Available online at www.sciencedirect.com
SciVerse ScienceDirect
Journal of Functional Analysis 262 (2012) 3495–3517
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Higher-order nondivergence elliptic and parabolic equations in Sobolev spaces and Orlicz spaces

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Received 10 June 2011; accepted 19 January 2012

Available online 8 February 2012

Communicated by Fang Hua-Lin

Abstract

In this paper we obtain the global regularity estimates of the solutions in Sobolev spaces and Orlicz spaces for higher-order elliptic and parabolic equations of nondivergence form in the whole space. We only need to focus on the parabolic case since the corresponding result in the elliptic case can be obtained as a corollary.

Keywords: Regularity; Sobolev; Orlicz spaces; Higher-order; Nondivergence; Elliptic, parabolic; Small BMO; The whole space

1. Introduction

L^p-type regularity found by Calderón–Zygmund [12] is one of the key a priori estimates in the theory of second-order elliptic and parabolic equations. Many authors [4,13,14,30,35] studied such estimates of strong solutions for second-order elliptic and parabolic equations with different assumptions on the coefficients and domain. Moreover, L^p-type regularity of weak solutions for second-order divergence elliptic and parabolic equations is extensively studied by many researchers [1,2,8,9,11,16–18,31,32]. Recently, some authors [18,27–29] obtained the global L^p-type regularity for second-order elliptic and parabolic equations in the whole space.

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2012.01.016

However, the corresponding regularity results for the higher-order elliptic and parabolic equations are less (for instance, see [19,21,23,24,26,33,34,36–38]). In this paper we consider the global regularity estimates in Sobolev spaces and Orlicz spaces for the higher-order parabolic and elliptic problems in the whole space.

Now we define

$$D_x^\nu u = \frac{\partial^{|\nu|} u}{\partial_{x_1}^{\nu_1} \ldots \partial_{x_n}^{\nu_n}},$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is a multiple index, $\nu_i \geqslant 0$ $(i = 1, 2, \ldots, n)$ and $|\nu| = \sum_{i=1}^n |\nu_i|$. For convenience, we often omit the subscript x in $D_x^\nu u$ and write $D^k u = \{D^\nu u\colon |\nu| = k\}$.

In this paper we are mainly concerned with global regularity estimates for the following higher-order parabolic problems

$$\mathcal{L}u =: u_t - \sum_{|\nu|=0}^{m} a_\nu(x,t) D^\nu u = f(x,t) \quad \text{in } \mathbb{R}_T^n =: \mathbb{R}^n \times (0,T), \tag{1.1}$$

$$u(x,0) = 0, \tag{1.2}$$

where T is a positive fixed time, m is a positive even integer and the coefficients a_ν satisfy

$$(-1)^{\frac{m}{2}-1} \sum_{|\nu|=m} a_\nu(x,t)\xi^\nu \geqslant \Lambda_1 \quad \text{for any } \xi \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \tag{1.3}$$

and

$$\sum_{|\nu|=0}^{m} \left|a_\nu(x,t)\right| \leqslant \Lambda_2 \tag{1.4}$$

for all $(x,t) \in \mathbb{R}_T^n$ and positive constants $\Lambda_1, \Lambda_2 > 0$. In fact, as a corollary of the parabolic case we can obtain the corresponding result in the elliptic case

$$\mathcal{M}u =: \sum_{|\nu|=0}^{m} a_\nu(x) D^\nu u = f(x) \quad \text{in } \mathbb{R}^n, \tag{1.5}$$

where m is a positive even integer and the coefficients $a_\nu(x)$ satisfy

$$(-1)^{\frac{m}{2}-1} \sum_{|\nu|=m} a_\nu(x)\xi^\nu \geqslant \Lambda_3 \quad \text{for any } \xi \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \tag{1.6}$$

and

$$\sum_{|\nu|=0}^{m} \left|a_\nu(x)\right| \leqslant \Lambda_4 \tag{1.7}$$

for all $x \in \mathbb{R}^n$ and positive constants $\Lambda_3, \Lambda_4 > 0$.

Especially when $m=2$, (1.1) and (1.5) are reduced to

$$a_{ij}u_{x_ix_j}+b_iu_{x_i}+c_iu=f \quad \text{and} \quad u_t-a_{ij}u_{x_ix_j}-b_iu_{x_i}-c_iu=f. \tag{1.8}$$

Definition 1.1. For $1<p<\infty$. The Sobolev space $W^{m,p}(\mathbb{R}^n)$ consists of all functions $u\in L^p(\mathbb{R}^n)$ for which the norm

$$\|u\|_{W^{m,p}(\mathbb{R}^n)}=\left\{\sum_{0\leqslant s\leqslant m}\left\|D^su\right\|^p_{L^p(\mathbb{R}^n)}\right\}^{1/p}<+\infty,$$

where

$$\|u\|_{L^p(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}|u|^p\,dx\right)^{1/p}.$$

Moreover, we denote by $H^m=W^{m,2}$, in the interest of simplicity.

Now we denote the distance in $\mathbb{R}^{n+1}$ as

$$\delta(P_1,P_2)=\max\left\{|x_1-x_2|,|t_1-t_2|^{1/m}\right\} \quad \text{for } P_1(x_1,t_1),P_2(x_2,t_2)\in\mathbb{R}^{n+1},$$

and the cylinders in $\mathbb{R}^{n+1}$ as

$$Q_R=B_R\times\left(-R^m,R^m\right] \quad \text{and} \quad Q_R(x,t)=Q_R+(x,t),\ (x,t)\in\mathbb{R}^{n+1},$$

where

$$B_R=\left\{x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n\colon |x|=\sqrt{\sum_{i=1}^n x_i^2}<R\right\} \quad \text{is an open ball in } \mathbb{R}^n.$$

Moreover, we define

$$W_p^{1,m}\left(\mathbb{R}^n_T\right)=\left\{u\colon u_t,D_x^i\in L^p\left(\mathbb{R}^n_T\right) \quad \text{for } 0\leqslant i\leqslant m\right\}.$$

Throughout this paper we assume that the coefficients of a_ν are in BMO space and their semi-norms are small enough. More precisely, we introduce the following definitions.

Definition 1.2 *(Small BMO condition).* (i) Elliptic case. We say that the coefficients $a_\nu(x)$ are (δ,R)-vanishing if for given $\delta>0$, there exists $R>0$ such that

$$\sup_{0<r\leqslant R}\sup_{x\in\mathbb{R}^n}\fint_{B_r(x)}\left|a_\nu(y)-\overline{a_\nu}_{B_r(x)}\right|dy\leqslant\delta, \tag{1.9}$$

where

$$\overline{a_\nu}_{B_r(x)} = \fint_{B_r(x)} a_\nu(y)\,dy.$$

(ii) Parabolic case. We say that the coefficients $a_\nu(x,t)$ are (δ, R)-vanishing if

$$\sup_{0<r\leqslant R}\ \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} \fint_{Q_r(x,t)} \left|a_\nu(y,s) - \overline{a_\nu}_{Q_r(x,t)}\right| dy\,ds \leqslant \delta,$$

where

$$\overline{a_\nu}_{Q_r(x,t)} = \fint_{Q_r(x,t)} a_\nu(y,s)\,dy\,ds.$$

Recently integrability of solutions for elliptic/parabolic problems with discontinuous coefficients of small BMO type have been extensively studied by many authors (see [5–10]). We would like to point out that if a function satisfies the VMO condition, then it satisfies the small BMO condition which we treat in this paper.

Suppose that $\{a_\nu^0\colon\ |\nu| = m\}$ is a collection of constants, where m is a positive even integer. Then we consider

$$\mathcal{L}_0 u =: u_t - \sum_{|\nu|=m} a_\nu^0 D^\nu u = f(x,t) \quad \text{in } \mathbb{R}^n_T, \tag{1.10}$$

$$u(x,0) = 0, \tag{1.11}$$

where the coefficients a_ν satisfies (1.3)–(1.4), that is to say,

$$(-1)^{\frac{m}{2}-1} \sum_{|\nu|=m} a_\nu^0 \xi^\nu \geqslant \Lambda_1 \quad \text{and} \quad \sum_{|\nu|=m} \left|a_\nu^0\right| \leqslant \Lambda_2 \tag{1.12}$$

for any $\xi \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and positive constants $\Lambda_1, \Lambda_2 > 0$. Then, by Fourier transform, we can obtain the following explicit solution of (1.10)–(1.11)

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y, t-s) f(y,s)\,dy\,ds, \tag{1.13}$$

where

$$\Gamma(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\left\{ i\xi\cdot x + (-1)^{\frac{m}{2}} t \sum_{|\nu|=m} a_\nu^0 \xi^\nu \right\} d\xi$$

is the fundamental solution of $\mathcal{L}_0 \Gamma = 0$. If u satisfies (1.10)–(1.11) with (1.12) and $f \in L^p(\mathbb{R}^n_T)$ for $p > 1$, from the elementary Calderón–Zygmund decomposition for parabolic equations and

Marcinkiewicz interpolation theorem (see [21,26,24,38]) we have

$$\left\| D^m u \right\|_{L^p(\mathbb{R}^n_T)} \leqslant C \|\mathcal{L}_0 u\|_{L^p(\mathbb{R}^n_T)} = C\|f\|_{L^p(\mathbb{R}^n_T)}. \tag{1.14}$$

Since the 1960s, the need to use wider spaces of functions than Sobolev spaces came from various practical problems. Orlicz spaces (see [3,5,7,6,10,15,25,40,41], etc.) have been considered as one of the most natural generalizations of Sobolev spaces. Here for the reader's convenience, we will give some definitions on the general Orlicz spaces.

Definition 1.3. A convex function $\phi : \mathbb{R} \to \mathbb{R}^+$ is said to be a Young function if

$$\phi(-t) = \phi(t), \qquad \phi(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = +\infty.$$

Definition 1.4. A Young function ϕ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if there exists a positive constant K such that for every $t > 0$,

$$\phi(2t) \leqslant K\phi(t). \tag{1.15}$$

Moreover, a Young function ϕ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if there exists a number $a > 1$ such that for every $t > 0$,

$$\phi(t) \leqslant \frac{\phi(at)}{2a}. \tag{1.16}$$

Examples 1.5.

(1) $\phi_1(t) = (1 + |t|)\log(1 + |t|) - |t| \in \Delta_2$, but $\phi_1(t) \notin \nabla_2$.
(2) $\phi_2(t) = e^{|t|} - |t| - 1 \in \nabla_2$, but $\phi_2(t) \notin \Delta_2$.
(3) $\phi_3(t) = |t|^p$, $\phi_4(t) = |t|^p \log(1 + |t|) \in \Delta_2 \cap \nabla_2$, $p > 1$.

Remark 1.6.

(1) In fact, if a function ϕ satisfies (1.15) and (1.16), then

$$\phi(\theta_1 t) \leqslant K\theta_1^{p_1}\phi(t) \quad \text{and} \quad \phi(\theta_2 t) \leqslant 2a\theta_2^{p_2}\phi(t), \tag{1.17}$$

for every $t > 0$ and $0 < \theta_2 \leqslant 1 \leqslant \theta_1 < \infty$, where $p_1 = \log_2 K$ and $p_2 = \log_a 2 + 1$.
(2) Under condition (1.17), it is easy to check that ϕ satisfies

$$\phi(0) = 0, \qquad \lim_{t\to\infty} \phi(t) = +\infty \quad \text{and} \quad \lim_{t\to 0+} \frac{\phi(t)}{t} = \lim_{t\to+\infty} \frac{t}{\phi(t)} = 0.$$

Definition 1.7. Let ϕ be a Young function. Then the Orlicz class $K^\phi(\mathbb{R}^n)$ is the set of all measurable functions $g : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^n} \phi\big(|g|\big)\, dx < \infty.$$

The Orlicz space $L^{\phi}(\mathbb{R}^n)$ is the linear hull of $K^{\phi}(\mathbb{R}^n)$.

Moreover, we define

$$W^{m,\phi}(\mathbb{R}^n)=\{u\colon\ D_x^i\in L^{\phi}(\mathbb{R}^n)\ \text{for}\ 0\leqslant i\leqslant m\}$$

and

$$W_{\phi}^{1,m}(\mathbb{R}_T^n)=\{u\colon\ u_t,D_x^i\in L^{\phi}(\mathbb{R}_T^n)\ \text{for}\ 0\leqslant i\leqslant m\}.$$

Lemma 1.8. *(See [3,10,25,40].) Let ϕ be a Young function satisfying $\phi\in\Delta_2\cap\nabla_2$. Then*

(1) $K^{\phi}(\mathbb{R}^n)=L^{\phi}(\mathbb{R}^n)$. *Especially when $\phi(s)=|s|^p$ for $p>1$, $L^{\phi}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$.*

(2)
$$\int_{\mathbb{R}^n}\phi(|g|)\,dx=\int_0^\infty\bigl|\{x\in\mathbb{R}^n\colon\ |g|>\mu\}\bigr|\,d[\phi(\mu)].$$

(3) *If $q\in(1,p_2)$, where p_2 is defined in* (1.17)*, then for any $b_1,b_2>0$ we have*

$$\int_0^\infty\frac{1}{\theta^q}\Biggl\{\int_{\{x\in\mathbb{R}^n\colon\ |g|>b_1\theta\}}|g|^q\,dx\Biggr\}\,d[\phi(b_2\theta)]\leqslant C(b_1,b_2,\phi)\int_{\mathbb{R}^n}\phi(|g|)\,dx.$$

When $a_\nu\in L^\infty\cap VMO$, Palagachev and Softova [36,37] have studied local L^p-type regularity for the following higher-order elliptic and parabolic equations

$$\sum_{|\nu|=m}a_\nu(x)D^\nu u=f(x)\quad\text{and}\quad u_t-\sum_{|\nu|=m}a_\nu(x,t)D^\nu u=f(x,t).$$

Moreover, Haller-Dintelmann, Heck and Hieber [23] extended the corresponding local result in [36] to the global case for the higher-order parabolic equation (1.1) when the leading coefficients a_ν for $|\nu|=m$ belong to the class $L^\infty\cap VMO$. Furthermore, Dong and Kim [19] have proved the global L^p-type regularity of (1.1) and (1.5) with small BMO leading coefficients in the whole space. Recently, Byun and Ryu [7] obtained the following results in Orlicz spaces

$$\int_{\Omega}\phi(|D_x^m u|^2)\,dx\leqslant C\Bigl(\int_{\Omega}\phi(|\mathbf{f}|^2)\,dx+1\Bigr),\quad\phi\in\Delta_2\cap\nabla_2$$

for the weak solutions of the higher-order elliptic problems in divergence form

$$D^\alpha\bigl(A_{\alpha\beta}(x)D^\beta u\bigr)=D^\alpha f_\alpha\quad x\in\Omega,\ |\alpha|=|\beta|=m.$$

Here $\mathbf{f}=\{f_\alpha\colon\ |\alpha|=m\}$. Moreover, Byun and Ryu [6] also proved the following local regularity estimates in Orlicz spaces

$$|\mathbf{f}|^2\in L_{loc}^{\phi}\quad\Longrightarrow\quad|D^m u|^2\in L_{loc}^{\phi}$$

for the weak solutions of the higher-order parabolic problems in divergence form

$$u_t + (-1)^m D^\alpha \big(A_{\alpha\beta}(x,t) D^\beta u\big) = D^\alpha f_\alpha, \quad (x,t) \in \Omega \times (0,T),\ |\alpha| = |\beta| = m. \tag{1.18}$$

In this paper we consider the global regularity theory in Orlicz spaces, and give the corresponding homogeneous estimates.

Theorem 1.9.

(1) *Assume that $\phi \in \Delta_2 \cap \nabla_2$ and the coefficients $a_\nu(x,t)$ satisfy* (1.3)–(1.4) *and small BMO condition for $|\nu| = m$. If $u \in W_\phi^{1,m}(\mathbb{R}_T^n)$ satisfies* (1.1)–(1.2) *and $f \in L^\phi(\mathbb{R}_T^n)$, then there exists a positive constant δ, depending on n, ϕ, T, Λ_1, Λ_2, such that*

$$\sum_{i=0}^{m} \int_{\mathbb{R}_T^n} \phi\big(\big|D_x^i u\big|\big)\, dx\, dt \leqslant C \int_{\mathbb{R}_T^n} \phi\big(|f|\big)\, dx\, dt, \tag{1.19}$$

where the constant C is independent of u and f.

(2) *Assume that $\phi \in \Delta_2 \cap \nabla_2$ and the coefficients $a_\nu(x)$ satisfy* (1.6)–(1.7) *and small BMO condition for $|\nu| = m$. If $f \in L^\phi(\mathbb{R}^n)$ and $u \in W^{m,\phi}(\mathbb{R}^n)$ satisfies*

$$\mathcal{M}u - \lambda u = f \quad \text{in } \mathbb{R}^n,$$

then there exist positive constants δ, λ_0, depending on n, Λ_4, such that, for any $\lambda \geqslant \lambda_0$ we have

$$\sum_{i=0}^{m} \int_{\mathbb{R}^n} \phi\big(\big|D_x^i u\big|\big)\, dx \leqslant C \int_{\mathbb{R}^n} \phi\big(|f|\big)\, dx, \tag{1.20}$$

where the constant C is independent of u and f.

Remark 1.10.

(1) Especially when $\phi(t) = |t|^p$ for $p > 1$, then (1.19)–(1.20) are reduced to L^p-type estimates.
(2) We remark that the global $\Delta_2 \cap \nabla_2$ condition in the theorem above is optimal (see [40]).

2. Regularity in Orlicz spaces

In this section we shall finish the proof of the main result: Theorem 1.9.

2.1. Auxiliary result

In this subsection we first prove the following auxiliary result, which will be a crucial ingredient in the proof of Theorem 1.9.

Theorem 2.1. *Assume that $u \in C_0^\infty(Q_R)$ is the solution of*

$$\mathcal{L}_1 u = u_t - \sum_{|\nu|=m} a_\nu D^\nu u = f \quad \text{in } \mathbb{R}^n_T, \tag{2.1}$$

$$u(x,0) = 0 \quad \text{in } \mathbb{R}^n, \tag{2.2}$$

where the coefficients $a_\nu(x,t)$ satisfy (1.3)–(1.4) *and small BMO condition. Then there exists a positive constant δ, depending only on n, ϕ, Λ_1, Λ_2, such that*

$$\int_{\mathbb{R}^n_T} \phi(|u_t|) + \phi(|D^m u|)\,dx\,dt \leqslant C \int_{\mathbb{R}^n_T} \phi(|f|)\,dx\,dt, \tag{2.3}$$

where the constant C is independent of u and f.

Let $u \in C_0^\infty(\mathbb{R}^n_T)$ be the solution of (1.10)–(1.11), i.e.,

$$\mathcal{L}_0 u = u_t - \sum_{|\nu|=m} a^0_\nu D^\nu u = f \quad \text{in } \mathbb{R}^n_T,$$

$$u(x,0) = 0 \quad \text{in } \mathbb{R}^n,$$

with the constant coefficients a^0_ν satisfying (1.12). From (1.14) we have

$$\|D^m u\|_{L^p(\mathbb{R}^n_T)} \leqslant C\|f\|_{L^p(\mathbb{R}^n_T)} = C\|\mathcal{L}_0 u\|_{L^p(\mathbb{R}^n_T)} \quad \text{for any } p > 1.$$

Moreover, let $w \in W^{2m,1}_p(\mathbb{R}^n_T)$ be the solution of

$$\mathcal{L}_0 w = \overline{\mathcal{L}_0 u} = \begin{cases} \mathcal{L}_0 u & \text{in } Q_{\kappa r} \cap \mathbb{R}^n_T,\ \kappa, r > 0, \\ 0 & \text{in } \mathbb{R}^n_T \setminus Q_{\kappa r}, \end{cases} \tag{2.4}$$

$$w(x,0) = 0. \tag{2.5}$$

Furthermore, if we define $h = u - w \in W^{2m,1}_p(\mathbb{R}^n_T)$, then we find that h satisfies

$$\mathcal{L}_0 h = h_t - \sum_{|\nu|=m} a^0_\nu D^\nu h = 0 \quad \text{in } Q_{\kappa r}. \tag{2.6}$$

In fact, we can suppose that w and u is vanishing in $\{t < 0\}$ since we only consider the space $\mathbb{R}^n_T$. Then from (1.14) we have

$$\int_{Q_{\kappa r}} |w_t|^p + |D^m w|^p\,dx\,dt \leqslant \int_{\mathbb{R}^n_T} |w_t|^p + |D^m w|^p\,dx\,dt$$
$$\leqslant C \int_{\mathbb{R}^n_T} |\overline{\mathcal{L}_0 u}|^p\,dx\,dt = C \int_{Q_{\kappa r}} |\mathcal{L}_0 u|^p\,dx\,dt. \tag{2.7}$$

Lemma 2.2. *For any $\kappa \geqslant 4$ and $r > 0$ we have*

$$\fint_{Q_r} \left|D_x^m h - \left(D_x^m h\right)_{Q_r}\right|^p dx\,dt \leqslant C\kappa^{-p}\left(\fint_{Q_{\kappa r}} |\mathcal{L}_0 u|^p\,dx\,dt + \fint_{Q_{\kappa r}} \left|D_x^m u\right|^p dx\,dt\right). \tag{2.8}$$

Proof. Using Poincaré's inequality, for any $\kappa \geqslant 4$ and $r > 0$ we find that

$$\begin{aligned}\fint_{Q_r} \left|D_x^m h - \left(D_x^m h\right)_{Q_r}\right|^p dx\,dt &\leqslant \sup_{Q_r}\left\{r^p\left|D_x^{m+1} h\right|^p + r^{pm}\left|D_t D_x^m h\right|^p\right\}\\ &\leqslant \sup_{Q_{\kappa r/4}} \kappa^{-p}\left\{\kappa^p r^p\left|D_x^{m+1} h\right|^p + \kappa^{pm} r^{pm}\left|D_t D_x^m h\right|^p\right\}.\end{aligned}$$

Moreover, from the elementary local estimates on derivatives whose proof is similar to that of Theorem 9 in Chapter 2 of [20] by using (1.13), and the fact that $D_x^m h$ still satisfies (2.6), we have

$$\fint_{Q_r} \left|D_x^m h - \left(D_x^m h\right)_{Q_r}\right|^p dx\,dt \leqslant C\kappa^{-p} \fint_{Q_{\kappa r}} \left|D_x^m h\right|^p dx\,dt,$$

which implies that

$$\fint_{Q_r} \left|D_x^m h - \left(D_x^m h\right)_{Q_r}\right|^p dx\,dt \leqslant C\kappa^{-p}\left\{\fint_{Q_{\kappa r}} \left|D_x^m u\right|^p dx\,dt + \fint_{Q_{\kappa r}} \left|D_x^m w\right|^p dx\,dt\right\}$$

since $h = u - w$. Furthermore, (2.7) implies that

$$\fint_{Q_r} \left|D_x^m h - \left(D_x^m h\right)_{Q_r}\right|^p dx\,dt \leqslant C\kappa^{-p}\left\{\fint_{Q_{\kappa r}} \left|D_x^m u\right|^p dx\,dt + \fint_{Q_{\kappa r}} |\mathcal{L}_0 u|^p\,dx\,dt\right\}.$$

This completes the proof. □

Lemma 2.3. *There exists a positive constant C, depending only on n, Λ_1, Λ_2, such that, for any $\kappa \geqslant 4$ and $r > 0$ we have*

$$\fint_{Q_r} \left|D_x^m u - \left(D_x^m u\right)_{Q_r}\right|^p dx\,dt \leqslant C\left\{\kappa^{m+n}\left(|\mathcal{L}_0 u|^p\right)_{Q_{\kappa r}} + \kappa^{-p}\left(\left|D_x^m u\right|^p\right)_{Q_{\kappa r}}\right\}. \tag{2.9}$$

Proof. Using Hölder's inequality, we find that

$$\fint_{Q_r} \left|D_x^m u - \left(D_x^m u\right)_{Q_r}\right|^p dx\,dt \leqslant C\left\{\fint_{Q_r} \left|D_x^m (u-h)\right|^p dx\,dt + \fint_{Q_r} \left|D_x^m h - \left(D_x^m h\right)_{Q_r}\right|^p dx\,dt\right.$$

$$
\begin{aligned}
&+ \fint_{Q_r} \left|\left(D_x^m h\right)_{Q_r} - \left(D_x^m u\right)_{Q_r}\right|^p dx\,dt \Bigg\} \\
&\leqslant C\Bigg\{ \fint_{Q_r} \left|D_x^m w\right|^p dx\,dt + \fint_{Q_r} \left|D_x^m h - \left(D_x^m h\right)_{Q_r}\right|^p dx\,dt \Bigg\} \\
&=: I_1 + I_2. \qquad (2.10)
\end{aligned}
$$

Estimate of I_1. (2.7) implies that

$$
I_1 \leqslant \frac{C}{|Q_r|} \int_{Q_{\kappa r}} |\mathcal{L}_0 u|^p \,dx\,dt \leqslant C\kappa^{m+n} \fint_{Q_{\kappa r}} |\mathcal{L}_0 u|^p \,dx\,dt.
$$

Estimate of I_2. From Lemma 2.2 we deduce that

$$
I_2 \leqslant C\kappa^{-p}\left(\fint_{Q_{\kappa r}} |\mathcal{L}_0 u|^p \,dx\,dt + \fint_{Q_{\kappa r}} \left|D_x^m u\right|^p dx\,dt \right).
$$

Thus, combining the estimates of I_1 and I_2, we have

$$
\fint_{Q_r} \left|D_x^m u - \left(D_x^m u\right)_{Q_r}\right|^p dx\,dt \leqslant C\left(\kappa^{m+n} \fint_{Q_{\kappa r}} |\mathcal{L}_0 u|^p \,dx\,dt + \kappa^{-p} \fint_{Q_{\kappa r}} \left|D_x^m u\right|^p dx\,dt \right), \qquad (2.11)
$$

which completes the proof. □

We shall use the Hardy–Littlewood maximal function $\mathcal{M}g$ and sharp maximal function $g^{\#}$ given by

$$
\mathcal{M}g(x,t) = \sup_{r>0} \fint_{Q_r(x,t)} \left|g(y,s)\right| dy\,ds
$$

and

$$
g^{\#}(x,t) = \sup_{r>0} \fint_{Q_r(x,t)} \left|g(y,s) - \bar{g}_{Q_r(x,t)}\right| dy\,ds,
$$

where

$$
\bar{g}_{Q_r(x,t)} = \fint_{Q_r(x,t)} g(y,s)\,dy\,ds.
$$

Now we recall the following Fefferman–Stein theorem.

Lemma 2.4. *(See [22].) Suppose $f \in L^p(\mathbb{R}^n)$ and $f^\# \in L^p(\mathbb{R}^n)$ for $p > 1$. Then*

$$\|f\|_{L^p(\mathbb{R}^n)} \leqslant \|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f^\#\|_{L^p(\mathbb{R}^n)}.$$

Lemma 2.5. *(See [39].) If $f \in L^p(\mathbb{R}^n)$ for $p > 1$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$ with the estimate*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}.$$

From Lemma 2.3, we obtain the following result.

Lemma 2.6. *Assume that $u \in C_0^\infty(Q_R)$ is the solution of (2.1)–(2.2), where the coefficients $a_\nu(x,t)$ satisfy (1.3)–(1.4) and small BMO condition. Then there exists a positive constant C, depending only on n, Λ_1, Λ_2, such that*

$$\left(D_x^m u\right)^\# \leqslant C\delta^{\frac{\mu\sigma}{p(1+\sigma)}}\left[\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)\right]^{\frac{1}{p(1+\sigma)}} + C\left[\mathcal{M}\left(|\mathcal{L}_1 u|^p\right)\right]^{\frac{\mu}{p}}\left[\mathcal{M}\left(\left|D_x^m u\right|^p\right)\right]^{\frac{1-\mu}{p}}, \quad (2.12)$$

for some small constant $\sigma > 0$ and $\mu = \frac{p}{n+m+p}$.

Proof. 1. Firstly, we shall prove the following inequality

$$\begin{aligned}
&\fint_{Q_r(x_0,t_0)} \left|D_x^m u - \left(D_x^m u\right)_{Q_r(x_0,t_0)}\right|^p dx\,dt \\
&\quad\leqslant C\left\{\kappa^{m+n}\left[\delta^{\frac{\sigma}{1+\sigma}}\left[\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)(x_0,t_0)\right]^{\frac{1}{1+\sigma}} + \mathcal{M}\left(|f|^p\right)(x_0,t_0)\right]\right\} \\
&\qquad + C\kappa^{-p}\mathcal{M}\left(\left|D_x^m u\right|^p\right)(x_0,t_0) \qquad (2.13)
\end{aligned}$$

for any $(x_0,t_0) \in \mathbb{R}^n_T$, $r > 0$ and $\kappa > 0$. When $\kappa \in (0,4)$, the proof of (2.13) is trivial since

$$\begin{aligned}
&\fint_{Q_r(x_0,t_0)} \left|D_x^m u - \left(D_x^m u\right)_{Q_r(x_0,t_0)}\right|^p dx\,dt \\
&\quad\leqslant \fint_{Q_r(x_0,t_0)} \left|D_x^m u\right|^p dx\,dt \leqslant 4^p\kappa^{-p}\mathcal{M}\left(\left|D_x^m u\right|^p\right)(x_0,t_0).
\end{aligned}$$

So we only need to consider the case that $\kappa \geqslant 4$. Fix $(x_0,t_0) \in \mathbb{R}^n_T$, $\kappa \geqslant 4$ and $r > 0$. We define

$$\mathcal{L}_0' u = u_t - \sum_{|\nu|=m} \overline{a_\nu} D^\nu u \quad \text{in } \mathbb{R}^n_T,$$

where

$$\overline{a_\nu} = \begin{cases} \overline{a_\nu}_{Q_{\kappa r}(x_0,t_0)} & \text{if } \kappa r \leqslant R, \\ \overline{a_\nu}_{Q_R} & \text{if } \kappa r > R. \end{cases}$$

From the fact that $u \in C_0^\infty(Q_R)$ and totally similar to the proof of Lemma 2.3, we obtain

$$\begin{aligned}
&\fint_{Q_r(x_0,t_0)} \big|D_x^m u - \big(D_x^m u\big)_{Q_r(x_0,t_0)}\big|^p\,dx\,dt \\
&\quad \leqslant C\big\{\kappa^{m+n}\big(\big|\mathcal{L}_0' u\big|^p\big)_{Q_{\kappa r}(x_0,t_0)} + \kappa^{-p}\big(\big|D_x^m u\big|^p\big)_{Q_{\kappa r}(x_0,t_0)}\big\}. \qquad (2.14)
\end{aligned}$$

Since $u \in C_0^\infty(Q_R)$ and

$$\mathcal{L}_0' u = \mathcal{L}_0' u - \mathcal{L}_1 u + f = \sum_{|\nu|=m} (a_\nu - \overline{a_\nu}) D^\nu u + f,$$

we have

$$\begin{aligned}
\fint_{Q_{\kappa r}(x_0,t_0)} \big|\mathcal{L}_0' u\big|^p\,dx\,dt &\leqslant C \fint_{Q_{\kappa r}(x_0,t_0)} \bigg|\sum_{|\nu|=m} (a_\nu - \overline{a_\nu}) D^\nu u\bigg|^p\,dx\,dt + C \fint_{Q_{\kappa r}(x_0,t_0)} |f|^p\,dx\,dt \\
&=: J_1 + J_2.
\end{aligned}$$

Estimate of J_1. We divide into two cases.

Case 1: $\kappa r \leqslant R$. Using Hölder's inequality, Definition 1.2 and (1.4), we have

$$\begin{aligned}
J_1 &\leqslant C \sum_{|\nu|=m} \bigg(\fint_{Q_{\kappa r}(x_0,t_0)} \big|(a_\nu - \overline{a_\nu}_{Q_{\kappa r}(x_0,t_0)})\big|^{\frac{p(1+\sigma)}{\sigma}}\,dx\,dt\bigg)^{\frac{\sigma}{1+\sigma}} \\
&\quad \times \bigg(\fint_{Q_{\kappa r}(x_0,t_0)} \big|D_x^m u\big|^{p(1+\sigma)}\,dx\,dt\bigg)^{\frac{1}{1+\sigma}} \\
&\leqslant C \sum_{|\nu|=m} \Lambda_2^{\frac{p+(p-1)\sigma}{\sigma+1}} \bigg(\fint_{Q_{\kappa r}(x_0,t_0)} \big|(a_\nu - \overline{a_\nu}_{Q_{\kappa r}(x_0,t_0)})\big|\,dx\,dt\bigg)^{\frac{\sigma}{1+\sigma}} \\
&\quad \times \bigg(\fint_{Q_{\kappa r}(x_0,t_0)} \big|D_x^m u\big|^{p(1+\sigma)}\,dx\,dt\bigg)^{\frac{1}{1+\sigma}} \\
&\leqslant C\delta^{\frac{\sigma}{1+\sigma}} \bigg(\fint_{Q_{\kappa r}(x_0,t_0)} \big|D_x^m u\big|^{p(1+\sigma)}\,dx\,dt\bigg)^{\frac{1}{1+\sigma}} \leqslant C\delta^{\frac{\sigma}{1+\sigma}} \big(\big|D_x^m u\big|^{p(1+\sigma)}\big)_{Q_{\kappa r}(x_0,t_0)}^{\frac{1}{1+\sigma}},
\end{aligned}$$

where C depends on Λ_2, n, σ.

Case 2: $\kappa r > R$. From the definition of $\overline{a_\nu}$ and the fact that $u \in C_0^\infty(Q_R)$, we observe that

$$J_1 \leqslant \frac{C}{|Q_{\kappa r}|} \int_{Q_{\kappa r}(x_0,t_0) \cap Q_R} \bigg|\sum_{|\nu|=m} (a_\nu - \overline{a_\nu}_{Q_R}) D^\nu u\bigg|^p\,dx\,dt.$$

Then, using Hölder's inequality, Definition 1.2 and (1.4), we have

$$
\begin{aligned}
J_1 &\leqslant C\sum_{|\nu|=m}\left(\frac{1}{|Q_{\kappa r}|}\int_{Q_{\kappa r}(x_0,t_0)\cap Q_R}\left|(a_\nu-\overline{a_\nu}_{Q_R})\right|^{\frac{p(1+\sigma)}{\sigma}}dx\,dt\right)^{\frac{\sigma}{1+\sigma}}\\
&\quad\times\left(\fint_{Q_{\kappa r}(x_0,t_0)}\left|D_x^m u\right|^{p(1+\sigma)}dx\,dt\right)^{\frac{1}{1+\sigma}}\\
&\leqslant C\sum_{|\nu|=m}\Lambda_2^{\frac{(p-1)\sigma+p}{\sigma+1}}\left\{\left(\fint_{Q_R}\left|(a_\nu-\overline{a_\nu}_{Q_R})\right|dx\,dt\right)^{\frac{\sigma}{1+\sigma}}\right.\\
&\quad\left.\times\left(\fint_{Q_{\kappa r}(x_0,t_0)}\left|D_x^m u\right|^{p(1+\sigma)}dx\,dt\right)^{\frac{1}{1+\sigma}}\right\}\\
&\leqslant C\delta^{\frac{\sigma}{1+\sigma}}\left(\fint_{Q_{\kappa r}(x_0,t_0)}\left|D_x^m u\right|^{p(1+\sigma)}dx\,dt\right)^{\frac{1}{1+\sigma}}\leqslant C\delta^{\frac{\sigma}{1+\sigma}}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)_{Q_{\kappa r}(x_0,t_0)}^{\frac{1}{1+\sigma}}.
\end{aligned}
$$

Combining the estimates of Case 1 and Case 2, we conclude that

$$
\fint_{Q_{\kappa r}(x_0,t_0)}\left|\mathcal{L}_0' u\right|^p dx\,dt\leqslant C\delta^{\frac{\sigma}{1+\sigma}}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)_{Q_{\kappa r}(x_0,t_0)}^{\frac{1}{1+\sigma}}+C\fint_{Q_{\kappa r}(x_0,t_0)}|f|^p\,dx\,dt. \tag{2.15}
$$

Thus, (2.14) and (2.15) imply that

$$
\begin{aligned}
&\fint_{Q_r(x_0,t_0)}\left|D_x^m u-\left(D_x^m u\right)_{Q_r(x_0,t_0)}\right|^p dx\,dt\\
&\quad\leqslant C\left\{\kappa^{m+n}\left[\delta^{\frac{\sigma}{1+\sigma}}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)_{Q_{\kappa r}(x_0,t_0)}^{\frac{1}{1+\sigma}}+\fint_{Q_{\kappa r}(x_0,t_0)}|f|^p\,dx\,dt\right]\right.\\
&\qquad\left.+\kappa^{-p}\left(\left|D_x^m u\right|^p\right)_{Q_{\kappa r}(x_0,t_0)}\right\}\\
&\quad\leqslant C\left\{\kappa^{m+n}\left[\delta^{\frac{\sigma}{1+\sigma}}\left[\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)(x_0,t_0)\right]^{\frac{1}{1+\sigma}}+\mathcal{M}\left(|f|^p\right)(x_0,t_0)\right]\right.\\
&\qquad\left.+\kappa^{-p}\mathcal{M}\left(\left|D_x^m u\right|^p\right)(x_0,t_0)\right\}
\end{aligned}
$$

for any $\kappa>0$, $r>0$ and some small constant σ, where C depends on Λ_2, n.

2. From the result of 1 and by taking the supremum with respect to $r>0$, we can easily obtain

$$
\begin{aligned}
\left[\left(D_x^m u\right)^{\#}(x_0,t_0)\right]^p&\leqslant C\left\{\kappa^{m+n}\left[\delta^{\frac{\sigma}{1+\sigma}}\left[\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)(x_0,t_0)\right]^{\frac{1}{1+\sigma}}+\mathcal{M}\left(|f|^p\right)(x_0,t_0)\right]\right.\\
&\quad\left.+\kappa^{-p}\mathcal{M}\left(\left|D_x^m u\right|^p\right)(x_0,t_0)\right\},
\end{aligned}
$$

which implies that

$$\left[\left(D_x^m u\right)^{\#}\right]^p \leqslant C\left\{\kappa^{m+n}\left[\delta^{\frac{\sigma}{1+\sigma}}\left(\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)\right)^{\frac{1}{1+\sigma}}+\mathcal{M}\left(|f|^p\right)\right]+\kappa^{-p}\mathcal{M}\left(\left|D_x^m u\right|^p\right)\right\}.$$

By minimizing with respect to κ we can obtain

$$\begin{aligned}\left[\left(D_x^m u\right)^{\#}\right]^p &\leqslant C\left[\delta^{\frac{\sigma}{1+\sigma}}\left(\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)\right)^{\frac{1}{1+\sigma}}+\mathcal{M}\left(|f|^p\right)\right]^{\mu}\left[\mathcal{M}\left(\left|D_x^m u\right|^p\right)\right]^{1-\mu}\\ &\leqslant C\delta^{\frac{\mu\sigma}{1+\sigma}}\left(\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)\right)^{\frac{\mu}{1+\sigma}}\left[\mathcal{M}\left(\left|D_x^m u\right|^p\right)\right]^{1-\mu}\\ &\quad+C\left[\mathcal{M}\left(|f|^p\right)\right]^{\mu}\left[\mathcal{M}\left(\left|D_x^m u\right|^p\right)\right]^{1-\mu},\end{aligned}$$

where $\mu=\frac{p}{m+n+p}$, which completes the proof of (2.12) since

$$\left[\mathcal{M}\left(\left|D_x^m u\right|^p\right)\right]^{\frac{1}{p}} \leqslant \left[\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)\right]^{\frac{1}{p(1+\sigma)}}. \qquad \square$$

Let

$$q=(1+p_2)/2>1, \tag{2.16}$$

where p_2 is defined in (1.17). In fact, in the subsequent proof we can choose any constant q with $1<q<p_2$. Moreover, we can select the fixed constants p and σ satisfying

$$1<p=(1+q)/2=(3+p_2)/4<q \quad \text{and} \quad p(1+\sigma)<q. \tag{2.17}$$

Lemma 2.7. *Assume that $u \in C_0^\infty(Q_R)$ is the solution of* (2.1)–(2.2)*, where the coefficients $a_\nu(x,t)$ satisfy* (1.3)–(1.4) *and small BMO condition. Then there exists a positive constant C, depending only on n, Λ_1, Λ_2, such that*

$$\|u_t\|_{L^q(\mathbb{R}_T^n)}+\left\|D_x^m u\right\|_{L^q(\mathbb{R}_T^n)} \leqslant C\|\mathcal{L}_1 u\|_{L^q(\mathbb{R}_T^n)}.$$

Proof. Using Lemma 2.4, Lemma 2.6 and Hölder's inequality, we arrive at

$$\begin{aligned}\left\|D_x^m u\right\|_{L^q(\mathbb{R}_T^n)} &\leqslant C\left\|\left(D_x^m u\right)^{\#}\right\|_{L^q(\mathbb{R}_T^n)}\\ &\leqslant C\delta^{\frac{\mu\sigma}{p(1+\sigma)}}\left\|\left[\mathcal{M}\left(\left|D_x^m u\right|^{p(1+\sigma)}\right)\right]^{\frac{1}{p(1+\sigma)}}\right\|_{L^q(\mathbb{R}_T^n)}\\ &\quad+C\left\|\left[\mathcal{M}\left(\left|\mathcal{L}_1 u\right|^p\right)\right]^{\frac{1}{p}}\right\|_{L^q(\mathbb{R}_T^n)}^{\mu}\left\|\left[\mathcal{M}\left(|D_x^m u|^p\right)\right]^{\frac{1}{p}}\right\|_{L^q(\mathbb{R}_T^n)}^{1-\mu}.\end{aligned}$$

Furthermore, from Lemma 2.5 and the fact that $q>p(1+\sigma)$ we obtain

$$\left\|D_x^m u\right\|_{L^q(\mathbb{R}_T^n)} \leqslant C_1\left\{\delta^{\frac{\mu\sigma}{p(1+\sigma)}}\left\|D_x^m u\right\|_{L^q(\mathbb{R}_T^n)}+\|\mathcal{L}_1 u\|_{L^q(\mathbb{R}_T^n)}^{\mu}\left\|D_x^m u\right\|_{L^q(\mathbb{R}_T^n)}^{1-\mu}\right\},$$

where $C_1 = C_1(n, \Lambda_1, \Lambda_2)$. Choosing δ small enough to ensure that $C_1\delta^{\frac{\mu\sigma}{p(1+\sigma)}} \leqslant 1/2$, we finally obtain

$$\left\| D_x^m u \right\|_{L^q(\mathbb{R}_T^n)} \leqslant C\|\mathcal{L}_1 u\|_{L^q(\mathbb{R}_T^n)}. \tag{2.18}$$

Since u satisfies $\mathcal{L}_1 u = u_t - \sum_{|\nu|=m} a_\nu D^\nu u$, from (1.4) and (2.18) we have

$$\|u_t\|_{L^q(\mathbb{R}_T^n)} \leqslant C\|\mathcal{L}_1 u\|_{L^q(\mathbb{R}_T^n)}. \qquad \square$$

Now we write

$$M_0 = \fint_{\mathbb{R}_T^n} \left|D^m u\right|^q \, dx\, dt + \frac{1}{\epsilon} \fint_{\mathbb{R}_T^n} |f|^q \, dx\, dt, \tag{2.19}$$

where $\epsilon > 0$ is a small enough constant. Set

$$u_\lambda = u/(M_0\lambda) \quad \text{and} \quad f_\lambda = f/(M_0\lambda) \tag{2.20}$$

for any $\lambda > 0$. Then u_λ is still the solution of (2.1)–(2.2) with f_λ replacing f.

Lemma 2.8. *(Cf. [10,40].) For any $\lambda > 0$, there exists a family of disjoint cylinders $\{Q_i^0\}_{i\in\mathbb{N}} = \{Q_{\rho_i}(x_i, t_i)\}_{i\in\mathbb{N}}$ with $(x_i, t_i) \in E_\lambda(1) =: \{(x, t) \in \mathbb{R}_T^n\colon |D^m u_\lambda|^q > 1\}$ and $\rho_i = \rho((x_i, t_i), \lambda) > 0$ such that*

$$J_\lambda\left[Q_i^0\right] = 1 \quad \textit{and} \quad J_\lambda\left[Q_\rho(x_i, t_i)\right] < 1 \quad \textit{for any } \rho > \rho_i, \tag{2.21}$$

where

$$J_\lambda[Q_\rho] = \fint_{Q_\rho} \left|D^m u_\lambda\right|^q \, dx\, dt + \frac{1}{\epsilon} \fint_{Q_\rho} |f_\lambda|^q \, dx\, dt. \tag{2.22}$$

Moreover, we have

$$E_\lambda(1) \subset \bigcup_{i\in\mathbb{N}} Q_i^1 \cup \textit{negligible set},$$

where

$$Q_i^j = 5jQ_i^0 =: B_{5j\rho_i}(x_i) \times \left(t_i - (5j\rho_i)^m, t_i + (5j\rho_i)^m\right], \quad j = 1, 2.$$

Moreover, we have

$$\left|Q_i^0\right| \leqslant C\left(\int_{\{(x,t)\in Q_i^0\colon |D^m u_\lambda|^q > 1/4\}} \left|D^m u_\lambda\right|^q \, dx\, dt + \frac{1}{\epsilon} \int_{\{(x,t)\in Q_i^0\colon |f_\lambda|^q > \epsilon/4\}} |f_\lambda|^q \, dx\, dt\right).$$

Similarly to [6,10,40], we can easily finish the proof of Theorem 2.1. In fact, we can totally omit the boundary part since $u \in C_0^\infty(Q_R)$.

2.2. Final proof

Observe that we have the artificial assumption $u \in C_0^\infty(Q_R)$ in the statement of Theorem 2.1. Next, motivated by [18,28,29], we shall show that this assumption is redundant and can be removed.

Lemma 2.9. *Under the same assumptions on the coefficients a_ν and f as those in Theorem* 1.9. *Let $u \in C_0^\infty(Q_R)$ be the solution of*

$$\mathcal{L}u + \lambda u =: u_t - \sum_{|\nu|=0}^{m} a_\nu(x,t) D^\nu u + \lambda u = f(x,t) \quad \text{in } \mathbb{R}_T^n, \tag{2.23}$$

$$u(x,0) = 0. \tag{2.24}$$

Then there exist positive constants δ, λ_0, depending only on n, ϕ, Λ_1, Λ_2, such that

$$\sum_{i=0}^{m} \lambda^{\frac{(m-i)p}{m}} \int_{\mathbb{R}_T^n} \phi\big(\big|D_x^i u\big|\big)\,dx\,dt \leqslant C \int_{\mathbb{R}_T^n} \phi\big(|f|\big)\,dx\,dt \tag{2.25}$$

for any $\lambda \geqslant \lambda_0$.

Proof. We divide into two cases.

Case 1: $|\nu| = m$. Let $u \in C_0^\infty(Q_R)$ be the solution of

$$\mathcal{L}_1 u + \lambda u =: u_t - \sum_{|\nu|=m} a_\nu(x,t) D^\nu u + \lambda u = f(x,t) \quad \text{in } \mathbb{R}_T^n, \tag{2.26}$$

$$u(x,0) = 0. \tag{2.27}$$

Assume that $\zeta(y) \in C_0^\infty(-R/2, R/2)$ is a cut-off function. We define

$$\widetilde{u}(z,t) = \widetilde{u}(x,y,t) = u(x,t)\zeta(y)\cos\big(\lambda^{\frac{1}{m}} y\big), \quad z = (x,y) \in \mathbb{R}^n \times \mathbb{R},$$

and

$$\widetilde{\mathcal{L}}_1 \widetilde{u}(z,t) =: \mathcal{L}_1 \widetilde{u}(x,t) - (-1)^{\frac{m}{2}-1} D_y^m \widetilde{u} = \widetilde{u}_t - \sum_{|\nu|=m} a_\nu(x,t) D_x^\nu \widetilde{u} - (-1)^{\frac{m}{2}-1} D_y^m \widetilde{u},$$

where $D_y^m = \frac{d^m}{dy^m}$. Therefore, it is easy to check that the coefficients $\widetilde{a}_\nu$ of the operator $\widetilde{L}$ still satisfy (1.3)–(1.4) and small BMO condition. Since u satisfies (2.26)–(2.27), we find that

$$\begin{cases} \widetilde{\mathcal{L}}_1 \widetilde{u}(z,t) = \widetilde{f}, \\ \widetilde{u}(z,0) = 0, \end{cases}$$

where

$$\widetilde{f}=f\zeta\cos\big(\lambda^{\frac{1}{m}}y\big)-(-1)^{\frac{m}{2}-1}u(x,t)\sum_{1\leqslant i\leqslant m}C_m^iD_y^i\zeta(y)D_y^{m-i}\cos\big(\lambda^{\frac{1}{m}}y\big),$$

where $C_m^i=\frac{m!}{(m-i)!i!}$. Therefore, from Theorem 2.1 and the fact that $\zeta(y)\in C_0^\infty(-R/2,R/2)$ we see that

$$\int_{\mathbb{R}_T^{n+1}}\phi\big(\big|D_z^m\widetilde{u}\big|\big)\,dx\,dy\,dt\leqslant C\int_{\mathbb{R}_T^{n+1}}\phi\big(|\widetilde{f}|\big)\,dx\,dy\,dt. \tag{2.28}$$

It follows from (1.17) that

$$\phi\big(\big|D^mu(x)\big|\big)\leqslant K\big(\big|\zeta(y)\cos\big(\lambda^{\frac{1}{m}}y\big)\big|\big)^{-p_1}\phi\big(\big|\zeta(y)\cos\big(\lambda^{\frac{1}{m}}y\big)D^mu(x)\big|\big).$$

Since $\cos x$ is a periodic function, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}_T^n}\phi\big(\big|D_x^mu\big|\big)\,dx\,dt\\
&\quad=\bigg(\int_{\mathbb{R}}\frac{1}{K}\big(\big|\zeta(y)\cos\big(\lambda^{\frac{1}{m}}y\big)\big|\big)^{p_1}\,dy\bigg)^{-1}\int_{\mathbb{R}^{n+1}}\frac{1}{K}\big(\big|\zeta(y)\cos\big(\lambda^{\frac{1}{m}}y\big)\big|\big)^{p_1}\phi\big(\big|D_x^mu(x)\big|\big)\,dx\,dy\\
&\quad\leqslant C\int_{\mathbb{R}_T^{n+1}}\phi\big(\big|\zeta(y)\cos\big(\lambda^{\frac{1}{m}}y\big)D_x^mu(x)\big|\big)\,dx\,dy\,dt\\
&\quad=C\int_{\mathbb{R}_T^{n+1}}\phi\big(\big|D_x^m\widetilde{u}\big|\big)\,dx\,dy\,dt\leqslant C\int_{\mathbb{R}_T^{n+1}}\phi\big(\big|D_z^m\widetilde{u}\big|\big)\,dx\,dy\,dt.
\end{aligned}$$

Similarly, from (1.17) we compute that

$$\begin{aligned}
\int_{\mathbb{R}_T^n}\phi\big(\big|D_x^{m-1}u\big|\big)\,dx\,dt&\leqslant C\int_{\mathbb{R}_T^{n+1}}\phi\big(\big|D_x^{m-1}u\cdot\zeta(y)\cdot\sin\big(\lambda^{\frac{1}{m}}y\big)\big|\big)\,dx\,dy\,dt\\
&\leqslant C\int_{\mathbb{R}_T^{n+1}}\phi\bigg(\frac{1}{\lambda^{\frac{1}{m}}}\big|D_x^{m-1}D_y\widetilde{u}-D_x^{m-1}u\cdot\zeta'(y)\cdot\cos\big(\lambda^{\frac{1}{m}}y\big)\big|\bigg)\,dx\,dy\,dt\\
&\leqslant\frac{C}{\lambda^{\frac{p_2}{m}}}\bigg(\int_{\mathbb{R}_T^{n+1}}\phi\big(\big|D_z^m\widetilde{u}\big|\big)\,dx\,dy\,dt+\int_{\mathbb{R}_T^n}\phi\big(\big|D_x^{m-1}u\big|\big)\,dx\,dt\bigg),
\end{aligned}$$

that is to say,

$$\lambda^{\frac{p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^{m-1}u\big|\big)\,dx\,dt \leqslant C \int_{\mathbb{R}^{n+1}_T} \phi\big(\big|D_z^m \widetilde{u}\big|\big)\,dx\,dy\,dt,$$

by taking $\lambda \geqslant \lambda_0 > 1$ large enough. Moreover, we can compute that

$$\begin{aligned}
\int_{\mathbb{R}^n_T} \phi\big(\big|D_x^{m-2}u\big|\big)\,dx\,dt &\leqslant C \int_{\mathbb{R}^{n+1}_T} \phi\bigg(\frac{1}{\lambda^{\frac{2}{m}}}\big|D_x^{m-2}D_{yy}^2\widetilde{u} + D_x^{m-2}u\big(2\lambda^{\frac{1}{m}}\zeta'(y)\cdot\sin\big(\lambda^{\frac{1}{m}}y\big) \\
&\quad - \zeta''(y)\cos\big(\lambda^{\frac{1}{m}}y\big)\big)\big|\bigg)\,dx\,dy\,dt \\
&\leqslant \frac{C}{\lambda^{\frac{2p_2}{m}}}\bigg(\int_{\mathbb{R}^{n+1}_T} \phi\big(\big|D_z^m\widetilde{u}\big|\big)\,dx\,dy\,dt + \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^{m-2}u\big|\big)\,dx\,dt\bigg),
\end{aligned}$$

that is to say,

$$\lambda^{\frac{2p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^{m-2}u\big|\big)\,dx\,dt \leqslant C \int_{\mathbb{R}^{n+1}_T} \phi\big(\big|D_z^m \widetilde{u}\big|\big)\,dx\,dy\,dt,$$

by taking $\lambda \geqslant \lambda_0 > 1$ large enough. In fact, for any $0 \leqslant i \leqslant m$ we can compute that

$$\lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u\big|\big)\,dx\,dt \leqslant C \int_{\mathbb{R}^{n+1}_T} \phi\big(\big|D_z^m \widetilde{u}\big|\big)\,dx\,dy\,dt \tag{2.29}$$

for $\lambda \geqslant \lambda_0 > 1$ large enough. Therefore, combining (2.28)–(2.29), we conclude that

$$\sum_{i=0}^m \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u\big|\big)\,dx\,dt \leqslant C \int_{\mathbb{R}^n_T} \phi\big(|\widetilde{f}|\big)\,dx\,dt,$$

which implies that

$$\sum_{i=0}^m \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u\big|\big)\,dx\,dt \leqslant C\bigg(\int_{\mathbb{R}^n_T} \phi\big(|f|\big)\,dx\,dt + \int_{\mathbb{R}^n_T} \phi\big(|u|\big)\,dx\,dt\bigg).$$

Therefore, we have

$$\sum_{i=0}^m \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u\big|\big)\,dx\,dt \leqslant C \int_{\mathbb{R}^n_T} \phi\big(|f|\big)\,dx\,dt = C \int_{\mathbb{R}^n_T} \phi\big(|\mathcal{L}_1 u + \lambda u|\big)\,dx\,dt$$

by taking $\lambda \geqslant \lambda_0 > 1$ large enough.

Case 2: $0 \leqslant |\nu| \leqslant m$. Let $u \in C_0^\infty(Q_R)$ be the solution of (2.23)–(2.24). Then we have

$$\mathcal{L}_1 u + \lambda u = u_t - \sum_{|\nu|=m} a_\nu(x,t) D^\nu u + \lambda u = f(x,t) + \sum_{|\nu|=0}^{m-1} a_\nu(x,t) D^\nu u.$$

Therefore, from the result of Case 1 we obtain

$$\begin{aligned}\sum_{i=0}^{m} \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u\big|\big)\,dx\,dt &\leqslant C \int_{\mathbb{R}^n_T} \phi\big(|\mathcal{L}_1 u|\big)\,dx\,dt \\ &= C\Bigg(\int_{\mathbb{R}^n_T} \phi\big(|f|\big)\,dx\,dt + \sum_{s=0}^{m-1} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^s u\big|\big)\,dx\,dt\Bigg),\end{aligned}$$

which implies that

$$\sum_{i=0}^{m} \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u\big|\big)\,dx\,dt \leqslant C \int_{\mathbb{R}^n_T} \phi\big(|f|\big)\,dx\,dt$$

by taking $\lambda \geqslant \lambda_0 > 1$ large enough. Thus we finish the proof. □

Furthermore, we can remove the additional condition $u \in C_0^\infty(Q_R)$ and then obtain the following result.

Lemma 2.10. *Under the same assumptions on the coefficients a_ν and f as those in Theorem* 1.9. *Let u be the solution of* (2.23)–(2.24)*, i.e.,*

$$\mathcal{L}u + \lambda u =: u_t - \sum_{|\nu|=0}^{m} a_\nu(x,t) D^\nu u + \lambda u = f(x,t) \quad \text{in } \mathbb{R}^n_T,$$

$$u(x,0) = 0.$$

Then there exist positive constants δ, λ_0, depending only on n, ϕ, Λ_1, Λ_2, such that

$$\sum_{i=0}^{m} \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u\big|\big)\,dx\,dt \leqslant C \int_{\mathbb{R}^n_T} \phi\big(|f|\big)\,dx\,dt$$

for any $\lambda \geqslant \lambda_0$.

Proof. By an elementary approximation argument, we may assume that

$$u^0(x,t) = u(x,t)\rho(x - x_0, t - t_0) =: u(x,t)\rho^0(x,t),$$

where $(x_0, t_0) \in \mathbb{R}^n_T$ and $\rho(x,t) \in C_0^\infty(Q_{R/2})$. Then we compute

$$\mathcal{L}u^0 + \lambda u^0 = f^0,$$

where

$$f^0 =: f\rho^0 + u(\rho^0)_t - \sum_{|\nu|=1}^{m} a_\nu(x,t) \sum_{\substack{1\leqslant|\gamma|\leqslant m\\ \gamma\leqslant\nu}} C_\nu^\gamma D_x^\gamma \rho^0(x,t) D_x^{\nu-\gamma} u(x,t).$$

Assume that $\lambda \geqslant \lambda_0 > 1$. From Lemma 2.9 we find that

$$\begin{aligned}
&\sum_{i=0}^{m} \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u^0\big|\big)\, dx\, dt \\
&\quad \leqslant C \int_{\mathbb{R}^n_T} \phi\big(\big|f^0\big|\big)\, dx\, dt \\
&\quad \leqslant C\left(\int_{\mathbb{R}^n_T} \phi\big(|f\chi_{Q_{R/2}(x_0,t_0)}|\big)\, dx\, dt + \sum_{j=0}^{m-1} \int_{\mathbb{R}^n_T} \phi\big(\big|\chi_{Q_{R/2}(x_0,t_0)} D_x^j u\big|\big)\, dx\, dt \right).
\end{aligned} \tag{2.30}$$

Moreover, if $1 \leqslant |\nu| = i \leqslant m$, then we have

$$D_x^\nu u^0 = \rho^0 D_x^\nu u + \sum_{\substack{1\leqslant|\gamma|\leqslant|\nu|=i\\ \gamma\leqslant\nu}} C_\nu^\gamma D_x^\gamma \rho^0(x,t) D_x^{\nu-\gamma} u,$$

which implies that

$$\int_{\mathbb{R}^n_T} \phi\big(\big|\rho^0 D_x^i u\big|\big)\, dx\, dt \leqslant C\left(\sum_{j=0}^{i-1} \int_{\mathbb{R}^n_T} \phi\big(\big|\chi_{Q_{R/2}(x_0,t_0)} D_x^j u\big|\big)\, dx\, dt + \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u^0\big|\big)\, dx\, dt \right).$$

Thus, from (2.30) and the inequality above we conclude that

$$\begin{aligned}
&\sum_{i=1}^{m} \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|\rho^0 D_x^i u\big|\big)\, dx\, dt \\
&\quad \leqslant C\left(\sum_{i=1}^{m} \lambda^{\frac{(m-i)p_2}{m}} \sum_{j=0}^{i-1} \int_{\mathbb{R}^n_T} \phi\big(\big|\chi_{Q_{R/2}(x_0,t_0)} D_x^j u\big|\big)\, dx\, dt + \sum_{i=1}^{m} \lambda^{\frac{(m-i)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|D_x^i u^0\big|\big)\, dx\, dt \right) \\
&\quad \leqslant C\left(\sum_{i=1}^{m} \sum_{j=0}^{i-1} \lambda^{\frac{(m-j-1)p_2}{m}} \int_{\mathbb{R}^n_T} \phi\big(\big|\chi_{Q_{R/2}(x_0,t_0)} D_x^j u\big|\big)\, dx\, dt \right.
\end{aligned}$$

$$+\int_{\mathbb{R}^n_T}\phi\big(|f\chi_{Q_{R/2}(x_0,t_0)}|\big)\,dx\,dt+\sum_{j=0}^{m-1}\int_{\mathbb{R}^n_T}\phi\big(\big|\chi_{Q_{R/2}(x_0,t_0)}D^j_x u\big|\big)\,dx\,dt\Bigg)$$

$$\leqslant C\Bigg(\int_{\mathbb{R}^n_T}\phi\big(|f\chi_{Q_{R/2}(x_0,t_0)}|\big)\,dx\,dt+\sum_{j=0}^{m-1}\lambda^{\frac{(m-j-1)p_2}{m}}\int_{\mathbb{R}^n_T}\phi\big(\big|\chi_{Q_{R/2}(x_0,t_0)}D^j_x u\big|\big)\,dx\,dt\Bigg).$$

Obviously, (2.30) implies that

$$\sum_{i=0}^{m}\lambda^{\frac{(m-i)p_2}{m}}\int_{\mathbb{R}^n_T}\phi\big(\big|\rho^0 D^i_x u\big|\big)\,dx\,dt$$
$$\leqslant C\Bigg(\int_{\mathbb{R}^n_T}\phi\big(|f\chi_{Q_{R/2}(x_0,t_0)}|\big)\,dx\,dt+\sum_{j=0}^{m-1}\lambda^{\frac{(m-j-1)p_2}{m}}\int_{\mathbb{R}^n_T}\phi\big(\big|\chi_{Q_{R/2}(x_0,t_0)}D^j_x u\big|\big)\,dx\,dt\Bigg).$$

Furthermore, by integrating with (x_0,t_0) over $\mathbb{R}^n_T$ and choosing proper $\lambda\geqslant\lambda_0>1$, we can obtain the desired result. □

Now we are ready to finish the proof of Theorem 1.9.

Proof of Theorem 1.9. (1) Let $\zeta(t)\in C_0^\infty(\mathbb{R})$ be a cut-off function and

$$v(x,t)=\zeta\left(\frac{t}{k}\right)u(x).$$

Since u satisfies

$$\mathcal{M}u-\lambda u=:\sum_{|\nu|=0}^{m}a_\nu(x)D^\nu u-\lambda u=f(x)\quad\text{in }\mathbb{R}^n,$$

we have

$$\mathcal{L}v+\lambda v=\widetilde{f}=:-\zeta\left(\frac{t}{k}\right)f+\frac{1}{k}\zeta'\left(\frac{t}{k}\right)u.$$

Thus, using Lemma 2.10, we obtain

$$\sum_{i=0}^{m}\lambda^{\frac{(m-i)p_2}{m}}\int_{\mathbb{R}^n_T}\phi\left(\left|\zeta\left(\frac{t}{k}\right)D^i_x u\right|\right)dx\,dt$$
$$\leqslant C\int_{\mathbb{R}^n_T}\phi\big(|\widetilde{f}|\big)\,dx\,dt$$

$$\leqslant C\left(\int_{\mathbb{R}^n_T} \phi\left(\left|\zeta\left(\frac{t}{k}\right)f\right|\right)dx\,dt + \int_{\mathbb{R}^n_T} \phi\left(\left|\frac{1}{k}\zeta'\left(\frac{t}{k}\right)u\right|\right)dx\,dt\right)$$

for any $\lambda \geqslant \lambda_0$. Selecting $\lambda = \lambda_0$ and $k > 0$ large enough, from (1.17) we can obtain

$$\sum_{i=0}^{m} \int_{\mathbb{R}^n_T} \phi\left(\left|D_x^i u\right|\right)dx\,dt \leqslant C \int_{\mathbb{R}^n_T} \phi\left(|f|\right)dx\,dt.$$

(2) Now we denote

$$w(x,t) = u(x,t)e^{-\lambda_0 t}.$$

Then w is the solution of

$$\mathcal{L}w + \lambda_0 w = e^{-\lambda_0 t} f,$$

since u is the solution of (1.1)–(1.2). Therefore, it follows from Lemma 2.10 that

$$\sum_{i=0}^{m} \lambda_0^{\frac{(m-i)p}{m}} \int_{\mathbb{R}^n_T} \phi\left(\left|D_x^i u\right|\right)dx\,dt \leqslant C \int_{\mathbb{R}^n_T} \phi\left(\left|e^{-\lambda_0 t} f\right|\right)dx\,dt,$$

where C depends on n, p, Λ_1, Λ_2. This completes our proof. □

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable comments and suggestions to improve the expressions. This work was done while the second author was visiting Department of Mathematics, University of Iowa. Moreover, the authors would like to thank Prof. Sun-sig Byun for many helpful discussions. This work is supported in part by the NSF (11001165), Research Fund for the Doctoral Program of Higher Education of China (20093108120003) and Shanghai Leading Academic Discipline Project (J50101).

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