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Research Note

# The complexity of approximating MAPs for belief networks with bounded probabilities

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**Abstract**

Probabilistic inference and *maximum a posteriori* (MAP) explanation are two important and related problems on Bayesian belief networks. Both problems are known to be NP-hard for both approximation and exact solution. In 1997, Dagum and Luby showed that efficiently approximating probabilistic inference is possible for belief networks in which all probabilities are bounded away from 0. In this paper, we show that the corresponding result for MAP explanation does not hold: finding, or approximating, MAPs for belief networks remains NP-hard for belief networks with probabilities bounded within the range $[l, u]$ for any $0 \leqslant l < 0.5 < u \leqslant 1$. Our results cover both deterministic and randomized approximation. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Bayesian belief networks [7,8] are an important probabilistic knowledge representation for reasoning under uncertainty. Two important and related evidential reasoning problems on Bayesian belief networks are probabilistic inference, also known as belief updating, and MAP explanation, also known as belief revision. In probabilistic inference, the objective is to evaluate the probability $P(X = x \mid \mathcal{E})$ of a given instantiation $x$ of a domain variable $X$ conditioned on some observed evidence $\mathcal{E}$; while in MAP explanation, the objective is to find the instantiation $\mathcal{A}$ with *maximum a posteriori* probability $P(\mathcal{A} \mid \mathcal{E})$. Complexity results for these two problems have until now mirrored one another. For singly-connected

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networks, both problems are solvable in linear-time by similar message-passing algorithms [8]. Probabilistic inference was shown to be NP-hard [2] in 1990; MAP explanation was shown to be NP-hard [9] in 1994. Approximating probabilistic inference was shown to be NP-hard [3] in 1993; approximating MAP explanation was shown to be NP-hard [1] in 1998.

In 1997, Dagum and Luby [4] showed that their previous NP-hardness result does not hold for the class of belief networks which do not contain *extreme* probabilities, that is, probabilities that come arbitrarily close to 0. This is significant as the majority of belief networks of interest for practical applications are likely to fall in this class. Unfortunately, as we shall show in this paper, the corresponding result does not hold for MAP explanation: we show that approximating MAP explanation remains NP-hard when network probabilities are bounded within any fixed interval $[l, u]$ where $0 \leqslant l < 0.5 < u \leqslant 1$. Our results will cover both deterministic and randomized approximation, and apply even for null evidence sets.

## 2. Bayesian belief networks

A *Bayesian belief network* is a triple $B = \langle V, E, P\rangle$, where $V$ is a finite set of nodes identified with random variables, $(V, E)$ is a directed acyclic graph which forms an independency map [8] over $V$, and $P = \{P_v \mid v \in V\}$ is a set of local conditional probability distributions, where each $P_v$ specifies the probability of each possible instantiation of $v$, given every possible instantiation of its parents $\pi(v)$. Let $u$ and $l$, respectively, be the largest and smallest probabilities present in the set of local probability distributions $\{P_v \mid v \in V\}$ for a given belief network $B$. Then, $B$ will be said to have a *local variance bound* (LVB) $\sigma$ equal to $u/l$.

**Definition.** For a given $\sigma \geqslant 1$, we will let $\mathcal{B}_\sigma$ denote the class of belief networks with LVB no greater than $\sigma$.

Based on the assumption that a belief network's underlying graph is an independency map of the network variables, the joint probability of any given full instantiation, $\mathcal{A}: V \mapsto v_1, \ldots, v_n$, of the network variables can be computed according to

$$P(v_1, v_2, \ldots, v_n) = \prod_{i=1}^{n} P\bigl(v_i \mid \pi(v_i)\bigr). \tag{1}$$

The *maximum a posteriori* (MAP) explanation problem is an optimization problem where we are given a belief network $B$ and partial assignment $\mathcal{E}$ of $B$ which represents "real-world" observations, or evidence, and we are required to find an instantiation $\mathcal{A}$ which has a *maximum a posteriori* probability $P(\mathcal{A} \mid \mathcal{E})$. In probabilistic inference, we are given a belief network $B$ and partial assignment $\mathcal{E}$ of $B$ and are required to find the probability $P(X = x \mid \mathcal{E})$, where $X$ is a node in the network and $x$ is a value from the domain of $X$.

## 3. Deterministic and randomized approximation

An approximation algorithm with a *ratio bound* $\rho$ for a maximization problem is an algorithm which returns a solution whose quality $s^{app}$ is guaranteed to be within a ratio $\rho$ of the quality $s^{opt}$ of an optimal solution,

$$\frac{s^{opt}}{s^{app}} \leqslant \rho. \tag{2}$$

For randomized approximation, we often seek an algorithm which will return an optimal solution, or a solution within a ratio bound $\rho$ of an optimal solution, with a probability that is guaranteed to be no less than some bound $\delta$.

In the following section, we prove that approximating the MAP problem with a constant ratio bound is NP-hard. Later, we show that randomized approximation in polynomial time is not possible unless RP $=$ NP.

We begin by describing the "known NP-complete problem" we will use in our transformation.

An instance of the ONE-IN-THREE 3-SAT problem consists of a set $U$ of variables, and a collection $C$ of clauses over $U$ such that each clause $c \in C$ has $|c| = 3$. The question is whether there exists a truth assignment for $U$ such that each clause has exactly one true literal. The ONE-IN-THREE 3-SAT problem is NP-complete and remains NP-complete if it is restricted such that no $c \in C$ contains a negated literal [5, p. 259]. It is this restricted no-negation ONE-IN-THREE 3-SAT problem (which we will abbreviate as NN3SAT for the remainder of the paper), that we will use in our proof.

An example of this problem is the set of variables $\{x_1, x_2, x_3, x_4, x_5\}$ and the set of clauses $\{\{x_1, x_4, x_5\}, \{x_2, x_3, x_5\}, \{x_1, x_2, x_5\}, \{x_3, x_4, x_5\}\}$, i.e.,

$$\phi = (x_1 \vee x_4 \vee x_5) \wedge (x_2 \vee x_3 \vee x_5) \wedge (x_1 \vee x_2 \vee x_5) \wedge (x_3 \vee x_4 \vee x_5). \tag{3}$$

This problem instance happens to be satisfiable: one satisfying assignment is $\{x_1 = \mathbf{T}, x_2 = \mathbf{F}, x_3 = \mathbf{T}, x_4 = \mathbf{F}, x_5 = \mathbf{F}\}$.

## 4. Proof of main result

In this section, we prove the following result:

**Theorem 4.1.** *Let $\sigma > 1$ be an arbitrary constant. Approximating the MAP problem for the class $\mathcal{B}_\sigma$ with a fixed ratio-bound $\rho$ is* NP-*hard for any $\rho \geqslant 1$.*

**Construction 4.1.** Let $\rho$ and $\sigma$ be fixed arbitrary constants such that $\rho \geqslant 1$ and $\sigma > 1$. Given an instance of $(U, C)$ of NN3SAT we construct a belief network $B$ over a set of binary variables $V$, and evidence $\mathcal{E}$, as follows:

(i) Let $n = |U|$ and $m = |C|$. Let

$$u = \frac{\sigma}{\sigma + 1}. \tag{4}$$

Note that $1 > u > 0.5$ since $\sigma > 1$. Let

$$d = \lceil \log_{2u} \rho \rceil + 1. \tag{5}$$

(ii) For each $x \in U$, construct a node $v_x \in V$ which will be called a *truth setting* node. All the truth setting nodes will be root nodes. For each truth setting node $v_x$, set $v_x$'s probability distribution to be

$$P(v_x = \mathbf{T}) = \tfrac{1}{2}. \tag{6}$$

(iii) For each $c \in C$, construct $d$ nodes $v_c^1, \ldots, v_c^d \in V$ which will be called *clause satisfaction* nodes. Construct, for each $c \in C$ and for all $i = 1, \ldots, d$, an edge from $v_x$ to $v_c^i$, for each $x \in c$. Therefore, each clause satisfaction node will have an in-degree of exactly three. Set the conditional distribution for each clause satisfaction node $v$ as follows:

$$P(v = \mathbf{T} \mid x_1, x_2, x_3) = \begin{cases} u, & \text{if exactly one of } x_1, x_2, \text{ and } x_3 \text{ is } \mathbf{T}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

(iv) Let $\mathcal{E}$ be $\emptyset$.

**Proof.** Let $A$ be a polynomial-time approximation algorithm for the MAP problem with ratio-bound $\rho$. For a given arbitrary instance $(U, C)$ of NN3SAT, we apply Construction 4.1 to produce an instance $(B, \mathcal{E})$ of the MAP problem and run Algorithm $A$. Let $\mathcal{A}'$ be the solution that is returned by Algorithm $A$. We claim that $(U, C)$ is satisfiable if and only if

$$P(\mathcal{A}') > \left(\tfrac{1}{2}\right)^n (u)^{(m-1)d} \left(\tfrac{1}{2}\right)^d. \tag{7}$$

Let $P(\widehat{\mathcal{A}})$ be the probability of an optimum solution $\widehat{\mathcal{A}}$, if $(U, C)$ is satisfiable; let $P(\widetilde{\mathcal{A}})$ be the probability of an optimum solution $\widetilde{\mathcal{A}}$, if $(U, C)$ is not satisfiable.

We can see that

$$P(\widehat{\mathcal{A}}) = \left(\tfrac{1}{2}\right)^n (u)^{md}, \tag{8}$$

and

$$P(\widetilde{\mathcal{A}}) \leqslant \left(\tfrac{1}{2}\right)^n (u)^{(m-1)d} \left(\tfrac{1}{2}\right)^d. \tag{9}$$

Consequently, since the unsatisfiability of $(U, C)$ implies $P(\mathcal{A}') \leqslant P(\widetilde{\mathcal{A}})$, Eq. (7) follows from

$$\frac{P(\widehat{\mathcal{A}})}{P(\widetilde{\mathcal{A}})} \geqslant \frac{\left(\frac{1}{2}\right)^n (u)^{md}}{\left(\frac{1}{2}\right)^n (u)^{(m-1)d} \left(\frac{1}{2}\right)^d}, \tag{10}$$

$$\geqslant \frac{(u)^d}{\left(\frac{1}{2}\right)^d}, \tag{11}$$

$$\geqslant (2u)^d > \rho. \tag{12}$$

Our transformation can be carried out in polynomial time. One truth setting node is constructed for each $x \in U$, and $d$ clause satisfaction nodes are constructed for each $c \in C$. The number of edges created is exactly $3md$ and each clause satisfaction node has exactly three inputs. In an explicit representation of the local probability distributions, the total number of conditional probability table entries is $2n + 16md$. Belief network B belongs to $\mathcal{B}_\sigma$ since Eq. (4) implies $\sigma = u/(1-u)$. $\square$

## 5. Restricted topology and randomized approximation

### 5.1. Restricted topology

Previous complexity results [1,9] have applied to belief networks with a maximum depth of 4 (i.e., every longest directed path has length 3), assuming functional specification of distributions. The results of this paper apply to belief networks in which every longest directed path has length 1, i.e., *bipartite* belief networks, without any assumptions on specification of distributions. Our results also apply for belief networks in which the maximum in-degree is bounded by $k$, for $k \geqslant 3$. However, they may not directly apply for networks in which the outdegree is bounded.

### 5.2. Randomized approximation

A *Monte Carlo algorithm* $A$ for a yes–no decision problem $\Pi$ is an algorithm which always terminates with an affirmative or negative answer such that

– The probability that $A$ answers affirmatively and the correct answer to $\Pi$ is "no" is 0;
– The probability that $A$ answers negatively and the correct answer to $\Pi$ is "yes" is less than $\frac{1}{2}$.

A problem $\Pi$ belongs to the complexity class RP if and only if it has a polynomial time Monte Carlo algorithm [6]. While it is known that $\mathrm{P} \subseteq \mathrm{RP} \subseteq \mathrm{NP}$, it remains open whether $\mathrm{P} \subset \mathrm{RP}$ and whether $\mathrm{RP} \subset \mathrm{NP}$.

**Theorem 5.1.** *Let $\sigma > 1$ be an arbitrary constant. If there exists a polynomial-time randomized algorithm that returns, with probability greater than some fixed $\delta$, solutions within a fixed ratio bound $\rho$ of the optimal for the MAP problem for the class $B_\sigma$, for any $\rho \geqslant 1$ and any $0 < \delta < 1$, then* $\mathrm{RP} = \mathrm{NP}$.

Let $A$ be such an algorithm; we use $A$ to construct a Monte Carlo algorithm $A'$ by running $A$ $k$ times and answering affirmatively if any of the $k$ runs returns a positive answer, where

$$k = \left\lceil \log_{(1-\delta)} \left(\tfrac{1}{2}\right) \right\rceil + 1. \tag{13}$$

Algorithm $A'$ runs in polynomial time and is a Monte Carlo algorithm since the probability of $k$ "false negatives" is

$$(1-\delta)^k < \tfrac{1}{2}. \tag{14}$$

## 6. Concluding remarks

Complexity results for probabilistic inference and MAP explanation have until now mirrored one another. For singly-connected networks, both problems are solvable in linear-time by similar message-passing algorithms [8]. For multiply-connected networks, probabilistic inference was shown to be NP-hard for exact solution [2] in 1990 and for approximation in 1993; MAP explanation was shown to be NP-hard [9] for exact solution and for approximation [1] in 1998.

In 1997, Dagum and Luby [4] showed that their previous NP-hardness result does not hold for the class of belief networks which contain probabilities that come arbitrarily close to zero in the case that the evidence set is empty or constant-sized. We have shown in this paper that approximating MAP explanation remains NP-hard even when probabilities are bounded within any fixed interval $[l, u]$, where $0 \leqslant l < 0.5 < u \leqslant 1$. Our result holds without restriction on the size of the evidence.

It is not clear why limiting the range of probability values makes approximating probabilistic inference easier but has no corresponding effect (at least in the worst-case complexity sense) on MAP explanation. This is an issue that bears further examination.

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