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PERFECT MEASURABLE SPACES*

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1. Introduction

During the recent years a good deal of attention was focused on the following (*measurable*) *union problem*: let $\mathscr{A}$ be a σ-algebra of subsets of a space X and $X_i \in \mathscr{A}$ $(i \in I)$. What conditions on $\mathscr{A}$ and X_i $(i \in I)$ guarantee the existence of a subfamily X_i $(i \in J)$ whose union is not measurable with respect to $\mathscr{A}$, i.e. does not belong to $\mathscr{A}$? Usually it is assumed that the family X_i $(i \in I)$ is point-finite, i.e. $\{i \in I : x \in X_i\}$ is finite for all $x \in X$, and that the X_i are small, i.e. belong to some σ-ideal. The main goal of the present paper is to give a general treatment of the union problem which yields most of the results in the literature as special cases. We begin by reviewing some of the previous work. This will enable us to present our result in a proper setting.

Our own work received impetus from an unpublished result of R. Solovay, which is as follows:

Theorem 1.1 (R. Solovay, 1970). *Let X_i $(i \in I)$ be a disjoint partition of the interval* $[0, 1]$ *where each X_i has Lebesgue measure zero. Then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\}$ is not Lebesgue measurable.*

As a corollary of Theorem 1.1 Solovay showed:

Theorem 1.2. *Let $f : [0, 1] \to X$ be Lebesgue measurable, where X is a metric space. Then there is a closed separable $Y \subseteq X$ such that $f^{-1}(Y)$ has Lebesgue measure one.*

Theorem 1.2 can also be restated as follows. Letting μ denote the Lebesgue measure on $[0, 1]$, we can define a Borel measure ν on X by setting

$$\nu(B) = \mu(f^{-1}(B))$$

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for $B \subseteq X$, B Borel. Theorem 1.2 is then equivalent to: there is a closed separable Y such that $\nu(Y) = 1$.

Another equivalent result is the following generalised Lusin theorem.

Theorem 1.3. *Under the assumptions of Theorem* 1.2, *there is a compact* $K \subseteq [0, 1]$ *such that* $\mu(K) > 0$ *and* $f \mid K$ *is continuous.*

Theorem 1.3 is well known when X is separable, being only a slight extension of the classical Lusin's theorem.

To pinpoint further the nature of the above results it is necessary to comment on their relationship to the problem of the existence of real-valued measurable (r.v.m.) cardinals, [31], [32]. Assuming there is no such cardinal Theorem 1.1 is easy to see (thus Theorems 1.2, 1.3 follow as well). This was pointed out by Kuratowski [20]. In outline, supposing that Theorem 1.1 is false, one defines a measure ν on $\mathscr{P}(I)$ by setting

$$\nu(J) = \mu(\bigcup \{X_i : i \in J\}) \quad (J \subseteq I).$$

Then ν is a countably additive diffuse probability measure on $\mathscr{P}(I)$, and the existence of a real-valued measurable cardinal follows.

Thus the importance of Theorem 1.1 lies in eliminating the assumption that there is no real-valued measurable cardinal. This situation is fairly typical. Thus one has, e.g., the following result of Marczewski and Sikorski (1947):

Theorem 1.4 [22]. *Suppose that there is no real-valued measurable cardinal. Let X be a metric space and* μ *a Borel probability measure on X. Then there is a closed separable subspace Y such that* $\mu(Y) = 1$.

Theorem 1.4 also easily implies Theorems 1.2 and 1.3 (in the absence of r.v.m. cardinals). The main part of the proof of Theorem 1.4 is the construction of a covering of a set of positive measure by disjoint measure zero sets, thereby creating the situation described at the outset.

Much work has been done in this direction (Varadarajan [33], Moran [23], Haydon [16], Gardner [14]). We mention, for instance,

Theorem 1.5 [16]. *Let X be a metacompact space and* μ *a regular Borel probability measure on X. Then either there is a closed Lindelöf subspace Y such that* $\mu(Y) = 1$, *or X has a discrete subspace whose cardinality is real-valued measurable.*

Once again, in the proof one considers a covering of a set of positive measure by open null sets. Exploiting the metacompactness one refines this covering to a *point-finite* one. Then, after further work, one succeeds in disjointizing the sets. Thus, while studying measures on topological spaces it usually suffices to work with disjoint families. This is also true for other cases [11, Theorem 3E]. However, point-finite coverings enter naturally and it is not always obvious how to accomplish an adequate disjointization. In an abstract situation, this constitutes a much more difficult problem than the disjoint case. Another interesting point is that if the disjointization is not possible, the connection with r.v.m. cardinals in no longer clear. Unlike Theorem 1.1, it is clear that the assumption that there is no r.v.m. cardinal cannot be omitted in Theorems 1.4 and 1.5 (just consider a discrete space X of r.v.m. cardinality). A moment of reflection, however, shows that Theorem 1.2 partially restores the validity of Theorem 1.4, without assuming the non-existence of r.v.m. cardinals (see also the comment following Theorem 1.2). A more general result is due to Koumoullis [18] and Pachl [26]:

Theorem 1.6. *Let X be a metric space and μ a* perfect *probability Borel measure on X. Then either μ is a Radon measure (that is, for every $\varepsilon > 0$ there is a compact subset K with $\mu(K) > 1 - \varepsilon$), or there is a closed discrete subset Y whose cardinal $|Y|$ is $\{0, 1\}$-measurable.*

Perfect measures were introduced by Gnedenko and Kolmogorov (see [15, p. 18]). Theorem 1.6 implies Lusin's Theorem for perfect measures on metric spaces and mappings into metric spaces with cardinality < the least measurable cardinal. For Radon measure spaces and mappings into any metric space, Lusin's Theorem was obtained by Fremlin [10]. In the present paper we generalize Theorem 1.6 to developable spaces (see Corollary 3.3). The key to Theorem 1.6 is the realization that *Theorem* 1.1 *above remains valid for any perfect measure if $|I| <$ the least* ($\{0, 1\}$-) *measurable cardinal.* An important advantage associated with the setting of perfect measures is that the cardinality restrictions are much less severe (the least measurable cardinal as opposed to $2^{\aleph_0}$ which is at least implicit in Theorems 1.1, 1.2, 1.3).

Having realized the significance of the union problem in measure theory it is natural to inquire what happens for other σ-algebras and ideals. For example, what happens when $\mathscr{A}$ is the σ-algebra of sets with the property of Baire and X_i ($i \in I$) are meager? Recall that a subset S of a topological space has the property of Baire if S can be expressed as $S = S_0 \triangle S_1$, where S_0 is open and S_1 is meager. Bukovsky proved:

Theorem 1.7 [4]. *Let X_i ($i \in I$) be a disjoint partition of the interval* [0, 1] *into meager sets. Then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\}$ does not have the property of Baire.*

Another interesting result of this type was recently obtained by Louveau and Simpson [21]. To describe their result, let $\mathbb{N}$ denote the space of natural numbers with the discrete topology. Then $2^{\mathbb{N}}$ traditionally denotes the space of non-empty closed subsets of $\mathbb{N}$ with the Vietoris topology. Louveau and Simpson consider the subspace Ω of $2^{\mathbb{N}}$ consisting of the *infinite* subsets of $\mathbb{N}$ and obtain the analogue of Theorem 1.7 for Ω. Actually, in their result X_i $(i \in I)$ is point-finite. But Fremlin [11] shows that in general if Theorem 1.7 holds for some space X, then it also holds when X_i $(i \in I)$ is a point-finite covering of X. The space Ω has recently come into prominence due to the work of Ellentuck [7] who showed that Ω plays an important role in generalizing Ramsey's Theorem. Such generalizations of Ramsey's Theorem have recently found applications in Banach space theory; see e.g. Odell [24].

Our results on measurable spaces generalize both Theorem 1.7 as well as the Louveau–Simpson result. (See also Fremlin [11, 6I] for a topological generalization). The analogue of Theorem 1.7 holds for other spaces (developable, Čech-complete [11] and K-analytic). These results are discussed in Section 3.

All results discussed so far were connected either with measure or category. A general theorem which includes both of these as special cases is due to Brzuchowski et al. [3]. In order to be able to state their result we need a definition. An ideal $\mathcal{I} \subseteq \mathcal{P}(X)$, where X is a topological space, is said to have a Borel base if every set in $\mathcal{I}$ is a subset of a Borel set belonging to $\mathcal{I}$.

Theorem 1.8 [3]. *Let X be a Polish space (that is, a complete separable metric space) and $\mathcal{I}$ be a σ-ideal with a Borel base. Let X_i $(i \in I)$ be a point-finite covering of X by sets from $\mathcal{I}$. Then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\}$ does not differ from a Borel set by a set from $\mathcal{I}$.*

Theorem 1.8 was obtained by Prikry [27] for the case of measure and category for all separable (*not necessarily complete*) metric spaces. These results are presented here (see Corollaries 4.1 and 4.2). They are not covered by the general theory of perfect measurable spaces whereas Theorem 1.8 above is.

In Section 2 we introduce and study in detail our main new concepts — those of perfect and weakly perfect measurable spaces. In Section 3 we examine perfect and weakly perfect measurable spaces when the underlying space is a topological space. In Section 4 we discuss the union problem for measurable spaces which are not necessarily perfect or weakly perfect.

Notations. For a set I, $\mathcal{P}(I)$ denotes the power set of I and $|I|$ denotes the cardinal of I. c is the cardinal of the continuum.

All topological spaces X are assumed to be at least Hausdorff. $\mathcal{B}(X)$ denotes the Borel σ-algebra on X, i.e., the σ-algebra generated by the closed sets in X. In Section 3, we also consider the Baire σ-algebra $\mathcal{B}_a(X)$ which is generated by the zero sets of continuous functions.

2. Perfect and weakly perfect measurable spaces

As usual, if $(X, \mathcal{A})$ is a measurable space and Y is a topological space, $f : X \to Y$ is $\mathcal{A}$-measurable if for all open $U \subseteq Y$ (equivalently, for all Borel

$U \subseteq Y$), $f^{-1}(A) \in \mathscr{A}$. Most of the subsequent discussion evolves around a triple $(X, \mathscr{A}, \mathscr{I})$ where $(X, \mathscr{A})$ is a measurable space and $\mathscr{I}$ is an ideal of subsets of X (i.e. $\mathscr{I}$ is closed under taking subsets and finite unions). Usually, we assume that $\mathscr{I}$ is a σ-ideal (i.e. $\mathscr{I}$ is closed under enumerable unions). In this case, if we set

$$\mathscr{A}_{\mathscr{I}} = \{A \triangle Y : A \in \mathscr{A},\ Y \in \mathscr{I}\}$$

then $(X, \mathscr{A}_{\mathscr{I}})$ is a measurable space. We say that a family $\mathscr{C} \subseteq \mathscr{P}(X)$ is a base for $\mathscr{I}$ if every set in $\mathscr{I}$ is contained in a set in $\mathscr{C} \cap \mathscr{I}$. If there is a base $\mathscr{C} \subseteq \mathscr{A}$ we say that $\mathscr{I}$ has an $\mathscr{A}$-base.

Definition 2.1. $(X, \mathscr{A}, \mathscr{I})$ is a *perfect* (resp. *weakly perfect*) *measurable space* if whenever $f: X \to \mathbb{R}$ is $\mathscr{A}$-measurable there is an analytic $A \subseteq f(X)$ such that $X - f^{-1}(A) \in \mathscr{I}$ (resp. $f^{-1}(A) \notin \mathscr{I}$). In addition, $(X, \mathscr{A}, \mathscr{I})$ is weakly perfect if $X \in \mathscr{I}$.

Note. Essentially the same concept of perfect measurable spaces was introduced independently by Fremlin [11] who used the term 'semi-perfect'. Our results on these spaces were obtained in 1981 while the first author was visiting the University of Minnesota.

Clearly, if $(X, \mathscr{A}, \mathscr{I})$ is perfect, then it is weakly perfect. Note that in the above definition $f^{-1}(A)$ does not necessarily belong to $\mathscr{A}$ (i.e. might be non-measurable). However, in most applications, $f^{-1}(A)$ belongs to $\mathscr{A}$, or at least to $\mathscr{A}_{\mathscr{I}}$. This is true if $\mathscr{A}$ is closed under the Suslin operation, or if $\mathscr{I}$ is a σ-ideal with an $\mathscr{A}$-base and $(X, \mathscr{A}, \mathscr{I})$ is perfect.

Before proceeding with the statements of our main results we shall give a variety of examples and counterexamples concerning perfect and weakly perfect measurable spaces. Most of them follow easily from well known results, while those for which the underlying space X is a topological space are also covered by the results of Section 3.

The seminal instance of perfect measurable spaces is connected with the concept of perfect measures (Gnedenko and Kolmogorov [15]). Here we shall give a well known equivalent definition: a probability measure space $(X, \mathscr{A}, \mu)$ is perfect if for every $\mathscr{A}$-measurable $f: X \to \mathbb{R}$ there is a Borel subset B of $\mathbb{R}$ such that $B \subseteq f(X)$ and $\mu(f^{-1}(B)) = 1$; see Sazonov [29]. The class of perfect measures includes Borel probability measures on separable complete metric spaces, or more generally Radon probability measures. This follows easily from Lusin's Theorem (see Section 1). Moreover, the completion of a perfect measure space is perfect, as is every $\{0, 1\}$-valued probability measure. It is easy to see that if $(X, \mathscr{A}, \mu)$ is a probability measure space, it is perfect iff $(X, \mathscr{A}, \mathscr{I})$ is perfect where $\mathscr{I}$ is the ideal of μ-measure zero sets. One direction follows from every Borel set being analytic and the other direction from every analytic set being universally measurable.

We can define $(X, \mathscr{A}, \mu)$ to be weakly perfect if the corresponding $(X, \mathscr{A}, \mathscr{I})$ is weakly perfect. For a probability Borel measure μ on a separable metric space we have that μ is perfect iff μ is Radon (see [29]) and that μ is weakly perfect iff there is a compact set of positive measure. Both of these results follow from the proof of Lemma 2.2.

The σ-algebra of sets with the property of Baire and the corresponding ideal — that of meager sets — also provide examples of perfect (resp. weakly perfect) measurable spaces, which we shall call perfect (resp. weakly perfect) *category spaces*. If the underlying space X is a complete separable metric space, then we have an example of a perfect category space. This can be seen by using the theorem that if $f: X \to \mathbb{R}$ has the the property of Baire, then there is a dense G_δ-set G such that $f \mid G$ is continuous [25, §8]. If, on the other hand, the underlying space X is Ellentuck's space [7] or [24], then we have an example of a weakly perfect category space (see Section 3) which is not perfect.

If X is an analytic separable metric space, then $(X, \mathscr{B}(X), \{\emptyset\})$ is perfect, and thus $(X, \mathscr{B}(X), \mathscr{I})$, for an arbitrary $\mathscr{I}$, is likewise perfect. More generally, $(X, \mathscr{A}, \{\emptyset\})$ clearly is perfect iff $f(X)$ is analytic for every $\mathscr{A}$-measurable $f: X \to \mathbb{R}$. These are exactly the smooth spaces of Falkner [8].

If $(X, \mathscr{A}, \mathscr{I})$ is perfect and $\mathscr{I}$ is a σ-ideal with an $\mathscr{A}$-base, then $(X, \mathscr{A}_\mathscr{I}, \mathscr{I})$ is perfect, and similarly for 'weakly perfect'; see Lemma 2.6. Hence in particular, if X is an analytic separable metric space and $\mathscr{I}$ is a σ-ideal with a Borel base, then $(X, \mathscr{B}(X)_\mathscr{I}, \mathscr{I})$ is perfect.

We are now ready to state our main results.

Theorem 2.1. *Let $(X, \mathscr{A}, \mathscr{I})$ be a perfect (resp. weakly perfect) measurable space and X_i $(i \in I)$ be a point-finite covering of X. Set*

$$\mathscr{A}' = \{J \subseteq I : \bigcup \{X_i : i \in J\} \in \mathscr{A}\},$$
$$\mathscr{I}' = \{J \subseteq I : \bigcup \{X_i : i \in J\} \in \mathscr{I}\}.$$

Then if $\mathscr{A}'$ is a σ-algebra, $(I, \mathscr{A}', \mathscr{I}')$ is a perfect (resp. weakly perfect) measurable space.

This theorem is relatively easy to prove if X_i $(i \in I)$ are disjoint. Then $\mathscr{A}'$ is automatically a σ-algebra. In general $\mathscr{A}'$ is not a σ-algebra; but it is always closed under enumerable unions. Our proof shows that in general the 'paved' space $(I, \mathscr{A}', \mathscr{I}')$ is perfect (resp. weakly perfect).

Theorem 2.2. *Let $(X, \mathscr{A})$ be a measurable space and $\mathscr{I}$ a σ-ideal on X. Let X_i $(i \in I)$ be a point-finite covering of X. Also suppose that one of the conditions* (a), (b) *below holds*:

(a) *$(X, \mathscr{A}, \mathscr{I})$ is perfect, $|I| <$ the least measurable cardinal and there is no $J \subseteq I$ such that $|J| \leq \aleph_0$ and $X - \bigcup \{X_i : i \in J\} \in \mathscr{I}$.*

(b) *$(X, \mathscr{A}, \mathscr{I})$ is weakly perfect, $|I| \leq c$ and $X_i \in \mathscr{I}$ for all $i \in I$ and $X \notin \mathscr{I}$.*
Then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\} \notin \mathscr{A}$. Moreover, if $\mathscr{I}$ has an $\mathscr{A}$-base, then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\} \notin \mathscr{A}_\mathscr{I}$.

Remarks. Theorem 2.2(a) extends [11, Proposition 3I] as $\mathcal{I}$ need not be ω_1-saturated. Other special cases of Theorem 2.2(a) include Theorems 1.1, 1.7, 1.8, [18, Theorem 2.5] and [9, Proposition 2.3].

The restriction on $|I|$ in Theorem 2.2(a) (resp. 2.2(b)) can be replaced by the following condition on $(X, \mathcal{A}, \mathcal{I})$: there is a thick set $R \subset X$ such that $|R| <$ the least measurable cardinal (resp. $|R| \leq c$). Here R is thick means that $R \cap A \neq \emptyset$ for all $A \in \mathcal{A} - \mathcal{I}$. Indeed, there is $J \subseteq I$ such that $R \subset Y = \bigcup \{X_i : i \in J\}$ and $|J| \leq |R|$. If $Y \notin \mathcal{A}$ we are done. If $Y \in \mathcal{A}$, then $(Y, \mathcal{P}(Y) \cap \mathcal{A}, \mathcal{P}(Y) \cap \mathcal{I})$ is perfect (resp. weakly perfect) and since $X - Y \in \mathcal{I}$, Theorem 2.2 applies.

However, the cardinality restrictions in Theorem 2.2 cannot be removed in general. For the perfect case the counterexample is provided by the space $(X, \mathcal{P}(X), \mathcal{I})$ where $|X| \geq$ the least measurable cardinal and $\mathcal{I}$ is a proper maximal σ-ideal containing all singletons. For the weakly perfect case the counterexample is $(X, \mathcal{P}(X), \mathcal{I})$ where $|X| > c$ and $\mathcal{I}$ is the ideal of at most enumerable subsets of X. In both cases the point-finite covering is the family of all singletons.

The cardinality restrictions in Theorem 2.2 can be removed in certain perfect product spaces as the next theorem shows (see also Section 3 for other cases). The proof combines an idea of Fremlin [10] with an application of Theorems 2.1 and 2.2.

Theorem 2.3. *Let $(Y_\alpha, \mathcal{A}_\alpha)$, $(\alpha \in \kappa)$, be measurable spaces, where κ is arbitrary, and $|Y_\alpha| <$ the least measurable cardinal. Set*

$$X = \prod_{\alpha \in \kappa} Y_\alpha$$

and let $\mathcal{A}$ be the usual product σ-algebra on X. Suppose that $\mathcal{I}$ is any σ-ideal such that $(X, \mathcal{A}, \mathcal{I})$ is perfect and X_i $(i \in I)$ a point-finite covering of X such that there is no $J \subseteq I$ with $|J| \leq \aleph_0$ and $X - \bigcup \{X_i : i \in J\} \in \mathcal{I}$. Then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\} \notin \mathcal{A}$. Moreover, if $\mathcal{I}$ has an $\mathcal{A}$-base, then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\} \notin \mathcal{A}_{\mathcal{I}}$.

Theorem 2.4. *If $(X, \mathcal{A}, \mathcal{I})$ is a perfect measurable space and X_i $(i \in I)$ is a point-finite covering of X such that $X_i \notin \mathcal{I}$ and $\bigcup \{X_i : i \in J\} \in \mathcal{A}$ for all $J \subseteq I$, then I is at most enumerable.*

We also have useful characterizations of perfect and weakly perfect measurable spaces (Theorems 2.5 and 2.6). Either one of Theorems 2.1 or 2.5 and 2.6 can serve as a basis for the development of the theory of these spaces.

Theorem 2.5. *For a triple $(X, \mathcal{A}, \mathcal{I})$ the following are equivalent:*

(a) *$(X, \mathcal{A}, \mathcal{I})$ is perfect.*

(b) *For every countably generated σ-algebra $\mathcal{A}' \subseteq \mathcal{A}$ there is some $f : X \to \mathbb{R}$*

and an analytic set $A \subseteq f(X)$ *such that*

(i) $\mathcal{A}' \subseteq f^{-1}(\mathcal{B}(\mathbb{R}))$, *and* (ii) $X - f^{-1}(A) \in \mathcal{I}$.

(c) *For every Suslin scheme* $A(s) \in \mathcal{A}$, *where s ranges over finite sequences from* $\mathbb{N}$, *there is an analytic* $C \subseteq \mathbb{N}^{\mathbb{N}}$ *such that*

$$\text{(i)} \bigcap_{n \in \mathbb{N}} A(\sigma \mid n) \neq \emptyset \quad \textit{for } \sigma \in C, \quad \textit{and} \quad \text{(ii)} \bigcup_{\sigma \notin C} \bigcap_{n \in \mathbb{N}} A(\sigma \mid n) \in \mathcal{I}.$$

(d) *This is the same as* (c), *but in addition*, $A(s)$ *are decreasing and disjoint in the sense that if* $\sigma, \tau \in \mathbb{N}^{\mathbb{N}}$ *and* $\sigma \neq \tau$, *then*

$$\bigcap_{n \in \mathbb{N}} A(\sigma \mid n) \cap \bigcap_{n \in \mathbb{N}} A(\tau \mid n) = \emptyset.$$

Theorem 2.6. *For a triple* $(X, \mathcal{A}, \mathcal{I})$, *where* $\mathcal{I}$ *is a proper ideal on* X, *the following are equivalent*:

(a) $(X, \mathcal{A}, \mathcal{I})$ *is weakly perfect.*

(b) *This is the same as in Theorem* 2.5(b), *except that* (ii) *becomes* $f^{-1}(A) \notin \mathcal{I}$.

(c) *The same as in Theorem* 2.5(c), *except that* (ii) *becomes*

$$\bigcup_{\sigma \in C} \bigcap_{n \in \mathbb{N}} A(\sigma \mid n) \notin \mathcal{I}.$$

(d) *The same as* (c) *with* $A(s)$ *decreasing and disjoint.*

We shall now prove Theorems 2.1–2.6. For the proof of Theorem 2.1, we need the following definition and Lemmas 2.1 and 2.2; we also assume that $\mathcal{I}$ is a *proper* ideal (i.e. $X \notin \mathcal{I}$), the other case being trivial.

Definition 2.2. Let $X = \bigcup \{X_i : i \in I\}$, where the family $\{X_i\}$ is point-finite. Also let $f: I \to [0, 1)$. Then the *associated many-valued function* g is defined as follows: For each $x \in X$, let $F_x = \{i \in I : x \in X_i\}$ (hence F_x is finite). Let $\langle r_0, r_1, \ldots, r_k \rangle$ enumerate $\{f(i) : i \in F_x\}$ in increasing order. We set $g(x) = \langle r_0, r_1, \ldots, r_k, 1, 1, \ldots \rangle \in [0, 1]^{\mathbb{N}}$. Thus $g: X \to [0, 1]^{\mathbb{N}}$.

Lemma 2.1. *Let* $(X, \mathcal{A})$ *be a measurable space and* X_i $(i \in I)$ *a point-finite covering of* X. *Set*

$$\mathcal{A}' = \{J \subseteq I : \bigcup \{X_i : i \in J\} \in \mathcal{A}\}.$$

Let $f: I \to [0, 1)$ *be* $\mathcal{A}'$*-measurable and* $g: X \to [0, 1]^{\mathbb{N}}$ *be the associated many-valued function. Then* g *is* $\mathcal{A}$*-measurable.*

Proof. Let $\pi_k : [0, 1]^{\mathbb{N}} \to [0, 1]$ be the k-th projection. It clearly suffices to show that for all k, $g_k = \pi_k \circ g$ is $\mathcal{A}$-measurable, that is, $g_k^{-1}([0, r)) \in \mathcal{A}$ for every $r \in [0, 1]$. We prove this by induction on k.

We have $g_0(x) < r$ iff $x \in X_i$ for some $i \in I$ such that $f(i) < r$ iff $x \in \bigcup \{X_i : f(i) <$

$r\}$. Since f is $\mathscr{A}'$-measurable, $f^{-1}([0, r)) \in \mathscr{A}'$. Hence $\bigcup\{X_i : f(i) < r\} \in \mathscr{A}$ and g_0 is $\mathscr{A}$-measurable.

For the induction step suppose that g_j $(j \leq k)$ are $\mathscr{A}$-measurable. For every open $U \subseteq [0, 1]$, set

$$C_U = \bigcup \{X_i : f(i) \in U\}.$$

As in the preceding step we conclude that $C_U \in \mathscr{A}$. In particular, for open intervals (s, r), $C_{(s,r)} \in \mathscr{A}$. We now have $g_{k+1}(x) < r$ iff

$$\exists s < r\ (s \in Q \wedge g_k(x) < s \wedge x \in C_{(s,r)}),$$

where Q is the set of rational numbers. Hence

$$g_{k+1}^{-1}([0, r)) = \bigcup_{\substack{s<r\\ s \in Q}} (g_k^{-1}([0, s)) \cap C_{(s,r)})$$

and we are done.

Lemma 2.2. *the following are equivalent*:

(a) $(X, \mathscr{A}, \mathscr{I})$ *is perfect* (*resp. weakly perfect*).

(b) *If* Y *is a separable metric space and* $f : X \to Y$ *is* $\mathscr{A}$*-measurable, then there is an analytic* $A \subseteq f(X)$ *such that* $X - f^{-1}(A) \in \mathscr{I}$ (*resp.* $f^{-1}(A) \notin \mathscr{I}$).

Proof. (b) implies (a) is trival. The other direction follows easily from the following well known fact (cf. [20, §36, III]):

If Y is a separable metric space, then there is a one-to-one function $f : Y \to \mathbb{R}$ such that $f^{-1} : f(Y) \to Y$ is continuous and for every open $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is F_σ in Y.

Proof of Theorem 2.1. Let $f : I \to (0, 1)$ be $\mathscr{A}'$-measurable. We shall show that there is an analytic $A' \subseteq f(I)$ such that $I - f^{-1}(A') \in \mathscr{I}$ (resp. $f^{-1}(A') \notin \mathscr{I}'$).

Let $g : X \to [0, 1]^{\mathbb{N}}$ be the associated many-valued function. Then g is $\mathscr{A}$-measurable by Lemma 2.1. Since $(X, \mathscr{A}, \mathscr{I})$ is perfect (resp. weakly perfect), there is, by Lemma 2.2, an analytic $A \subseteq G(X)$ such that $X - g^{-1}(A) \in \mathscr{I}$ (resp. $g^{-1}(A) \notin \mathscr{I}$). Hence all $\pi_k(A)$ are analytic subsets of $[0, 1]$ and so is $\bigcup_{k \in \mathbb{N}} \pi_k(A)$. Moreover, setting

$$A' = \bigcup_{k \in \mathbb{N}} \pi_k(A) - \{1\},$$

we have that A' is analytic and $A' \subseteq f(I)$.

We claim that $X_i \cap g^{-1}(A) = \emptyset$ for all $i \in I - f^{-1}(A')$. Indeed, if $x \in X_i \cap g^{-1}(A)$ then, by the definition of g, we have $f(i) = \pi_k(g(x))$ for some k and $f(i) \neq 1$. Since $g(x) \in A$, $f(i) \in \pi_k(A) - \{1\} \subseteq A'$.

If $X - g^{-1}(A) \in \mathscr{I}$, then by the claim

$$\bigcup \{X_i : i \in I - f^{-1}(A')\} \in \mathscr{I},$$

that is, $I - f^{-1}(A') \in \mathscr{I}'$. Similarly, if $g^{-1}(A) \notin \mathscr{I}'$. This completes the proof.

For the proof of Theorem 2.2 we need to analyze perfect and weakly perfect spaces of the form $(X, \mathcal{P}(X), \mathcal{I})$.

Lemma 2.3. *If* $(X, \mathcal{P}(X), \mathcal{I})$ *is perfect, then* $\mathcal{I}$ *is* ω_1*-saturated (i.e. there is no uncountable disjoint family of sets not belonging to* $\mathcal{I}$*).*

Proof. Let X_i $(i \in I)$ be a disjoint family of sets not belonging to $\mathcal{I}$ and assume that $|I| \leq c$. It suffices to show that $|I| \leq \aleph_0$. Let Z be a totally imperfect (i.e. containing no nonempty perfect set) subset of $\mathbb{R}$ with $|Z| = c$ (see [20, §40]). By a well known theorem, every analytic subset of Z is at most enumerable. Now consider a function $f: X \to \mathbb{R}$ with $f(X) \subseteq Z$, $f \mid X_i$ constant and $f(X_i) \neq f(X_j)$ for every i and $j \in I$, $i \neq j$. Since $(X, \mathcal{P}(X), \mathcal{I})$ is perfect and f is trivially $\mathcal{P}(X)$-measurable, there is an analytic $A \subseteq f(X)$ such that $X - f^{-1}(A) \in \mathcal{I}$. Thus $|A| \leq \aleph_0$ and $X_i \subseteq f^{-1}(A)$. Thus $|I| \leq \aleph_0$ as required.

Lemma 2.4. *The following are equivalent if* $\mathcal{I}$ *is a* σ*-ideal on* X:

(a) $(X, \mathcal{P}(X), \mathcal{I})$ *is perfect.*

(b) *For every partition of* X, X_i $(i \in I)$ *where* $|I| \leq c$, *there is an at most enumerable* $J \subseteq I$ *such that* $X - \bigcup \{X_i : i \in J\} \in \mathcal{I}$.

(c) *The same as* (b) *except that* $|I| <$ *the least measurable cardinal.*

Proof. (a)$\to$(b). As in the previous lemma we consider a totally imperfect set Z of reals and a function $f: X \to \mathbb{R}$ with $f(X) \subseteq Z$, $f \mid X_i$ constant and $f(X_i) \neq f(X_j)$ for every i and $j \in I$, $i \neq j$. Since $(X, \mathcal{P}(X), \mathcal{I})$ is perfect, there is an analytic $A \subseteq f(X)$ such that $X - f^{-1}(A) \in \mathcal{I}$. Then $|A| \leq \aleph_0$ and setting

$$J = \{i \in I : f(X_i) \subseteq A\}$$

we have $|J| \leq \aleph_0$ and $X - \bigcup \{X_i : i \in J\} \in \mathcal{I}$.

(b)$\to$(a). For every function $f: X \to \mathbb{R}$, we apply (b) for the partition $\{f^{-1}(\{y\}) : y \in f(X)\}$. Thus there is an at most enumerable, hence also analytic, $A \subseteq f(X)$ such that $X - f^{-1}(A) \in \mathcal{I}$.

(c)$\to$(b) is trivial. So it remains to show (a)$\to$(c). Assume that $(X, \mathcal{P}(X), \mathcal{I})$ is perfect and let X_i $(i \in I)$ be a partition of X, where $|I| <$ the least measurable cardinal. By Lemma 2.3, $\mathcal{I}$ is ω_1-saturated, so $J = \{i \in I : X_i \notin \mathcal{I}\}$ is at most enumerable. We claim that

$$X_1 = X - \bigcup \{X_i : i \in J\} \in \mathcal{I}.$$

Suppose that the claim is false and consider the perfect space

$$(X_1, \mathcal{P}(X_1), \mathcal{I} \cap \mathcal{P}(X_1)),$$

where $\mathcal{I} \cap \mathcal{P}(X_1)$ is a proper ideal on X_1. This space can be 'extended' back to X by putting $X - X_1$ into the new proper ideal on X, the resulting space being perfect. So we can assume that $J = \emptyset$, i.e., $X_i \in \mathcal{I}$ for all $i \in I$, and that $X \notin \mathcal{I}$.

Define

$$\mathscr{I}' = \{J \subseteq I : \bigcup \{X_i : i \in J\} \in \mathscr{I}\}.$$

By Theorem 2.1, $(I, \mathscr{P}(I), \mathscr{I}')$ is perfect and Lemma 2.3 implies that $\mathscr{I}'$ is ω_1-saturated. Moreover, we have $\{i\} \in \mathscr{I}'$ for all $i \in I$ and $I \notin \mathscr{I}'$. Hence, by Ulam's Theorem [17, p. 300], either there exists some $J \subseteq I$ such that $\mathscr{I}' \cap \mathscr{P}(J)$ is a proper maximal σ-ideal on J, or there exists a partition of I into $\leqslant c$ sets in $\mathscr{I}'$. If J is as in the first alternative of Ulam's Theorem, let λ be the additivity degree of the ideal $\mathscr{I}' \cap \mathscr{P}(J)$, i.e., the least cardinal such that there are sets $I_\alpha \in \mathscr{I}' \cap \mathscr{P}(I)$ $(\alpha \in \lambda)$ with $J = \bigcup \{I_\alpha : \alpha \in \lambda\}$. It is known that λ is a measurable cardinal, which is a contradiction since $\lambda \leqslant |J| \leqslant |I|$. If the second alternative occurs we come again to a contradiction because $(I, \mathscr{P}(I), \mathscr{I}')$ is perfect and we already know that (a) $\rightarrow$ (b). This completes the proof of Lemma 2.4.

Lemma 2.5. *The following are equivalent if $\mathscr{I}$ is a proper σ-ideal on X:*

(a) *$(X, \mathscr{P}(X), \mathscr{I})$ is weakly perfect.*

(b) *For every partition of X, X_i $(i \in I)$ where $|I| \leqslant c$, there is an at most enumerable $J \subseteq I$ such that $\bigcup \{X_i : i \in J\} \notin \mathscr{I}$.*

The proof is similar as that of Lemma 2.4, (a) $\leftrightarrow$ (b).

Proof of Theorem 2.2. We start by proving the first conclusion of Theorem 2.2. We define a σ-ideal $\mathscr{I}'$ on I as in the statement of Theorem 2.1. Supposing that case (a) is false it follows from Theorem 2.1 that $(I, \mathscr{P}(I), \mathscr{I}')$ is perfect. Applying Lemma 2.4 for the partition of I into singletons, we conclude that $I - J \in \mathscr{I}'$ for some at most enumerable J. Hence it easily follows that $X - \bigcup \{X_i : i \in J\} \in \mathscr{I}$, a contradiction. The proof of case (b) follows similarly by contradiction applying Lemma 2.5.

The second conclusion follows from the first and Lemma 2.6, concluding the proof of Theorem 2.2.

Lemma 2.6. *Let $(X, \mathscr{A}, \mathscr{I})$ be perfect (resp. weakly perfect), where $\mathscr{I}$ is a σ-ideal with an $\mathscr{A}$-base. Then $(X, \mathscr{A}_\mathscr{I}, \mathscr{I})$ is perfect (resp. weakly perfect).*

Proof. Let $f : X \rightarrow \mathbb{R}$ be $\mathscr{A}_\mathscr{I}$-measurable. Lemma 2.6 clearly follows if we can find an $\mathscr{A}$-measurable $g : X \rightarrow \mathbb{R}$ such that $\{x : f(x) \neq g(x)\} \in \mathscr{I}$. To find such a g we proceed as follows. Let U_n $(n \in \mathbb{N})$ be an open base for $\mathbb{R}$. Since f is $\mathscr{A}_\mathscr{I}$-measurable and $\mathscr{I}$ has an $\mathscr{A}$-base we can pick an $S_n \in \mathscr{A} \cap \mathscr{I}$ such that $f^{-1}(U_n) \cap (X - S_n) \in \mathscr{A}$. Then $S = \bigcup \{S_n : n \in \mathbb{N}\} \in \mathscr{A} \cap \mathscr{I}$. It now suffices to set

$$g(x) = \begin{cases} f(x), & \text{if } x \notin S, \\ 0, & \text{if } x \in S. \end{cases}$$

Proof of Theorem 2.3. As in Theorem 2.2 the second conclusion follows from the first and Lemma 2.6. So we shall prove only the first conclusion. Suppose that this is false. We proceed as in the proof of Lemma 2.4(a)$\to$(c). So we can assume that $X_i \in \mathcal{I}$ for all $i \in I$ and that $X \notin \mathcal{I}$. Then we define

$$\mathcal{I}' = \{J \subseteq I : \bigcup \{X_i : i \in J\} \in \mathcal{I}\}$$

so that $(I, \mathcal{P}(I), \mathcal{I}')$ is perfect and $\mathcal{I}'$ is ω_1-saturated (Theorem 2.1 and Lemma 2.3). Again by Ulam's Theorem, either there exists some $J \subseteq I$ such that $\mathcal{I}' \cap \mathcal{P}(J)$ is a proper maximal ideal on J, or there exists a partition of I into $\leqslant c$ sets in $\mathcal{I}'$. By Theorem 2.2 the second alternative in Ulam's Theorem cannot occur.

Now if J is as in the first alternative, let λ be the additivity degree of $\mathcal{I}' \cap \mathcal{P}(J)$. Then λ is a measurable cardinal and there are disjoint sets $I_\alpha \in \mathcal{I}' \cap \mathcal{P}(J)$ $(\alpha \in \lambda)$ such that

$$J = \bigcup \{I_\alpha : \alpha \in \lambda\}.$$

We consider the family

$$X'_\alpha = \bigcup \{X_i : i \in I_\alpha\} \quad (\alpha \in \lambda).$$

This is a point-finite family of sets from $\mathcal{I}$, the union of every subfamily of X_α belongs to $\mathcal{A}$ and

$$X_1 = \bigcup \{X'_\alpha : \alpha \in \lambda\} \notin \mathcal{I}.$$

Now we can extend the perfect space

$$(X_1, \mathcal{P}(X_1) \cap \mathcal{A}, \mathcal{I} \cap \mathcal{P}(X_1))$$

back to X, putting $X - X_1$ to a new ideal, the resulting space being perfect. This means that we can assume that $I = \lambda$ and $\mathcal{I}'$ is λ-complete.

Now we set $Z_\xi = \bigcup_{\eta \geqslant \xi} X_\eta$ $(\xi \in \lambda)$. Since $\mathcal{I}'$ is λ-complete $Z_\xi \notin \mathcal{I}$. Since $\mathcal{A}$ is a product σ-algebra and $Z_\xi \in \mathcal{A}$, we can pick an enumerable $E_\xi \subseteq \lambda$ such that

$$Z_\xi = C_\xi \times \prod \{Y_\alpha : \alpha \in \lambda - E_\xi\}$$

and

$$C_\xi \subseteq \prod \{Y_\alpha : \alpha \in E_\xi\}$$

for every $\xi \in \lambda$.

By the Δ-system of Erdös and Rado (see [6]), there is some $E \subseteq \lambda$ and $J \subseteq \lambda$ such that $|J| = \lambda$ and for all ξ and $\eta \in \lambda$, if $\xi \neq \eta$ then $E_\xi \cap E_\eta = E$. Pick $t_\xi \in C_\xi$ for $\xi \in J$. Since $|\prod \{Y_\alpha : \alpha \in E\}| <$ the least measurable cardinal, there is some $K \subseteq J$ and $t \in \prod \{Y_\alpha : \alpha \in E\}$ such that $|K| = \lambda$ and $t_\xi \mid E = t$ for $\xi \in K$. It is now easy to find an $x \in X$ such that $x \mid E_\xi = t_\xi$ for all $\xi \in K$. Hence $x \in Z_\xi$ for all $\xi \in K$, contradicting the point-finiteness and the proof is complete.

Proof of Theorem 2.4. Define

$$\mathcal{I}' = \{J \subseteq I : \bigcup \{X_i : i \in J\} \in \mathcal{I}\}.$$

By Theorem 2.1, $(I, \mathcal{P}(X), \mathcal{I}')$ is perfect and Lemma 2.3 implies that $\mathcal{I}'$ is ω_1-saturated. Since $\{i\} \notin \mathcal{I}'$ for every $i \in I$, it follows that I is at most enumerable.

Proof of Theorem 2.5. (a)$\to$(b). Let $\mathcal{A}' \subseteq \mathcal{A}$ be a countably generated σ-algebra. Then there is some $f: X \to \mathbb{R}$ such that $\mathcal{A}' = f^{-1}(\mathcal{B}(\mathbb{R}))$ (cf. [5, Theorem 0.1]). Hence f is $\mathcal{A}$-measurable and there is an analytic $A \subseteq f(X)$ such that $X - f^{-1}(A) \in \mathcal{I}$.

(b)$\to$(c). Let $\mathcal{A}'$ be the σ-algebra generated by all the $A(s)$. Let f and A now be as in (b). It suffices to show that

$$C = \left\{\sigma \in \mathbb{N}^{\mathbb{N}} : \bigcap_n A(\sigma \mid n) \cap f^{-1}(A) \neq \emptyset\right\}$$

is analytic.

Let $g: \mathbb{N}^{\mathbb{N}} \to A$ be continuous and onto. Pick $B(s) \in \mathcal{B}(\mathbb{R})$ such that $A(s) = f^{-1}(B(s))$. We show that

$$C = \left\{\sigma \in \mathbb{N}^{\mathbb{N}} : \bigcap_n g^{-1}(B(\sigma \mid n)) \neq \emptyset\right\}.$$

First, if $\tau \in \bigcap_n g^{-1}(B(\sigma \mid n))$, then $g(\tau) \in \bigcap_n B(\sigma \mid n)$. Since $g(\tau) \in A \subset f(X)$, there is an $x \in X$ such that $g(\tau) = f(x)$. Hence

$$(*) \qquad x \in \bigcap_n A(\sigma \mid n) \cap f^{-1}(A)$$

Secondly, if $(*)$ holds, then $f(x) \in A = g(\mathbb{N}^{\mathbb{N}})$. Hence $f(x) = g(\tau)$ for some τ. Since $x \in \bigcap_n f^{-1}(B(\sigma \mid n))$, it follows that $g(\tau) = f(x) \in \bigcap_n B(\sigma \mid n)$, so

$$\tau \in \bigcap_n g^{-1}(B(\sigma \mid n)).$$

Hence it now follows that C is the projection onto the second coordinate of the Borel set

$$\bigcap_n \bigcup_{|s|=n} g^{-1}(B(s)) \times I(s),$$

where $I(s) = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma \text{ extends } s\}$.

(c)$\to$(d) is trivial.

(d)$\to$(a). Let $f: X \to \mathbb{N}^{\mathbb{N}}$ be $\mathcal{A}$-measurable. We shall show that there is an analytic $C \subseteq f(X)$ such that $X - f^{-1}(C) \in \mathcal{I}$. This suffices because $\mathbb{N}^{\mathbb{N}}$ is Borel isomorphic to $\mathbb{R}$.

Let $A(s) = f^{-1}(I(s))$ where $I(s)$ is as above. The Suslin scheme $A(s)$ is decreasing and disjoint. Let C be as guaranteed by (d). Then C works.

The proof of Theorem 2.6 is similar to that of Theorem 2.5.

As an application of the characterization Theorems 2.5 and 2.6 we shall give another proof of Theorem 2.1 for the case $\mathscr{A}' = \mathscr{P}(I)$ — the only case we have found a use for.

With the notation and the assumptions of Theorem 2.1, we shall show that $(I, \mathscr{P}(I), \mathscr{I}')$ satisfies (d) of Theorem 2.5, resp. 2.6. Let $B(s)$ be a decreasing Suslin scheme of subsets of I. Set

$$A(s) = \bigcup \{X_i : i \in B(s)\}$$

This is a Suslin scheme in $\mathscr{A}$ since $\mathscr{A}' = \mathscr{P}(I)$. Since $(X, \mathscr{A}, \mathscr{I})$ is perfect, resp. weakly perfect, by Theorem 2.5, resp. 2.6, there is an analytic $C \subseteq \mathbb{N}^{\mathbb{N}}$ such that

(a) $\quad \bigcap_n A(\sigma \mid n) \neq \emptyset \quad (\sigma \in C)$

and

(b) $\quad \bigcup_{\sigma \notin C} \bigcap_n A(\sigma \mid n) \in \mathscr{I}, \qquad$ resp. $\bigcup_{\sigma \in C} \bigcap_n A(\sigma \mid n) \notin \mathscr{I}.$

Using the fact that $\{X_i : i \in I\}$ is point-finite,

$$\bigcap_n A(\sigma \mid n) = \bigcup \left\{X_i : i \in \bigcap_n B(\sigma \mid n)\right\}.$$

Hence if $\sigma \in C$, then by (a), $\bigcap_n B(\sigma \mid n) \neq \emptyset$. Also by (b),

$$\bigcup_{\sigma \notin C} \bigcup \left\{X_i : i \in \bigcap_n B(\sigma \mid n)\right\} \in \mathscr{I},$$

resp.,

$$\bigcup_{\sigma \in C} \bigcup \left\{X_i : i \in \bigcap_n B(\sigma \mid n)\right\} \notin \mathscr{I}.$$

Hence

$$\bigcup_{\sigma \notin C} \bigcap_n B(\sigma \mid n) \in \mathscr{I}', \qquad \text{resp.,} \quad \bigcup_{\sigma \in C} \bigcap_n B(\sigma \mid n) \notin \mathscr{I}'.$$

Hence (d) holds.

We conclude this section with a measurable selection theorem for perfect measurable spaces. First we need the following concept of measurability: if F is a function from a measurable space $(X, \mathscr{A})$ into the family $\mathscr{C}(Y)$ of non-empty compact subsets of a topological space Y, we say that F is $\mathscr{A}$-measurable if for all open $U \subseteq Y$, $\{x \in X : F(x) \cap U \neq \emptyset\} \in \mathscr{A}$. For point-valued functions this is clearly equivalent to the ordinary measurability. A selection for F is a function $f : X \to Y$ such that $f(x) \in F(x)$ for every x.

Theorem 2.7. *Let $(X, \mathscr{A})$ be a measurable space, $\mathscr{I}$ a σ-ideal on X with an $\mathscr{A}$-base and Y a metric space. Assume that $(X, \mathscr{A}, \mathscr{I})$ is perfect and $|Y| <$ the least measurable cardinal. Then if $F : X \to \mathscr{C}(Y)$ is $\mathscr{A}_{\mathscr{I}}$-measurable, there is a separable $\tilde{Y} \subseteq Y$ such that $X - \{x : F(x) \subseteq \tilde{Y}\} \in \mathscr{I}$ and F admits an $\mathscr{A}_{\mathscr{I}}$-measurable selection.*

The proof of this theorem is similar to that of Theorems 1 and 2 in [19] except that Theorem 3 and Lemma 4 in [19] are replaced by Theorems 2.2 and 2.4, respectively.

Theorem 2.7 generalizes [9, Theorem 4.1]; for other related results see [11, §9].

3. Some examples

The results of the previous section deal with arbitrary perfect and weakly perfect measurable spaces. Here we study such spaces when the underlying space X is a topogical space; the σ-algebra is usually the family $\mathscr{B}(X)$ of Borel sets. We find conditions on X so that for an ideal $\mathscr{I}$ on X, $(X, \mathscr{B}(X), \mathscr{I})$ is perfect or weakly perfect. When this is done, the results of the previous section apply and yield more concrete results, where the notions of perfect and weakly perfect measurable spaces are at least implicit. Such results were stated in Section 1.

Theorem 3.1. *Let X be a metric, or more generally a developable space with $|X| <$ the least measurable cardinal and let $\mathscr{I}$ be a σ-ideal on X. Then $(X, \mathscr{B}(X), \mathscr{I})$ is perfect iff there is an analytic $A \subseteq X$ such that $X - A \in \mathscr{I}$.*

Proof. First suppose that there is an analytic A such that $X - A \in \mathscr{I}$. Then if $f: X \to \mathbb{R}$ is $\mathscr{B}(X)$-measurable it is easily seen that $f(A)$ is analytic and the result follows.

To prove the opposite direction, we shall consider the metric case separately since this is the most interesting case and the proof is simpler. Thus suppose that X is metric and also separable at first. Let $f: X \to \mathbb{R}$ be as in the proof of Lemma 2.2. Hence there is an analytic $A' \subseteq f(X)$ such that $X - f^{-1}(A') \in \mathscr{I}$. It suffices to set $A = f^{-1}(A')$.

We shall now reduce the general metric case to the separable one. We set

$$\mathscr{U} = \{U \subseteq X : U \text{ is open and } U \in \mathscr{I}\}.$$

Set $Y = \bigcup \mathscr{U}$. We shall show that $Y \in \mathscr{I}$. Suppose not. By Stone's theorem, we can pick a σ-disjoint open refinement $\mathscr{V} = \bigcup \{\mathscr{V}_n : n \in \mathbb{N}\}$ of $\mathscr{U}$, where each family $\mathscr{V}_n$ is disjoint. We can now pick $n \in \mathbb{N}$ such that $Y_n = \bigcup \mathscr{V}_n \notin \mathscr{I}$. Let $\mathscr{V}_n = \{U_i : i \in I\}$ be a one-to-one indexing of sets from $\mathscr{V}_n$. As usual, we set

$$\mathscr{I}' = \{J \subseteq I : \{U_i : i \in J\} \in \mathscr{I}\}.$$

Clearly $(I, \mathscr{P}(I), \mathscr{I}')$ is perfect and all singletons belong to $\mathscr{I}'$ while $I \notin \mathscr{I}'$ and $|I| <$ the least measurable cardinal. This contradicts Lemma 2.4. Hence $Y \in \mathscr{I}$. Set $F = X - Y$. F is closed, hence in $\mathscr{B}(X)$ and thus we can suppose without loss of generality that $X = F$. Hence if $\emptyset \neq U \subseteq X$ and U is open, then $U \notin \mathscr{I}$.

Since every c.c.c. metric space is separable, it suffices to show that X is c.c.c. Let U_i $(i \in I)$ be a family of non-empty disjoint open subsets. Then if $\mathscr{I}'$ is defined as above, we have $\mathscr{I}' = \{\emptyset\}$. Hence $(I, \mathscr{P}(I), \{\emptyset\})$ is perfect, and thus by Lemma 2.3, $|I| \leq \aleph_0$.

This completes the proof of Theorem 3.2 when X is metric.

Let us recall that a Hausdorff space X is developable if there is a sequence $\mathscr{U}_n$ $(n \in \mathbb{N})$ of open covers such that for every $x \in X$, the family

$$\bigcup \{U \in \mathscr{U}_n : x \in U\} \quad (n \in \mathbb{N}),$$

is neighbourhood basis at x.

Fix a sequence $\mathscr{U}_n$ $(n \in \mathbb{N})$ as above. By Bing [1, Theorem 9], we can pick $\mathscr{F}_n$, a closed σ-discrete refinement of $\mathscr{U}_n$. The union of an arbitrary subfamily of $\mathscr{F}_n$ is an F_σ, hence Borel. By using Lemma 2.4 similarly as in the proof of the metric case we can find $\mathscr{E}_n \subseteq \mathscr{F}_n$ such that $|\mathscr{E}_n| \leq \aleph_0$ and $X - \bigcup \mathscr{E}_n \in \mathscr{I}$. Let $\mathscr{V}_n \subseteq \mathscr{U}_n$ be at most enumerable and such that for every $E \in \mathscr{E}_n$ there is some $U \in \mathscr{V}_n$ containing E. Set

$$Y = \bigcap_n \left(\bigcup \mathscr{V}_n\right).$$

Clearly $X - Y \in \mathscr{I}$. Moreover Y is second countable since

$$\bigcup_n \{U \cap Y : U \in \mathscr{V}_n\}$$

is an open base for Y, by the definition of developability. Since $Y \in \mathscr{B}(X)$ and $X - Y \in \mathscr{I}$, we have a reduction of the general case to the second countable case. This is handled similarly as the separable metric case. This is because every second countable Hausdorff space is Borel isomorphic to a subset of $\mathbb{R}$, by an open mapping onto its range.

Theorem 3.2. *Let X be a developable space with $|X| \leq c$ and $\mathscr{I}$ be a proper σ-ideal on X. Then $(X, \mathscr{B}(X), \mathscr{I})$ is weakly perfect iff there is an analytic $A \subseteq X$ such that $A \notin \mathscr{I}$.*

Proof. If there is an analytic $A \subseteq X$ such that $A \notin \mathscr{I}$, the result follows as in the previous theorem. The opposite direction follows from Lemma 2.2 and Lemma 3.1, concluding the proof of Theorem 3.2.

Lemma 3.1. *Let X be a developable space with $|X| \leq c$. Then there is a Borel measurable function $f: X \to \mathbb{R}^{\mathbb{N}}$ such that:*

(a) *for every second countable $Y \subseteq X$, $f|Y: Y \to f(Y)$ is a Borel isomorphism; and*

(b) *for every analytic $A \subseteq f(X)$, $f^{-1}(A)$ is analytic.*

Proof. We claim that there are partitions $\mathscr{E}_n$, $(n \in \mathbb{N})$, of X such that:

(i) the union of every subfamily of $\mathscr{E}_n$ is a Borel set (actually an F_σ-set);

(ii) every open set in X is the union of some members of $\bigcup \{\mathscr{E}_n : n \in \mathbb{N}\}$;

(iii) for every $Y \subseteq X$, Y is second countable iff

$$|\{E \in \mathscr{E}_n : E \cap Y \neq \emptyset\}| \leq \aleph_0$$

for every $n \in \mathbb{N}$.

To see this, let $\mathscr{F}_n$ ($n \in \mathbb{N}$) be as in proof of Theorem 3.1. Then

$$\mathscr{F}_n = \bigcup \{\mathscr{F}_{n,m} : m \in \mathbb{N}\},$$

where each $\mathscr{F}_{n,m}$ is a discrete family of closed sets. Now we set

$$\mathscr{E}_{n,m} = \mathscr{F}_{n,m} \cup \{X - \bigcup \mathscr{F}_{n,m}\}$$

and rearrange $\mathscr{E}_{n,m}$ ($n \in \mathbb{N}$, $m \in \mathbb{N}$) as $\mathscr{E}_n$ ($n \in \mathbb{N}$). It is easy to see that (i)–(iii) hold. The metric case is simpler: we consider a σ-discrete base $\mathscr{V} = \bigcup \{\mathscr{V}_n : n \in \mathbb{N}\}$ and set

$$\mathscr{E}_n = \mathscr{V}_n \cup \{X - \bigcup \mathscr{V}_n\}.$$

Now we proceed as follows. We choose a totally imperfect set of reals Z with $|Z| = c$. Since $|\mathscr{E}_n| \leq c$, there is $f_n : X \to \mathbb{R}$ such that $f_n(X) \subseteq Z$, $f_n \mid E$ is constant and $f_n(E) \neq f_n(E')$ for every E and $E' \in \mathscr{E}_n$, $E \neq E'$. Define

$$f : X \to \mathbb{R}^{\mathbb{N}}$$

by

$$f(x) = \langle f_n(x) : n \in \mathbb{N} \rangle.$$

Then f is Borel measurable by (i). Since X is Hausdorff, (ii) implies that $\bigcup \{\mathscr{E}_n : n \in \mathbb{N}\}$ separates points and so f is one-to-one.

By the definition of f, it is easy to see that for every $E \in \bigcup \{\mathscr{E}_n : n \in \mathbb{N}\}$ there is a closed $F \subseteq \mathbb{R}^{\mathbb{N}}$ such that $E = f^{-1}(F)$. Now, if $Y \subseteq X$ is second countable, then by (iii)

$$|\{E \in \mathscr{E}_n : E \cap Y \neq \emptyset\}| \leq \aleph_0$$

for every $n \in \mathbb{N}$; so by (ii) every relatively open set in Y has the form $Y \cap f^{-1}(C)$ for some F_σ set C in $\mathbb{R}^N$. This shows that $f \mid Y$ is a Borel isomorphism, concluding the proof of (a).

To prove (b) let A be an analytic subset of $f(X)$ and let $\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ denote the n-th projection. Then $\pi_n(A) \subseteq Z$ is analytic and by the choice of Z it follows that $\pi_n(A)$ is at most enumerable. So we have

$$|\{E \in \mathscr{E}_n : E \cap f^{-1}(A) \neq \emptyset\}| \leq \aleph_0$$

for every n and, by (iii), $f^{-1}(A)$ is second countable. Now (a) implies that $f^{-1}(A)$ is Borel isomorphic to A and so (b) follows.

Remarks. The proof of Lemma 3.1 actually shows that for every second countable $Y \subseteq X$, $f \mid Y$ is a Borel isomorphism of class 1, in the sense that both functions $f \mid Y$ and $(f \mid Y)^{-1}$ map open sets to F_σ sets.

The cardinality restrictions in Theorems 3.1 and 3.2 are necessary. Counterexamples are those mentioned after the statement of Theorem 2.2, where X is considered as a discrete metric space.

However, for the special case of Theorem 3.1 when $\mathscr{I} = \{\emptyset\}$, or more generally, if the union of any discrete (equivalently, σ-discrete) family of closed sets from $\mathscr{I}$ belongs to $\mathscr{I}$, no restriction on X is necessary. We indicate the necessary modification in the proof of Theorem 3.2 for developable X. Since $\mathscr{F}_n$ is σ-discrete,

$$\bigcup \{F \in \mathscr{F}_n : F \in \mathscr{I}\} \in \mathscr{I}.$$

Set $\mathscr{E}_n = \{F \in \mathscr{F}_n : F \notin \mathscr{I}\}$. Then

$$X - \bigcup \mathscr{E}_n \in \mathscr{I},$$

and using Lemma 2.3, $|\mathscr{E}_n| \leq \aleph_0$. The rest is as above.

Hence we have the following result, due to Falkner [8, Theorem 4.6] for metric spaces.

Corollary 3.1. *Let X be a developable space. Then $(X, \mathscr{B}(X), \{\emptyset\})$ is perfect ("$(X, \mathscr{B}(X))$ is smooth" in Falkner's terminology) iff X is analytic.*

Corollary 3.2. *Let $(X, \mathscr{A}, \mathscr{I})$ be perfect (resp. weakly perfect) and $f: X \to Y$ be $\mathscr{A}$-measurable, where Y is developable and $|Y| <$ the least measurable cardinal (resp. $|Y| \leq c$). Then there is an analytic $A \subseteq Y$ such that $X - f^{-1}(A) \in \mathscr{I}$ (resp. $f^{-1}(A) \notin \mathscr{I}$ if $\mathscr{I}$ is a proper ideal).*

Proof. Set $Z \in \mathscr{I}'$ iff $f^{-1}(Z) \in \mathscr{I}$. Then $(Y, \mathscr{B}(Y), \mathscr{I}')$ is perfect (resp. weakly perfect). Hence the result follows from Theorems 3.1 and 3.2.

For measure and category spaces we have:

Corollary 3.3. *Let X be a developable space with $|X| <$ the least measurable cardinal (resp. $|X| \leq c$). A probability Borel measure μ on X is perfect (resp. weakly perfect) iff μ is a Radon measure (resp. there is a compact subset of X with positive μ-measure).*

Proof. If $A \subseteq X$ is analytic, then A is measurable with respect to any Borel measure on X and every Borel measure on A is a Radon measure (see [30, Ch. II, Theorem 10]). Hence the result follows from Theorems 3.1 and 3.2.

Remark. Corollary 3.3 for perfect measures was proved in [18, Theorem 4.13] when X is in addition regular and weakly metacompact, generalizing Theorem 1.6.

Corollary 3.4. *Let X be a complete metric space with $|X| <$ the least measurable cardinal (resp. $|X| \leqslant c$) and let $\mathcal{I}$ denote the ideal of meager sets in X. Then $(X, \mathcal{B}(X), \mathcal{I})$ is perfect (resp. weakly perfect) iff X is separable (resp. there exists a nonempty open separable subset of X).*

Proof. Assume that $(X, \mathcal{B}(X), \mathcal{I})$ is perfect and $|X| <$ least measurable cardinal. By Theorem 3.1, there is an analytic $A \subseteq X$ such that $X - A$ is meager. Then $X - \bar{A}$ is open meager, hence $\bar{A} = X$. Since A is separable, so is X. The converse is obvious since X is now assumed to be a Polish space.

Now assume that $(X, \mathcal{B}(X), \mathcal{I})$ is weakly perfect and $|X| \leqslant c$. By Theorem 3.2, there is an analytic set $A \subseteq X$ of the second category. Since A is separable, the interior of $\bar{A}$ is a nonempty open separable set. Again the converse is obvious since any nonempty open separable set in X is a Polish space.

We shall now present several results concerning perfect measurable spaces when the underlying space X is K-analytic.

Lemma 3.2. *Let X be a topological space and $\mathcal{I}$ a proper ideal on X. Then $(X, \mathcal{B}_a(X), \mathcal{I})$ is perfect (resp. weakly perfect) iff for every continuous $f: X \to Y$, where Y is separable metric, there is an analytic $A \subseteq f(X)$ such that $X - f^{-1}(A) \in \mathcal{I}$ (resp. $f^{-1}(A) \notin \mathcal{I}$).*

Proof. The implication from left to right follows by Lemma 2.2. In the opposite direction it suffices to show that (b) of Theorem 2.5, (resp. 2.6) holds. Let $A_n \in \mathcal{B}_a(X)$ ($n \in \mathbb{N}$) and $\mathcal{A}'$ be the σ-algebra generated by the sets A_n. For every n, let $f_n: X \to \mathbb{R}^{\mathbb{N}}$ be a continuous function such that $A_n = f_n^{-1}(B_n)$ for some $B_n \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$. Define

$$f: X \to (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$$

by

$$f(x) = \langle f_n(x) : n \in \mathbb{N} \rangle$$

Then f is continuous and by our assumption there is an analytic $A \subseteq f(X)$ such that $X - f^{-1}(A) \in \mathcal{I}$, (resp. $f^{-1}(A) \notin \mathcal{I}$). Moreover, we have

$$\mathcal{A}' \subseteq f^{-1}(\mathcal{B}((\mathbb{R}^{\mathbb{N}})^{\mathbb{N}})).$$

Since $(\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ is Borel isomorphic to $\mathbb{R}$, we have (b) of Theorem 2.5 (resp. 2.6).

We recall that a Hausdorff space X is K-analytic, if there is an upper semicontinuous f from $\mathbb{N}^{\mathbb{N}}$ into the family of compact subsets of X such that

$$X = \bigcup \{f(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\}.$$

We refer to [28] for results on K-analytic spaces. Here we shall use the following facts: (i) The class of K-analytic spaces is preserved under taking continuous images, closed subspaces, countable unions and countable intersections; (ii)

Every regular K-analytic space is Lindelöf (hence, every regular K-analytic space is completely regular); and (iii) Every metrizable K-analytic space is analytic. Lemma 3.2 in conjunction with (i) and (ii) yield immediately the following:

Lemma 3.3 [8, Proposition 4.4]. *If X is K-analytic, then $(X, \mathscr{B}_a(X))$ is smooth, hence $(X, \mathscr{B}_a(X), \mathscr{I})$ is perfect for every ideal $\mathscr{I}$.*

Theorem 3.3. *Let X be a K-analytic regular space and $\mathscr{I}$ the ideal of meager sets. Then $(X, \mathscr{B}(X), \mathscr{I})$ is perfect iff X has a closed comeager c.c.c. subset.*

Recall that a space Y is c.c.c. if every disjoint family of open sets in Y is at most enumerable.

Proof. Suppose $(X, \mathscr{B}(X), \mathscr{I})$ is perfect. Set

$$\mathscr{U} = \{U \subseteq X : U \text{ open and meager}\}, \qquad V = \bigcup \mathscr{U}.$$

By Banach's category theorem [25, §16], V is meager. Set $Y = X - V$. Then if $\emptyset \neq U \subseteq Y$, and U is relative open, $U \notin \mathscr{I}$. Hence by Theorem 2.4 (or an easy argument using Lemma 2.3), Y is c.c.c.

Now for the converse let Y be closed, c.c.c. and comeager. It suffices to show that $(Y, \mathscr{B}(Y), \mathscr{I}')$, where $\mathscr{I}' = \mathscr{I} \cap \mathscr{P}(Y)$, is perfect. Y is K-analytic as a closed subspace of a K-analytic space. Hence by Lemma 3.3, $(Y, \mathscr{B}_a(Y), \mathscr{I})$ is perfect. Thus by Lemma 2.6, it suffices to show that $\mathscr{B}(Y)$ is included in the $\mathscr{I}'$-completion of $\mathscr{B}_a(Y)$.

Let G be an open subset of Y. Let $\mathscr{V}$ be a maximal family of non-empty pairwise disjoint cozero subsets of G. Then $\mathscr{V}$ is at most enumerable and thus $V = \bigcup \mathscr{V} \in \mathscr{B}_a(X)$ and is dense in G by the complete regularity. Hence $G - V \in \mathscr{I}'$ and the proof is complete.

Corollary 3.5. *Let X be either* (i) *a regular K-analytic Baire space or* (ii) *a Cech-complete space. Then $(X, \mathscr{B}(X), \mathscr{I})$, where $\mathscr{I}$ is the ideal of meager sets, is perfect iff X is c.c.c.*

Proof. First note that a Cech-complete space is also a Baire space. As in Theorem 3.4, this fact alone suffices to show that if $(X, \mathscr{B}(X), \mathscr{I})$ is perfect, then X is c.c.c.

For the opposite direction first suppose (i). Then X is a closed comeager c.c.c. subset of itself. Hence the result follows from Theorem 3.3.

Now suppose (ii), i.e., X is a G_δ-subset of βX, the Stone–Čech compactification of X. Let $\mathscr{I}'$ be the ideal of meager subsets of βX. By Theorem 3.3,

$$(\beta X, \mathscr{B}(\beta X), \mathscr{I}')$$

is perfect. Hence since $X \in \mathscr{B}(\beta X)$, $(X, \mathscr{B}(X), \mathscr{I}' \cap \mathscr{P}(X))$ is perfect. But since X is comeager in βX, $\mathscr{I}' \cap \mathscr{P}(X) = \mathscr{I}$.

The next two corollaries follow directly from Theorem 2.2 (resp. Theorem 2.4) and Lemma 3.3. Corollary 3.7 follows also from [13, Lemma 1].

Corollary 3.6. *Let X be a K-analytic space and $\mathcal{I}$ a proper σ-ideal on X with a Baire base. Let X_i $(i \in I)$ be a point-finite covering of X by sets from $\mathcal{I}$, where $|I| <$ the least measurable cardinal. Then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\}$ does not differ from a Baire set by a set from $\mathcal{I}$.*

Corollary 3.7. *Let X be a K-analytic space and X_i $(i \in I)$ be a point-finite covering of X by nonempty sets such that $\bigcup \{X_i : i \in J\} \in \mathcal{B}_a(X)$ for all $J \subseteq I$. Then I is at most enumerable.*

Case (ii) of the next corollary is due to Fremlin [11].

Corollary 3.8. *Let X be either* (i) *a regular K-analytic Baire c.c.c. space or* (ii) *a Cech-complete c.c.c. space. Let X_i $(i \in I)$ be a point-finite covering of X by meager sets. Then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\}$ does not have the Baire property.*

Proof. As in Corollary 3.6, the result follows directly from Theorem 2.2 and Corollary 3.5, if we assume that $|I| <$ the least measurable cardinal. Here the cardinality restriction is dropped in view of a theorem of Fremlin [11, Theorem 3H] because $(X, \mathcal{B}(X)_{\mathcal{I}}, \mathcal{I})$, where $\mathcal{I}$ is the ideal of meager sets, is $K\alpha$-regular; i.e. for every $E \in \mathcal{B}(X)_{\mathcal{I}}$ there is a K-analytic set $H \subseteq E$ such that $E - H \in \mathcal{I}$. We prove this fact for case (i) only (see [11] for case (ii)). If E is closed, take $H = E$. If E is open the argument of the last paragraph of the proof of Theorem 3.3 shows that there is a Baire set H in X such that $E - H \in \mathcal{I}$; but a Baire set in a K-analytic space is K-analytic. Finally, it is easy to see that this property of E is preserved by countable unions and countable intersections, completing the proof.

Remarks. It is clear from Lemma 3.3 that for a K-analytic space X, $(X, \mathcal{B}(X))$ is smooth if $\mathcal{B}(X) = \mathcal{B}_a(X)$. The problem whether $\mathcal{B}(X) = \mathcal{B}_a(X)$ whenever X is regular K-analytic and $(X, \mathcal{B}(X))$ is smooth, is undecidable in ZFC. This follows from Theorem 44H, (xiii) $\Rightarrow$ (ii), and Example 44J(b) in [12].

Let $\mathcal{I}$ be the ideal of meager sets in a K-analytic space X. By Theorem 3.3, $(X, \mathcal{B}(X), \mathcal{I})$ is weakly perfect if there is a closed non-meager c.c.c. subset. We don't know if the converse is also true.

Fremlin [11] proved that the analogue of Corollary 3.8 holds for Radon measure spaces $(X, \mathcal{A}, \mathcal{I})$. That is, X is a topolotical space, $\mathcal{A}$ is the σ-algebra of μ-measurable sets for some nonzero Radon measure μ on X and $\mathcal{I}$ is the ideal of μ-measure zero sets. As in Corollary 3.8, ths follows from the (easily seen) fact that $(X, \mathcal{A}, \mathcal{I})$ is $K\alpha$-regular [11]; see also [19] for another proof. Still another proof of this result follows from Theorem 2.3. Indeed, if $X = \{0, 1\}^{\kappa}$, for arbitrary κ, and μ is the usual product measure, the result is a special case of Theorem 2.3 because μ is perfect. As observed by Fremlin the general case reduces to the above (see [10, Proposition 1D]).

We conclude this section with a useful example of a weakly perfect category space. The underlying space is Ellentuck's space [7]. We shall use the following notation. If $A \subseteq \mathbb{N}$ is infinite, let

$$\Omega(A) = \{B \subseteq A : B \text{ is infinite}\}.$$

For A, B subsets of $\mathbb{N}$ define $A < B$ if $n < m$ for every $n \in A$ and $m \in B$. If $s \subseteq \mathbb{N}$ is finite and $A \subseteq \mathbb{N}$ is infinite, let

$$\Omega(s, A) = \{B \in \Omega(\mathbb{N}) : s \subseteq B \subseteq s \cup A \text{ and } s < B - s\}.$$

Thus $\Omega(\emptyset, A) = \Omega(A)$. The Ellentuck's space if $\Omega = \Omega(\mathbb{N})$ with the topology which has for a base all sets of the form $\Omega(s, A)$.

A set $P \subseteq \Omega$ is called completely Ramsey if for every finite $s \subseteq \mathbb{N}$ and every infinite $A \subseteq \mathbb{N}$, there is $B \in \Omega(A)$ such that $\Omega(s, B) \subseteq P$ or $\Omega(s, B) \subseteq \Omega - P$.

Lemma 3.4 (Ellentuck [7]). *A set $P \subseteq \Omega$ is completely Ramsey iff P has the property of Baire.*

Ellentuck [7] also proves that every meager set in Ω is nowhere dense. Thus Ω is a Baire space.

Theorem 3.4. *If $\mathscr{I}$ is the ideal of meager sets in Ω, then $(\Omega, \mathscr{B}(\Omega), \mathscr{I})$ is weakly perfect.*

Proof. Let $f : \Omega \to (0, 1)$ be a Borel measurable function. Let $n_0 = 0$ and $A_0 = \mathbb{N}$. Suppose we have defined $n_0, n_1, \ldots, n_{k-1}$ in $\mathbb{N}$ and $A_0, A_1, \ldots, A_{k-1}$ infinite subsets of $\mathbb{N}$. Then we choose $A_k \in \Omega(A_{k-1})$ such that $\{n_{k-1}\} < A_k$ and for all $s \subseteq \{n_0, \ldots, n_k\}$

(*) diameter of $f(\Omega(s, A_k)) < 1/2^k$.

We can do this by considering a finite partition $\{T_1, \ldots, T_m\}$ of $(0, 1)$ to intervals of length $<1/2^k$. Then Lemma 3.4 implies that each $f^{-1}(T_i)$ $(i = 1, \ldots, m)$ is completely Ramsey and A_k is found easily by applying the Ramsey property finitely many times. Now we set $n_k = \min A_k$.

By the above construction

$$n_0 < n_1 < \cdots < n_k < \cdots,$$

$$A_0 \supset A_1 \supset \cdots \supset A_k \supset \cdots$$

and $n_j \in A_k$ for all $j \geqslant k$. Let $A = \{n_0, n_1, \ldots\}$. Since $\Omega(A) \notin \mathscr{I}$ it suffices to show that $f(\Omega(A))$ is analytic.

Let $Z = \Omega(A)$ endowed with the relative topology from $\{0, 1\}^{\mathbb{N}}$, where $\{0, 1\}^{\mathbb{N}}$ has the usual product topology. Observe that Z is G_δ in $\{0, 1\}^{\mathbb{N}}$, so $Z \times (0, 1)$ is a Polish space. If we show that the graph of $f \mid Z$, $\mathrm{Gr}(f \mid Z)$, is closed in $Z \times (0, 1)$, then $f(\Omega(A))$, being the projection of $\mathrm{Gr}(f \mid Z)$ to $(0, 1)$, must be analytic.

To do this, let $\{X_l\}$ be a sequence in Z with $X_l \to X \in Z$ and $f(X_l) \to t \in (0, 1)$. It suffices to show that $t = f(X)$. X is of the form

$$X = \{n_{k_0} < n_{k_1} < \cdots\} \subseteq A.$$

Let $\varepsilon > 0$ and choose $l_0 \in \mathbb{N}$ such that

$$1/2^{k_{l_0}} < \varepsilon/2 \quad \text{and} \quad |f(X_l) - t| < \varepsilon/2 \quad \text{for all } l \geq l_0.$$

Since $X_l \to X$, there is some $l \geq l_0$ such that

$$s = \{n_{k_0}, n_{k_1}, \ldots, n_{k_{l_0-1}}\}$$

is the set of the first l_0 elements of X_1. Then X and X_l belong to $\Omega(s, A_{k_{l_0}})$, so (*) implies that $|f(X_l) - f(X)| < 1/2^{k_{l_0}}$. Thus we have

$$|f(X) - t| \leq |f(X) - f(X_l)| + |f(X_l) - t| < \varepsilon.$$

Therefore $t = f(X)$.

The following corollary, due to Louveau and Simpson [21], is now immediate from Theorems 2.2 and 3.4. See also Fremlin [11] for another proof.

Corollary 3.9. *If X_i $(i \in I)$ is a point-finite covering of Ω by meager sets, then there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\}$ does not have the property of Baire, hence is not Ramsey.*

Remark. Given a triple $(X, \mathcal{A}, \mathcal{I})$, where $\mathcal{I}$ is a proper σ-ideal on X, a sufficient condition in order that $(X, \mathcal{A}, \mathcal{I})$ be weakly perfect is the existence of some $Y \in \mathcal{A} - \mathcal{I}$ such that $(Y, \mathcal{A} \cap \mathcal{P}(Y), \mathcal{I} \cap \mathcal{P}(Y))$ is perfect. We have already used this fact when X is developable or K-analytic. However, this condition is not necessary in general. The weakly perfect space $(\Omega, \mathcal{B}(\Omega), \mathcal{I})$ of Theorem 3.4 provides a counterexample. Indeed, let $Y \in \mathcal{B}(\Omega)$ such that $(Y, \mathcal{B}(\Omega) \cap \mathcal{P}(Y), \mathcal{I} \cap \mathcal{P}(Y))$ is perfect. Since Y is completely Ramsey (Lemma 3.4), it follows easily that $Y - \mathrm{int}(Y)$ is nowhere dense. Since Ω is a Baire space, Y is c.c.c. (see Theorem 2.4). But Ω is nowhere c.c.c., hence $\mathrm{int}(Y) = \emptyset$. It now follows that Y is nowhere dense, hence $Y \in \mathcal{I}$.

4. Related results

In this section we prove a result on the union problem for not necessarily perfect, or weakly perfect, measurable spaces (Theorem 4.1). We assume instead that $(X, \mathscr{A}, \mathscr{I})$ satisfies a countability condition and has a Fubini-type property defined below. We also relate the union problem to the rectangle problem (Theorem 4.2).

First we need some notation. If $\mathscr{C}$ is a family of subsets of X, define $\mathscr{C}_\alpha$ $(\alpha < \omega_1)$ as follows: $\mathscr{C}_0 = \mathscr{C}$; for α odd (resp. even) let $\mathscr{C}_\alpha$ be the family of intersections (resp. unions) of at most enumerable subfamilies of $\bigcup_{\beta<\alpha} \mathscr{C}_\beta$.

For each $Z \subseteq X \times X$ and $x \in X$, let Z_x (resp. Z^x) be the vertical (resp. horizontal) section at x of Z. As usual, $\pi_i : X \times X \to X$ $(i = 1, 2)$ denote the first and second projection.

Definition 4.1. A triple $(X, \mathscr{A}, \mathscr{I})$ has the *weak Fubini property* if for every Z in the product σ-algebra $\mathscr{A} \otimes \mathscr{A}$ with $Z_x \in \mathscr{I}$ for all $x \in X$ and $\pi_2(Z) \notin \mathscr{I}$, there exists some $y \in \pi_2(Z)$ such that $Z^y \in \mathscr{I}$.

Theorem 4.1. *Let $(X, \mathscr{A})$ be a measurable space and $\mathscr{I}$ a proper σ-ideal on X. Assume that*:

(a) *there exists $\mathscr{C} \subseteq \mathscr{A}$ with $|\mathscr{C}| \leq \aleph_0$ and some $0 < \alpha < \omega_1$ such that $\mathscr{C}_\alpha$ is a base for $\mathscr{I}$; and*

(b) *$(X, \mathscr{A}_\mathscr{I}, \mathscr{I})$ satisfies the weak Fubini property.*

Then for every point-finite covering X_i $(i \in I)$ of X with $X_i \in \mathscr{I}$ there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\} \notin \mathscr{A}_\mathscr{I}$.

Proof. Suppose that the theorem is false and let X_ξ $(\xi \in \kappa)$ be a counterexample to the theorem. Let ρ be minimal such that

$$\tilde{X} = \bigcup \{X_\xi : \xi \in \rho\} \notin \mathscr{I}.$$

Then

$$(\tilde{X}, \mathscr{A}_\mathscr{I} \cap \tilde{X}, \mathscr{I} \cap \tilde{X})$$

satisfies (a) and (b) and

$$\bigcup \{X_\xi : \xi \in T\} \in \mathscr{A}_\mathscr{I} \cap \tilde{X}$$

for all $T \subseteq \rho$. Thus without loss of generality we assume that $\rho = \kappa$, so that $\tilde{X} = X$ and

$$\bigcup \{X_\xi : \xi \in \tau\} \in \mathscr{I}$$

for all $\tau < \kappa$. Obviously, we can also assume that $X \in \mathscr{C}$ and $\mathscr{C}$ is closed under finite unions and finite intersections; so each $\mathscr{C}_\beta$ $(\beta < \omega_1)$ has the same properties.

For every $\xi \in \kappa$, let

$$Y_\xi = \bigcup_{\eta<\xi} X_\eta$$

and, by (b), choose $C_\xi \in \mathscr{C}_\alpha \cap \mathscr{I}$ such that $Y_\xi \subseteq C_\xi$. We set

$$Z = \bigcup_{\xi\in\kappa} (X_\xi \times C_\xi)$$

and claim that

$$Z \in (\mathscr{A}_{\mathscr{I}} \times \mathscr{C})_{\alpha}$$

where

$$\mathscr{A}_{\mathscr{I}} \times \mathscr{C} = \{A \times C : A \in \mathscr{A}_{\mathscr{I}},\ C \in \mathscr{C}\}.$$

The claim is proved by induction on α (cf. [2, Theorem 3]).

If $\alpha = 1$, we set

$$A_C = \bigcup \{X_\xi : C \subseteq C_\xi\} \in \mathscr{A}_{\mathscr{I}}$$

for every $C \in \mathscr{C}$. Then we have

$$Z = \bigcup \{A_C \times C : C \in \mathscr{C}\} \in (\mathscr{A}_{\mathscr{I}} \times \mathscr{C})_{\alpha}.$$

Now assume that $1 < \alpha < \omega_1$ and the claim is true for every β, $0 < \beta < \alpha$.

Case 1: α is even.

We fix a sequence γ_n ($n \in \mathbb{N}$) of ordinals less than α as follows: if α is limit, then $\sup\{\gamma_n : n \in \mathbb{N}\} = \alpha$ and if α is nonlimit, say $\alpha = \alpha_0 + 1$, then $\gamma_n = \alpha_0$ for all $n \in \mathbb{N}$. By our assumption on $\mathscr{C}$, for every $\xi \in \kappa$ there is a decreasing sequence $C_{\xi,n}$ ($n \in \mathbb{N}$) such that $C_{\xi,n} \in \mathscr{C}_{\gamma_n}$ and

$$C_\xi = \bigcap_{n=0}^{\infty} C_{\xi,n}.$$

Then

$$Z = \bigcup_{\xi \in \kappa} \left(X_\xi \times \bigcap_{n=0}^{\infty} C_{\xi,n} \right) = \bigcap_{n=0}^{\infty} \bigcup_{\xi \in \kappa} (X_\xi \times C_{\xi,n}),$$

where the last equality is proved as follows: if (x, y) belongs to the right side, then there are $\xi_n \in \kappa$ ($n \in \mathbb{N}$) such that $x \in X_{\xi_n}$ and $y \in C_{\xi,n}$ for all $n \in \mathbb{N}$. Since X_ξ ($\xi \in \kappa$) is point-finite, there is an infinite $M \subseteq \mathbb{N}$ and some $\xi \in \kappa$ such that $\xi_n = \xi$ for all $n \in M$. Hence $(x, y) \in X_\xi \times C_{\xi,n}$ for $n \in M$. But $C_{\xi,n}$ is decreasing, so $(x, y) \in X_\xi \times C_{\xi,n}$ for all n. The other direction is obvious. By the induction hypothesis, it follows that $Z \in (\mathscr{A}_{\mathscr{I}} \times \mathscr{C})_{\alpha}$.

Case 2: α is odd.

We have $\alpha = \alpha_0 + 1$. Let $C_{\xi,n}$ ($n \in \mathbb{N}$) be a sequence in $\mathscr{C}_{\alpha_0}$ such that

$$C_\xi = \bigcup_{n=0}^{\infty} C_{\xi,n}.$$

Then

$$Z = \bigcup_{\xi \in \kappa} \left(X_\xi \times \bigcup_{n=0}^{\infty} C_{\xi,n} \right) = \bigcup_{n=0}^{\infty} \bigcup_{\xi \in \kappa} (X_\xi \times C_{\xi,n}) \in (\mathscr{A}_{j} \times \mathscr{C})_{\alpha}.$$

Thus the proof of the claim is complete.

Now if $x \in X$ and

$$\{\xi \in \kappa : x \in X_\xi\} = \{\xi_1, \xi_2, \ldots, \xi_l\},$$

then it is easy to see that

$$Z_x = \bigcup_{j=1}^{l} C_{\xi_j}.$$

Thus $Z_x \in \mathscr{I}$ for every $x \in X$.

Since $\pi_2(Z) = X \notin \mathscr{I}$, by (b) there is some $y \in X$ such that $Z^y \in \mathscr{I}$. Let $\eta \in \kappa$ with $y \in X_\eta$. For every $\xi > \eta$ we have $y \in Y_\xi \subseteq C_\xi$, hence $X_\xi \times \{y\} \subseteq X_\xi \times C_\xi$. Therefore

$$\bigcup_{\xi>\eta} X_\xi \times \{y\} \subseteq Z,$$

i.e., $\bigcup_{\xi>\eta} X_\xi \subseteq Z^y$. Since $\bigcup_{\xi\leq\eta} X_\xi \in \mathscr{I}$, this contradicts the fact that $Z^y \in \mathscr{I}$ and completes the proof of the theorem.

We now present two applications of Theorem 4.1 for measure and category spaces; Corollary 4.1 is also obtained in [11].

Corollary 4.1. *Let $(X, \mathscr{A}, \mu)$ be a probability measure space, where $\mathscr{A}$ is countably generated. Then for every point-finite covering X_i $(i \in I)$ of X with $\mu^*(X_i) = 0$, there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\}$ is not μ-measurable.*

Proof. We apply Theorem 4.1, when $\mathscr{I} = \{A \subseteq X : \mu^*(A) = 0\}$ and $\mathscr{A}$ is the σ-algebra of μ-measurable sets. Since condition (b) of Theorem 4.1 is trivially a special case of Fubini's Theorem, it suffices to verify condition (a).

Since $\mathscr{A}$ is countably generated, there is $f: X \to \mathbb{R}$ such that $\mathscr{A} = f^{-1}(\mathscr{B}(\mathbb{R}))$ (cf. [5, Theorem 0.1]). Let $f(\mu)$ be the image measure on $\mathbb{R}$ defined by

$$f(\mu)(B) = \mu(f^{-1}(B))$$

for every $B \in \mathscr{B}(\mathbb{R})$ and let $\mathscr{V}$ be a countable base for the topology of $\mathbb{R}$. Since $f(\mu)$ is regular, $\mathscr{V}_2$ is a base for the ideal of $f(\mu)$-measure zero sets. Thus, if we set $\mathscr{C} = f^{-1}(\mathscr{V})$, then $\mathscr{C}_2$ is a base for $\mathscr{I}$.

Corollary 4.2. *Let X be a separable metric space, which is not meager in itself. Then for every point-finite covering X_i $(i \in I)$ of X with each X_i meager, there is some $J \subseteq I$ such that $\bigcup \{X_i : i \in J\}$ does not have the property of Baire.*

Proof. Let $\mathscr{A} = \mathscr{B}(X)$ and $\mathscr{I}$ be the ideal of meager sets in X, so that $\mathscr{A}_\mathscr{I}$ is the σ-algebra of sets with the property of Baire. Condition (b) of Theorem 4.1 follows from Kuratowski–Ulam Theorem [25, §15], the category analogue of Fubini's Theorem. Moreover, using the fact that every meager set is included in an F_σ meager set and that X is second countable, we conclude that condition (a) holds. Thus the result follows from Theorem 4.1.

Remarks. Neither of the conditions (a) and (b) of Theorem 4.1 can be dropped as the following examples (i) and (ii) show.

(i) Assume that κ is a real-valued measurable cardinal and μ is a probability measure defined on all subsets of κ and vanishing on singletons. Then $(\kappa, \mathscr{P}(\kappa), \mathscr{I})$, where $\mathscr{I} = \{A \subseteq \kappa : \mu(A) = 0\}$, has the weak Fubini property but the conclusion of Theorem 4.1 fails.

(ii) Let X be a non-enumerable subset of $\mathbb{R}$ and $\mathscr{I}$ be the σ-ideal of at most enumerable subsets of X. If $\mathscr{C}$ is an enumerable base for the relative topology on X, then $\mathscr{I} \subseteq \mathscr{C}_2$. Thus condition (a) for $(X, \mathscr{P}(X), \mathscr{I})$ holds, but the conclusion of Theorem 4.1 fails.

We don't know if condition (a) in Theorem 4.1 can be replaced by

(a′) There exists $\mathscr{C} \subseteq \mathscr{A}$ such that $|\mathscr{C}| \leq \aleph_0$ and $\mathscr{C}_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathscr{C}_\alpha$ is a base for $\mathscr{I}$.

Corollary 4.1 (resp. 4.2) can be stated in the more general setting of Theorem 2.2. Thus, instead of saying that each X_i is of measure zero (resp. meager) we may assume that there is no $J \subseteq I$ such that $|J| \leq \aleph_0$ and $X - \bigcup \{X_i : i \in J\}$ is of measure zero (resp. meager). This is because the ideal considered is ω_1-saturated.

Let $(X, \mathscr{A})$ be a measurable space and $\mathscr{E}$ a point-finite covering of X. Let $\mathscr{E}_\mathrm{d}$ denote the family of all finite intersections of elements from $\mathscr{E}$. It is clear that $\mathscr{E}_\mathrm{d}$ is also a point-finite covering of X. Thus we may consider the union problem for each of the coverings $\mathscr{E}$ and $\mathscr{E}_\mathrm{d}$. The next theorem shows that the equivalence of the two problems depends on the rectangle property for the cardinal of $\mathscr{E}$. Recall that a cardinal κ has the rectangle property if every subset of $\kappa \times \kappa$ belongs to the σ-algebra generated by the rectangles $A_1 \times A_2$, where $A_i \subseteq \kappa$ $(i = 1, 2)$; see [2]. It is easy to see that if κ has the rectangle property and $n \in \mathbb{N}$, then every subset of κ^n belongs to the σ-algebra generated by the sets $A_1 \times A_2 \times \cdots \times A_n$, where $A_i \subseteq \kappa$ $(i = 1, 2, \ldots, n)$.

Theorem 4.2. *Let $(X, \mathscr{A})$ be a measurable space and $\mathscr{E}$ a point-finite covering of X such that $|\mathscr{E}|$ has the rectangle property. Then the following are equivalent:*

(a) *There exists $\mathscr{B} \subseteq \mathscr{E}$ such that $\bigcup \mathscr{B} \notin \mathscr{A}$.*

(b) *There exists $\mathscr{B} \subseteq \mathscr{E}_\mathrm{d}$ such that $\bigcup \mathscr{B} \notin \mathscr{A}$.*

Moreover, if $|\mathscr{E}|$ does not have the rectangle property the result fails.

For a family of sets $\mathscr{E}$, let $\mathscr{E}_\mathrm{u}$ be the family of all unions of elements from $\mathscr{E}$ and $\sigma(\mathscr{E})$ be the σ-algebra of sets generated by $\mathscr{E}$. Then we have the following lemma which immediately implies Theorem 4.2.

Lemma 4.1. (a) *Let $\mathscr{E}$ be a point-finite covering of a set X such that $\kappa = |\mathscr{E}|$ has the rectangle property. Then*

$$(*) \qquad \sigma(\mathscr{E}_{\mathrm{du}}) = \sigma(\mathscr{E}_\mathrm{u}).$$

(b) *If κ is a cardinal without the rectangle property, then there is a point-finite covering $\mathscr{E}$ of a set X such that $|X| = |\mathscr{E}| = \kappa$ and $(*)$ fails.*

Proof. (a) Clearly, $\sigma(\mathscr{E}_{\mathrm{du}}) \supseteq \sigma(\mathscr{E}_{\mathrm{u}})$, so it suffices to show that $\mathscr{E}_{\mathrm{du}} \subseteq \sigma(\mathscr{E}_{\mathrm{u}})$. Let

$$\mathscr{E} = \{A_\xi : \xi \in \kappa\}$$

and define

$$\mathscr{S} = \{S \in \mathscr{P}(\kappa \times \kappa) : \bigcup_{(\xi,\eta)\in S} (A_\xi \cap A_\eta) \in \sigma(\mathscr{E}_{\mathrm{u}})\}.$$

We claim that $\mathscr{S}$ has the following properties:

(i) If S_1 and $S_2 \in \mathscr{P}(\kappa)$, then $S_1 \times S_2 \in \mathscr{S}$.

(ii) If $S_i \in \mathscr{S}$ $(i \in \mathbb{N})$, then $\bigcup_i S_i \in \mathscr{S}$.

(iii) If $S_i \in \mathscr{S}$ and $S_i \supseteq S_{i+1}$ $(i \in \mathbb{N})$, then $\bigcap_i S_i \in \mathscr{S}$.

To prove (i)–(iii), it is enough to observe that

$$\bigcup_{(\xi,\eta)\in S_1\times S_2} (A_\xi \cap A_\eta) = \Big(\bigcup_{\xi\in S_1} A_\xi\Big) \cap \Big(\bigcup_{\eta\in S_2} A_\eta\Big),$$

$$\bigcup_{(\xi,\eta)\in \bigcup_i S_i} (A_\xi \cap A_\eta) = \bigcup_i \bigcup_{(\xi,\eta)\in S_i} (A_\xi \cap A_\eta),$$

and

$$\bigcup_{(\xi,\eta)\in \bigcap_i S_i} (A_\xi \cap A_\eta) = \bigcap_i \bigcap_{(\xi,\eta)\in S_i} (A_\xi \cap A_\eta)$$

where S_i are as in (i), (ii) and (iii), respectively. We prove only the inclusion '$\supseteq$' in the last equation; the rest is verified directly without using the point-finiteness. Let

$$x \in \bigcap_i \bigcup_{(\xi,\eta)\in S_i} (A_\xi \cap A_\eta).$$

Choose $(\xi_i, \eta_i) \in S_i$ such that $x \in A_{\xi_i} \cap A_{\eta_i}$. By the point-finiteness of $\mathscr{E}$, the same (ξ_i, η_i) occurs infinitely often and since S_i is decreasing, there is $(\xi, \eta) \in \bigcap_i S_i$ such that $x \in A_\xi \cap A_\eta$. Hence

$$x \in \bigcup_{(\xi,\eta)\in \bigcap_i S_i} (A_\xi \cap A_\eta).$$

Now, since the family of finite unions of rectangles is an algebra, it follows from (i)–(iii) that $\mathscr{S}$ contains the σ-algebra generated by rectangles. But κ has the rectangle property, so $\mathscr{S} = \mathscr{P}(\kappa \times \kappa)$. Thus we have proved that

$$\bigcup_{(\xi_1,\dots,\xi_n)\in S} (A_{\xi_1} \cap A_{\xi_2} \cap \cdots \cap A_{\xi_n}) \in \sigma(\mathscr{E}_{\mathrm{u}})$$

holds for every $S \subseteq \kappa^n$ when $n = 2$. By the comments before Theorem 4.2, this is proved similarly for every finite $n > 2$, while the case $n < 2$ is trivial.

Finally, observe that an arbitrary member of $\mathscr{E}_{\mathrm{du}}$ has the form

$$\bigcup_{n=0}^{\infty} \bigcup_{(\xi_1,\dots,\xi_n)\in S_n} (A_{\xi_1} \cap A_{\xi_2} \cap \cdots \cap A_{\xi_n}),$$

where $S_n \subseteq \kappa^n$. Hence, by the above, it belongs to $\sigma(\mathscr{E}_{\mathrm{u}})$.

(b) Let $X = \kappa \times \kappa$ and

$$\mathscr{E} = \{\{\xi\} \times \kappa : \xi \in \kappa\} \cup \{\kappa \times \{\eta\} : \eta \in \kappa\}$$

Then, $|X| = |\mathscr{E}| = \kappa$ and $\mathscr{E}$ is point-finite. (In fact, every subfamily of with nonempty intersection has cardinal ≤ 2.) Moreover, $\sigma(\mathscr{E}_{\mathrm{du}}) = \mathscr{P}(\kappa \times \kappa)$ and $\sigma(\mathscr{E}_{\mathrm{u}})$ is the σ-algebra generated by the rectangles in $\kappa \times \kappa$. Since κ does not have the rectangle property, $\sigma(\mathscr{E}_{\mathrm{u}}) \neq \sigma(\mathscr{E}_{\mathrm{du}})$.

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