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The Idempotents of the Symmetric Group and Nakayama's Conjecture

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1. INTRODUCTION, IDEMPOTENTS AT CHARACTERISTIC ZERO

A previous article of the author [4] described the construction of an orthogonal basis for a Specht module over a field of characteristic zero, giving rise to Young's seminormal representation of $\mathbf{S}_n$, the symmetric group on n letters. We now extend the formalism to the case where the field has arbitrary characteristic. A complete set of orthogonal idempotents of the group-algebra can now be constructed, which are primitive if the characteristic is zero. The primitive central idempotents can also be constructed, which gives a particularly simple proof of the well-known Nakayama Conjecture [3].

We take K to be an algebraic number field of characteristic zero (except in Theorem 1.9), p an arbitrary prime, R the ring of p-integral elements of K, and $\bar{K}$ the residue field $\bar{K} = R/P$, where P is the unique maximal ideal in R, so that $\bar{K}$ has characteristic p. S^μ is the Specht module corresponding to a partition $\mu = (\mu_1, \mu_2, \ldots)$ of n. Where necessary, the ground-ring will be distinguished by a suffix, e.g., S^μ_K.

Generally we shall follow the notation of [4], with some modifications. The class of the (i, j) node in the Young diagram $[\mu]$ is the difference $j - i$, and the p-class or p-residue is the residue modulo p of the class. The classes of the nodes of a proper diagram determine the shape of the diagram exactly, though the p-classes do not. If $[\mu]$ and $[\lambda]$ have the same p-classes then we write $\mu \sim_p \lambda$; this is equivalent to the statement that $[\mu]$ and $[\lambda]$ have the same p-core. We may extend this equivalence relation to tableaux by defining $t \sim_p \bar{t}$ if every $u \leqslant n$ occupies a node of the same p-class in both t and $\bar{t}$. If it is always a node of the same class, then we write $t \sim \bar{t}$, and obviously t and $\bar{t}$ must correspond to the same partition; in fact, [4, Lemma 2.2] shows that if $t \sim \bar{t}$, then either the tableaux are identical or at least one is non-standard.

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The standard μ-tableaux in the ordering of [4, p. 288] are $t_1^\mu, t_2^\mu, \ldots, t_d^\mu$ and the corresponding standard Specht polynomials are $e_1^\mu, e_2^\mu, \ldots, e_d^\mu$. The class of the node occupied by u in t_i^μ is α_{ui}^μ. If

$$L_u = (1, u) + (2, u) + \cdots + (u-1, u), \qquad u = 2, 3, \ldots, n,$$

where (u, v) denotes a transposition, then S_K^μ has a K-basis $f_1^\mu, f_2^\mu, \ldots, f_d^\mu$, orthogonal with respect to the bilinear form $\langle , \rangle$ defined in [2, p. 14], where [4, p. 291]

$$f_i^\mu = E_i^\mu e_i^\mu, \tag{1.1}$$

$$E_i^\mu = \prod_{c=-n+1}^{n-1} \prod_{\{u \mid \alpha_{ui}^\mu \neq c\}} \frac{c - L_u}{c - \alpha_{ui}^\mu}, \tag{1.2}$$

and

$$(L_u - \alpha_{ui}^\mu) f_i^\mu = 0, \qquad u \leqslant n. \tag{1.3}$$

For any u, v, L_u and L_v commute. Since $\{\alpha_{ui}^\mu \mid u \leqslant n\}$ uniquely determines t_i^μ, it follows from (1.2) and (1.3) that

$$E_i^\lambda f_j^\mu = \delta_{ij}\, \delta_{\lambda\mu} f_j^\mu, \tag{1.4}$$

where δ is the Kronecker delta; one if its subscripts are equal, zero, otherwise.

1.5 THEOREM. *$\{E_i^\mu \mid \mu$ is a partition of n, $i = 1, 2, \ldots, \dim(S^\mu)\}$ is a complete set of primitive orthogonal idempotents of KS_n.*

Proof. The Specht modules corresponding to partitions of n comprise a complete set of irreducible KS_n modules, so that the annihilator of all the orthogonal basis elements is $\mathrm{rad}(KS_n)$; since KS_n is semi-simple, this radical is zero. Consequently we may conclude from (1.4) that for all i, j, the following are all zero,

$$E_i^\mu E_j^\lambda \qquad \text{if} \quad t_i^\mu \neq t_j^\lambda,$$

$$E_i^\mu E_i^\mu - E_i^\mu,$$

$$1 - \sum_{i,\mu} E_i^\mu,$$

so that $\{E_i^\mu\}$ is a complete set of orthogonal idempotents. To show that E_i^μ is primitive, it is sufficient to show that $E_i^\mu KS_n E_i^\mu$ is a skew-field [1, p. 167]. However, from (1.4) we see that $E_i^\mu KS_n E_i^\mu = KE_i^\mu \cong K$.

These idempotents are identical with those constructed recursively by

Thrall (see [5, p. 27]), though the form is different. The method of the theorem has a wider applicability; for example, from (1.3) and (1.4) we have

$$(L_u - \alpha^{\mu}_{ui}) E^{\mu}_i = 0, \qquad u \leqslant n, \tag{1.6}$$

so that, summing over i, μ,

$$L_u = \sum_{i,\mu} \alpha^{\mu}_{ui} E^{\mu}_i . \tag{1.7}$$

Thus the K-module spanned by the idempotents $\{E^{\mu}_i\}$, which is easily shown to be a maximal commutative submodule of KS_n, is identical with the ring of polynomials in $\{L_u\}$ over K. Notice also that if t is a non-standard tableau, we can construct an operator E_t in analogy with (1.2); however, this ddoes not give anything new, since either there is a $t^{\mu}_i \sim t$, so that $E^{\mu}_i = E_t$, or $E_t = 0$, since it annihilates all the orthogonal basis elements. In particular, if u and $u-1$ are in the same row or column of t^{μ}_i and $t = (u, u-1)t^{\mu}_i$ then $E_t = 0$.

1.8 LEMMA. *Any polynomial in $L_2, L_3, \ldots, L_n$ which is symmetric in L_{u-1} and L_u commutes with $(u, u-1)$. If $u-1$ and u are in the same row or column of t^{μ}_i then $(u, u-1)$ commutes with E^{μ}_i, or otherwise, with $E^{\mu}_i + E^{\mu}_j$, where $t^{\mu}_j = (u, u-1)\, t^{\mu}_i$.*

Proof. It is easy to verify that $(u, u-1)$ commutes with $L_u + L_{u-1}, L_u L_{u-1}$, and with L_v for $v \neq u-1, u$. Since any polynomial symmetric in L_{u-1} and L_u can be expressed in terms of these operators, it must commute with $(u, u-1)$. Let $t = (u, u-1)t^{\mu}_i$; then $E^{\mu}_i + E_t$ is symmetric in L_{u-1} and L_u, and so commutes with $(u, u-1)$; $E_t = 0$ if $u-1$ and u are in the same row or column of t^{μ}_i; $E_t = E^{\mu}_j$ otherwise.

It is an immediate consequence of this lemma that the sum of the orthogonal idempotents corresponding to a particular partition belongs to $Z(KS_n)$, the centre of KS_n. The next Theorem gives a useful characteristic-free characterisation of the centre of the group-algebra.

1.9 THEOREM. *Let K be an arbitrary integral domain. Then $Z(KS_n)$ is the ring of completely symmetric polynomiale in the operators $L_2, L_3, \ldots, L_n$.*

Proof. Such a symmetric polynomial must commute with every transposition $(u, u-1)$ by the previous lemma, and therefore belongs to $Z(KS_n)$. It remains to show that $Z(KS_n)$ is spanned by symmetric polynomials. The K-dimension of $Z(KS_n)$ is equal to the number of conjugacy classes of S_n, or equivalently, proper partitions of n. For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$, let X^μ be the sum of all distinct products of the form

$$(L_{u_1})^{\mu_1 - 1}(L_{u_2})^{\mu_2 - 1} \cdots (L_{u_s})^{\mu_s - 1}, \tag{$*$}$$

where $u_1, u_2, \ldots, u_s$ runs over all sets of s elements from $1, 2, \ldots, n$. X^μ is clearly a symmetric polynomial; the restriction to distinct terms avoids duplication when μ has equal parts. Each permutation contributing to X^μ is a product of $n - s$ transpositions, though some may be simplified. Let $\hat{\mu}_i$ be the sum of the first i parts of μ, i.e., $\hat{\mu}_i = \mu_1 + \mu_2 + \cdots + \mu_i$, and let σ^μ be the product of the disjoint cycles

$$(n - \hat{\mu}_{i-1}, n - \hat{\mu}_{i-1} + 1, \ldots, n - \hat{\mu}_i + 1), \qquad i = 1, 2, \ldots, s,$$

where $\hat{\mu}_0 = 0$. σ^μ occurs with coefficient 1 in the expansion of the term

$$(L_n)^{\mu_1 - 1}(L_{n - \hat{\mu}_1})^{\mu_2 - 1}(L_{n - \hat{\mu}_2})^{\mu_3 - 1} \cdots (L_{\mu_s})^{\mu_s - 1}$$

and in no other term, so that X^μ is non-zero. Now suppose that σ^λ occurs in the expansion of X^μ with non-zero coefficient, where λ is a partition of n with s' parts. Since σ^λ can be written as a product of $n - s'$ transpositions, but not fewer, we must have $s' \geqslant s$. If $s' = s$ then σ^λ occurs in the expansion of a term of the form $(*)$ without any cancellations, in which case the ith factor contributes $\mu_i - 1$ transpositions involving u_i to some cycle of σ^λ, so that each $\lambda_j - 1$ is the sum of one or more terms of the form $\mu_i - 1$. Consequently, for each i, $\hat{\mu}_i \leqslant \hat{\lambda}_i$, i.e., $\mu \trianglelefteq \lambda$, where $\triangleleft$ is the dominance relation on partitions [2, p. 10]. We may order partitions of n so that for any λ, μ, λ precedes μ if $s' > s$ or if $s' = s$ and $\mu \triangleleft \lambda$, so that σ^μ appears in X^μ but not in any X^λ where λ precedes μ. Consequently, $\{X^\mu\}$ is linearly independent, and so spans a K-submodule of $Z(KS_n)$, M, say, of the same dimension as $Z(KS_n)$. It remains to show that M is a pure submodule of the torsion-free module $Z(KS_n)$. Let Y be an arbitrary element of $Z(KS_n)$; by dimensions, there is a number $k \in K$ such that $kY \in M$, and kY may be expanded in terms of the basis elements of M. Suppose that one of the coefficients in this expansion is not divisible by k; let X^μ be the latest term in the above ordering having this property. Obviously the coefficient of σ^μ is also not divisible by k, which is impossible, since k must divide the coefficient of each element of S_n occurring in kY. Consequently, k divides each coefficient in the expansion of kY, so that $Y \in M$, and therefore $M = Z(KS_n)$.

2. Idempotents at Arbitrary Characteristic

We now turn our attention to the case where the ground-field has arbitrary characteristic; i.e., we consider the field $\bar{K}$. Rather than work directly over $\bar{K}$

it is convenient to take the ring R, since $S_{\bar{K}}^{\mu} \simeq S_R^{\mu}/PS_R^{\mu}$, and S_R^{μ} is a submodule of S_K^{μ}. We denote residues modulo P or PS_R^{μ} by a bar, e.g., $\bar{X}$; if $X \in RS_n$ is idempotent, then so is $\bar{X} \in \bar{K}S_n$. Notice that the analysis of the last section now fails simply because the denominator in (1.2) is not usually prime to p.

The relation $\sim_p$ divides the partitions of n into equivalence classes, say, $B_1, B_2, \ldots$, and further divides the tableaux; let the equivalence classes of tableaux corresponding to partitions in B_i be $T_1^i, T_2^i, \ldots$, in some arbitrary order. We now set out to construct idempotents in RS_n; let

$$F_j^i = \sum_{\{u,k \mid t_k^{\mu} \in T_j^i\}} E_k^{\mu}, \qquad H^i = \sum_j F_j^i.$$

2.1 THEOREM. *$\{F_j^i\}$ is a complete set of orthogonal idempotents of RS_n; similarly for $\{\bar{F}_j^i\}$, $\bar{K}S_n$.*

Proof. All that is needed is to prove that $F_j^i \in RS_n$; the rest follows from Theorem 1.5, since a sum of orthogonal idempotents is idempotent. For some $t_k^{\mu} \in T_j^i$, let

$$F^* = \prod_{c=-n+1}^{n-1} \prod_{\{u \mid \alpha_{uk}^{\mu} \not\equiv c (\mathrm{mod}\, p)\}} \frac{c - L_u}{c - \alpha_{uk}^{\mu}}.$$

Obviously $F^* \in RS_n$. The numerator depends only on the class T_j^i, while the denominator depends only on the partition μ; neither depends on the particular tableau chosen. Let us write w^{μ} for this denominator; from (1.4) we have

$$\begin{aligned} F^* f_l^{\lambda} &= (w^{\lambda}/w^{\mu}) f_l^{\lambda}, &&\text{if } t_l^{\lambda} \in T_j^i, \\ &= 0, &&\text{otherwise,} \end{aligned}$$

so that $F^* = \sum_{\{\lambda, l \mid t_l^{\lambda} \in T_j^i\}} (w^{\lambda}/w^{\mu}) E_l^{\lambda}$.

Now $1 - w^{\lambda}/w^{\mu} \equiv 0 \pmod p$ if $t_l^{\lambda} \in T_j^i$, and since it is rational it must be a multiple of p. If p^m is the largest power of p dividing the denominator of any E_l^{λ} with $t_l^{\lambda} \in T_j^i$, we have $(1 - w^{\lambda}/w^{\mu})^m E_l^{\lambda} \in RS_n$. Therefore

$$(F_j^i - F^*)^m = \sum_{\{\lambda, l \mid t_l^{\lambda} \in T_j^i\}} (1 - w^{\lambda}/w^{\mu})^m E_l^{\lambda} \in RS_n.$$

But F_j^i is idempotent, and $F_j^i F^* = F^*$, so that

$$(F_j^i - F^*)^m = F_j^i - 1 + (1 - F^*)^m$$

by the binomial expansion; comparison of these equations yields $F^i_j \in RS_n$, as required. By taking residues modulo P, we can replace R by $\bar{K}$ and F^i_j by $\bar{F}^i_j$.

The idempotents are not, however, primitive. S^μ_R is decomposed by the action of $\{F^i_j\}$ into a direct sum of orthogonal R-submodules of the form $F^i_j S^\mu_R$; we may choose a basis which reflects this decomposition. From [4, Theorem 3.10] we know that there are numbers $a_{kl} \in K$ such that

$$e^\mu_k = \sum_{\{l \mid t^\mu_l \trianglelefteq t^\mu_k\}} a_{kl} f^\mu_l, \tag{2.2}$$

where $a_{kk} = 1$, and $\lhd$ is defined for tableaux as in [4]. Consequently, if we define $g^\mu_k = F^i_j e^\mu_k$ for $t^\mu_k \in T^i_j$, then

$$g^\mu_k = \sum_{\{l \mid t^\mu_l \leqslant t^\mu_k, t^\mu_l \sim_p t^\mu_k\}} a_{kl} f^\mu_l, \tag{2.3}$$

and since transformation (2.3), like (2.2), is unimodular, $g^\mu_1, g^\mu_2, \ldots$ is an R-basis for S^μ_R, and $\langle g^\mu_k, g^\mu_l \rangle = 0$ unless $t^\mu_k \sim_p t^\mu_l$. The transformation also partitions the Gram matrix [4, p. 294] of S^μ_R into a direct sum of submatrices corresponding to the orthogonal subspaces of S^μ_R. Combining (2.3) with (1.4) and the definitions of F^i_j and H^i gives

$$\begin{aligned} F^i_j g^\mu_k &= g^\mu_k, && \text{if } t^\mu_k \in T^i_j, \\ &= 0, && \text{otherwise,} \end{aligned} \tag{2.4}$$

$$\begin{aligned} H^i g^\mu_k &= g^\mu_k, && \text{if } \mu \in B_i, \\ &= 0, && \text{otherwise.} \end{aligned} \tag{2.5}$$

Thus if ξ^μ is an arbitrary element of S^μ_R, we have from (2.5)

$$\begin{aligned} H^i \xi^\mu &= \xi^\mu, && \text{if } \mu \in B_i, \\ &= 0, && \text{otherwise.} \end{aligned} \tag{2.6}$$

Taking residues modulo P, we see also that $\bar{g}^\mu_1, \bar{g}^\mu_2, \ldots$ is a $\bar{K}$-basis for $S^\mu_{\bar{K}}$, and obtain the analogues of (2.4) to (2.6).

Finally we turn our attention to the construction of the primitive central or block idempotents of $\bar{K}S_n$. The main result here is Nakayama's Conjecture, which is proved quite simply in the final theorem. First we examine the action of $Z(RS_n)$ and $Z(\bar{K}S_n)$ on the Specht modules.

2.7 LEMMA. *Let ξ^μ be an arbitrary element of S^μ_R, and X an arbitrary* *element of $Z(RS_n)$; then $X\xi^\mu = x^\mu \xi^\mu$, where $x^\mu \in R$ depends only on μ, and $\bar{X}\bar{\xi}^\mu = \bar{x}^i \bar{\xi}^\mu$, where $\bar{x}^i \in \bar{K}$ depends only on the equivalence class B_i of μ.*

Proof. X can be represented as a symmetric polynomial over R, say, $\varphi(L_2, L_3, ..., L_n)$, and for any j,

$$Xf_j^\mu = \varphi(\alpha_{2j}^\mu, \alpha_{3j}^\mu, ...,) f_j^\mu = x^\mu f_j^\mu,$$

where by the symmetry of φ, x^μ depends only on the classes of $[\mu]$, and not on the particular choice of j. Since ξ^μ can be expanded in terms of the orthogonal basis of S_R^μ, we have $X\xi^\mu = x^\mu \xi^\mu$. Moreover, $\bar{x}^\mu$ depends only on the p-classes of $[\mu]$, so that if $\mu \in B_i$, then we may set $\bar{x}^i = \bar{x}^\mu$, and by taking residues modulo p, obtain $\bar{X}\bar{\xi}^\mu = \bar{x}^i \bar{\xi}^\mu$ as required.

2.8 THEOREM (Nakayama's Conjecture). *$\{\bar{H}^i\}$ is a complete set of primitive orthogonal central idempotents of $\bar{K}S_n$, and $S_{\bar{K}}^\mu$, $S_{\bar{K}}^\lambda$ belong to the same block of $\bar{K}S_n$ if and only if $\mu \sim_p \lambda$.*

Proof. From Lemma 1.8 we know that $H^i \in Z(RS_n)$, since it commutes with every transposition $(u-1, u)$, and from Theorem 2.1 that $\{H^i\}$ is a complete set of orthogonal idempotents of RS_n; by taking residues modulo P we see that $\{\bar{H}^i\}$ is a complete set of central idempotents of $\bar{K}S_n$. It remains to prove that $\bar{H}^i$ is centrally primitive. Suppose that $\bar{X}$ is a central primitive idempotent for the block containing $S_{\bar{K}}^\mu$, and let $\bar{\xi}^\mu$ be an arbitrary element of $S_{\bar{K}}^\mu$; then $\bar{X}\bar{\xi}^\mu = \bar{\xi}^\mu$. If $S_{\bar{K}}^\lambda$ belongs to a different block, and $\bar{\xi}^\lambda \in S_{\bar{K}}^\lambda$, then $\bar{X}\bar{\xi}^\lambda = 0$, so that by Lemma 2.7, $\mu \nsim_p \lambda$; consequently, if $\mu \sim_p \lambda$ then $S_{\bar{K}}^\mu$ and $S_{\bar{K}}^\lambda$ belong to the same block. On the other hand, if $\mu \in B_i$, then from (2.6),

$$\begin{aligned} \bar{H}^i \bar{\xi}^\lambda &= \bar{\xi}^\lambda, \qquad \text{if } \mu \sim_p \lambda, \\ &= 0, \qquad \text{otherwise}, \end{aligned}$$

from which we may conclude that $S_{\bar{K}}^\lambda$ and $S_{\bar{K}}^\mu$ belong to the same block if and only if $\mu \sim_p \lambda$, and that $\bar{H}^i$ is the corresponding block idempotent, and therefore primitive.

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