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Entropy bound of local quantum field theory with generalized uncertainty principle

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ABSTRACT

We study the entropy bound for local quantum field theory (LQFT) with generalized uncertainty principle. The generalized uncertainty principle provides naturally a UV cutoff to the LQFT as gravity effects. Imposing the non-gravitational collapse condition as the UV–IR relation, we find that the maximal entropy of a bosonic field is limited by the entropy bound $A^{3/4}$ rather than A with A the boundary area.

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1. Introduction

The counting of degrees of freedom of local quantum field theory (LQFT) including gravity effects is an important issue. For example, statistical mechanics implies that a thermal photon gas could be described by LQFT without gravity. Its entropy takes the form of $S \sim T^3 L^3$ when it is confined to a box of size L as an IR cutoff. If the temperature T could be an arbitrarily chosen parameter, one finds that the system has an entropy proportional to the volume L^3 . However, this temperature has to be limited by the energy bound $E \sim T^4 L^3 \leq E_{\text{BH}} \sim L$, or else the system will undergo collapse to form a black hole when considering the gravity effects. Applying this energy bound to the entropy, one finds the entropy bound of $S \leq S_{\text{max}} \sim A^{3/4}$, where $A \sim L^2$ is the boundary area of the system. The derivation above is firstly given by 't Hooft in [1]. The entropy bound $A^{3/4}$ for LQFT was also discussed by other authors [2–5].

However, there are still controversies over this topic. Starting from a bosonic field model and imposing the gravitational stability condition, a holographic entropy bound of $S \leq S_{\text{max}} \sim A$ appears as the covariant entropy bound. This may not describe the local quantum field theory including gravity effects because the LQFT

requires a more stricter entropy bound rather than the holographic entropy bound.

In order to obtain a compact relation from the energy and entropy bounds, the UV cutoff Λ is necessarily introduced to regularize the LQFT [2]. Explicitly, the LQFT with $E_\Lambda \simeq \Lambda^4 L^3$ and $S_\Lambda \simeq \Lambda^3 L^3$ is able to describe a thermodynamic system at temperature T , provided that $T \leq \Lambda$. If $T \gg 1/L$, the energy and entropy will be those for a thermal photon gas: $E_R \simeq T^4 L^3$ and $S_R \simeq T^3 L^3$. In this case, the modes with momentum more than the UV cutoff have been excluded from consideration. However, this cutoff could be justified in an average sense, and hence the states with momentum $p > \Lambda$ should be accounted properly.

On the other hand, one believes that the generalized uncertainty principle (GUP) arises from the Heisenberg uncertainty principle when gravity effect is taken into account [6–9]. Its commutation relation of

$$[\vec{x}, \vec{p}] = i\hbar(1 + \beta^2 \vec{p}^2) \quad (1)$$

leads to the generalized uncertainty relation [10]

$$\Delta x \geq \frac{\hbar}{2} \left(\frac{1}{\Delta p} + \beta \Delta p \right) \quad (2)$$

which implies the presence of a minimal length scale

$$\Delta x_{\text{min}} = \hbar \sqrt{\beta}. \quad (3)$$

Thus, the leading order correction to the standard formula is expected to be proportional to the Planck length l_P , if one chooses

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$\beta = G/c^3\hbar$ with the Newton's constant $G = c^3 l_p^2/\hbar$. Hence the GUP plays the same role as the UV cutoff Λ does show because the UV cutoff also determines a minimal detectable length.

In this Letter, we show that the entropy bound for LQFT is $A^{3/4}$ rather than A by using the GUP and non-gravitational collapse condition, instead of the UV cutoff introduced by hand.

2. State counting with UV cutoff

We consider a massless scalar field confined to three-dimensional spacelike cube of size L in Minkowski space, as has been done in [11–14]. The modes of the field are then the solution to the wave equation $\nabla^2\phi = 0$ with periodic boundary conditions. Any mode of quantized wave vector \vec{k} could be labelled by three positive integers $\vec{m} = (m_x, m_y, m_z)$ as $\vec{k} = \frac{\pi}{L}\vec{m}$. The corresponding energy of the mode is

$$E_{\vec{m}} = \hbar\omega_{\vec{m}} = \hbar c|\vec{k}| = \frac{\hbar\pi c}{L}\sqrt{\vec{m}\cdot\vec{m}} \quad (4)$$

where we used the relation of $\omega = ck = p/\hbar$. Hereafter we use the Planck units of $G = \hbar = c = k_B = 1$, which implies a simple relation of $p = \omega = k$. The total number of these quantized modes is calculated by the replacement

$$N = \sum_{\vec{k}} 1 \rightarrow \frac{L^3}{(2\pi)^3} \int d^3\vec{k} = \frac{L^3}{2\pi^2} \int_0^\Lambda \omega^2 d\omega = \frac{\Lambda^3 L^3}{6\pi^2}, \quad (5)$$

where Λ is introduced to be the UV energy cutoff of the LQFT. See Fig. 1 for the uniform distribution up to $\omega = \Lambda$. Thanks to the UV cutoff, N is finite and thus there exists one-to-one correspondence between the wave vector \vec{k} and a character i with $i \in [1, N]$. Upon quantization of a massless scalar field which obey the Bose–Einstein statistics, we can construct the Fock states by assigning occupying number n_i to these N different modes

$$|\Psi\rangle = |n(\vec{k}_1), n(\vec{k}_2), \dots, n(\vec{k}_N)\rangle \rightarrow |n_1, n_2, \dots, n_N\rangle, \quad (6)$$

where the normalized state contains $n(\vec{k}_1)$ particles with momentum \vec{k}_1 , $n(\vec{k}_2)$ particles with momentum \vec{k}_2 , and so on. Thus, the basis of the Hilbert space \mathcal{H} of the system is spanned by each different set of $\{n_i\}$, and the number of occupancies $\{n_i\}$ gives the corresponding dimension of the Hilbert space ($\dim\mathcal{H}$). Usually, the dimension of the Hilbert space is infinite for bosons unless the number of particles in each mode i is constrained by a finite bound. However, the non-gravitational collapse requirement makes the permissible dimension of Hilbert space finite as

$$E = \sum_{i=1}^N n_i \omega_i \leq E_{\text{BH}} = L. \quad (7)$$

The number of solutions or occupancies $\{n_i\}$ satisfying the above bound gives the dimension $W \equiv \dim\mathcal{H}$ of physically permitted Hilbert space. In other words, in order to determine the dimension of Hilbert space, one has to know the number of admissible solutions $\{n_i\}$ satisfying Eq. (7). This corresponds to the knapsack or counting lattice points problem. When confining $\{n_i\}$ to a Cartesian coordinate system $\{x_i\}$, the question refers to the counting lattice points contained within the convex polytopes determined by

$$\sum_{i=1}^N x_i \omega_i \leq E_{\text{BH}}, \quad x_i \geq 0, \quad (8)$$

with right-angle side lengths

$$L_i = \frac{E_{\text{BH}}}{\omega_i}, \quad \text{with } i \in [1, N]. \quad (9)$$

Actually, it is difficult to find an exact solution to this question. For $L_i \gg 1$, one may use the volume of the corresponding polytopes to approximately evaluate the number of lattice points within them.

We note that an N particle state with one particle occupying one mode ($n_i = 1$) corresponds to the lowest energy state with N modes simultaneously excited. In this case, it should satisfy the gravitational stability condition of Eq. (7). Hence, the energy bound is given by

$$E \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \omega^3 d\omega = \frac{\Lambda^4 L^3}{8\pi^2} \leq E_{\text{BH}}. \quad (10)$$

The last inequality implies the UV–IR relation

$$\Lambda^2 \leq \frac{1}{L}. \quad (11)$$

On the other hand, the entropy associated with the system is given by

$$S = - \sum_{j=1}^W \rho_j \ln \rho_j, \quad (12)$$

where ρ_j is the possible distribution of the Hilbert state basis. It is clear that the maximum value of the entropy is realized by taking a uniform distribution of $\rho_j = 1/W$. Then, the maximum entropy is given by

$$S_{\text{max}} = - \sum_{j=1}^W \frac{1}{W} \ln \frac{1}{W}, \quad (13)$$

where the bound of W is determined by

$$W = \dim\mathcal{H} < \sum_{m=0}^N \frac{z^m}{(m!)^2} \leq \sum_{m=0}^{\infty} \frac{z^m}{(m!)^2} = I_0(2\sqrt{z}) \sim \frac{e^{2\sqrt{z}}}{\sqrt{4\pi\sqrt{z}}}. \quad (14)$$

Here I_0 is the zeroth-order Bessel function of the second kind. Since z is given by

$$z = \sum_{i=1}^N L_i \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \left[\frac{E_{\text{BH}}}{\omega} \right] \omega^2 d\omega = \frac{\Lambda^2 L^4}{4\pi^2}, \quad (15)$$

we find the bound

$$z \leq L^3, \quad (16)$$

where we used the UV–IR relation in Eq. (11). Therefore, we have the bound for the maximum entropy

$$S_{\text{max}} = \ln W \leq A^{3/4}. \quad (17)$$

This is a brief derivation of the entropy bound by using the LQFT.

3. State counting with GUP

The GUP relation of Eq. (2) has an effect on the density of states in momentum space [10] as

$$\frac{d^3\vec{p}}{(1 + \beta\vec{p}^2)^3} \quad (18)$$

with an important factor of $1/(1 + \beta\vec{p}^2)^3$, which effectively cuts off the integral beyond $p = 1/\sqrt{\beta}$. Intuitively, this can be understood from the observation that the right-hand side of Eq. (1) includes a \vec{p} -dependent term and thus affect the cell size in phase space as “being \vec{p} -dependent”. Rigorously, making use of the Liouville theorem, one could show that the invariant weighted phase space volume under time evolution is given by [10]

$$\frac{d^3\vec{x}d^3\vec{p}}{(1 + \beta\vec{p}^2)^3}, \quad (19)$$

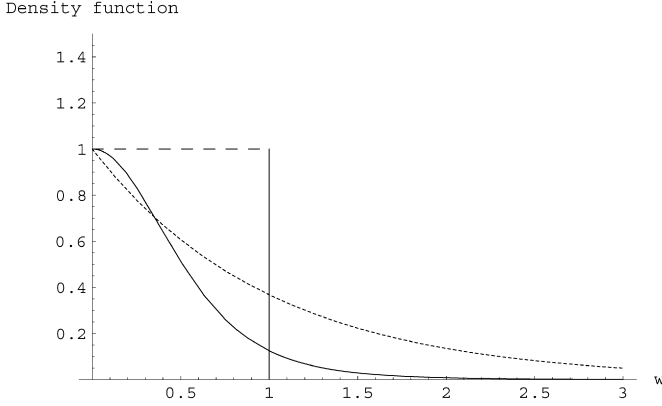


Fig. 1. Density functions for regularization of LQFT as function of ω . The dashed line denotes “uniform density function” as the UV cutoff, while the solid curve represents the GUP density function in Eq. (18) with $\beta = 1$. If one considers the solid curve within the box, it provides the density function for UV cutoff and GUP for Section 4. The dotted curve denotes density function for an all-order result in Appendix A. Here we choose $\Lambda = 1/\sqrt{\beta} = 1$.

where the classical commutation relations corresponding to the quantum commutation relation of Eq. (1), as $\{x_i, p_j\} = (1 + \beta p^2)\delta_{ij}$, $\{p_i, p_j\} = 0$, and $\{x_i, x_j\} = 2\beta(p_i x_j - p_j x_i)$, are used. This factor plays a role of the UV cutoff of the consequent momentum integration, as shown in Fig. 1.

For a massless scalar field confined to three-dimensional spacelike cube of size L , the total number of modes can be calculated as

$$N = \frac{L^3}{2\pi^2} \int_0^\infty \frac{\omega^2 d\omega}{(1 + \beta\omega^2)^3} = \frac{1}{32\pi} \frac{L^3}{\beta^{3/2}} \sim \frac{L^3}{\beta^{3/2}}. \tag{20}$$

As expected, due to strong suppression of density of state at high momenta, the total number is rendered finite with $1/\sqrt{\beta}$ acting effectively as the UV cutoff. This result is in strong contrast to the previous calculations where the UV cutoff Λ is an arbitrary scale which must be introduced by hand, and where one must assume that the physics beyond this cutoff does not contribute (see Fig. 1).

The energy bound is given by

$$E \rightarrow \frac{L^3}{2\pi^2} \int_0^\infty \frac{\omega^3 d\omega}{(1 + \beta\omega^2)^3} = \frac{1}{8\pi^2} \frac{L^3}{\beta^2} \leq E_{\text{BH}}. \tag{21}$$

The last inequality implies the important UV–IR relation

$$L \leq \beta. \tag{22}$$

In order to calculate the maximum entropy, we need to know z which is calculated to be

$$z \rightarrow \frac{L^3}{2\pi^2} \int_0^\infty \left[\frac{E_{\text{BH}}}{\omega} \right] \frac{\omega^2 d\omega}{(1 + \beta\omega^2)^3} = \frac{1}{8\pi^2} \frac{L^4}{\beta}. \tag{23}$$

Using Eq. (22), one has the bound for z as

$$z \leq L^3. \tag{24}$$

Finally, we arrive at the same bound for the maximum entropy

$$S_{\text{max}} = \ln W \leq A^{3/4}. \tag{25}$$

4. State counting with UV cutoff and GUP

For a massless scalar field confined to 3-dimensional spacelike cube of size L , the mode counting method is changed to include the GUP effect (see the solid curve in the box in Fig. 1)

$$N \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \frac{\omega^2 d\omega}{(1 + \beta\omega^2)^3} \approx \frac{L^3}{2\pi^2} \left(\frac{\Lambda^3}{3} - \frac{3\Lambda^5\beta}{5} \right). \tag{26}$$

That is, the total number of states is decreased when we include the GUP effect.

The energy bound is modified up to β

$$E \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \frac{\omega^2 d\omega}{(1 + \beta\omega^2)^3} \approx \frac{L^3}{2\pi^2} \left(\frac{\Lambda^4}{4} - \frac{\beta\Lambda^6}{2} \right) \leq E_{\text{BH}}. \tag{27}$$

The last inequality implies

$$\Lambda^2 \leq \frac{1}{L} \left(1 + \frac{\beta}{L} \right). \tag{28}$$

The maximum entropy is given by

$$S_{\text{max}} = - \sum_{j=1}^W \frac{1}{W} \ln \frac{1}{W}, \tag{29}$$

with $W \sim e^{2\sqrt{z}}$. Since z is given up to β by

$$z \rightarrow \frac{L^3}{2\pi^2} \int_0^\Lambda \left[\frac{E_{\text{BH}}}{\omega} \right] \frac{\omega^2 d\omega}{(1 + \beta\omega^2)^3} \approx \frac{L^4}{2\pi^2} \left(\frac{\Lambda^2}{2} - \frac{3\beta\Lambda^4}{4} \right), \tag{30}$$

one finds the bound when using Eq. (28)

$$z \leq L^3 - \frac{\beta L^2}{2}. \tag{31}$$

Therefore, we have a modified bound for the maximum entropy

$$S_{\text{max}} = \ln W \leq A^{3/4} - \frac{\beta}{4} A^{1/4} \tag{32}$$

which shows clearly that the upper bound is decreased due to the GUP.

5. Discussions

In this Letter, we show how gravity effects offer a way to calculate the maximal entropy bound of the LQFT. Gravity effects provide UV and IR cutoffs to the LQFT. If the GUP is really considered as a reflection of gravity effects, it gives a UV cutoff which makes the total number of modes N finite. On the other hand, the energy bound implies that the number of particles n_i is limited and the total energy of the system is less than that of the same-sized black hole. The former makes the dimension of Hilbert space finite, while the latter leads to the bound as the UV–IR relation. Then, we obtain the maximal entropy bound of $A^{3/4}$. This is consistent with the picture that the gravity effects make the dimension of Hilbert space finite.

We wish to emphasize why our work is meaningful by comparing it with two known approaches. Without the UV cutoff, we could derive the entropy bound of $S \leq S_{\text{max}} \sim A^{3/4}$, as suggested by 't Hooft in [1], if the UV cutoff is much larger than the temperature and the energy bound is imposed. However, this corresponds to a heuristic derivation because the LQFT was not explicitly used for calculation. As was briefly reviewed in Section 2, we introduce UV and IR cutoffs to calculate the entropy bound of a massless field when using the LQFT. This means that we need both UV-control

and IR-control to get an important UV–IR relation of Eq. (11). However, as is shown Fig. 1, the UV cutoff Λ may correspond to an *ad hoc* density function because it was introduced by hand. In this work, we introduce the new cutoff β based on the GUP which effectively cuts off the short distance region. This case provides a more natural derivation of $A^{3/4}$ than using the UV cutoff Λ because the GUP is considered as a meaningful extension of the first principle “Heisenberg uncertainty principle” when taking into gravity effects account.

We mention that as was shown in Eq. (20), the total number of modes N is clearly determined by imposing the GUP. In order to confirm this, we introduce an all-order result to the Heisenberg uncertainty relation. As is shown in Appendix A, we choose the commutation relation as a way of implementing all-order GUP corrections. Then, the total number N of modes in Eq. (35) takes a similar form as in Eq. (20). This supports that the GUP corrections to the Heisenberg uncertainty relation is equivalent to a UV cutoff to the LQFT.

Finally, we note that the non-gravitational collapse condition plays the important role: it makes the dimension of Hilbert space finite and thus provides the bound of maximal entropy.

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Appendix A. An all-order result in GUP

The GUP commutation relation in Eq. (1) can be extended into [15]

$$[\vec{x}, \vec{p}] = i\hbar e^{\beta^2 \vec{p}^2}, \quad (33)$$

which includes all order corrections to the Heisenberg uncertainty principle. In this case, the weighting factor is given by [16]

$$d^3\vec{p} e^{\beta^2 \vec{p}^2}. \quad (34)$$

The total number of modes is calculated to be

$$N \rightarrow \frac{L^3}{2\pi^2} \int_0^\infty \omega^2 e^{-\beta^2 \omega^2} d\omega = \frac{L^3}{8\pi^{3/2} \beta^3} \sim \frac{L^3}{\beta^3} \quad (35)$$

without any ambiguity. The energy bound is obtained as

$$E \rightarrow \frac{L^3}{2\pi^2} \int_0^\infty \omega^3 e^{-\beta^2 \omega^2} d\omega = \frac{L^3}{4\pi^2 \beta^4} \leq E_{\text{BH}} = L, \quad (36)$$

where the last equality implies the UV–IR relation as the bound

$$L \leq \beta^2. \quad (37)$$

In order to compute the maximal entropy, one has to know the variable z

$$z \rightarrow \frac{L^3}{2\pi^2} \int_0^\infty \left[\frac{E_{\text{BH}}}{\omega} \right] \omega^2 e^{-\beta^2 \omega^2} d\omega = \frac{L^4}{4\pi^2 \beta^2}. \quad (38)$$

Using the energy bound of Eq. (36), one finds the bound for z as

$$z \leq L^3. \quad (39)$$

Thus, the maximum entropy bound is confirmed to be

$$S_{\text{max}} = \ln W \leq A^{3/4}. \quad (40)$$

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