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Periodic Solutions of Systems of Second-Order Differential Equations

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1. INTRODUCTION

Let E^n denote n-dimensional real Euclidean space and let I denote the compact interval $[0, T]$, $T > 0$. Let $f(t, x, y)$ be a continuous function with domain $I \times E^n \times E^n$ and range in E^n. In this paper, we provide sufficient conditions in order that the second-order differential system

$$x'' = f(t, x, x'), \qquad \left(' = \frac{d}{dt}\right) \tag{1.1}$$

has at least one periodic solution, i.e., a solution $x(t)$ which satisfies the periodic boundary conditions

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{1.2}$$

Problems of this type have been approached in a great variety of ways. Based on the fundamental paper of Cesari [2], Knobloch [6, 7], using differential inequalities and the Miranda Fixed Point Theorem [16], has studied globally Lipschitzian equations. Using boundary-value problem techniques, this author [17] has generalized some of Knobloch's results. Marlin and Ullrich [10], again using the Miranda Theorem, have studied second-order dissipative systems without damping; their work contains, as a particular case, the results obtained by Seifert [18] for the scalar equation

$$x'' + g(x) = p(t),$$

where g is a general restoring term.

The approach used by Seifert has the advantage that it may be employed to obtain rather general results for systems, as has been demonstrated by Mawhin in a series of papers [11–15].

In this paper, we use a combination of the above approaches. We establish existence-uniqueness results (in some cases, uniqueness will not be required) for solutions $x(t)$ of (1.1) which satisfy the boundary conditions

$$x(0) = y = x(T) \tag{1.3}$$

and then study the vector field

$$U(y) = x'(0; y) - x'(T; y) \tag{1.4}$$

($x(t; y)$ is the unique solution of (1.1), (1.3).)

Before proceeding to particular cases, we consider a very general existence result and then give several conditions to which this general situation applies.

Throughout this paper, we shall use, as a norm in E^n, the norm induced by the inner product, i.e., if $x = (x_1, ..., x_n)$, then

$$| x |^2 = \sum_{i=1}^{n} x_i^2.$$

2. Existence of Periodic Solutions

In this section, we establish conditions under which the unique solvability of the boundary-value problem (1.1), (1.3) implies the existence of at least one periodic solution of (1.1).

Lemma 2.1. *Let Ω be a convex homeomorphic image of the open solid n-sphere $\sigma^n = \{x : | x | < 1\}$ with $\bar{\Omega}$ compact, and assume that for each $y \in \bar{\Omega}$,* (1.1), (1.3) *has a unique solution $x(t; y)$. Further, let there exist a constant N (depending on Ω and f) such that*

$$| x(t; y)|, \qquad | x'(t; y)| \leqslant N, \qquad 0 \leqslant t \leqslant T, \quad y \in \bar{\Omega}.$$

Then $x(t; y)$ anx $x'(t; y)$ are continuous in y for each $t \in I$.

Proof. A straightforward application of the Ascoli–Arzela Theorem.

Theorem 2.2. *Let the hypotheses of Lemma* 2.1 *hold and let $\bar{\Omega}$ be symmetric about a point $z \in \Omega$. For each $y \in \bar{\Omega}$, define $U(y) = x'(0; y) - x'(T; y)$. Let A be the continuous involution of $\partial\Omega$ (the boundary of Ω) onto itself which maps each $y \in \partial\Omega$ onto the point which is symmetric to y with respect to z. Further assume that for all $y \in \partial\Omega$ for which $U(y) \neq 0$, $U(y)$ and $U(Ay)$ do not have the same direction. Then there exists a periodic solution of* (1.1).

Proof. If there exists $y \in \partial\Omega$ such that $U(y) = 0$, there is nothing to prove. Assume therefore that the vector field U is nondegenerate on $\partial\Omega$. Since $\bar{\Omega}$ is convex and symmetric about z, there exists a homeomorphism g of $\bar{\sigma}^n$ onto $\bar{\Omega}$ such that

$$g(0) = z, \qquad g^{-1}Ag(r) = -r$$

for all $r \in \partial\sigma^n$, i.e., $g^{-1}Ag$ is the antipodal map of $\partial\sigma^n$.

Define the vector field φ on $\bar{\sigma}^n$ by

$$\varphi(r) = U(g(r)).$$

Then, since U is nondegenerate on $\partial\Omega$, φ will be nondegenerate on $\partial\sigma^n$. Further, the assumptions on U and A imply that $\varphi(r)$ and $\varphi(-r)$ will not have the same direction, i.e.,

$$\frac{\varphi(-r)}{|\varphi(-r)|} \neq \frac{\varphi(r)}{|\varphi(r)|}.$$

Hence the vector field

$$\psi(r, \lambda) = \frac{\varphi(r)}{|\varphi(r)|} - \lambda \frac{\varphi(-r)}{|\varphi(-r)|}$$

is nondegenerate on $\partial\sigma^n$ for $0 \leqslant \lambda \leqslant 1$. Therefore $\psi(r, 0)$ and $\psi(r, 1)$ are homotopic, which implies that the topological degrees of $\psi(r, 0)$ and $\psi(r, 1)$ with respect to the origin are identical. On the other hand, $\psi(r, 1)$ is an odd vector field and thus its degree is an odd integer (see [1, p. 46] or [9, p. 75]). Thus by homotopy, the degree of $\psi(r, 0)$ is an odd integer, which implies that $\varphi(r)$ must have a zero in σ^n and hence U must have a zero in Ω, proving the existence of a periodic solution of (1.1).

Remark. The hypotheses on Ω and A may be replaced by the more general condition that A be a fixed-point free involution on $\partial\Omega$ and that there exists a homeomorphism $g : \bar{\sigma}^n \to \bar{\Omega}$ such that $g^{-1}Ag$ is the antipodal map on $\partial\sigma^n$. However, for the applications we give, the conditions of Theorem 2.2 will suffice.

3. Some Applications

In the first application considered below, we shall assume that for each positive constant M, there exists a positive constant N such that whenever $x(t)$ is a solution of (1.1) defined on I with $|x(t)| \leqslant M$, then $|x'(t)| \leqslant N$.

Further there exists another constant $P \geqslant N$ such that if $x(t)$ is a solution (defined on I) of the perturbed equation

$$x'' = f(t, x, x') + \epsilon x, \qquad 0 < \epsilon \leqslant \epsilon_0, \tag{3.1}$$

with $|x(t)| \leqslant M$, then $|x'(t)| \leqslant P$. Various conditions exist which imply the above assumption, for example, if f is independent of x' or satisfies a Nagumo Condition with respect to x' on compact (t, x) sets with Nagumo function φ and $\int^\infty s\, ds/[\epsilon_0 M + \varphi(s)] = \infty$. We refer the interested reader to Hartman [4, 5] and Heimes [6] where Nagumo-type conditions are assumed.

We further adopt the following convention: If $x, \bar{x}, y, \bar{y}$ are vectors in E^n, we denote by

$$\Delta x = x - \bar{x}, \qquad \Delta y = y - \bar{y}, \qquad \Delta f = f(t, x, y) - f(t, \bar{x}, \bar{y}).$$

THEOREM 3.1. *Let there exist a positive constant R such that*

$$x \cdot f(t, x, y) + |y|^2 \geqslant 0 \qquad \textit{if} \quad x \cdot y = 0, \qquad |x| = R \tag{3.2}$$

and

$$\Delta x \cdot \Delta f + |\Delta y|^2 \geqslant 0 \tag{3.3}$$

for any $x, \bar{x} \in E^n$, $|x|$, $|\bar{x}| \leqslant R$, $x \neq \bar{x}$ and $y, \bar{y} \in E^n$ with $\Delta x \cdot \Delta y = 0$.
Then there exists a periodic solution $x(t)$ of (1.1) *with $|x(t)| \leqslant R$.*

Proof. Let $\epsilon > 0$ be given with $\epsilon \leqslant \epsilon_0$. Consider the auxiliary (perturbed) Eq. (3.1) and let its right side be denoted by F.

Then,

$$x \cdot F + |y|^2 = x \cdot f + \epsilon |x|^2 + |y|^2 \geqslant \epsilon |x|^2 > 0 \tag{3.4}$$

if $x \cdot y = 0$ and $|x| = R$.

Further,

$$\Delta x \cdot \Delta F + |\Delta y|^2 = \Delta x \cdot \Delta f + \epsilon |\Delta x|^2 + |\Delta y|^2 \geqslant \epsilon |\Delta x|^2 > 0 \tag{3.5}$$

if $\Delta x \cdot \Delta y = 0$, $\Delta x \neq 0$.

Conditions (3.4) and (3.5) imply (see Hartman [3; or 4, pp. 427–433]) that the boundary-value problem

$$x'' = F(t, x, x', \epsilon), \tag{3.6}$$

$$x(0) = z = x(T), \tag{3.7}$$

has a unique solution $x(t; z, \epsilon)$ with

$$| x(t; z, \epsilon)| \leqslant R, \tag{3.8}$$

for any z, $| z | \leqslant R$.

Let $\Omega = \{x : | x | < R\}$. Then by assumption, there exists a constant $P > 0$ such that $| x'(t; z, \epsilon)| \leqslant P$ for any $z \in \bar{\Omega}$.

Hence by Lemma 2.1, the vector field

$$U(z, \epsilon) = x'(0; z, \epsilon) - x'(T; z, \epsilon)$$

is continuous on $\bar{\Omega}$. Again assume that $U(z, \epsilon)$ is nondegenerate on $\partial\Omega$. Letting $r(t) = \frac{1}{2} | x(t; z, \epsilon)|^2$, we find

$$r'(t) = x(t; z, \epsilon) \cdot x'(t; z, \epsilon) \tag{3.9}$$

and

$$r''(t) = x \cdot F + | x' |^2. \tag{3.10}$$

It follows from (3.8)–(3.10) and (3.4) that

$$r'(0) < 0 < r'(T)$$

for any $z \in \partial\Omega$, which, in turn, implies that $z \cdot U(z, \epsilon) < 0$ for $z \in \partial\Omega$. We conclude therefore that

$$\frac{U(z, \epsilon)}{| U(z, \epsilon)|} \neq \frac{U(-z, \epsilon)}{| U(-z, \epsilon)|}, \quad z \in \partial\Omega,$$

i.e., $U(z, \epsilon)$ and $U(-z, \epsilon)$ cannot have the same direction. Using Theorem 2.2, we conclude that $U(z, \epsilon)$ will have a zero in Ω for every ϵ, $0 < \epsilon < \epsilon_0$, i.e., there exists a periodic solution $x(t, \epsilon)$ of (3.1). We now obtain a periodic solution $x(t)$ of (1.1) by a standard limiting argument.

Remark. We applied Theorem 2.1 to the perturbed system (3.1) rather than the original system, because the conditions of Theorem 3.1 are not sufficient to guarantee unique solvability of boundary-value problems for (1.1).

Using a result from [6] which is based on the functional analytic method of Cesari [2], Knobloch [8] established a theorem very similar to our Theorem 3.1; the only difference being that Knobloch assumes a local Lipschitz condition on f with respect to (x, x') which we do not require; on the other hand, our assumption (3.3) is not required in [8].

COROLLARY 3.2. *Let A be a positive definite $n \times n$ matrix, h a Lipschitz-*

continuous bounded function on E^n into E^n and let φ be continuous on I into E^n. Then the equation

$$x'' = Ax + h(x') + \varphi(t) \tag{3.11}$$

has a periodic solution, provided that

(i) *$h(x')$ has the same or opposite direction as x',*

(ii) *$k \leqslant 2\sqrt{\mu}$, where μ is the least eigenvalue of A and k is the Lipschitz constant associated with h.*

Proof. We verify that (3.2) and (3.3) are satisfied for some $R > 0$.

$$\begin{aligned} &Ax \cdot x + h(y) \cdot x + \varphi \cdot x + |y|^2 \\ &\quad \geqslant \mu |x|^2 + h(y) \cdot x + \varphi \cdot x + |y|^2 \\ &\quad \geqslant \mu |x|^2 - |\varphi| |x| + |y|^2, \qquad \text{if} \quad x \cdot y = 0 \\ &\quad \geqslant |x|(\mu |x| - |\varphi|) + |y|^2. \end{aligned}$$

Choosing $R > 0$ large enough so that $\mu R \geqslant |\varphi|$, we see that (3.2) is satisfied. To verify (3.3), let $x, \bar{x}$ ($x \neq \bar{x}$) be such that $|x|, |\bar{x}| \leqslant R$ and let $y, \bar{y}$ be any vectors in E^n. Then

$$\begin{aligned} &A\Delta x \cdot \Delta x + \Delta h \cdot \Delta x + |\Delta y|^2 \\ &\quad \geqslant \mu |\Delta x|^2 - k |\Delta y| |\Delta x| + |\Delta y|^2 \\ &\quad \geqslant 2\sqrt{\mu} |\Delta x| |\Delta y| - k |\Delta x| |\Delta y| \geqslant 0. \end{aligned}$$

4. Differential Inequalities—More Applications

In this section, we consider some applications of differential inequalities together with Theorem 2.2 to obtain the existence of periodic solutions of (1.1). The results we establish are extensions of some special cases of results of Knobloch [6] and the author [17] obtained for one-dimensional second-order equations. In order to avoid unnecessary complications in the proofs, we shall assume that f is independent of x'. The results, however, remain valid if f depends on x' and satisfies a suitable growth condition (for example, a Nagumo condition) with respect to x'.

In E^n, we consider the usual partial ordering $x \leqslant y$ if and only if $x_i \leqslant y_i$, $i = 1,\ldots, n$ and $x < y$ if and only if $x_i < y_i$, $i = 1,\ldots, n$. A function $\alpha : I \to E^n$ which is twice-continuously differentiable, will be called a *lower solution* of (1.1) if

$$\alpha''(t) \geqslant f(t, \alpha(t)), \qquad t \in I. \tag{4.1}$$

Similarly, a twice-continuously differentiable function $\beta : I \to E^n$ is called an *upper solution* of (1.1) if

$$\beta''(t) \leqslant f(t, \beta(t)), \qquad t \in I. \tag{4.2}$$

Further, $f(t, x)$ is said to be of type D on a set $K \subset I \times E^n$ if for every (t, x), $(t, y) \in K$ with $x \leqslant y$, $x_i = y_i$

$$f_i(t, x) \geqslant f_i(t, y).$$

THEOREM 4.1. *Let there exist lower and upper solutions* α *and* β *of* (1.1) *with* $\alpha(t) \leqslant \beta(t)$, $t \in I$, *and let* f *be of type* D *on the set*

$$w = \{(t, x) : \alpha(t) \leqslant x \leqslant \beta(t), t \in I\}.$$

Then for every y *and* z, $\alpha(0) \leqslant y \leqslant \beta(0)$, $\alpha(T) \leqslant z \leqslant \beta(T)$, *the boundary-value problem*

$$\begin{cases} x'' = f(t, x) \\ x(0) = y, \qquad x(T) = z \end{cases} \tag{4.3}$$

has a solution $x(t)$ *with* $(t, x(t)) \in w$.

Proof. Define $F(t, x)$ on $I \times E^n$ in the following way. For each $i = 1,\ldots, n$, let

$$F_i(t, x) = \begin{cases} f_i(t, \bar{x}) + \dfrac{x_i - \beta_i(t)}{1 + |x_i|}, & \text{if } x_i > \beta_i(t) \\ f_i(t, \bar{x}), & \text{if } \alpha_i(t) \leqslant x_i \leqslant \beta_i(t) \\ f_i(t, \bar{x}) + \dfrac{x_i - \alpha_i(t)}{1 + |x_i|}, & \text{if } x_i < \alpha_i(t), \end{cases}$$

where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$, and

$$\bar{x}_j = \begin{cases} \beta_j(t), & \text{if } x_j > \beta_j(t) \\ x_j, & \text{if } \alpha_j(t) \leqslant x_j \leqslant \beta_j(t) \\ \alpha_j(t), & \text{if } x_j < \alpha_j(t), \end{cases}$$

$j = 1,\ldots, n$.

Then $F(t, x)$, so defined, is bounded on $I \times E^n$.

Let y and z be as in the statement of the theorem and consider the boundary-value problem

$$\begin{cases} x'' = F(t, x), \\ x(0) = y, \qquad x(T) = z. \end{cases} \tag{4.4}$$

Since F is bounded, (4.4) has a solution $x(t)$ [4, p. 424]. We next prove that $(t, x(t)) \in w$ and hence show that $x(t)$ is a solution of (4.3). To establish this claim, we show that $\alpha(t) \leqslant x(t)$. The proof that $x(t) \leqslant \beta(t)$ follows similarly.

Assume that $\alpha(t) \nleqslant x(t)$. Then there exists an index i, $1 \leqslant i \leqslant n$ and points t_1, t_2, $0 \leqslant t_1 < t_2 \leqslant T$, such that

$$\alpha_i(t_j) = x_i(t_j), \qquad j = 1, 2$$

and

$$\alpha_i(t) > x_i(t), \qquad t_1 < t < t_2 .$$

The difference $\alpha_i(t) - x_i(t)$ therefore will assume a positive maximum at some point $t_0 \in (t_1, t_2)$, where $\alpha_i'(t_0) = x_i'(t_0)$. At t_0, however (note that $\alpha'' \geqslant F(t, \alpha)$),

$$\begin{aligned} \alpha_i''(t_0) - x_i''(t_0) &\geqslant F_i(t_0, \alpha(t_0)) - F_i(t_0, x(t_0)) \\ &\geqslant f_i(t_0, \alpha(t_0)) - f_i(t_0, \bar{x}(t_0)) - \frac{x_i(t_0) - \alpha_i(t_0)}{1 + |x_i(t_0)|} \\ &\geqslant f_i(t_0, \alpha(t_0)) - f_i(t_0, \alpha(t_0)) + \frac{\alpha_i(t_0) - x_i(t_0)}{1 + |x_i(t_0)|} \\ &> 0. \end{aligned}$$

(Here we have used the fact that $f_i(t_0, \bar{x}(t_0)) \leqslant f_i(t_0, \alpha(t_0))$). The last inequality, however, is a contradiction of the fact that $\alpha_i(t) - x_i(t)$ has a positive maximum at t_0.

THEOREM 4.2. *Let the hypotheses of Theorem* 4.1 *hold and let* α *and* β *be such that*

$$\alpha(0) = \alpha(T), \qquad \beta(0) = \beta(T), \qquad \alpha'(0) \geqslant \alpha'(T), \qquad \beta'(0) \leqslant \beta'(T). \quad (4.5)$$

Further, assume that for every y, $\alpha(0) \leqslant y \leqslant \beta(0)$, (4.3) (*with* $y = z$) *has at most one solution* $x(t)$ *such that* $(t, x(t)) \in w$. *Then there exists a periodic solution of* (1.1).

Proof. Let $[\alpha, \beta] = \{y : \alpha(0) \leqslant y \leqslant \beta(0)\}$. Then by Theorem 4.1 and hypothesis, there exists a unique solution $x(t; y)$ of (1.1) such that

$$x(0; y) = x(T; y)$$

and $\alpha(t) \leqslant x(t; y) \leqslant \beta(t)$, for every $y \in [\alpha, \beta]$. Further, there exists a constant $N > 0$ (depending on $[\alpha, \beta]$) such that $|x(t; y)|$, $|x'(t; y)| \leqslant N$, for every $y \in [\alpha, \beta]$. The vector field

$$U(y) = x'(0; y) - x'(T; y)$$

is, therefore, continuous on $[\alpha, \beta]$. If $\alpha(0) = \beta(0)$, it follows from (4.5) that $U(\alpha(0)) = 0$ and the proof is complete. In fact, if there exists i, $1 \leqslant i \leqslant n$ such that $\alpha_i(0) = \beta_i(0)$, then it follows again from (4.5) that the i-th component of $U(y)$, $U(y)_i$, is zero for every $y \in [\alpha, \beta]$. We therefore only need to consider the components $U(y)_j$ for those integers j for which $\alpha_j(0) < \beta_j(0)$. We assume now that $\alpha_j(0) < \beta_j(0)$ for $j = 1,..., n$, the contrary situation will follow from above and by a repetition of the argument to follow in a lower dimensional setting.

Let $\Omega = \{y : \alpha(0) < y < \beta(0)\}$. If there exists $y \in \partial\Omega$ such that $U(y) = 0$, the proof is complete. Otherwise, we may assume that U is nondegenerate on $\partial\Omega$. Let A be the involution on $\partial\Omega$ mapping y into the point symmetric to y about $[\alpha(0) + \beta(0)]/2$.

Let h_i be a continuous function on $[\alpha_i(0), \beta_i(0)]$ such that $h_i(\beta_i(0)) < 0 < h_i(\alpha_i(0))$ and let $\epsilon > 0$ be given. Consider the vector field $\Phi(y; \epsilon)$ defined on $\bar{\Omega}$ by

$$\Phi(y; \epsilon) = U(y) + \epsilon(h_i(y_i),..., h_n(y_n)).$$

Then if y is on the face $y_i = \alpha_i(0)$, then

$$x_i'(0; y) \geqslant \alpha_i'(0) \geqslant \alpha_i'(T) \geqslant x_i'(T; y),$$

and hence $U(y)_i \geqslant 0$, therefore

$$\Phi(y; \epsilon)_i = U(y)_i + \epsilon h_i(\alpha(0)) > 0.$$

And if $\bar{y} = Ay$ (y, as above) then $\bar{y}$ lies on the face $y_i = \beta_i(0)$ and we obtain (using the properties of β) that

$$\Phi(\bar{y}, \epsilon)_i = U(\bar{y})_i + \epsilon h_i(\beta_i(0)) \leqslant \epsilon h_i(\beta_i(0)) < 0.$$

Hence $\Phi(y; \epsilon)$ and $\Phi(Ay; \epsilon)$ do not have the same direction, which implies that $\Phi(y; \epsilon)$ must have a zero in Ω. Pick a sequence $\epsilon_1 > \epsilon_2 > \cdots$ converging monotonically to zero and get a sequence $\{y^n\}$ of zeros of $\Phi(y; \epsilon_n)$ in Ω. This sequence must have a convergent subsequence converging to a zero of $U(y)$ in $\bar{\Omega}$.

COROLLARY 4.3. *Assume there exist constant vectors* α *and* β, $\alpha \leqslant \beta$, *such that*

$$f(t, \alpha) \leqslant 0 \leqslant f(t, \beta)$$

and f *is of type D on* $\{x : \alpha \leqslant x \leqslant \beta\}$ *for each* $t \in I$. *Let* $f(t, x)$ *be Lipschitz continuous with respect to* x, *i.e., there exists a constant* $L > 0$ *such that*

$$|f(t, x) - f(t, y)| \leqslant L \,| x - y |,$$

for all $\alpha \leqslant x,\ y \leqslant \beta$. *Then* (1.1) *has a periodic solution provided* $L < 8/T^2$.

Proof. By Theorem 4.1, there exists a solution $x(t)$ of (1.1) such that $x(0) = y = x(T)$, with $\alpha \leqslant x(t) \leqslant \beta$, for every y, $\alpha \leqslant y \leqslant \beta$. Suppose x and $\bar{x}$ are two such solutions. Then both x and $\bar{x}$ are fixed points of the operator S, defined by

$$(Sx)(t) = \int_0^T G(t, s) f(s, x(s))\, ds + y,$$

where $G(t, s)$ is the Green's function associated with this boundary-value problem. If we define $\| x \| = \max_I | x(t)|$, then

$$\| Sx - S\bar{x} \| \leqslant \frac{T^2}{8} L \| x - \bar{x} \|.$$

Hence S is a contraction mapping and consequently $x = \bar{x}$. We may now apply Theorem 4.2 to obtain the conclusion.

Remark. If the constant vectors α and β of Corollary 4.3 are such that $\alpha_j = \beta_j$ for some j, $1 \leqslant j \leqslant n$, then the hypotheses on f imply that $f_j(t, \alpha) = f_j(t, \beta) = 0$.

EXAMPLE. Consider the two-dimensional second-order system

$$\begin{cases} x'' = x^3 - y + p(t), \\ y'' = -x + y^3 + q(t), \end{cases} \tag{4.6}$$

where $| p(t)|$, $| q(t)| \leqslant a$. Let $b > 0$ be chosen so that $b^3 - b - a \geqslant 0$. Then letting $\beta = (b, b)$ and $\alpha = (-b, -b)$, we find that β is an upper and α is a lower solution of (4.6). Further the right side of (4.6) is of type D for $-b \leqslant x,\ y \leqslant b$, and is Lipschitz-continuous on that region. Hence, we may conclude that for T sufficiently small, the hypotheses of Corollary 4.3 are satisfied.

The hypothesis that f be of type D on the region defined by α and β may be dropped provided we change the concept of lower and upper solutions. The next result covers such a case. We shall omit its proof since it is similar to the proof of Theorem 4.2.

THEOREM 4.4. *Let* α *and* β *be twice-continuously differentiable functions satisfying* (4.5) *with* $\alpha(t) \leqslant \beta(t)$, $t \in I$. *Further assume that*

$$\begin{cases} \alpha_i''(t) \geqslant f_i(t, x_1, \ldots, x_{i-1}, \alpha_i(t), x_{i+1}, \ldots, x_n) \\ \beta_i''(t) \leqslant f_i(t, x_1, \ldots, x_{i-1}, \beta_i(t), x_{i+1}, \ldots, x_n) \end{cases} \tag{4.7}$$

for $\alpha_j(t) \leqslant x_j \leqslant \beta_j(t), j \neq i, i = 1, \ldots, n$.

Then (1.1) *has a periodic solution provided that for every* y, $\alpha(0) \leqslant y \leqslant \beta(0)$, (4.3) (*with* $y = z$) *has at most one solution* $x(t)$ *such that* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$.

EXAMPLE. Consider the two-dimensional second-order system

$$\begin{cases} x'' = x + \sin y + p(t) \\ y'' = y + \cos x + q(t) \end{cases} \tag{4.8}$$

where $|p(t)|$, $|q(t)| \leqslant a$. Let $b > 0$ be a constant such that $b \geqslant 1 + a$. Taking $\beta = (b, b)$ and $\alpha = (-b, -b)$, we see that (4.7) is satisfied. Again the right side of (4.8) satisfies a Lipschitz condition, so we may conclude that (4.8) has a periodic solution provided T is sufficiently small.

Another class (essentially the class considered by Heimes [5]) of differential equations may be shown to have periodic solutions by means of Theorem 2.2 and the differential inequality approach used in this section.

THEOREM 4.5. *Let* $f(t, x)$ *have the property that for each* $i = 1,..., n$, $f_i(t, x) \geqslant f_i(t, \bar{x})$ *whenever* $x_i - \bar{x}_i = \max_j(x_j - \bar{x}_j) \geqslant 0$. *Let* $\alpha(t)$ *and* $\beta(t)$ *be lower and upper solutions of* (1.1), *respectively, satisfying* (4.3) *with* $\alpha(0) \leqslant \beta(0)$. *Then there exists a periodic solution* $x(t)$ *of* (1.1) *with* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$.

Proof. It follows from the results of Heimes [5] that $\alpha(t) \leqslant \beta(t)$, $t \in I$ and that (1.1) has a unique solution $x(t; y)$, satisfying $x(0; y) = x(T; y)$ and $\alpha(t) \leqslant x(t; y) \leqslant \beta(t)$ for every y, $\alpha(0) \leqslant y \leqslant \beta(0)$. The proof is completed by following the steps used in the proof of Theorem 4.2.

COROLLARY 4.6. *For each* $i = 1,..., n$, *let* $g_i(t, x)$ *and* $h_i(t, x)$ *be continuous functions on* $I \times E^n$ *into* E^n *having the properties that there exist constants* $\delta_i > 0$ *and* $\Delta_i > 0$ *such that* $g_i(t, x) \geqslant \delta_i$ *and* $|h_i(t, x)| \leqslant \Delta_i$. *Then the system*

$$x_i'' = x_i g_i(t, x) + h_i(t, x), \qquad i = 1,..., n \tag{4.9}$$

has a periodic solution.

Proof. Let $a_i > 0$ be chosen so that $a_i\delta_i \geqslant \Delta_i$. Let $\beta = (a_1, ..., a_n)$, $\alpha = -\beta$, and $G_i = \max\{g_i(t, x) : t \in I, |x_j| \leqslant a_j, j = 1,..., n\}$. Further let $\mathscr{B} = \{\varphi \in C^1[0, T] : \varphi(0) = \varphi(T), \varphi'(0) = \varphi'(T)\}$, and for $\varphi \in \mathscr{B}$, let $\|\varphi\| = \max\{\max |\varphi(t)|, \max |\varphi'(t)|\}$. Then $(\mathscr{B}, \|\cdot\|)$ is a Banach space. The set $\mathscr{C}$ defined by

$$\mathscr{C} = \{\varphi \in \mathscr{B} : |\varphi_i(t)| \leqslant a_i, |\varphi_i'(t)| \leqslant (a_iG_i + \Delta_i)T,$$
$$|\varphi_i'(t) - \varphi_i'(s)| \leqslant (a_iG_i + \Delta_i) |t - s|, t, s \in I\}$$

then is a compact convex subset of $\mathscr{B}$. For each $\varphi \in \mathscr{C}$, we consider the linear system

$$x_i'' = x_i g_i(t, \varphi(t)) + h_i(t, \varphi(t)), \qquad i = 1,..., n. \tag{4.10}$$

The hypotheses of Theorem 4.5 are satisfied for the system (4.10) with α and β chosen as above. Hence, for each $\varphi \in \mathscr{C}$, system (4.10) has a periodic solution $x(t, \varphi)$ with $\alpha \leqslant x(t, \varphi) \leqslant \beta$. Furthermore, $x \in \mathscr{C}$. It follows from Theorem 2 of [19] that $x(t, \varphi)$ is unique. Define the operator S on $\mathscr{C}$ into itself by

$$(S\varphi)(t) = x(t, \varphi).$$

One easily verifies that the Schauder Fixed-Point Theorem is applicable, from which one concludes that the operator S has a fixed point in $\mathscr{C}$, and hence that (4.9) has a periodic solution.

Remark. We note that if it is known *a priori* that (4.3) has at most one solution for Eq. (4.9) with $y = z$, then Corollary 4.6 follows immediately from Theorem 4.4. In the absence of such an assumption, however, Theorem 4.4 does not apply. A result similar to, but independent from, Corollary 4.6 has recently been obtained by Mawhin [15].

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