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The loop orbifold of the symmetric product

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Abstract

Using the loop orbifold of the symmetric product, we give a formula for the Poincaré polynomial of the free loop space of the Borel construction of the symmetric product. We also show that the Chas–Sullivan orbifold product structure in the homology of the free loop space of the Borel construction of the symmetric product induces a ring structure in the homology of the inertia orbifold of the symmetric product. For a general almost complex orbifold, we define a new ring structure on the cohomology of its inertia orbifold which we call the virtual intersection ring. Finally we show that under Poincaré duality in the case of the symmetric product orbifold, both ring structures are isomorphic.

MSC: 55P35; 55N91

1. Introduction

The (naive) symmetric product of a space X is often defined as the *topological space*

$$X^n/\mathfrak{S}_n = X \times \cdots \times X/\mathfrak{S}_n.$$

We find that it is better to study instead the *orbispace*

$$[X^n/\mathfrak{S}_n] = [X \times \cdots \times X/\mathfrak{S}_n],$$

namely, the category whose objects are n-tuples $(x_1, \ldots, x_n)$ of points in X and whose arrows are elements of the form $(x_1, \ldots, x_n; \sigma)$ where $\sigma \in \mathfrak{S}_n$. The arrow $(x_1, \ldots, x_n; \sigma)$ has as its source $(x_1, \ldots, x_n)$, and as its target $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. This category is a groupoid, for the inverse of $(x_1, \ldots, x_n; \sigma)$ is $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; \sigma^{-1})$. For this reason we can think of $[X^n/\mathfrak{S}_n]$ as an orbispace [11,18]. We call it the *symmetric product of* X. We will assume that the reader is familiar with the theory of orbifolds as presented in [18].

In this paper we study the basic properties of the topology of the loop orbispace of the symmetric product $[X^n/\mathfrak{S}_n]$. By this we do not mean the free loop space $\mathcal{L}(X^n/\mathfrak{S}_n)$ of the naive symmetric product, but rather the orbifold $\mathsf{L}[X^n/\mathfrak{S}_n]$ whose objects are functors $[\mathbb{R}/\mathbb{Z}] \to [X^n/\mathfrak{S}_n]$ and whose morphisms are natural transformations.

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In [14] the first two authors have shown that the map L defines a functor

$$\mathsf{L}: \textbf{Orbifolds} \longrightarrow S^1\text{-}\textbf{Orbifolds}, \quad \mathsf{X} \mapsto \mathsf{LX},$$

from orbifolds to infinite dimensional orbifolds with actions of S^1. This functor when restricted to smooth manifolds becomes the ordinary free loop space functor $M \mapsto \mathcal{L}M$, where the S^1 action is given by rotating the loops. More interestingly, the action of S^1 on LX has as a fixed suborbifold $I(\mathsf{X})$ which is known as the inertia orbifold of X (cf. [6]).

The infinite dimensional orbifold LX not only is a very simple and natural object to consider, it also has the property that its geometrical realization is homotopically equivalent to the free loop space of the geometrical realization of the orbifold X [16, Theorem 2.1.1]; i.e., $\mathcal{L}B\mathsf{X} \simeq B\mathsf{LX}$, where the letter B stands for the functor that associates to a category its geometrical realization. This fact in the case of global quotient orbifolds ($\mathsf{X} = [Y/G]$, with G finite), allowed us in [16] to define a ring structure on $H_*(\mathcal{L}B\mathsf{X}; \mathbb{R})$, thus generalizing the ring structure on $H_*(\mathcal{L}M; \mathbb{Z})$ due to Chas and Sullivan [3,5].

In this paper we give the first steps to understanding the ring structure on $H_*(\mathcal{L}B\mathsf{X}; \mathbb{R})$ for the particular case of the symmetric product of a manifold $\mathsf{X} = [M^n/\mathfrak{S}_n]$. We start by giving a description of the orbifold $\mathsf{L}[M^n/\mathfrak{S}_n]$ that depends only on the loop space $\mathcal{L}M$ and the centralizers of elements in $\mathfrak{S}_n$. This description allows us to construct a generating function for the Poincaré polynomials of the spaces $\mathcal{L}(M \times_{\mathfrak{S}_n} E\mathfrak{S}_n)$ (Corollary 8). Similar arguments permit us to show that the ring structure on $H_*(\mathsf{L}[M^n/\mathfrak{S}_n]; \mathbb{R})$ induces a ring structure on $H_*(I[M^n/\mathfrak{S}_n]; \mathbb{R})$ (Theorem 13).

This ring structure on the homology of the inertia orbifold of the symmetric product attracted our attention, as it shares many formal properties with the dual of the Chen–Ruan product structure on the orbifold cohomology of the symmetric product [4,9,23]. Upon closer examination, the product structure turns out to be different, which is somewhat surprising.

We construct an associative product on the cohomology of the inertia orbifold (Definition 19) using what we call the *pull-push* formalism [13] and a criterion of Fantechi and Göttsche. We call this product the *virtual intersection product* on the cohomology of the inertial orbifold. We compare the virtual intersection product with the definition of Fantechi and Göttsche [9] of the Chen–Ruan product, and we conclude that the two products are different. The reason for this is essentially that the virtual intersection product is associated to transversal intersection, while the Chen–Ruan product is associated to holomorphic transversal intersection.

The main result of this paper (Proposition 23) establishes that the ring structure on the homology of the inertia orbifold of the symmetric product obtained in Theorem 13 is isomorphic, under Poincaré duality, with the ring structure of Definition 19. We also hope to have clarified the main differences between the construction of Chen and Ruan, which is complex geometric, and ours, which is topological.

2. The symmetric product

2.1. Poincaré polynomials

Let X be a topological space whose cohomology $H^i(M, \mathbb{R})$ is finitely generated for every $i > 0$. Let $b^i(X) := \dim H^i(M, \mathbb{R})$ be the i-th Betti number of X and denote by $\phi(X, y)$ the Poincaré polynomial of X, i.e.

$$\phi(X, y) = \sum_i b^i(X) y^i.$$

Macdonald [17] proved the following formula,

$$\sum_{n=0}^{\infty} \phi(X^n/\mathfrak{S}_n, y) q^n = \prod_{i \geq 0} \frac{\left(1 + q y^{2i+1}\right)^{b_{2i+1}(X)}}{\left(1 - q y^{2i}\right)^{b_{2i}(X)}}. \tag{1}$$

With y equal to -1 this is the famous formula for the Euler characteristic of the symmetric product:

$$\sum_{n=0}^{\infty} \chi(X^n/\mathfrak{S}_n) q^n = (1 - q)^{-\chi(X)}.$$

To obtain the formula (1) one uses the fact that when a finite group G acts on a CW-complex Y, the cohomology of the quotient Y/G is isomorphic to the G invariant part of the cohomology of Y, i.e.

$$H^*(Y/G;\mathbb{R}) \cong H^*(Y;\mathbb{R})^G.$$

Note also that there is no restriction on the cohomological dimension of X.

2.2. Orbifold Euler characteristic

There is a similar formula associated to the orbifold Euler characteristic $\chi_{\mathfrak{S}_n}$ of the symmetric product. This formula is obtained using the $\mathfrak{S}_n$-equivariant K-theory of X^n by means of the following expression,

$$\chi_{\mathfrak{S}_n}(X^n) := \text{Rank } K^0_{\mathfrak{S}_n}(X^n) - \text{Rank } K^1_{\mathfrak{S}_n}(X^n).$$

There is a formula due to Segal [21] that allows one to calculate the torsion free part of $K^*_G(Y)$ (where G acts on Y and G is a finite group) by localizing on the prime ideals of the representation ring of G. Using this formula of Segal we can obtain the following formula [8,22] for the generating function of the Euler characteristic:

$$\sum_{n=0}^{\infty} \chi_{\mathfrak{S}_n}(X^n) q^n = \prod_{j>0} \left(1-q^j\right)^{-\chi(X)}. \tag{2}$$

From Segal's result one also obtains the following isomorphism of graded vector spaces [1,2,12]:

$$K^*_G(Y) \otimes \mathbb{C} \cong \bigoplus_{(g)} K^*(Y^g)^{C(g)} \otimes \mathbb{C} \tag{3}$$

where (g) runs over the conjugacy classes of elements in G, Y^g is the fixed point locus of g and $C(g)$ is the centralizer of g in G acting on Y^g.

We now apply formula (3) to the symmetric product. Any permutation $\tau \in \mathfrak{S}_n$ can be written as a product of disjoint cycles. Define n_j as the number of cycles of length j in this presentation. One has that $\sum_{j>0} jn_j = n$ and therefore the n_j's form a partition of n. It is easy to see that if $\sigma \in \mathfrak{S}_n$ gives rise to the same partition of n as τ, then τ and σ must be conjugate. Hence the conjugacy classes of elements in $\mathfrak{S}_n$ are in one-to-one correspondence with partitions of n. We will denote the partition by the expression

$$\sum_{j>0} jn_j = n.$$

The fixed point set $(X^n)^\tau$ is isomorphic to $X^{\sum_j n_j}$ because each cycle contributes one copy of X. Further, $C(\tau) \cong \prod_j \mathfrak{S}_{n_j} \ltimes (\mathbb{Z}/j)^{n_j}$ because permuting the cycles of the same length of τ commutes with τ, and rotation of each cycle also commutes with τ. Since the cyclic groups act trivially on the fixed point set of the corresponding cycle, then the cyclic groups act trivially on $K^*(X^{\sum_j n_j})$. Therefore the following decomposition holds,

$$K^*_{\mathfrak{S}_n}(X^n) \otimes \mathbb{C} \cong \bigoplus_{(\tau)} K^*((X^n)^\tau)^{C(\tau)} \otimes \mathbb{C} \cong \bigoplus_{\sum jn_j = n} \bigotimes_j K^*(X^{n_j})^{\mathfrak{S}_{n_j}} \otimes \mathbb{C} \tag{4}$$

and formula (2) can be obtained by applying the Chern character isomorphism to $K^*(X^{n_j})^{\mathfrak{S}_{n_j}} \otimes \mathbb{C}$ and then formula (1).

2.3. Orbifold cohomology

For an orbifold $[Y/G]$ (viewed as a topological groupoid [18]) its *orbifold cohomology* is defined as the cohomology of the inertia orbifold $I[Y/G]$, i.e. $H^*_{\text{orb}}([Y/G]) := H^*(I[Y/G])$, where the inertia orbifold is defined as

$$I[Y/G] := \left[\left(\bigsqcup_{g \in G} Y^g \times \{g\} \right) \Big/ G \right]$$

and the action is given by

$$G \times \left(\bigsqcup_{g \in G} Y^g \times \{g\} \right) \to \left(\bigsqcup_{g \in G} Y^g \times \{g\} \right) \tag{5}$$
$$(h, (x, g)) \mapsto (xh, h^{-1} g h).$$

There is another (Morita equivalent) presentation of the inertia orbifold of $[Y/G]$ given by

$$I[M/G] \cong \bigsqcup_{(g)} [Y^g / C(g)]$$

where as before (g) runs over the conjugacy classes, Y^g is the fixed point locus and $C(g)$ is the centralizer. Then we have

$$H^*_{\mathrm{orb}}([Y/G]; \mathbb{R}) = \left(\bigoplus_{g \in G} H^*(Y^g; \mathbb{R}) \right)^G \cong \bigoplus_{(g)} H^*(Y^g; \mathbb{R})^{C(g)},$$

and

$$K^*_G(Y) \otimes \mathbb{C} \cong \bigoplus_{(g)} K^*(Y^g) \otimes \mathbb{C} \cong \bigoplus_{(g)} H^*(Y^g; \mathbb{R})^{C(g)} \cong H^*_{\mathrm{orb}}([Y/G]; \mathbb{C}) \tag{6}$$

where the middle isomorphism is given by the Chern character map.

We can define the orbifold Poincaré polynomial

$$\phi_{\mathrm{orb}}([Y/G], y) = \sum b^i_{\mathrm{orb}}([Y/G]) y^i$$

where the orbifold Betti number $b^i_{\mathrm{orb}}([Y/G])$ is the rank of $H^i_{\mathrm{orb}}([Y/G]; \mathbb{R})$. For the symmetric product, viewed as an orbifold $[X^n/\mathfrak{S}_n]$, we have that

$$H^*_{\mathrm{orb}}([X^n/\mathfrak{S}_n]; \mathbb{R}) \cong \bigoplus_{\sum j n_j = n} \bigotimes_j H^*(X^{n_j}; \mathbb{R})^{\mathfrak{S}_{n_j}} \tag{7}$$

and calculating the orbifold Poincaré polynomial one finds that

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi_{\mathrm{orb}}([X^n/\mathfrak{S}_n], y) q^n &= \sum_{n=0}^{\infty} q^n \left(\sum_{\sum j n_j = n} \prod_j \phi(X^{n_j}/\mathfrak{S}_{n_j}, y) \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{\sum j n_j = n} \prod_j \phi(X^{n_j}/\mathfrak{S}_{n_j}, y) (q^j)^{n_j} \right) \\
&= \prod_{j>0} \left(\sum_{n=0}^{\infty} \phi(X^n/\mathfrak{S}_n, y) q^{jn} \right) \\
&= \prod_{j>0} \frac{\prod_i (1 + q^j y^{2i+1})^{b^{2i+1}(X)}}{\prod_i (1 - q^j y^{2i})^{b^{2i}(X)}}.
\end{aligned} \tag{8}$$

When we set the variable y equal to -1, we get the formula (2) for the orbifold Euler characteristic. Again, for the previous formulæ to be valid one only needs that the cohomology of X is finitely generated at each i.

Remark 1. In algebraic geometry the Chen–Ruan orbifold cohomology of $[Y/G]$ is also defined as the cohomology of the inertia orbifold $I[Y/G]$ but has a shift in grading, which is called *age* by Reid [20], *shifting number* by Chen–Ruan [4] and *fermionic shift* by physicists. If the canonical divisor of Y is left invariant under the G action, the shiftings are all even numbers. Therefore in that case, the orbifold Euler chracteristic that we defined in Section 2.2 and the Euler characteristic of the Chen–Ruan orbifold cohomology agree, though their Poincaré polynomials disagree. This fact has caused many misunderstandings among topologists and algebraic geometers, and we hope that this article will clarify the differences (see Section 5). Let us emphasize then, that here we do not change the grading because we are dealing with topological properties of orbifolds and not with complex geometric properties.

Remark 2. The *ghost loop space* $\mathcal{L}_s(Y \times_G EG)$ of a G-space Y is defined as the subspace of its free loop space consisting of all those maps $S^1 \to Y \times_G EG$ that when composed with the natural projection $Y \times_G EG \to Y/G$ are constant. The first two authors have proved in [15] that there is a homotopy equivalence

$$\left(\bigsqcup_{g\in G} Y^g \times \{g\}\right) \times_G EG \simeq \mathcal{L}_s(Y \times_G EG).$$

Therefore we could state the theorems of this paper replacing everywhere the inertia orbifold by the ghost loop space.

3. The loop orbifold of the symmetric product

For an orbifold $[Y/G]$ the loop orbifold $\mathsf{L}[Y/G]$ has been defined in [14,16] and for the case of a global quotient it has a very simple description: $\mathsf{L}[Y/G] = [\mathcal{P}_G Y/G]$ where $\mathcal{P}_G Y = \sqcup_{g\in G} \mathcal{P}_g Y \times \{g\}$ with $\mathcal{P}_g Y = \{f : [0,1] \to Y | f(0)g = f(1)\}$, and the G action is given by

$$G \times \bigsqcup_{g\in G} \mathcal{P}_g Y \times \{g\} \to \bigsqcup_{g\in G} \mathcal{P}_g Y \times \{g\}$$
$$(h, (f, g)) \mapsto (f \cdot h, h^{-1}gh)$$

with $f \cdot h(t) := f(t)h$. The loop orbifold has another (Morita equivalent) presentation given by

$$\mathsf{L}[Y/G] \cong \bigsqcup_{(g)} [\mathcal{P}_g Y/C(g)]$$

where $C(g)$ acts on $\mathcal{P}_g Y$ in the natural way. It is a theorem proved in [16] that $B\mathsf{L}[Y/G] \simeq \mathcal{L}B[Y/G]$, i.e. the geometrical realization of the loop orbifold is homotopically equivalent to the free loop space of the geometrical realization of the orbifold, which in terms of the Borel construction states:

$$\mathcal{P}_G Y \times_G EG \simeq \bigsqcup_{(g)} \left(\mathcal{P}_g Y \times_{C(g)} EC(g)\right) \simeq \mathcal{L}(Y \times_G EG),$$

where $\mathcal{L}(Z) := \{\gamma : S^1 \to Z | \gamma \text{ is } C^0\}$ denotes the the free loop space of Z.

Since the cohomology of the finite groups is all torsion, when using real coefficients one gets the isomorphism

$$\begin{aligned} H^*(\mathcal{L}(Y \times_G EG); \mathbb{R}) &\cong H^*(\mathsf{L}[Y/G]; \mathbb{R}) \\ &\cong H^*(\mathcal{P}_G Y; \mathbb{R})^G \\ &\cong \bigoplus_{(g)} H^*(\mathcal{P}_g Y; \mathbb{R})^{C(g)}. \end{aligned} \tag{9}$$

In this section, we will use the isomorphism (9) to construct a generating function that calculates the Poincaré polynomial of $\mathcal{L}(X^n \times_{\mathfrak{S}_n} E\mathfrak{S}_n)$.

In the case of the symmetric product, one gets

$$\mathsf{L}[X^n/\mathfrak{S}_n] \cong \bigsqcup_{(\tau)} [\mathcal{P}_\tau X^n/C(\tau)],$$

and in cohomology

$$H^*(\mathcal{L}(X^n \times_{\mathfrak{S}_n} E\mathfrak{S}_n); \mathbb{R}) \cong \bigoplus_{(\tau)} H^*(\mathcal{P}_\tau X^n; \mathbb{R})^{C(\tau)}.$$

For the conjugacy class (τ) associated to the partition $\sum_j jn_j = n$, one has

Lemma 3. *The space $\mathcal{P}_\tau X^n$ is homeomorphic to the space $\prod_j (\mathcal{L}X)^{n_j}$, and the induced action of $C(\tau) \cong \prod_j \mathfrak{S}_{n_j} \ltimes (\mathbb{Z}/j)^{n_j}$ acts componentwise on $\prod_j (\mathcal{L}X)^{n_j}$; i.e $\mathfrak{S}_{n_j}$ acts on $(\mathcal{L}X)^{n_j}$ by permutations, and each $\mathbb{Z}/j$ acts on $\mathcal{L}X$ rotating the free loops by an angle that is a multiple of $\frac{2\pi}{j}$.*

Proof. When (τ) is represented by the product $\tau_1^1 \cdots \tau_1^{n_1} \tau_2^1 \cdots \tau_2^{n_2} \cdots$ of disjoint cycles, with τ_j^i the i-th cycle of size j, and $\sum j n_j = n$, then

$$\mathcal{P}_\tau X^n \cong \prod_j \prod_{i=1}^{n_j} \mathcal{P}_{\tau_j^i} X^j \cong \prod_j (\mathcal{P}_{\sigma_j} X^j)^{n_j}$$

where σ_j is the cycle $(1, 2, \ldots, j)$. Now, the space $\mathcal{P}_{\sigma_j} X^j$ consists of j-tuples $f = (f_1, \ldots, f_j)$ of paths $f_i \colon [0, 1] \to X$ such that $f(0)\sigma_j = f(1)$, i.e. $f_i(0) = f_{\sigma_j(i)}(1)$, which imply that the paths f_i could be concatenated into a loop $\tilde{f}$ which belongs to $\mathcal{L}X$. The map $\mathcal{P}_{\sigma_j} X^j \to \mathcal{L}X$, $f \mapsto \tilde{f}$ is clearly a homeomorphism.

The action of $\mathbb{Z}/j$ on $\mathcal{P}_{\sigma_j} X$, takes $(f_1, \ldots, f_j)$ to $(f_j, f_1, \ldots, f_{j-1})$. This induces a $\mathbb{Z}/j$ action on $\mathcal{L}X$ that takes $\tilde{f}$ to $\tilde{f}(\cdot + \frac{2\pi}{j})$. $\square$

We have then

$$[\mathcal{P}_\tau X^n / C(\tau)] \cong \prod_j [(\mathcal{P}_{\sigma_j} X^j)^{n_j} / \mathfrak{S}_{n_j} \ltimes (\mathbb{Z}/j)^{n_j}] \cong \prod_j [(\mathcal{L}X)^{n_j} / \mathfrak{S}_{n_j} \ltimes (\mathbb{Z}/j)^{n_j}].$$

As $\mathbb{Z}/j$ acts trivially on $H^*(\mathcal{L}X; \mathbb{R})$, one has

Corollary 4.

$$H^*(\mathcal{L}(X^n \times_{\mathfrak{S}_n} E\mathfrak{S}_n); \mathbb{R}) \cong \bigoplus_{(\tau)} H^*(\mathcal{P}_\tau X^n; \mathbb{R})^{C(\tau)} \cong \bigoplus_{\sum j n_j = n} \bigotimes_j H^*((\mathcal{L}X)^{n_j}; \mathbb{R})^{\mathfrak{S}_{n_j}}.$$

At this point we can see some similarities between the loop orbifold of the symmetric product of X, and the inertia orbifold of the symmetric product of $\mathcal{L}X$, namely that their real cohomologies agree.

Proposition 5. *The cohomologies with real coefficients of the orbifolds $\mathsf{L}[X^n/\mathfrak{S}_n]$ and $I[(\mathcal{L}X)^n/\mathfrak{S}_n]$ are isomorphic.*

Proof. By formula 7 we have that

$$H^*_{\text{orb}}([(\mathcal{L}X)^n/\mathfrak{S}_n]; \mathbb{R}) \cong \bigoplus_{\sum j n_j = n} \bigotimes_j H^*((\mathcal{L}X)^{n_j}; \mathbb{R})^{\mathfrak{S}_{n_j}}$$

which is isomorphic by (9) and Corollary 4 to $H^*(\mathsf{L}[X^n/\mathfrak{S}_n]; \mathbb{R})$. $\square$

Remark 6. For $n > 1$ and $\dim(X) > 0$, the orbifolds $\mathsf{L}[X^n/\mathfrak{S}_n]$ and $I[(\mathcal{L}X)^n/\mathfrak{S}_n]$ cannot be isomorphic because the actions of the cyclic groups $\mathbb{Z}/j$ are different. On the one hand, for $\mathsf{L}[X^n/\mathfrak{S}_n]$, we just argued that the action of the cyclic groups is by rotation on $\mathcal{L}X$ (coming from the action of σ_j on $\mathcal{P}_{\sigma_j} X^j$). And on the other, for $I[(\mathcal{L}X)^n/\mathfrak{S}_n]$, the action of the cyclic groups is trivial, because the copies of $\mathcal{L}X$ come from the fixed point loci of the group action generated by the cycle σ_j on $(\mathcal{L}X)^j$. Therefore, on the one hand one has the orbifold $[\mathcal{L}X/(\mathbb{Z}/j)]$ with the rotation action, and on the other, one has the orbifold $[\mathcal{L}X/(\mathbb{Z}/j)]$ with the trivial action.

In the case when $X = S^1$ and $n = 2$. Then

$$\mathsf{L}[(S^1)^2/\mathfrak{S}_2] = [(\mathcal{L}S^1)^2/\mathfrak{S}_2] \sqcup [\mathcal{L}S^1/(\mathbb{Z}/2)]$$

where the action of $\mathbb{Z}/2$ in the second component is by rotation, and

$$I[(\mathcal{L}S^1)^2/\mathfrak{S}_2] = [(\mathcal{L}S^1)^2/\mathfrak{S}_2] \sqcup [\mathcal{L}S^1/(\mathbb{Z}/2)]$$

where the action of $\mathbb{Z}/2$ in the second component is the trivial one.

As $\mathcal{L}S^1 \simeq S^1 \times \mathbb{Z}$ it is easy to see that in the first case the induced action of $\mathbb{Z}/2$ on $S^1 \times \mathbb{Z}$ is by the antipodal action on S^1, and in the second case is the trivial one. Therefore the geometrical realization of $[\mathcal{L}S^1/(\mathbb{Z}/2)]$ is homotopically equivalent to

$$(S^1 \times \mathbb{Z})$$

in the first case, and to

$$S^1 \times \mathbb{Z} \times \mathbb{R}P^\infty$$

in the second case. We can conclude that the orbifolds $\mathsf{L}[(S^1)^2/\mathfrak{S}_2]$ and $I[(\mathcal{L}S^1)^2/\mathfrak{S}_2]$ are not isomorphic because their geometrical realizations are not homotopically equivalent.

From formula (6) and Proposition 5 we have

Corollary 7.

$$K^*_{\mathfrak{S}_n}((\mathcal{L}X)^n) \otimes \mathbb{C} \cong H^*(\mathcal{L}(X^n \times_{\mathfrak{S}_n} E\mathfrak{S}_n); \mathbb{C}).$$

From Proposition 5 and formula (8), one gets

Corollary 8. *Let X be such that $H^i(\mathcal{L}X; \mathbb{R})$ is finitely generated. Then*

$$\sum_{n=0}^{\infty} \phi(\mathcal{L}(X^n \times_{\mathfrak{S}_n} E\mathfrak{S}_n), y) q^n = \prod_{j>0} \frac{\prod_i (1 + q^j y^{2i+1})^{b^{2i+1}(\mathcal{L}X)}}{\prod_i (1 - q^j y^{2i})^{b^{2i}(\mathcal{L}X)}}$$

where $b_i(\mathcal{L}X)$ is the i-th Betti number of $\mathcal{L}X$.

Remark 9. The fact that the cohomologies of $I[\mathcal{L}X^n/\mathfrak{S}_n]$ and $\mathsf{L}[X^n/\mathfrak{S}_n]$ agree is a feature of the symmetric product. In general, for an orbifold $[Y/G]$, the cohomologies of $I[\mathcal{L}Y/G]$ and $\mathsf{L}[Y/G]$ do not have to agree. Take for example the $\mathbb{Z}/2$ action on S^2 by rotating π radians along the z-axis. In this case

$$I[\mathcal{L}S^2/(\mathbb{Z}/2)] = [\mathcal{L}S^2/(\mathbb{Z}/2)] \sqcup [\mathcal{L}(S^2)^\xi/(\mathbb{Z}/2)]$$

where ξ generates the group $\mathbb{Z}/2$, and therefore $\mathcal{L}(S^2)^\xi$ is the set of two points, the north and south poles. Hence

$$H^*(I[\mathcal{L}S^2/(\mathbb{Z}/2)]; \mathbb{R}) \cong H^*(\mathcal{L}S^2; \mathbb{R}) \oplus \mathbb{R}^{\oplus 2}.$$

On the other hand

$$\mathsf{L}[S^2/(\mathbb{Z}/2)] = [\mathcal{L}S^2/(\mathbb{Z}/2)] \sqcup [\mathcal{P}_\xi S^2/(\mathbb{Z}/2)]$$

with cohomology

$$H^*(\mathsf{L}[S^2/(\mathbb{Z}/2)]; \mathbb{R}) \cong H^*(\mathcal{L}S^2; \mathbb{R}) \oplus H^*(\mathcal{L}S^2; \mathbb{R})$$

because $\mathcal{P}_\xi S^2$ is homeomorphic to $\mathcal{L}S^2$ (see [16] for the explicit homeomorphism). We can conclude that

$$H^*(I[\mathcal{L}S^2/(\mathbb{Z}/2)]; \mathbb{R}) \ncong H^*(\mathsf{L}[S^2/(\mathbb{Z}/2)]; \mathbb{R}).$$

4. Ring structure in the homology of the loop orbifold

In [16] we have shown that for orbifolds of the type $[Y/G]$ with Y oriented, smooth, without boundary and compact, and G acting by orientation preserving diffeomorphisms, the homology of the loop orbifold $H_*(\mathsf{L}[Y/G]; \mathbb{R})$ has a ring structure. In this section we will study the ring structure of $H_*(\mathsf{L}[M^n/\mathfrak{S}_n]; \mathbb{R})$, and we will show that it induces a ring structure in the homology of $I[M^n/\mathfrak{S}_n]$ in such a way that $H_*(I[M^n/\mathfrak{S}_n]; \mathbb{R})$ becomes a sub ring of $H_*(\mathsf{L}[M^n/\mathfrak{S}_n]; \mathbb{R})$.

Let's start by recalling from [16] how the ring structure in $H_*(\mathsf{L}[M^n/\mathfrak{S}_n]; \mathbb{R})$ is defined. In this section all homologies will have integer coefficients, unless some other coefficients are explicitly stated.

For all $\tau \in \mathfrak{S}_n$ consider the evaluation maps $ev_1 : \mathcal{P}_\tau M^n \to M^n$, $f \mapsto f(1)$ and $ev_0 : \mathcal{P}_\tau M^n \to M^n$, $f \mapsto f(0)$, and the space

$$\mathcal{P}_\tau M^n {}_1\times_0 \mathcal{P}_\sigma M^n := \{(h,k) \in \mathcal{P}_\tau M^n \times \mathcal{P}_\sigma M^n | h(1) = k(0)\}$$

together with the map $ev_\infty : \mathcal{P}_\tau M^n {}_1\times_0 \mathcal{P}_\sigma M^n \to M^n, (h,k) \mapsto h(1)$. Notice that the following diagram is a pullback square

$$\begin{tikzcd}
\mathcal{P}_\tau M^n {}_1\times_0 \mathcal{P}_\sigma M^n \arrow[r] \arrow[d, "ev_\infty"] & \mathcal{P}_\tau M^n \times \mathcal{P}_\sigma M^n \arrow[d, "ev_1 \times ev_0"] \\
M^n \arrow[r] & M^n \times M^n,
\end{tikzcd}$$

where the map in the bottom row is of codimension nd with $d = \dim(M)$. Therefore we can perform the Thom–Pontryagin construction on the top row, defining a homomorphism

$$H_*(\mathcal{P}_\tau M^n \times \mathcal{P}_\sigma M^n) \to H_{*-nd}(\mathcal{P}_\tau M^n {}_1\times_0 \mathcal{P}_\sigma M^n)$$

that composed with the natural map

$$H_*(\mathcal{P}_\tau M^n {}_1\times_0 \mathcal{P}_\sigma M^n) \to H_*(\mathcal{P}_{\tau\sigma} M^n)$$

of composition of paths, defines a product denoted by $\star$

$$\begin{aligned}
&\star : H_p(\mathcal{P}_\tau M^n) \times H_q(\mathcal{P}_\sigma M^n) \to H_{p+q-nd}(\mathcal{P}_{\tau\sigma} M^n) \\
&(\alpha, \beta) \mapsto \alpha \star \beta.
\end{aligned}$$

This product extends to all

$$\bigoplus_\tau H_*(\mathcal{P}_\tau M^n) \times \{\tau\}$$

and is graded associative shifted by $-nd$.

By taking the induced product on the $\mathfrak{S}_n$ invariant part

$$\left(\bigoplus_\tau H_*(\mathcal{P}_\tau M^n; \mathbb{R}) \times \{\tau\}\right)^{\mathfrak{S}_n} \cong H_*(\mathsf{L}[M^n/\mathfrak{S}_n], \mathbb{R})$$

we have defined thus a ring structure in the homology of the loop orbifold of the symmetric product.

Lemma 10. $(H_*(\mathsf{L}[M^n/\mathfrak{S}_n], \mathbb{R}), \star)$ *is a graded ring (with degree shifted by $-nd$).*

Now let's study what is the behavior of the evaluation and inclusion of constant maps. Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{P}_\tau M^n \arrow[r, "ev"] & M^n \\
(M^n)^\tau \arrow[u, "i^\tau"] \arrow[ur, "f^\tau"'] &
\end{tikzcd} \tag{10}$$

where $(M^n)^\tau$ is the fixed point loci of τ, f^τ is the inclusion of the fixed point loci, i^τ is the inclusion of constant loops, and ev is the evaluation at 0. Then we have

Lemma 11. *The image in homology of ev_* is equal to the image in homology of f^τ_*.*

Proof. Restricting the diagram (10) to one of the cycles σ of size l that defines τ, it becomes

$$\begin{tikzcd} \mathcal{P}_\sigma M^l = \mathcal{L}M \arrow[r, "ev"] & M^l \\ (M^l)^\sigma = M \arrow[u, "i^\sigma"] \arrow[ur, "f^\sigma"'] & \end{tikzcd}$$

where f^σ is the diagonal inclusion $M \to M^l$ and the evaluation map ev takes a loop $\alpha : S^1 \to M$ and maps it to $ev(\alpha) = (\alpha(0), \alpha(\frac{2\pi}{l}), \ldots, \alpha(\frac{2(l-1)\pi}{l}))$. Defining the homotopy $ev^t(\alpha) = (\alpha(0), \alpha(\frac{2\pi t}{l}), \ldots, \alpha(\frac{2(l-1)\pi t}{l}))$ one sees that $ev^1 = ev$ and ev^0 are homotopic, and as $ev^0(\alpha) = f^\sigma(\alpha(0))$, the lemma follows. $\square$

As the inclusion map f^τ is a composition of several diagonal inclusions, it then induces an injective homomorphism $f^\tau_* : H_*((M^n)^\tau) \to H_*(M^n)$.

Definition 12. For each $\tau \in \mathfrak{S}_n$, let $H^\tau_*(M^n) := im(f^\tau_*) \subset H_*(M^n)$.

By Lemma 11 we get that the diagonal arrow in the diagram

$$\begin{tikzcd} H_*(\mathcal{P}_\tau M^n) \arrow[r, "ev_*"] & H^\tau_*(M^n) \\ H_*((M^n)^\tau) \arrow[u, "i^\tau_*"] \arrow[ur, "f^\tau_*"', "\cong"] & \end{tikzcd}$$

is an isomorphism.

Therefore we can define a ring structure in $\left(\bigoplus_\tau H^\tau_*(M^n) \times \{\tau\}\right)$ in the following way

$$\begin{aligned} &\bullet : (H^\tau_*(M^n) \times \{\tau\}) \times (H^\sigma_*(M^n) \times \{\sigma\}) \to (H^{\tau\sigma}_{*-nd}(M^n) \times \{\tau\sigma\}) \\ &((\alpha, \tau), (\beta, \sigma)) \mapsto (\alpha \bullet \beta, \tau\sigma) \end{aligned} \tag{11}$$

where

$$\alpha \bullet \beta = ev_* \left(\left(i^\tau_* \circ (f^\tau_*)^{-1} \alpha \right) \star \left(i^\sigma_* \circ (f^\sigma_*)^{-1} \beta \right) \right)$$

and $\star$ is the product structure in the loop orbifold. Using the isomorphisms f^τ_* we also have a ring structure in $\bigoplus_\tau H_*((M^n)^\tau) \times \{\tau\}$ that we will also denote by $\bullet$. We have the compatibility of all the products

$$\begin{tikzcd} H_*((M^n)^\tau) \times H_*((M^n)^\sigma) \arrow[r, "\bullet"] \arrow[d, "i^\tau_* \times i^\sigma_*"] \arrow[dd, bend right, "\cong"'] & H_*((M^n)^{\tau\sigma}) \arrow[d, "i^{\tau\sigma}_*"] \arrow[dd, bend left, "\cong"] \\ H_*(\mathcal{P}_\tau M^n) \times H_*(\mathcal{P}_\sigma M^n) \arrow[r, "\star"] \arrow[d, "ev_* \times ev_*"] & H_*(\mathcal{P}_{\tau\sigma} M^n) \arrow[d, "ev_*"] \\ H^\tau_*(M^n) \times H^\sigma_*(M^n) \arrow[r, "\bullet"] & H^{\tau\sigma}_*(M^n). \end{tikzcd}$$

Thus we can induce a product structure on the $\mathfrak{S}_n$ invariant part

$$\left(\bigoplus_\tau H_*((M^n)^\tau, \mathbb{R}) \times \{\tau\} \right)^{\mathfrak{S}_n} \cong H_*(I[M^n/\mathfrak{S}_n], \mathbb{R}),$$

and we can conclude,

Theorem 13. *The real homology of the inertia orbifold $(H_*(I[M^n/\mathfrak{S}_n]; \mathbb{R}), \bullet)$ becomes a graded ring (shifted by $-nd$). Moreover, the inclusion of constant loops $i : I[M^n/\mathfrak{S}_n] \to \mathsf{L}[M^n/\mathfrak{S}_n]$ and the evaluation maps induce ring* *homomorphisms that make the following diagram commutative*

$$\begin{tikzcd} & H_*(\mathsf{L}[M^n/\mathfrak{S}_n];\mathbb{R}) \arrow[dr, "ev_*"] & \\ H_*(I[M^n/\mathfrak{S}_n];\mathbb{R}) \arrow[ur, "i_*"] \arrow[rr, "\cong"'] & & \left(\bigoplus_\tau H_*^\tau(M^n;\mathbb{R})\times\{\tau\}\right)^{\mathfrak{S}_n}. \end{tikzcd}$$

Remark 14. The inclusion of the inertia orbifold into the loop orbifold, in general does not induce an injective homomorphism in homology. Take the example of Remark 9, namely the action of $\mathbb{Z}/2$ in S^2 by rotation along the z-axis. If the generator of $\mathbb{Z}/2$ is ξ, then the fixed point set $(S^2)^\xi$ consist of two points, the north and the south pole. The inclusion of the inertia orbifold into the loop orbifold is then $(S^2)^\xi \to \mathcal{P}_\xi S^2$, where $\mathcal{P}_\xi S^2 = \{f : [0,1] \to S^2 | f(0)\xi = f(1)\}$. But as the space $\mathcal{P}_\xi S^2$ is connected (homeomorphic to $\mathcal{L}S^2$), then the homomorphism $H_*((S^2)^\xi) \to H_*(\mathcal{P}_\xi S^2)$ cannot be injective.

Remark 15. We have seen how to define a ring structure on the homology of $I[M^n/\mathfrak{S}_n]$ using the structure of the homology of the loop orbifold. It is easy to see that the product $\bullet$ we have defined in formula (11) can be equivalently defined at the cycle level by using intersection of cycles in M^n. Namely, for cycles in $(M^n)^\tau$ and $(M^n)^\sigma$, their transversal intersection in M^n is a cycle in $(M^n)^{\langle\tau,\sigma\rangle}$, and therefore the intersection can be pushed forward to a cycle in $(M^n)^{\tau\sigma}$. The associativity (at the homology level) follows directly from the fact that the ordinary intersection product is associative in homology.

5. The virtual intersection product of an orbifold

For (almost) complex orbifolds, we would like to compare the product structure that we have defined in Section 4 on the inertia orbifold to other products that exist on the same space, in particular the Chen–Ruan product [4]. For this purpose we summarize a criterion of Fantechi and Göttsche [9] on how to define a product on the cohomology of the inertia orbifold. We will recall first a particular case of the *excess intersection formula* [19, Prop. 3.3]. In this section all orbifolds will be (almost) complex and compact.

Let S be a manifold and let S_1 and S_2 be closed submanifolds that intersect *cleanly*; that is, $U := S_1 \cap S_2$ is a submanifold of S and at each point x of U the tangent space of U is the intersection of the tangent spaces of S_1 and S_2. Let $E(S, S_1, S_2)$ be the *excess* bundle of the intersection, i.e., the vector bundle over U which is the quotient of the tangent bundle of S by the sum of the tangent bundles of S_1 and S_2 restricted to U. Thus $E(S, S_1, S_2) = 0$ if and only if S_1 and S_2 intersect transversally. In the Grothendieck group of vector bundles over U the excess bundle becomes

$$E(S, S_1, S_2) = T_S|_U + T_U - T_{S_1}|_U - T_{S_2}|_U.$$

Denote by $e(S, S_1, S_2)$ the top Chern class of $E(S, S_1, S_2)$ and by

$$\begin{tikzcd} U \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, "h"] & S_1 \arrow[d, "j_1"] \\ S_2 \arrow[r, "j_2"'] & S \end{tikzcd} \tag{12}$$

the relevant inclusion maps. Then for any cohomology class $\alpha \in H^*(S_1)$ the following *excess intersection formula* [19, Prop. 3.3] holds in the cohomology ring of S_2:

$$j_2^* j_{1*}\alpha = i_{2*}\left(e(S, S_1, S_2) i_1^*(\alpha)\right). \tag{13}$$

With formula (13) we can show the following result.

Lemma 16. *Let $\alpha \in H^*(S_1)$ and $\beta \in H^*(S_2)$, then*

$$h_*\left(i_1^*(\alpha) i_2^*(\beta) e(S, S_1, S_2)\right) = (j_{1*}\alpha)(j_{2*}\beta).$$

Proof. For inclusions $f : N \to M$ of manifolds, the "umkehrungs" homomorphism

$$f_* : H^*(N) \to H^{*+q}(M), \quad q = \dim M - \dim N$$

has the properties of functoriality, i.e.,

$$(f \circ g)_* = f_* \circ g_*$$

and this is a homomorphism of $H^*(M)$-modules, i.e.,

$$f_*(vf^*(u)) = (f_*v)u.$$

As the degree of $e(S, S_1, S_2)$ is even, we can move it around without changing any sign. Therefore by applying the functoriality to h, the excess intersection formula (13) and the module structure for $i_{2*}$and j_{2*}, we can conclude that

$$\begin{aligned} h_*\left(i_1^*(\alpha)i_2^*(\beta)e(S, S_1, S_2)\right) &= j_{2*}i_{2*}\left((e(S, S_1, S_2)i_1^*(\alpha))i_2^*(\beta)\right) \\ &= j_{2*}\left(i_{2*}\left(e(S, S_1, S_2)i_1^*(\alpha)\right)\beta\right) \\ &= j_{2*}\left(j_2^* j_{1*}(\alpha)\beta\right) \\ &= (-1)^{|\alpha||\beta|} j_{2*}\left(\beta j_2^* j_{1*}(\alpha)\right) \\ &= (-1)^{|\alpha||\beta|}(j_{2*}\beta)(j_{1*}\alpha) \\ &= (j_{1*}\alpha)(j_{2*}\beta). \quad \square \end{aligned}$$

Now we summarize Fantechi and Göttsche's criterion. Consider the almost complex orbifold $[Y/G]$ where Y is an almost complex manifold and G acts preserving the almost complex structure. Define the groups

$$H^*(Y, G) := \bigoplus_{g \in G} H^*(Y^g) \times \{g\}$$

where Y^g is the fixed point set of the element g. The group G acts in the natural way as in (5). Denote by $Y^{g,h} = Y^g \cap Y^h$ and suppose that we have G invariant cohomology classes $c(g, h) \in H^*(Y^{g,h})$; i.e. such that $v^*c(k^{-1}gk, k^{-1}hk) = c(g, h)$ where $v : Y^{k^{-1}gk, k^{-1}hk} \to Y^{g,h}$ takes x to $v(x) := xk$. Define the map

$$\begin{aligned} &\times : H^*(Y^g) \times H^*(Y^h) \to H^*(Y^{gh}) \\ &(\alpha, \beta) \mapsto i_*\left(\alpha|_{Y^{g,h}} \cdot \beta|_{Y^{g,h}} \cdot c(g, h)\right) \end{aligned}$$

where $i : Y^{g,h} \to Y^{gh}$ is the natural inclusion.

Lemma 17 (*[9, Lemma 1.17]*). *A sufficient condition for the map $\times$ to define an associative product on $H^*(Y, G)$ is that for every ordered triple of elements $(g, h, k) \in G$ the following relation holds in the cohomology of $W = Y^g \cap Y^h \cap Y^k$:*

$$c(g, h)|_W \cdot c(gh, k)|_W \cdot e(Y^{gh}, Y^{g,h}, Y^{gh,k}) = c(g, hk)|_W \cdot c(h, k)|_W \cdot e(Y^{hk}, Y^{g,hk}, Y^{h,k}). \tag{14}$$

In particular, we have

Lemma 18. *The cohomology classes $c(g, h) = e(Y, Y^g, Y^h)$ satisfy condition (14), and therefore the map $\times$ defines an associative product on $H^*(Y, G)$.*

Proof. As $e(E + F) = e(E)e(F)$, we just need to check the equality of (14) for $c(g, h) = e(Y, Y^g, Y^h)$ in the Grothendieck ring of vector bundles over W. The top row of (14) is then

$$\begin{aligned} &E(Y, Y^g, Y^h)|_W + E(Y, Y^{gh}, Y^k)|_W + E(Y^{gh}, Y^{g,h} Y^{gh,k}) \\ &\quad = T_Y + T_{Y^{g,h}} - T_{Y^g} - T_{Y^h} + T_Y + T_{Y^{gh,k}} - T_{Y^{gh}} - T_{Y^k} + T_{Y^{gh}} + T_{Y^{g,h,k}} - T_{Y^{g,h}} - T_{Y^{gh,k}} \end{aligned}$$

(all the bundles are restricted to W) and after a reordering one can see that this is equal to

$$E(Y, Y^g, Y^{hk})|_W + E(Y, Y^h, Y^k)|_W + E(Y^{hk}, Y^{g,hk}, Y^{h,k}),$$

which is the bottom row of (14). $\square$

Definition 19. With the classes $c(g, h) = e(Y, Y^g, Y^h)$, we will call the product $\times$ in $H^*(Y, G)$ the *virtual intersection product*. Given that $H^*(Y, G; \mathbb{R})^G \cong H^*(I[Y/G]; \mathbb{R})$, the product $\times$ induces a ring structure on the orbifold cohomology of $[Y/G]$. We will also call this ring product the virtual intersection product.

A heuristic reason why the virtual intersection product is different from the Chen–Ruan orbifold product [4,9, 23] is that the latter intersects cycles *holomorphically* and therefore there is less room to do perturbation theory. Algebraically one can see this fact clearly, because the degree of the classes $c_{CR}(g, h)$ for the Chen–Ruan product is no greater than the degree of the classes $c(g, h)$.

Remark 20. The virtual intersection ring of a general almost complex orbifold $\mathcal{X}$ is actually a *Frobenius algebra*. There are two interesting limiting cases:

- When $G = 1$ and the orbifold is actually a manifold, the virtual intersection ring coincides with the usual intersection ring of a smooth manifold, which is a Frobenius algebra.
- When $Y = \{\bullet\}$ is a point, the virtual intersection ring becomes the Dijkgraaf–Witten Frobenius algebra associated to a finite group [7].

We will return to the proof and discussion of this remark elsewhere.

6. Poincaré duality on the inertia orbifold of the symmetric product

In the case of the symmetric product $[M^n/\mathfrak{S}_n]$ with M an (almost) complex manifold,

$$\deg(c(\tau, \sigma)) = d[n + \mathcal{O}(\langle\tau, \sigma\rangle) - \mathcal{O}(\langle\tau\rangle) - \mathcal{O}(\langle\sigma\rangle)],$$

where $d = \dim_{\mathbb{R}}(M)$ and $\mathcal{O}(\Gamma)$ is the number of orbits of the action of $\Gamma \subset \mathfrak{S}_n$ on $\{1, 2, \dots, n\}$, and

$$\deg(c_{CR}(\tau, \sigma)) = \frac{d}{2}[n + 2\mathcal{O}(\langle\tau, \sigma\rangle) - \mathcal{O}(\langle\tau\rangle) - \mathcal{O}(\langle\sigma\rangle) - \mathcal{O}(\langle\tau\sigma\rangle)],$$

see [9, Cor. 3.4] or [23, Prop. 4.19].

Remark 21. As $\langle\tau\sigma\rangle$ is a subgroup of $\langle\tau, \sigma\rangle$, we have $\mathcal{O}(\langle\tau, \sigma\rangle) \le \mathcal{O}(\langle\tau\sigma\rangle)$, and therefore

$$\deg(c(\tau, \sigma)) \ge 2\deg(c_{CR}(\tau, \sigma)).$$

In the symmetric product we will show that the product $\times$ we have defined on the cohomology of the inertia orbifold is just the Poincaré dual of the product $\bullet$ we defined in (11). Let $f_*^\tau : H_*((M^n)^\tau) \xrightarrow{\cong} H_*^\tau(M^n)$ be the isomorphism defined in Definition 12, $f^{\tau,\sigma} : (M^n)^{\tau,\sigma} \to M^n$ the inclusion and $f_*^{\tau,\sigma} : H_*((M^n)^{\tau,\sigma}) \to H^{\tau,\sigma}(M^n) := \mathrm{Im}(f_*^{\tau,\sigma})$ the induced isomorphism. Denote by

$$D_\tau : H^{d\mathcal{O}(\langle\tau\rangle)-p}((M^n)^\tau) \xrightarrow{\cong} H_p((M^n)^\tau)$$

the Poincaré duality isomorphism on $(M^n)^\tau$ (respectively $D_{\tau,\sigma}$) and by D the Poincaré duality isomorphism in M^n. Define the isomorphisms

$$A_\tau : H^{d\mathcal{O}(\langle\tau\rangle)-p}((M^n)^\tau) \xrightarrow{\cong} H_p^\tau(M^n) \quad \text{with } A_\tau := f_*^\tau \circ D_\tau$$

(respectively $A_{\tau,\sigma}$). Then we have

Theorem 22. *The isomorphisms $\{A_\tau\}_\tau$ induce an isomorphism of rings*

$$A : \left(\left(\bigoplus_\tau H^{d\mathcal{O}(\langle\tau\rangle)-*}((M^n)^\tau) \times \{\tau\}\right), \times\right) \xrightarrow{\cong} \left(\left(\bigoplus_\tau H_*((M^n)^\tau) \times \{\tau\}\right), \bullet\right).$$

Proof. We only need to check that the following diagram is commutative:

$$\begin{tikzcd}
H^{d\mathcal{O}(\langle\tau\rangle)-p}((M^n)^\tau) \times H^{d\mathcal{O}(\langle\sigma\rangle)-q}((M^n)^\sigma) \arrow[r, "A_\tau \times A_\sigma", "\cong"'] \arrow[d, "|_{(M^n)^{\langle\tau,\sigma\rangle}}"] \arrow[ddd, bend right=60, "\times"'] & H_p^\tau(M^n) \times H_q^\sigma(M^n) \arrow[dd, "\pitchfork"'] \arrow[ddd, bend left=60, "\bullet"] \\
H^{d\mathcal{O}(\langle\tau\rangle)+d\mathcal{O}(\langle\sigma\rangle)-p-q}((M^n)^{\tau,\sigma}) \arrow[d, "\cup e"] & \\
H^{dn+d\mathcal{O}(\langle\tau,\sigma\rangle)-p-q}((M^n)^{\tau,\sigma}) \arrow[r, "A_{\tau,\sigma}", "\cong"'] \arrow[d, "\text{pushforward}"] & H^{\tau,\sigma}_{p+q-nd}(M^n) \arrow[d, "\text{inclusion}"'] \\
H^{dn+d\mathcal{O}(\langle\tau\sigma\rangle)-p-q}((M^n)^{\tau\sigma}) \arrow[r, "A_{\tau\sigma}", "\cong"'] & H^{\tau\sigma}_{p+q-nd}(M^n).
\end{tikzcd}$$

The left column is by definition the product $\times$, the right column is the product $\bullet$, $\pitchfork$ denotes the ordinary intersection product in homology and $e := e(M^n, (M^n)^\tau, (M^n)^\sigma)$ is the excess intersection class. The commutativity of the bottom square is by definition of the pushforward in cohomology.

As we will use Lemma 16, we invoke the notation of diagram (12). Let $S = M^n$, $S_1 = (M^n)^\tau$, $S_2 = (M^n)^\sigma$ and $U = (M^n)^{\tau,\sigma}$. For the commutativity of the top square we need to show that $A_{\tau,\sigma}(i_1^*(\alpha)i_2^*(\beta)e) = (A_\tau\alpha) \pitchfork (A_\sigma\beta)$ for $\alpha \in H * ((M^n)^\tau)$ and $\beta \in H^*((M^n)^\sigma)$.

By definition of Poincaré duality we have that $h_* = D^{-1} \circ A_{\tau,\sigma}$, $j_{1*} = D^{-1} \circ A_\tau$ and $j_{2*} = D^{-1} \circ A_\sigma$. Then

$$\begin{aligned}
D^{-1}A_{\tau,\sigma}(i_1^*(\alpha)i_2^*(\beta)e) &= h_*(i_1^*(\alpha)i_2^*(\beta)e) \\
&= (j_{1*}\alpha)(j_{2*}\beta) \\
&= (D^{-1}A_\tau\alpha)(D^{-1}A_\sigma\beta) \\
&= D^{-1}((A_\tau\alpha) \pitchfork (A_\sigma\beta)),
\end{aligned}$$

where the equality of the first line with the second is from Lemma 16. As D is an isomorphism, the commutativity of the entire diagram follows. $\square$

Moreover, the isomorphism A becomes $\mathfrak{S}_n$-equivariant if we equip both sides with an action of the same variance. This is easily done and follows from the commutativity of the following diagram:

$$\begin{tikzcd}
H^*((M^n)^\tau) \arrow[r, "\sigma_*", "\cong"'] \arrow[d, "D_\tau"', "\cong"] & H^*((M^n)^{\tau\sigma}) \arrow[d, "\cong"', "D_{\tau\sigma}"] \\
H_*((M^n)^\tau) \arrow[r, "(\sigma^{-1})^*", "\cong"'] & H_*((M^n)^{\tau\sigma}).
\end{tikzcd}$$

Therefore taking $\mathfrak{S}_n$ invariants on both sides of Theorem 22, we arrive at

Proposition 23. *The isomorphism A of Theorem 22 induces an isomorphism of rings*

$$(H^*(I[M^n/\mathfrak{S}_n]; \mathbb{R}), \times) \cong (H_*(I[M^n/\mathfrak{S}_n]; \mathbb{R}), \bullet).$$

We conclude that in the case of the orbifold $[M^n/\mathfrak{S}_n]$ for M a compact complex manifold, there is a ring structure on the cohomology of the inertia orbifold $I[M^n/\mathfrak{S}_n]$ which is in general different from the one constructed by Chen–Ruan. This virtual intersection product boils down to pairwise transversal intersection of cycles.

Here it may be worthwhile to mention that the same theorems are valid if we use K-theory rather than cohomology (cf. [10]). The proofs are very similar.

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