5. \tag{2.6}$$

The asymptotic behavior $T(z) \sim z^{-4}$ at infinity implies that the accessory parameters satisfy the following linear conditions

$$\sum_{i=1}^5 c_i = 0, \quad \sum_{i=1}^5 (c_i z_i + \epsilon_i) = 0, \quad \sum_{i=1}^5 (c_i z_i^2 + 2\epsilon_i z_i) = 0. \tag{2.7}$$

Fixing $z_1 = 0, z_4 = 1, z_5 = \infty$ the above conditions are solved as

$$\begin{aligned} c_1 &= -c_2(1 - z_2) - c_3(1 - z_3) + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5, \\ c_4 &= -c_2 z_2 - c_3 z_3 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 + \epsilon_5, \quad c_5 = 0, \end{aligned} \tag{2.8}$$

where accessory parameters c_2 and c_3 are taken as independent variables. Using (2.8) function $T(z)$ takes the form

$$\begin{aligned} T(z) &= \frac{\epsilon_1}{z^2} + \frac{\epsilon_2}{(z - z_2)^2} + \frac{\epsilon_3}{(z - z_3)^2} + \frac{\epsilon_4}{(z - 1)^2} + \\ &+ c_2 \frac{z_2(z_2 - 1)}{z(z - z_2)(z - 1)} + c_3 \frac{z_3(z_3 - 1)}{z(z - z_3)(z - 1)} - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5}{z(z - 1)}. \end{aligned} \tag{2.9}$$

2.1. Heavy-light approximation

Classical conformal blocks in the heavy-light limit contain two heavy operators while others are light [2]. Let $\epsilon_4 = \epsilon_5 \equiv \epsilon_h$ be the dimension of two heavy fields, while fields with dimensions $\epsilon_1, \epsilon_2, \epsilon_3$ be light. It means that the dimension of heavy operators is fixed in the semiclassical limit while those of light operators tend to zero. The equation (2.4) can then be solved perturbatively. Let us expand all functions as

$$\begin{aligned} \psi(z) &= \psi^{(0)}(z) + \psi^{(1)}(z) + \psi^{(2)}(z) + \dots, \\ T(z) &= T^{(0)}(z) + T^{(1)}(z) + T^{(2)}(z) + \dots, \\ c_2(z) &= c_2^{(0)}(z) + c_2^{(1)}(z) + c_2^{(2)}(z) + \dots, \\ c_3(z) &= c_3^{(0)}(z) + c_3^{(1)}(z) + c_3^{(2)}(z) + \dots, \end{aligned} \tag{2.10}$$

where expansion parameters are light conformal dimensions. We assume that the accessory parameter expansion starts with terms linear in the conformal dimensions so that $c_{2,3}^{(0)} = 0$. In the case of the heavy-light conformal blocks it is sufficient to consider just the first order corrections (see [2,6] for the discussion of the 4-pt case). The lowest order equations are then

$$\left(\frac{d^2}{dz^2} + T^{(0)}(z)\right)\psi^{(0)}(z) = 0, \quad \left(\frac{d^2}{dz^2} + T^{(0)}(z)\right)\psi^{(1)}(z) = -T^{(1)}\psi^{(0)}(z), \quad (2.11)$$

where the stress–energy tensor components are directly read off from (2.9),

$$\begin{aligned} T^{(0)}(z) &= \frac{\epsilon_h}{(z-1)^2}, \\ T^{(1)}(z) &= \frac{\epsilon_1}{z^2} + \frac{\epsilon_2}{(z-z_2)^2} + \frac{\epsilon_3}{(z-z_3)^2} + c_2 \frac{z_2(z_2-1)}{z(z-z_2)(z-1)} + c_3 \frac{z_3(z_3-1)}{z(z-z_3)(z-1)} \\ &\quad - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{z(z-1)}. \end{aligned} \quad (2.12)$$

Here and below we use notation $c_2(z)$ for $c_2^{(1)}(z)$ and $c_3(z)$ for $c_3^{(1)}(z)$ as they are the only coefficients which will come in further calculations.

The two branches in the zeroth order are given by

$$\psi_{\pm}^{(0)}(z) = (1-z)^{\gamma_{\pm}}, \quad \gamma_{\pm} = \frac{1 \pm \alpha}{2}, \quad \alpha = \sqrt{1-4\epsilon_h}. \quad (2.13)$$

Using the method of variation of parameters and noting that the Wronskian of the zeroth order solutions is $W(z) = \alpha$ we find the first order corrections

$$\begin{aligned} \psi_+^{(1)}(z) &= \frac{1}{\alpha} \psi_+^{(0)}(z) \int dz \psi_-^{(0)}(z) T^{(1)}(z) \psi_+^{(0)}(z) \\ &\quad - \frac{1}{\alpha} \psi_-^{(0)}(z) \int dz \psi_+^{(0)}(z) T^{(1)}(z) \psi_+^{(0)}(z), \\ \psi_-^{(1)}(z) &= \frac{1}{\alpha} \psi_+^{(0)}(z) \int dz \psi_-^{(0)}(z) T^{(1)}(z) \psi_-^{(0)}(z) \\ &\quad - \frac{1}{\alpha} \psi_-^{(0)}(z) \int dz \psi_+^{(0)}(z) T^{(1)}(z) \psi_-^{(0)}(z). \end{aligned} \quad (2.14)$$

Since both $\psi_{\pm}^{(0)}(z)$ and $T^{(1)}(z)$ have poles then $\psi_{\pm}^{(1)}(z)$ has branch points identified with the punctures at z_2 and z_3 .

2.2. Contour integration and monodromy

To find the monodromy we evaluate the following integrals

$$\begin{aligned} I_{++}^{(k)} &= \frac{1}{\alpha} \oint_{\gamma_k} dz \psi_-^{(0)}(z) T^{(1)}(z) \psi_+^{(0)}(z), & I_{+-}^{(k)} &= -\frac{1}{\alpha} \oint_{\gamma_k} dz \psi_+^{(0)}(z) T^{(1)}(z) \psi_+^{(0)}(z), \\ I_{-+}^{(k)} &= \frac{1}{\alpha} \oint_{\gamma_k} dz \psi_-^{(0)}(z) T^{(1)}(z) \psi_-^{(0)}(z), & I_{--}^{(k)} &= -\frac{1}{\alpha} \oint_{\gamma_k} dz \psi_+^{(0)}(z) T^{(1)}(z) \psi_-^{(0)}(z), \end{aligned} \quad (2.15)$$

over two contours γ_2 and γ_3 enclosing points $\{0, z_2\}$ and $\{0, z_2, z_3\}$ respectively. We find

$$\begin{aligned}
 I_{+-}^{(2)} &= +\frac{2\pi i}{\alpha} \left[+\alpha\epsilon_1 + c_2(1 - z_2) - \epsilon_2 + c_3(1 - z_3) - \epsilon_3 \right. \\
 &\quad \left. - (1 - z_2)^\alpha [c_2(1 - z_2) - \epsilon_2(1 + \alpha)] \right], \\
 I_{-+}^{(2)} &= -\frac{2\pi i}{\alpha} \left[-\alpha\epsilon_1 + c_2(1 - z_2) - \epsilon_2 + c_3(1 - z_3) - \epsilon_3 \right. \\
 &\quad \left. - (1 - z_2)^{-\alpha} [c_2(1 - z_2) - \epsilon_2(1 - \alpha)] \right], \\
 I_{++}^{(2)} &= -I_{--}^{(2)} = \frac{2\pi i}{\alpha} [c_3(1 - z_3) - \epsilon_3],
 \end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
 I_{+-}^{(3)} &= +\frac{2\pi i}{\alpha} \left[+\alpha\epsilon_1 + c_2(1 - z_2) - \epsilon_2 + c_3(1 - z_3) - \epsilon_3 - \right. \\
 &\quad \left. - (1 - z_2)^\alpha [c_2(1 - z_2) - \epsilon_2(1 + \alpha)] - (1 - z_3)^\alpha [c_3(1 - z_3) - \epsilon_3(1 + \alpha)] \right], \\
 I_{-+}^{(3)} &= -\frac{2\pi i}{\alpha} \left[-\alpha\epsilon_1 + c_2(1 - z_2) - \epsilon_2 + c_3(1 - z_3) - \epsilon_3 - \right. \\
 &\quad \left. - (1 - z_2)^{-\alpha} [c_2(1 - z_2) - \epsilon_2(1 - \alpha)] - (1 - z_3)^{-\alpha} [c_3(1 - z_3) - \epsilon_3(1 - \alpha)] \right], \\
 I_{++}^{(3)} &= I_{--}^{(3)} = 0.
 \end{aligned} \tag{2.17}$$

Within the heavy-light expansion two monodromy matrices $\mathbb{M} = \{M_{ij}, i, j = \pm\}$ associated with the contours γ_2 and γ_3

$$\begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix} \rightarrow \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix} \tag{2.18}$$

are expanded as

$$\mathbb{M} = \mathbb{M}_0 + \mathbb{M}_1 + \mathbb{M}_2 + \dots \tag{2.19}$$

The first order \mathbb{M}_0 defines the monodromy of $\psi^{(0)}(z)$. As the contours do not enclose the point $z = 1$ it follows that $\mathbb{M}_0 = \mathbb{I}$. In the linear order the monodromy matrices are given by

$$\mathbb{M}(\gamma_2) = \begin{pmatrix} 1 + I_{++}^{(2)} & I_{+-}^{(2)} \\ I_{-+}^{(2)} & 1 - I_{++}^{(2)} \end{pmatrix}, \quad \mathbb{M}(\gamma_3) = \begin{pmatrix} 1 & I_{+-}^{(3)} \\ I_{-+}^{(3)} & 1 \end{pmatrix}. \tag{2.20}$$

On the other hand, the monodromy matrices over contours γ_2 and γ_3 are defined by the conformal dimensions of the fields in the intermediate channels. Up to similarity transformation they are

$$\tilde{\mathbb{M}}(\gamma_2) = -\begin{pmatrix} e^{+\pi i \Lambda_1} & 0 \\ 0 & e^{-\pi i \Lambda_1} \end{pmatrix}, \quad \tilde{\mathbb{M}}(\gamma_3) = -\begin{pmatrix} e^{+\pi i \Lambda_2} & 0 \\ 0 & e^{-\pi i \Lambda_2} \end{pmatrix}, \tag{2.21}$$

where $\Lambda_1 = \sqrt{1 - 4\tilde{\epsilon}_1}$ and $\Lambda_2 = \sqrt{1 - 4\tilde{\epsilon}_2}$ parametrize intermediate dimensions. Computing corresponding eigenvalues we arrive at the following equation system

$$\sqrt{I_{++}^{(2)} I_{++}^{(2)} + I_{+-}^{(2)} I_{-+}^{(2)}} = 2\pi i \tilde{\epsilon}_1, \quad \sqrt{I_{+-}^{(3)} I_{-+}^{(3)}} = 2\pi i \tilde{\epsilon}_2. \tag{2.22}$$

3. Solving the monodromic equations

In what follows we solve the monodromic equations (2.22) for the special choice of the parameters

$$\epsilon_1 = \epsilon_2, \quad \tilde{\epsilon}_1 = \tilde{\epsilon}_2. \tag{3.1}$$

Equations (2.22) being squared take the form

$$\begin{aligned} (X + (1 - a)Y)X &= aI^2, \\ (X + (1 - b)Y + bE)(bX + a(1 - b)Y + aE) &= abI^2, \end{aligned} \tag{3.2}$$

where

$$X = (1 - a)(x - \epsilon_1) + (1 + a)\alpha\epsilon_1, \quad Y = (y - \epsilon_3), \quad E = \alpha\epsilon_3, \quad I = \alpha\tilde{\epsilon}_1, \tag{3.3}$$

and

$$\begin{aligned} x &= (1 - z_2)c_2, \quad y = (1 - z_3)c_3, \\ a &= (1 - z_2)^\alpha, \quad b = (1 - z_3)^\alpha. \end{aligned} \tag{3.4}$$

Thus, the monodromic equation system consists of two quadratic equations on two accessory parameters. Following the Bézout’s theorem the system has at most four different solutions (more precisely, none, two or four). Each equation describes a conic and their intersection points (at most four) are the solutions. The first equation in (3.2) is easily solved as $Y = Y(X)$ and thereby isolating variable X one arrives at a fourth order equation that can be solved explicitly using the well-known formulas for roots. However, the corresponding expressions containing square and cubic radicals are too complicated. One can also check that the discriminant is not zero so that all roots are different.

Exact solutions to the monodromic system are yet to be found. Instead, we propose to use an expansion which treats a solution as a deformation of some seed solution with respect to one of the conformal dimensions. As a seed solution we take the 4-point accessory parameter while the deformation parameter is the third conformal dimension ϵ_3 . Thus, the 5-point accessory parameters are expanded as

$$\begin{aligned} c_2 &= c_2^{(0)} + \epsilon_3 c_2^{(1)} + \epsilon_3^2 c_2^{(2)} + \dots, \\ c_3 &= \epsilon_3 c_3^{(1)} + \epsilon_3^2 c_3^{(2)} + \dots, \end{aligned} \tag{3.5}$$

where the zeroth-order $c_2^{(0)}$ is the 4-point accessory parameter [2,6], while $c_{2,3}^{(k)}$ are corrections, $k = 1, 2, \dots$ (see our notational comment below (2.12)). Taking $\epsilon_3 = 0$ yields $c_3 = 0$ so that two equations (3.2) are reduced to a single equation

$$X^2 - aI^2 = 0. \tag{3.6}$$

Recalling variable changes (3.4) and (3.3) we solve the above equation as

$$X = \pm\sqrt{a\alpha}\tilde{\epsilon}_1 : \quad x = \epsilon_1 + \frac{(a + 1)}{(a - 1)}\alpha\epsilon_1 \pm \alpha\frac{\sqrt{a}}{(a - 1)}\tilde{\epsilon}_1, \quad c_2^{(0)} = \frac{x}{1 - z_2}, \tag{3.7}$$

where the conformal block asymptotics fixes the sign to be +.

Having exact solution $c_2^{(0)}$ and using the expansion procedure one can easily generate higher order corrections to find series (3.5). The point is that substituting the power series into the

monodromic equations we find out that the zeroth order term satisfies the quadratic equation (3.6) while all higher order terms are subjected to some linear equations. In terms of variables x and y (3.4) the perturbation series is

$$x = \sum_{n=0}^{\infty} \epsilon_3^n x_n, \quad y = \sum_{n=1}^{\infty} \epsilon_3^n y_n, \tag{3.8}$$

where x_0 is the exact seed solution (3.7). All corrections up to the third order are given by

$$\begin{aligned} x_0 &= \epsilon_1 + \epsilon_1 \alpha \frac{(a+1)}{(a-1)} + \tilde{\epsilon}_1 \alpha \frac{\sqrt{a}}{a-1}, & x_1 &= \frac{\alpha}{2} \frac{a+b^2}{a-b^2}, \\ x_2 &= \frac{\alpha}{2\tilde{\epsilon}_1} \left[\frac{b\sqrt{a}(a-2ab+b^2)(a-2b+b^2)}{(a-b^2)^3} + \frac{(a-1)(a+b^2)^2}{4\sqrt{a}(a-b^2)^2} \right], \\ x_3 &= \frac{\alpha}{2\tilde{\epsilon}_1^2} \left[\frac{ab(b-1)(a-2ab+b^2)(a-2b+b^2)(a-3ab+3b^2-b^3)}{(a-b^2)^5} \right], \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} y_1 &= 1 - \alpha \frac{a+b^2}{a-b^2}, & y_2 &= \frac{\alpha}{\tilde{\epsilon}_1} \left[\frac{b\sqrt{a}(-a+2ab-b^2)(a-2b+b^2)}{(a-b^2)^3} \right], \\ y_3 &= \frac{\alpha}{2\tilde{\epsilon}_1^2} \left[\frac{b(a-2ab+b^2)(a-2b+b^2)(a^2+a^3-8a^2b+6ab^2+6a^2b^2-8ab^3+b^4+ab^4)}{(a-b^2)^5} \right]. \end{aligned} \tag{3.10}$$

Corrections $c_{2,3}^{(k)}$ at $k = 1, 2, \dots$ in (3.5) can be reproduced using the variable change (3.4).

4. Bulk geodesic approach

In this section we shortly discuss the basic ingredients of the multi-particle mechanics in the bulk used to compute classical conformal blocks in the 4-point case [6] and in the n -point case [7]. Let us consider a massive point particle propagating in the three-dimensional space with a conical singularity. The corresponding metric is given by

$$ds^2 = \frac{\alpha^2}{\cos^2 \rho} \left(dt^2 + \sin^2 \rho d\phi^2 + \frac{1}{\alpha^2} d\rho^2 \right), \tag{4.1}$$

where $t \in \mathbb{R}$, $\rho \in \mathbb{R}^+$, $\phi \in [0, 2\pi)$. The angle deficit is $2\pi(1 - \alpha)$ where $\alpha \in [0, 1]$. Here, $\alpha = 1$ corresponds to the global AdS_3 space, $\alpha = 0$ corresponds to the BTZ black hole threshold, while $\alpha = 1/n$, where $n \in \mathbb{Z}_+$ gives AdS_3/\mathbb{Z}_n conifold. The angle deficit metric (4.1) describes the constant negative curvature space with the Ricci scalar $R = -8$, while the conformal boundary is reached at $\rho \rightarrow \pi/2$.

The action associated with the interval (4.1) reads

$$S = \epsilon \int_{\lambda'}^{\lambda''} d\lambda \sqrt{g_{tt} \dot{t}^2 + g_{\phi\phi} \dot{\phi}^2 + g_{\rho\rho} \dot{\rho}^2}, \tag{4.2}$$

where ϵ is a classical conformal dimension identified with a mass, the metric coefficients are read off from (4.1), λ is the evolution parameter and the dot stands for the corresponding derivative.

As the action (4.2) is reparameterization invariant the evolution parameter can be conveniently chosen so that the proper velocity is unit. We observe that the boundary coordinates ϕ and t are cyclic, that is the corresponding Lagrange function has no explicit time and angular dependence

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial t} = 0, \tag{4.3}$$

and therefore both angular and time momenta are conserved $\dot{p}_\phi = \dot{p}_t = 0$. The particle dynamics can be reduced to the level surface characterized by constant values of the conserved momenta. However, the corresponding Rauth function is rather complicated. It can be simplified by taking either $p_\phi = 0$ and/or $p_t = 0$ in which case the Rauth function is just the Lagrange function (4.2) at $\dot{\phi} = 0$ and/or $\dot{t} = 0$.

In our case we are interested in $2n - 5$ massive particles subjected to particular boundary conditions meaning that $n - 2$ of their worldlines puncture the conformal boundary at fixed points w_1, \dots, w_{n-2} , where $w = \phi + it$. In general, each particle travels on geodesic characterized by its own angular and time momenta. There are two natural configurations of boundary attachment points when either time or angular momenta of all particles can be simultaneously set to zero. In the $t = const$ case the surface level is a space-like disk with a number of attachment points placed on the boundary circle. Heavy operators can be visualized as sitting in the center of the disk. In the $\phi = const$ case the surface level is a time-like infinite strip with a number of attachment points placed on the boundary line. Heavy operators are on the opposite edge at infinitely separated points. In each case both external and intermediate worldlines have different constant angular/time momenta according to their attachment points and vertex positions.

We note that both the disk and the infinite strip turn out to be constant negative curvature two-dimensional surfaces with equal Ricci scalars $R = -2$. Both metrics can be cast into the Poincaré disk form $d\tilde{s}^2 = \frac{1}{4} \frac{dud\bar{u}}{(1-u\bar{u})^2}$, where u, \bar{u} are images of the complexified coordinates $\phi \pm i\rho$ or $\rho \pm it$, and the conformal boundary is mapped to the circle $u\bar{u} = 1$. It follows that the dynamics in both slices can be described quite uniformly. In particular, the length of a given geodesic graph in a constant angle slice can be explicitly computed and be shown that it is related to the length of the same graph in a constant time slice by virtue of Wick rotation of attachment points coordinates.

In the sequel we consider a fixed time slice with the boundary attachments arranged circle-wise. In this case the Rauth function describes a massive particle moving in the disk $d\tilde{s}^2 = \frac{d\rho^2 + \alpha^2 \sin^2 \rho d\phi^2}{\cos^2 \rho}$. The geodesic length of a given segment characterized by a constant angular parameter $s \equiv |p_\phi|/\alpha$ is the disk line element integrated along a given path

$$S = \epsilon \ln \frac{\sqrt{\eta}}{\sqrt{1 + \eta} + \sqrt{1 - s^2 \eta}} \Big|_{\eta'}^{\eta''}, \tag{4.4}$$

where $\eta' = \cot^2 \rho'$ and $\eta'' = \cot^2 \rho''$ are initial/final radial positions. As the metric blows up at the conformal boundary $\rho = \pi/2$ the length is to be regularized by introducing a near boundary cutoff. Then, the geodesic length of a multi-line graph is obtained by summing (4.4) with various angular parameters. On the other hand, the angular parameters are uniquely fixed by conditions that follow from extremizing the total action with fixed boundary attachments.

As an example we consider a geodesic between two boundary points 0 and ϕ . Using (4.4) one finds that up to the boundary cutoff the geodesic length is given by $S(\phi) = \ln \sin(\alpha\phi/2)$. As the angular coordinate ϕ runs from 0 to 2π we find out that the geodesic is the arc stretched be-

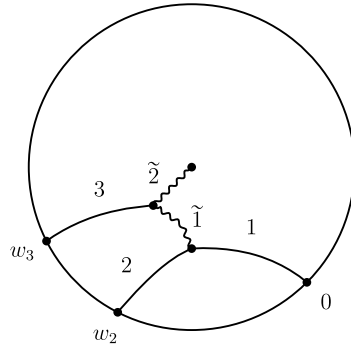


Fig. 2. A constant time disk. Solid lines 1, 2, 3 represent external operators with dimensions ϵ_i , $i = 1, 2, 3$, wavy lines $\tilde{1}, \tilde{2}$ represent intermediate operators with dimensions $\tilde{\epsilon}_j$, $j = 1, 2$.

tween the boundary points.² The arc along with a radial geodesic line and bulk-to-bulk geodesic segments are the basic elements of any n -point geodesic graph.

The geodesic graph in the 5-point case consists of five segments three of which have boundary attachments 0, w_2 , w_3 and meet two inner segments in two vertices, see Fig. 2. Positions of the vertices are defined through the equilibrium conditions on the angular and radial momenta

$$\tilde{s}_2 = 0, \quad \epsilon_3 s_3 - \tilde{\epsilon}_1 \tilde{s}_1 = 0, \quad \epsilon_1 s_1 - \epsilon_2 s_2 - \tilde{\epsilon}_1 \tilde{s}_1 = 0, \tag{4.5}$$

$$\epsilon_1 \sqrt{1 - s_1^2 \eta_1} + \epsilon_2 \sqrt{1 - s_2^2 \eta_1} = \tilde{\epsilon}_1 \sqrt{1 - \tilde{s}_1^2 \eta_1}, \quad \epsilon_3 \sqrt{1 - s_3^2 \eta_2} + \tilde{\epsilon}_1 \sqrt{1 - \tilde{s}_1^2 \eta_2} = \tilde{\epsilon}_2, \tag{4.6}$$

and the angular conditions saying that angular separations between the first and second lines and between the first and the third lines are w_2 and w_3 , respectively,

$$e^{i\alpha w_2} = \frac{(\sqrt{1 - s_1^2 \eta_1} - i s_1 \sqrt{1 + \eta_1})(\sqrt{1 - s_2^2 \eta_1} - i s_2 \sqrt{1 + \eta_1})}{(1 - i s_1)(1 - i s_2)}, \tag{4.7}$$

$$e^{i\alpha w_3} = \frac{(\sqrt{1 - s_3^2 \eta_2} - i s_3 \sqrt{1 + \eta_2})(\sqrt{1 - \tilde{s}_1^2 \eta_2} - i \tilde{s}_1 \sqrt{1 + \eta_2})(\sqrt{1 - s_1^2 \eta_1} - i s_1 \sqrt{1 + \eta_1})}{(1 - i s_3)(\sqrt{1 - \tilde{s}_1^2 \eta_1} - i \tilde{s}_1 \sqrt{1 + \eta_1})(1 - i s_1)}, \tag{4.8}$$

where s_1, s_2, s_3 and \tilde{s}_1, \tilde{s}_2 are respectively external and intermediate angular parameters, and η_1 and η_2 are radial positions of the vertices. In total, we have seven algebraic (irrational) equations on seven variables.

Modulo regulator terms the total multi-particle action is given by a sum of geodesic segments

$$S(w_2, w_3) = \epsilon_1 L_1(w_2, w_3) + \epsilon_2 L_2(w_2, w_3) + \epsilon_3 L_3(w_2, w_3) + \tilde{\epsilon}_1 L_{\tilde{1}}(w_2, w_3) + \tilde{\epsilon}_2 L_{\tilde{2}}(w_2, w_3), \tag{4.9}$$

² Let us note that function $S(\phi)$ is periodic with the period $4\pi/\alpha$ which is greater than the period of the angular variable ϕ . This gives rise to the interesting case of the so-called “long geodesics” which are the curves stretched between the two points and winding around the conical singularity. In the case of the conifold AdS_3/\mathbb{Z}_n they are the images of the true geodesic lines in AdS_3 and compute the entwinement [18]. It would be crucial to consider the long geodesics as new elements of n -point bulk graphs and find an interpretation in terms of the boundary conformal theory.

where

$$\begin{aligned}
 L_{1,2} &= -\ln \frac{\sqrt{\eta_1}}{\sqrt{1+\eta_1} + \sqrt{1-s_{1,2}^2\eta_1}}, & L_3 &= -\ln \frac{\sqrt{\eta_2}}{\sqrt{1+\eta_2} + \sqrt{1-s_3^2\eta_2}}, \\
 L_{\bar{1}} &= \ln \frac{\sqrt{\eta_1}}{\sqrt{1+\eta_1} + \sqrt{1-\tilde{s}_1^2\eta_1}} - \ln \frac{\sqrt{\eta_2}}{\sqrt{1+\eta_2} + \sqrt{1-\tilde{s}_1^2\eta_2}}, & L_{\bar{2}} &= \ln \frac{\sqrt{\eta_2}}{1 + \sqrt{1+\eta_2}}.
 \end{aligned}
 \tag{4.10}$$

Solving equations (4.5)–(4.8) we find all variables as functions of boundary attachment positions w_1, \dots, w_{n-2} and all classical conformal dimensions. However, exact solutions are not known yet and therefore we shall use the expansion method which computes a solution as a perturbative series over some exactly known seed solution.

5. Monodromy vs geodesic approach

Finding the geodesic length can be viewed from a slightly different perspective. As the geodesic length is just the multi-particle action with particular boundary conditions given by fixed attachment points of external lines one finds out that corresponding canonical momenta are partial derivatives of the action with respect to boundary attachment coordinates. The resulting system of partial differential equations can be explicitly integrated so that its solution certainly coincides with geometric computation of the geodesic length given in Section 4.

It follows that both the monodromy and the geodesic methods deal with potential vector fields whose components are identified with accessory or external angular parameters, while potentials are identified with respectively the classical conformal block or the geodesic length. Given vector field components as functions of puncture coordinates we easily find the corresponding potential function. However, finding the components which are subjected to algebraic equation system turns out to be quite complicated problem. Within the monodromy approach such a system arises from studying monodromy properties of the conformal block while within the geodesic approach analogous system follows from studying geometric characteristic of geodesic segments. Below we analyze the respective algebraic equation systems on the both sides and show that they have a common root while the systems themselves are different. The two systems are then related by virtue of a combined transformation that can be viewed as the AdS/CFT correspondence.

Let us consider the action (4.2) describing a geodesic segment attached somewhere to the boundary. The on-shell variation is $\delta S = p_\mu \delta x^\mu$, where p_μ is the momentum in the attachment point, while δx^μ is the variation of the boundary coordinate. In our case there are three boundary attachments $w_1 = 0$ and w_2, w_3 (see Fig. 2) so that

$$\alpha \epsilon_2 s_2(w_2, w_3) = \frac{\partial S(w_2, w_3)}{\partial w_2}, \quad \alpha \epsilon_3 s_3(w_2, w_3) = \frac{\partial S(w_2, w_3)}{\partial w_3}. \tag{5.1}$$

As noted in [6] in the 4-point case the angular momenta relations (5.1) are analogous to the definition of the accessory parameters (2.6). Indeed, the accessory parameters are defined in much the same way as

$$c_2(z_2, z_3) = \frac{\partial f(z_2, z_3)}{\partial z_2}, \quad c_3(z_2, z_3) = \frac{\partial f(z_2, z_3)}{\partial z_3}, \tag{5.2}$$

cf. (2.6). Note that there are the following consistency conditions³

$$\frac{\partial c_3}{\partial z_2} - \frac{\partial c_2}{\partial z_3} = 0, \quad \epsilon_3 \frac{\partial s_3}{\partial w_2} - \epsilon_2 \frac{\partial s_2}{\partial w_3} = 0. \tag{5.3}$$

Formally, the two systems above define potential vector fields in two dimensions which can be related to each other by means of a coordinate transformation along with a particular transformation of the respective potentials. A transition from the boundary insertion points $z_{1,2,3}$ to the attachment points $w_{1,2,3}$ in the bulk is given by

$$w_m = i \ln(1 - z_m), \quad m = 1, 2, 3. \tag{5.4}$$

In particular, $z_1 = 0$ maps to $w_1 = 0$, cf. Figs. 1 and 2. Naturally, (5.4) maps the complex plane to the cylinder interpreted as conformal boundary of the asymptotically AdS₃ space. The transformation is diagonal in the sense of the corresponding diagonal Jacobian matrix. The multi-particle action and the classical conformal block are related as

$$f(z_2, z_3) = S(w_2, w_3) + i\epsilon_2 w_2 + i\epsilon_3 w_3, \tag{5.5}$$

where coordinates are related by (5.4). We find that three accessory parameters and three external angular parameters are related as

$$c_m = \epsilon_m \frac{1 \pm i\alpha s_m(w)}{1 - z_m}, \quad m = 1, 2, 3, \tag{5.6}$$

where \pm depends on the direction of the s_m flow. According to Fig. 2 and relations (4.5) momenta s_2, s_3 and s_1 have opposite signs in (5.6), it is $-$ and $+$ respectively. Note that the accessory parameter c_1 is implicit within the monodromic approach since only two of the accessory parameters are left independent, cf. (2.8). On the other hand, relations (4.5) say that $\epsilon_1 s_1 = \epsilon_2 s_2 + \epsilon_3 s_3$, and using (5.6) it is exactly mapped to (2.8).

As discussed above both accessory parameters and angular momenta are subjected to algebraic equation systems (2.7), (2.22) and (4.5)–(4.8). Within the monodromy approach there are five variables c_1, \dots, c_5 (accessory parameters) subjected to three linear and two irrational equations $M_\alpha(c) = 0, \alpha = 1, \dots, 5$. Within the geodesic approach there are seven variables $s_1, s_2, s_3, \tilde{s}_1, \tilde{s}_2$ (external/intermediate angular momenta) and η_1, η_2 (radial vertex positions) subjected to three linear and four irrational equations $G_I(s, \tilde{s}, \eta) = 0, I = 1, \dots, 7$. In principle, one might expect that after isolating two independent bulk variables (say, s_2 and s_3) the residual two geodesic equations should match exactly with the monodromic equations (2.22) provided (5.6). However, this is not the case. Instead, a weaker version of the equivalence turns out to be true – the systems are required to have at least one common root.

For future reference, recalling the definition of quantities a and b (3.4) and introducing angle positions $\theta_2 = \alpha w_2/2$ and $\theta_3 = \alpha w_3/2$ we find out that two types of variables are related as

$$a = \exp(-2i\theta_2), \quad b = \exp(-2i\theta_3). \tag{5.7}$$

Parameters $\theta_{2,3}$ proved to be convenient when describing solutions of the bulk equations and now we see that they are directly related to the quantities a and b naturally introduced within the monodromic framework.

³ It is worth noting that the first condition can be represented as the Cauchy–Euler type equation $a \frac{\partial y}{\partial a} = b \frac{\partial x}{\partial b}$, where a, b and x, y are given by (3.4). Recalling expressions for the accessory parameters (3.9) and (3.10) we see that this is indeed the case in each particular order of the ϵ_3 expansion.

5.1. 4-point case

The geodesic equations read off from the general system in the 4-pt case with coinciding conformal dimensions $\epsilon_1 = \epsilon_2$ (3.1) are given by

$$2\epsilon_1\sqrt{1-s^2}\eta = \tilde{\epsilon}_1, \quad (\sqrt{1-s^2}\eta - is\sqrt{1+\eta}) = \frac{(1-is)}{\sqrt{a}}, \tag{5.8}$$

where we denoted $s_1 = s_2 \equiv s$. The first equation here expresses η as a function of s . Isolating independent variable s one can show that the second equation has the polynomial representation

$$(s+i)(s+i\frac{(a+1)-\sqrt{a}\chi}{1-a}) = 0, \quad \chi = \frac{\tilde{\epsilon}_1}{\epsilon_1}. \tag{5.9}$$

We see that modulo pre-factor $i\alpha\epsilon_1$ the second root of this residual geodesic equation coincides with the root (3.7) of the monodromic quadratic equation (3.6), i.e., $x - \epsilon_1 = -i\alpha\epsilon_1s$, where x is related to the accessory parameter c_2 via redefinition (3.4). The final relation between c_2 and s is exactly (5.6).

It is instructive to have both monodromic (3.6) and geodesic (5.9) equations expressed in the same notation. By making use of the change (5.6) the bulk and boundary quadratic equations are uniformly given by

$$\begin{aligned} \text{Monodromic equation:} \quad & \left(s+i\frac{(a+1)-\sqrt{a}\chi}{1-a}\right)^2 = 0, \\ \text{Geodesic equation:} \quad & (s+i)(s+i\frac{(a+1)-\sqrt{a}\chi}{1-a}) = 0. \end{aligned} \tag{5.10}$$

Indeed, we see that the above equations do not coincide but have a common root. Note the second root in the geodesic equation $s = -i$. Recalling that s defines the angular momentum as $p_\phi = \alpha s$ we find out that the second root yields a pure imaginary momentum $p_\phi = -i\alpha$. The corresponding vertex does not admit an original geometric interpretation since the radial position ρ defined by variable $\eta = \cot^2 \rho$ from (5.8) is also imaginary by virtue of the constraints for classical conformal dimensions [7]. Nonetheless, it would be interesting to study both formal and physical meaning of the corresponding solution for $S(w)$ and $f(z)$ elsewhere.

5.2. 5-point case

A polynomial representation of the geodesic equations in the 5-point case analogous to (5.9) is unknown yet. Instead, from the previous discussion we learn that computing all the roots is superfluous. Analyzing the first angular equation (4.7) we obtain exact results, while the second angular equation (4.8) can be considered only perturbatively by virtue of the expansion method used to find solutions to the monodromic equations. In particular, we show that, roughly speaking, the first and the second monodromic equations in (3.2) follow from respectively the first and the second angular equations. Indeed, seven bulk variables include two vertex positions which can be explicitly solved in terms of five external and intermediate angular parameters, $\eta_{1,2} = \eta_{1,2}(s, \tilde{s}|\epsilon, \tilde{\epsilon})$. Then, the remaining five geodesic equations include three linear relations (4.5) and two irrational angular equations (4.7) and (4.8). This structure is analogous to that of the monodromic equations which include three linear equations (2.7) and two irrational equations (2.22) on five accessory parameters.

Exact relations. By a direct computation we show that the second equation in (4.6) and the first angular equation (4.7) can be conveniently represented as a third order equation on variables $s_1 + s_2$ and $s_1 s_2$ [7]. Two of three roots should be discarded while the remaining root yields the following constraint

$$s_2 = \frac{\sigma + \cos 2\theta_2 + s_1 \sin 2\theta_2}{-s_1 + s_1 \cos 2\theta_2 - \sin 2\theta_2}, \quad \sigma = 1 - \frac{\tilde{\epsilon}_1^2}{2\epsilon_1^2}, \tag{5.11}$$

while relations (4.5) in this case reduce to $s_1 = s_2 + \frac{\epsilon_3}{\epsilon_1} s_3$. Equation (5.11) is the second order equation in variables s_2 and s_3 . It is not equivalent to the first angular equation in the bulk but instead represents one of its roots. Parameterizing coefficients in (5.11) as $\cos 2\theta_2 = (1 + a^2)/2a$, $\sin 2\theta_2 = (1 - a^2)/2ia$, and $\cot \theta_2 = i(1 + a)/(1 - a)$ (see (5.7)), and denoting

$$X = -i\alpha\epsilon_1(1 - a)(s_2 + \cot \theta_2), \quad Y = -i\alpha\epsilon_3 s_3, \tag{5.12}$$

we show that quadratic equation (5.11) can be cast into the form

$$X^2 + (1 - a)XY - aI^2 = 0, \tag{5.13}$$

which is exactly the first monodromic equation (3.2), while the coordinate changes (5.12) and (3.4) yield relation (5.6). In this way we see that the equivalence can be achieved just for roots of the algebraic equations and not for the equations themselves.

Expanding over the root. The remaining part of the geodesic equations is given by the first vertex equation in (4.6) and the second angular equation (4.8) which is much more complicated than the first angular equation. For convenience, we use the first angular equation (4.7) to represent the second angular equation (4.8) as follows

$$\begin{aligned} & e^{2i(\theta_3 - \theta_2)}(1 - is_3)(\sqrt{1 - s_2^2 \eta_1} - is_2 \sqrt{1 + \eta_1})(\sqrt{1 - v^2 s_3^2 \eta_1} - iv s_3 \sqrt{1 + \eta_1}) \\ & = (1 - is_2)(\sqrt{1 - s_3^2 \eta_2} - is_3 \sqrt{1 + \eta_2})(\sqrt{1 - v^2 s_3^2 \eta_2} - iv s_3 \sqrt{1 + \eta_2}), \end{aligned} \tag{5.14}$$

where the angular positions of the vertices defined by equations (4.6) are given by simple functions $\eta_{1,2} = \eta_{1,2}(s_2, s_3 | \epsilon, \tilde{\epsilon})$ along with $s_1 = s_2 + \nu s_3$, where $\nu = \epsilon_3/\tilde{\epsilon}_1$ and $\varkappa = \tilde{\epsilon}_1/\epsilon_1$ (see [7] for more details). In this form the second angular equation depends on s_2 and s_3 only. Using the coordinate change (5.12) the above equation can be rewritten as irrational equation $f(X, Y) = 0$. Therefore, combining it with (5.13) we arrive at a couple of algebraic equations, quadratic and irrational ones which should have a common root with the monodromic equations (3.2). By analogy with (5.11) we expect that of all roots of equation (5.14) considered as $f(X, Y) = 0$ we shall find a root $X = X(Y)$ which is to be treated as the second monodromic equation (3.2).

Getting rid of the radicals in the second angular equation produces some higher order polynomial equation which defies exact solution. Using the super-light approximation we expand the angular equations in small parameter ϵ_3 around the seed solutions obtained by setting $\epsilon_3 = 0$ in (5.13) and (5.14). The expansion of angular momenta up to the third order is given by

$$\begin{aligned} s_2 &= s_2^{(0)} + \nu s_2^{(1)} + v^2 s_2^{(2)} + v^3 s_2^{(3)} + \dots, \\ s_3 &= s_3^{(0)} + \nu s_3^{(1)} + v^2 s_3^{(2)} + \dots, \end{aligned} \tag{5.15}$$

where we used for convenience a rescaled deformation parameter $\nu = \epsilon_3/\tilde{\epsilon}_1$. Note that the above power series expansions have different highest orders in ϵ_3 . This is because the angular parameter s_3 enters the left-hand-side of the second equation in (5.1) with ϵ_3 prefactor so that to have the

mechanical action $S = S(w_2, w_3)$ up to terms of order ϵ_3^m it is sufficient to have s_2 and s_3 up to terms of order ϵ_3^m and ϵ_3^{m-1} , respectively.⁴

The expansion coefficients are found to be

$$\begin{aligned}
 s_2^{(0)} &= -\cot \theta_2 + \kappa \frac{1}{2 \sin \theta_2}, & s_2^{(1)} &= \frac{\kappa}{2} \cot(2\theta_3 - \theta_2), \\
 s_2^{(2)} &= \kappa \{ [9 \cos(2\theta_3) + 7 \cos(2\theta_2 - 2\theta_3) - \cos(4\theta_2 - 6\theta_3) + \cos(2\theta_2 - 6\theta_3) \\
 &\quad - 4 \cos(2\theta_2 - 4\theta_3) - 12] \} / \{ 32 \sin^3(\theta_2 - 2\theta_3) \}, \\
 s_2^{(3)} &= \kappa \frac{\sin \theta_3 [\sin(\theta_2 - 3\theta_3) - 3 \sin(\theta_2 - \theta_3)]}{8 \sin^5(\theta_2 - 2\theta_3)} \\
 &\quad \times [3 + \cos(2\theta_2 - 4\theta_3) - 2 \cos(2\theta_2 - 2\theta_3) - 2 \cos(2\theta_3)],
 \end{aligned} \tag{5.16}$$

and⁵

$$\begin{aligned}
 s_3^{(0)} &= -\cot(2\theta_3 - \theta_2), & s_3^{(1)} &= \frac{1}{2} \csc^3(\theta_2 - 2\theta_3) [\sin^2 \theta_2 + 4 \sin^2(\theta_2 - \theta_3) \sin^2 \theta_3], \\
 s_3^{(2)} &= -\frac{1}{16} \csc^5(\theta_2 - 2\theta_3) [6 \cos \theta_2 + \cos(\theta_2 - 4\theta_3) + \cos(3\theta_2 - 4\theta_3) - 8 \cos(\theta_2 - 2\theta_3)] \\
 &\quad \times [3 + \cos(2\theta_2 - 4\theta_3) - 2 \cos(2\theta_2 - 2\theta_3) - 2 \cos(2\theta_3)].
 \end{aligned} \tag{5.17}$$

One can also check that the consistency condition (5.3) rewritten in terms of variables θ_2 and θ_3 as $\epsilon_2 \partial s_2 / \partial \theta_3 = \epsilon_3 \partial s_3 / \partial \theta_2$ is satisfied. To conclude, we note that in each order in ν or equivalently ϵ_3 angular momenta corrections (5.16)–(5.17) are related to the accessory parameters corrections (3.9)–(3.10) according to the general relation (5.6). In this way we have shown that the monodromic equation system does have the same root as the geodesic equation system.

6. Classical conformal block vs multi-particle action

Below we present the resulting expressions for the classical heavy-light conformal block obtained on the boundary and in the bulk using respectively the monodromy and the geodesic approaches. The results obtained are in agreement with the computation by means of the AGT combinatorial representation [15] performed for the lower level coefficients of the coordinate expansion of the classical conformal block. The corresponding expressions are collected in the Appendix A.

6.1. Classical conformal block

The power series expansion of the 5-point classical conformal block $f(z)$ (2.1) is given by

$$f(z) = f^{(0)}(z) + \epsilon_3 f^{(1)}(z) + \epsilon_3^2 f^{(2)}(z) + \epsilon_3^3 f^{(3)}(z) + \dots \tag{6.1}$$

Using explicit expressions for the accessory parameters (3.4) and (3.9)–(3.10) and integrating (5.2) we find that expansion coefficients in (6.1) are given by

⁴ In principle, this fact must follow directly from the geodesic length formula (4.9)–(4.10). We explicitly checked that substituting $s_3 = s_3^{(0)} + \nu s_3^{(1)} + \nu^2 s_3^{(2)} + \nu^3 s_3^{(3)} + \dots$ into the length formula gives the same answer in the third order in ν exactly as if one sets $s_3^{(3)} = 0$.

⁵ In [7] we used another but equivalent representation of the first expansion coefficient $s_3^{(1)}$.

$$\begin{aligned}
 f^{(0)} &= -2\epsilon_1 \ln \left[\frac{a-1}{2\sqrt{a}} \right] + \tilde{\epsilon}_1 \ln \left[\frac{\sqrt{a}-1}{\sqrt{a}+1} \right] - \frac{\epsilon_1}{\alpha} \ln a, \\
 f^{(1)} &= -\ln \left[\frac{a-b^2}{2\sqrt{ab}} \right] - \frac{1}{\alpha} \ln b, \\
 f^{(2)} &= -\frac{(a+b^2)(a+a^2-4ab+b^2+ab^2)}{4\tilde{\epsilon}_1\sqrt{a}(a-b^2)^2}, \\
 f^{(3)} &= \frac{(b-1)b(a-b)(a+b^2)(a+a^2-4ab+b^2+ab^2)}{2\tilde{\epsilon}_1^2(a-b^2)^4}, \tag{6.2}
 \end{aligned}$$

where $a = (1 - z_2)^\alpha$ and $b = (1 - z_3)^\alpha$, see (3.4). The leading contribution $f^{(0)}$ is the 4-point classical heavy-light conformal block [2,6].

6.2. Multi-particle action

The power series expansion of the bulk multi-particle action $S(w)$ (4.9) is given by

$$S(w) = S^{(0)}(w) + \epsilon_3 S^{(1)}(w) + \epsilon_3^2 S^{(2)}(w) + \epsilon_3^3 S^{(3)}(w) + \dots \tag{6.3}$$

Using explicit expressions for the angular momenta (5.16)–5.17 and integrating (5.1) we find that expansion coefficients in (6.3) are given by

$$\begin{aligned}
 S_0(\theta) &= -2\epsilon_1 \ln \sin \theta_2 + \tilde{\epsilon}_1 \ln \tan \frac{\theta_2}{2}, & S_1(\theta) &= -\ln \sin(2\theta_3 - \theta_2), \\
 S_2(\theta) &= -\frac{\cos \theta_2 + 2 \csc^2(\theta_2 - 2\theta_3) \sin(\theta_2 - \theta_3) \sin \theta_3}{2\tilde{\epsilon}_1}, \\
 S_3(\theta) &= -\frac{\cos \theta_2 + 2 \csc^2(\theta_2 - 2\theta_3) \sin(\theta_2 - \theta_3) \sin \theta_3}{2\tilde{\epsilon}_1} \\
 &\quad \times \frac{4 \csc^2(\theta_2 - 2\theta_3) \sin(\theta_2 - \theta_3) \sin \theta_3}{2\tilde{\epsilon}_1}, \tag{6.4}
 \end{aligned}$$

where we switched to θ_2 and θ_3 , see (5.7). Expansion coefficients (6.4) are related to (6.2) according to the general identification formula (5.5). The same results follow from the explicit geodesic length formula (4.9)–(4.10). We note that (5.1) as well as (5.2) allow to define the action (the conformal block) up to some coordinate independent constant, while explicit expressions (4.9)–(4.10) allow to fix the constant.

7. Conclusion

We have computed the 5-point heavy-light conformal block in the super-light approximation up to the third order with respect to the conformal dimension of one of the three light fields. The computation has been done in two independent ways: using the monodromy and the geodesic approaches. The resulting expressions coincide. We observe different aspects of the correspondence between the two methods. In particular, we find that the boundary variables and equations find their counterparts in the bulk consideration. There is also a precise relation between the accessory parameters and the conserved angular momenta of the different geodesic segments.

The similarity between bulk and boundary computations leads to the natural assumption that in the present context the AdS₃/CFT₂ correspondence is to be understood in a strong sense,

i.e. as two different descriptions of the same Liouville theory. Indeed, in the semiclassical limit the correlators of the primary fields are dominated by the stationary-point configuration which is given by the Liouville equation with the delta-function sources corresponding to the heavy fields. The light fields do not affect the stationary-point configuration and are considered as propagating in the background geometry formed by the heavy ones. In our case there are two heavy operators with equal conformal weights so that we are dealing with the dynamics of the light fields propagating in the geometry of the Poincaré disk with two singularities inside. One can see that this geometry is nothing other than the two-dimensional slice geometry of the angle deficit/BTZ black hole in three dimensions. In particular, we have seen that the geodesic method deals with a constant time or angle slices which are Poincaré disks.

This link can be traced also in the opposite direction. Indeed, provided the Brown–Henneaux boundary condition is satisfied, the three-dimensional gravity with the negative cosmological constants is described by the Liouville theory [23]. Another way to see this theory inside the three-dimensional gravity is to represent the bulk metric as a stack of the Poincaré disks so that the exponential scale factor satisfies the residual Einstein equation which is again the Liouville equation [24].⁶ Adding light particles moving in the background geometry generated by static heavy sources one arrives at the Liouville equation with the heavy source terms supplemented with the standard geodesic equations for interacting light particles. This corresponds to decomposing the semiclassical correlation functions into conformal blocks and introducing intermediate channels.

Following the above heuristic arguments we tend to conclude that the Lagrangian description of the classical Liouville theory has to allow for establishing literal matching of the boundary and the bulk descriptions in the semiclassical limit. Current developments strongly indicate that this is indeed the case but an exact method of deriving the bulk geodesic description starting from the semiclassical Liouville theory is not yet available.

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Appendix A. The classical block from the AGT representation

Here we compare the results of Section 6 with those coming from the semiclassical computation for the combinatorial AGT representation [15] of the conformal blocks sketched in [7]. Introducing parameters q_1 and q_2 which define coordinates of the 5-point block as $z_2 = q_1 q_2$ and $z_3 = q_2$ the classical conformal block in the q -representation is given by

$$\tilde{\mathcal{F}}(q_1, q_2) = e^{-\frac{c}{6}\tilde{f}(q_1, q_2)}, \quad c \rightarrow \infty, \quad (\text{A.1})$$

where c is the central charge and $\tilde{\mathcal{F}}(q_1, q_2)$ is the conformal block in the AGT representation normalized as $\tilde{\mathcal{F}}(0) = 1$ so that $\tilde{f}(0) = 0$. We note that the classical conformal block in (2.1) has different asymptotic behavior: $f(z) \sim \log z$ at $z \rightarrow 0$.

⁶ The above Einstein/Liouville relation conforms with the well-known correspondence between two-dimensional Virasoro conformal blocks and physical states in three-dimensional quantum $SL(2, \mathbb{C})$ Chern–Simons theory [25]. See, *e.g.*, Refs. [26–30] for more discussion of quantum 3d gravity and the Liouville theory.

Using the power series expansion of the classical conformal block in the new coordinates

$$\tilde{f}(q) = \tilde{f}^{(0)}(q) + \epsilon_3 \tilde{f}^{(1)}(q) + \epsilon_3^2 \tilde{f}^{(2)}(q) + \epsilon_3^3 \tilde{f}^{(3)}(q) + \dots, \quad (\text{A.2})$$

and applying the AGT combinatorial computational scheme we can find the expansion coefficients $\tilde{f}^{(k)}(q)$ in the form of q_1 and q_2 power series expansions. It is convenient to introduce an auxiliary parameter t such that t^m term takes into account all contributions of the terms $q_1^{m_1} q_2^{m_2}$ with $m = m_1 + m_2$. Up to $m = 5$ we get

$$\begin{aligned} \tilde{f}^{(0)}(q_1, q_2|t) = & -\frac{1}{2}(\tilde{\epsilon}_1 q_1 q_2) t^2 + \frac{1}{48}(-4\epsilon_1 q_1^2 q_2^2 - 10\tilde{\epsilon}_1 q_1^2 q_2^2 + 4\epsilon_1 q_1^2 q_2^2 \alpha^2 \\ & + \tilde{\epsilon}_1 q_1^2 q_2^2 \alpha^2) t^4 + \frac{1}{48}(-4\epsilon_1 q_1^3 q_2^3 - 6\tilde{\epsilon}_1 q_1^3 q_2^3 + 4\epsilon_1 q_1^3 q_2^3 \alpha^2 \\ & + \tilde{\epsilon}_1 q_1^3 q_2^3 \alpha^2) t^6 + \mathcal{O}(t^7), \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \tilde{f}^{(1)}(q_1, q_2|t) = & -\frac{1}{2}(q_1 + q_2)t + \frac{1}{24}(-3q_1^2 + 6q_1 q_2 - 7q_2^2 + 4q_2^2 \alpha^2) t^2 \\ & + \frac{1}{24}(-q_1^3 - 3q_1^2 q_2 + q_1 q_2^2 - 5q_2^3 - 4q_1 q_2^2 \alpha^2 + 4q_2^3 \alpha^2) t^3 \\ & + \frac{1}{2880}(-45q_1^4 - 180q_1^3 q_2 + 330q_1^2 q_2^2 + 60q_1 q_2^3 - 469q_2^4 + 120q_1^2 q_2^2 \alpha^2 \\ & - 240q_1 q_2^3 \alpha^2 + 440q_2^4 \alpha^2 - 16q_2^4 \alpha^4) t^4 + \mathcal{O}(t^5), \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \tilde{f}^{(2)}(q_1, q_2|t) = & -\frac{1}{16\tilde{\epsilon}_1}(q_1^2 + q_2^2 \alpha^2) t^2 + \frac{1}{16\tilde{\epsilon}_1}(-q_1^3 + q_1^2 q_2 + q_1 q_2^2 \alpha^2 - q_2^3 \alpha^2) t^3 \\ & + \frac{1}{192\tilde{\epsilon}_1}(-9q_1^4 + 6q_1^3 q_2 - q_1^2 q_2^2 - 11q_1^2 q_2^2 \alpha^2 + 6q_1 q_2^3 \alpha^2 - 11q_2^4 \alpha^2 \\ & + 2q_2^4 \alpha^4) t^4 + \mathcal{O}(t^5), \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \tilde{f}^{(3)}(q_1, q_2|t) = & \frac{1}{128\tilde{\epsilon}_1^2}(-q_1^4 + 2q_1^2 q_2^2 \alpha^2 - q_2^4 \alpha^4) t^4 \\ & + \frac{1}{64\tilde{\epsilon}_1^2}(-q_1^5 + q_1^4 q_2 + q_1 q_2^4 \alpha^4 - q_2^5 \alpha^4) t^5 + \mathcal{O}(t^6). \end{aligned} \quad (\text{A.6})$$

Setting $t = 1$ yields expansion coefficients in the decomposition (A.2). We find out that up to coordinate independent terms

$$S(\theta_2, \theta_3) = \tilde{f}(q_2, q_3) - \epsilon_1 \ln(1 - q_1 q_2) - \epsilon_3 \ln(1 - q_2) + (2\epsilon_1 - \tilde{\epsilon}_1) \ln q_1 q_2 - \epsilon_3 \ln q_2. \quad (\text{A.7})$$

Recalling the difference in the asymptotic behavior of $\tilde{f}(q)$ and $f(z)$ (modulo the standard prefactor $z_2^{2\epsilon_1 - \tilde{\epsilon}_1} z_3^{-\epsilon_3}$) we see that the above identification conforms (5.5).

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