Additivity of Integrals on Generalized Measure Spaces

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It is proved that the integral is additive on simple summable functions of a generalized measure space. © 1985 Academic Press, Inc.

1. INTRODUCTION

Generalized measure spaces, first introduced by Suppes [10], have now been studied by several authors [2–6, 9, 10, 12]. There are applications of this subject for quantum mechanics, elementary length physics, pattern recognition, and computer science [6]. A generalized measure space (GMS) is a triplet $(\Omega, \mathcal{C}, \mu)$, where $\mathcal{C}$ is a $\sigma$-class (or Dynkin system) of subsets of $\Omega$, that is,

(i) $\Omega \in \mathcal{C}$

(ii) if $A \in \mathcal{C}$, then $A^c \in \mathcal{C}$ (where $A^c$ is the complement of $A$),

(iii) if $A_i \in \mathcal{C}$ are mutually disjoint, then $\bigcup A_i \in \mathcal{C}$, $i = 1, 2, \ldots$;

$\mu$ is a measure on $\mathcal{C}$, that is, a nonnegative set function such that $\mu(\emptyset) = 0$ and $\mu(\bigcup A_i) = \sum \mu(A_i)$ if $A_i$ are mutually disjoint elements of $\mathcal{C}$.

For an example of a $\sigma$-class which is not a $\sigma$-algebra, let $\lambda > 0$ and let $n$ be a positive integer. Then the collection of measurable subsets of the interval $[0, n\lambda]$ whose Lebesgue measures are an integer multiple of $\lambda$ forms a $\sigma$-class. This is an example of an elementary length space [2, 5] and $\lambda$ represents the elementary length. For another example, we consider axiomatic quantum mechanics. In the quantum logic approach [7, 8, 11],
the quantum propositions are represented by elements of a \( \sigma \)-orthomodular poset \( P \) and the states are represented by probability measures on \( P \). A state is \textit{dispersion-free} if its only values are 0 and 1. It is shown in [4] that \( P \) is isomorphic to a \( \sigma \)-class if and only if the set of dispersion-free states on \( P \) is order determining. This result has implications to "hidden variable" theories in quantum mechanics.

Let \((\Omega, \mathcal{F}, \mu)\) be a GMS and let \( \mathcal{B} \) be the Borel \( \sigma \)-algebra on \( R \). A function \( f: \Omega \to R \) is \textit{measurable} if \( \mathcal{F}(f) = f^{-1}(\mathcal{B}) \) is a sub-\( \sigma \)-algebra of \( \mathcal{F} \). Then \((\Omega, \mathcal{F}(f), \mu)\) is a measure space, and \( \int f \, d\mu \) is defined in the usual way. Two functions \( f, g \) are \textit{compatible} if they are measurable with respect to a common sub-\( \sigma \)-algebra of \( \mathcal{F} \).

A fundamental result in classical integration theory is that the sum of two measurable functions is measurable and the integral is additive on the sum. It is shown in [4] that the sum of two measurable functions in a GMS need not be measurable. However, if the sum is measurable (the functions are then said to be \textit{summable}), the author asked if it follows that the integral must be additive on the sum. A counterexample is given in [6, 12] which answers this question negatively. While this counterexample is not complicated, the idea which makes it work requires the functions to be nonsimple (have an infinite number of values). It is therefore of interest to inquire whether the integral is additive on simple summable functions. The answer to this question could also be important in applications [4, 5]. This paper answers this question positively.

These results indicate a major difference between classical and generalized measure theory. In the classical theory, the standard proof of the additivity of the integral establishes that result for simple functions and then passes to the general case via the limit. In a GMS, the integral is additive on simple summable functions but this does not extend to the general case. In classical theory the proof of the additivity of the integral on simple functions is quite elementary. This same question in a GMS has proved to be much more difficult, and investigation has led the authors to some fascinating combinatorial patterns.

Our additivity result is, in a certain sense, analogous to an important result due to Gleason [1]. This result states that corresponding to any probability measure \( \mu \) on the lattice \( L \) of orthogonal projections on a separable Hilbert space \( H (\dim H > 2) \) there exists a unique positive trace class operator \( T \) on \( H \) such that \( \mu(P) = \text{tr}(TP) \) for all \( P \in L \). In this context, which is important in quantum mechanics, self-adjoint operators correspond to quantum mechanical observables. If \( A \) is a self-adjoint operator with spectral resolution \( P^A(\cdot) \), the \textit{integral} (or \textit{expectation}) of \( A \) relative to \( \mu \) is

\[
\int \lambda \, \mu[P^A(d\lambda)].
\]
Now let $A$ and $B$ be bounded self-adjoint operators (we consider bounded operators to avoid tedious domain problems). Then $A + B$ is self-adjoint and we ask whether the integral $I_\mu$ is additive; that is, does

$$I_\mu(A + B) = I_\mu(A) + I_\mu(B).$$

It follows from Gleason's theorem that the answer is yes. Indeed,

$$I_\mu(A + B) = \text{tr}[T(A + B)] = \text{tr}(TA) + \text{tr}(TB) = I_\mu(A) + I_\mu(B).$$

Conversely, one can show that if $I_\mu$ is additive, then Gleason's theorem follows fairly easily.

The additivity result presented here is in two parts. First the result is established for a special case, using combinatorial arguments. Second, using a limiting procedure, the special case is shown to imply the general result.

2. Special Case

Let $f$, $g$ be simple summable functions on a GMS $(\Omega, \mathscr{E}, \mu)$, each of which assumes the $N$ values, 0, 1, 2,..., $N - 1$. It follows that $R_i = \{ f = i \}$, $C_j = \{ g = j \}$, $i, j = 0, \ldots, N - 1$, and $D_t = \{ f + g = t \}$, $t = 0, 1, \ldots, 2N - 2$, are measurable sets of $\mathscr{E}$. Letting $r_i = \mu(R_i)$, $c_j = \mu(C_j)$, and $d_t = \mu(D_t)$, clearly $\sum r_i = \sum c_j = \sum d_t = \mu(\Omega)$. The question of interest is what further restrictions does the structure of the $\sigma$-class impose on these measures. Specifically, must

$$\sum ir_i + \sum jc_j = \sum td_t.$$

This equation (equivalent to $\int f \, d\mu + \int g \, d\mu = \int (f + g) \, d\mu$) we call the additivity equation.

The choice of notation for $R_i$, $C_j$, and $D_t$ is deliberate as these constructs can be effectively modeled with the $N$ rows, $N$ columns, and $2N - 1$ lower left to upper right diagonals of an $N \times N$ matrix. For later reference, let $B_{ij} = \{ f = i, g = j \}$, which corresponds to the $i, j$th block of $\Omega$. Of course, $B_{ij}$ need not be in $\mathscr{E}$. Consider the $\sigma$-class generated by the collection $\mathscr{B} = \{ R_i, C_j, D_t, i, j = 0, \ldots, N - 1, t = 0, \ldots, 2N - 1 \}$. Without loss of generality we will henceforth call this $\sigma$-class $\mathscr{E}$. 
In the sequel, $0 < \beta < 1$ will be a fixed irrational number and $\lambda$ will be a variable in the interval $[0, 1]$. Define the sequence

$$n_j(\lambda) = \lfloor j\lambda + \beta \rfloor, \quad j = 0, 1, \ldots,$$

where $\lfloor \cdot \rfloor$ is the greatest integer part function. Each $\lambda \in [0, 1]$ determines a subset $S_\lambda$ of $\Omega$ which is defined as

$$S_\lambda = \bigcup B_{i,j},$$

where $0 \leq i, j \leq N - 1$ are such that

$$n_i(\lambda) + n_j(\lambda) > n_{i+1}(\lambda) + n_{j-1}(\lambda)$$

for some $0 \leq s < i+j$.

**Lemma 1.** For fixed $t$ and $\lambda$, the set

$$\{n_i(\lambda) + n_{i-j}(\lambda): i = 0, 1, \ldots, t\}$$

consists either of a single integer or two consecutive integers.

**Proof.** There exist numbers $0 \leq \delta_1, \delta_2 < 1$ such that

$$n_i(\lambda) + n_{i-j}(\lambda) = \lfloor i\lambda + \beta \rfloor + \lfloor (t-i)\lambda + \beta \rfloor$$

$$= (i\lambda + \beta) - \delta_1 + (t-i)\lambda + \beta - \delta_2 = t\lambda + 2\beta - (\delta_1 + \delta_2).$$

Hence, for every $i = 0, \ldots, t$, $n_i(\lambda) + n_{i-j}(\lambda)$ is an integer in the interval $(t\lambda + 2\beta - 2, t\lambda + 2\beta]$ and this interval contains only two integers. \[\square\]

**Lemma 2.** (a) For fixed $\lambda \in [0, 1]$, at most one number in the sequence $j\lambda + \beta$, $j = 0, 1, \ldots$, is an integer. (b) As $\lambda$ increases from 0 to 1, the sequence

$$(n_j(0)) = (0, 0, 0, \ldots)$$

monotonically increases to the sequence

$$(n_j(1)) = (0, 1, 2, \ldots)$$

and precisely one of the integers in a sequence increases by 1 with each sequence change.

**Proof.** (a) Suppose $j_1\lambda + \beta = n_1$ and $j_2\lambda + \beta = n_2$, where $j_1 \neq j_2$ and $n_1, n_2$ are integers. Then

$$\beta = \frac{j_1n_2 - j_2n_1}{j_1 - j_2}.$$
which contradicts the fact that $\beta$ is irrational. (b) As $\lambda$ increases, the terms of the sequence $(n_j(\lambda))$ remain constant until one of the numbers $j\lambda + \beta$ becomes an integer. This terms then increases by 1 and by part (a) is the only term that does so.

**Lemma 3.** Let $0 \leq \lambda < 1$ and assume $(n_j(\lambda)) \neq (n_j(1))$. Strictly increase $\lambda$ until exactly one of the numbers, say $j_0\lambda + \beta$, $j_0 \leq N - 1$, becomes an integer. Call the $\lambda$ at which this occurs, $\lambda_0$. Then $S_{j_0}$ can be constructed from $S_\lambda$ by the following sequence of set operations.

1. Disjoint union with $C_{j_0}$.
2. Proper subtraction of $D_{2j_0}$.
3. Disjoint union with $R_{j_0}$.
4. Proper subtraction of all possible $D_j$'s.

**Proof.** By the choice of $\lambda_0$, we have

\[ [j_0\lambda + \beta] = [j_0\lambda_0 + \beta] - 1 \]
\[ [j\lambda + \beta] = [j\lambda_0 + \beta] \quad \text{for all } j \neq j_0. \]

Applying Lemma 1, for $s \neq j_0$ we have

\[ 2[j_0\lambda_0 + \beta] - 2 < [s\lambda_0 + \beta] + [(2j_0 - s)\lambda_0 + \beta]. \]

Hence, from (1) and (2) we obtain

\[ 2[j_0\lambda + \beta] < [s\lambda + \beta] + [(2j_0 - s)\lambda + \beta]. \]  
(3)

Similarly from Lemma 1, for $j \neq j_0$, we have

\[ [j\lambda_0 + \beta] + [j_0\lambda + \beta] - 1 \leq [s\lambda_0 + \beta] + [(j + j_0 - s)\lambda_0 + \beta]. \]

Hence, from (1) and (2) we obtain

\[ [j\lambda + \beta] + [j_0\lambda + \beta] \leq [s\lambda + \beta] + [(j + j_0 - s)\lambda + \beta]. \]
(4)

Now (3) and (4) show that $S_\lambda$ has the proper configuration for the specified set operations to be performed. Specifically, (3) shows that

\[ D_{2j_0} - B_{j_0,j_0} \subseteq S_\lambda, \quad B_{j_0,j_0} \cap S_\lambda = \emptyset. \]

This and (4) show that

\[ C_{j_0} \cap S_\lambda = R_{j_0} \cap S_\lambda = \emptyset. \]

An examination of the diagonal sets shows that $S_{j_0}$ is the set that results
from applying the specified set operations to $S_\lambda$. First consider $D_{2j_0}$. From Lemma 1 and (3), for $s \neq j_0$ we have

$$2[j_0\lambda + \beta] + 1 = [s\lambda + \beta] + [(2j_0 - s)\lambda + \beta].$$

Hence, from (1) we obtain

$$2[j_0\lambda_0 + \beta] - 1 = [s\lambda_0 + \beta] + [(2j_0 - s)\lambda_0 + \beta].$$

Therefore, $B_{j_0, j_0} \subseteq S_{\lambda_0}$, while

$$(D_{2j_0} - B_{j_0, j_0}) \cap S_{\lambda_0} = \emptyset.$$ 

This corresponds to the configuration of $D_{2j_0}$ after the set operations have been applied. Note that when $D_{2j_0}$ is the degenerate diagonal consisting only of $B_{j_0, j_0}$, $D_{2j_0}$ is removed in Step 4. However, in this case, $D_{2j_0} \cap S_{\lambda_0} = \emptyset$, since there are no other values with which to compare $2[j_0\lambda_0 + \beta]$ (see definition of $S_\lambda$), and the same conclusion applies.

Second, consider $D_{j + j_0}$, for $j \neq j_0$, and use (4) again:

(i) Suppose there is at least one $s$, $s \neq j_0$, such that equality holds in [4]. After the increase to $\lambda_0$,

$$[s\lambda_0 + \beta] + [(j + j_0 - s)\lambda_0 + \beta]$$

does not change, while

$$[j\lambda_0 + \beta] + [j_0\lambda_0 + \beta] = [j\lambda + \beta] + [j_0\lambda + \beta] + 1.$$ 

Hence,

$$S_{\lambda_0} \cap D_{j + j_0} = (S_\lambda \cap D_{j + j_0}) \cup B_{j_0, j} \cup B_{j, j_0}.$$ 

This corresponds exactly to the configuration of a diagonal which would not be removed in Step 4.

(ii) Otherwise

$$[j\lambda + \beta] + [j_0\lambda + \beta] < [s\lambda + \beta] + [(j + j_0 - s)\lambda + \beta]$$

for all $s \neq j_0$. When $\lambda$ is increased to $\lambda_0$, all of the values $[s\lambda_0 + \beta] + [(j + j_0 - s)\lambda_0 + \beta]$, including $s = j_0$, are equal. Hence, $D_{j + j_0} \cap S_{\lambda_0} = \emptyset$. This corresponds exactly to a diagonal which would be removed in Step 4.

\textbf{Corollary.} $B_\lambda \in \mathcal{C}$ if and only if $B_{\lambda_0} \in \mathcal{C}$. 

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**Lemma 4.** For $f$ and $g$ defined above, we have

$$\int f \, d\mu + \int g \, d\mu = \int (f + g) \, d\mu.$$

**Proof.** Since

$$(n_j(0)) = (0, 0, 0, \ldots)$$

$$(n_j(1)) = (0, 1, 2, \ldots)$$

it is easy to check that $S_0 = S_1 = \emptyset$. Lemma 3 shows that by letting $\lambda$ increase in a finite sequence of prescribed steps from 0 to 1, $S_1$ can be constructed from $S_0$ by adjoining row and column sets, and deleting diagonal sets. Each time the $j$th element of $(n_0(\lambda), n_1(\lambda), \ldots)$ increases by 1, $C_j$ is adjoined. Since $n_j(0) = 0$ and $n_j(1) = j$, $C_j$ is adjoined $j$ times. A similar count shows that $R_i$ is adjoined $i$ times. We next show that $D_i$ is deleted $t$ times. Note that $D_i = \bigcup B_{i,j}$, where $i + j = t$, $0 \leq i, j \leq N - 1$. Now $B_{i,j}$ is adjoined $i$ times as a subset of $R_i$ and $j$ times as a subset of $C_j$. Since $S_1 = \emptyset$, $B_{i,j}$ must be deleted $i + j = t$ times. Since the only way it can be deleted is as a subset of $D_i$, $D_i$ must be deleted $t$ times. It follows that

$$0 = \mu(S_1) = \sum ir_i + \sum jc_j - \sum td_i.$$

### 3. Main Result

We now use the special case proved in Lemma 4 to prove the result for arbitrary summable simple functions. But first we need

**Lemma 5.** Let $A$ be a $k \times k$ matrix with rational entries and let $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ satisfy $Ay = 0$. Then for any $\varepsilon > 0$, there exists a $z = (z_1, \ldots, z_k)$ such that $Az = 0$, $z_i$ are rational, and $|z_i - y_i| < \varepsilon$, $i = 1, \ldots, k$.

**Proof.** Let $Q^k$ be the vector space of rational $k$-tuples. Then $A$ defines a linear operator $A'$ on $Q^k$. Since rank$(A)$ equals the number of linearly independent columns of $A$, rank$(A) = $ rank$(A')$. It follows that dim $\eta(A) = $ dim $\eta(A')$. Let $e_1, \ldots, e_n$ be a basis for $\eta(A')$. Then $e_1, \ldots, e_n$ is a basis for $\eta(A)$. Hence, $\eta(A')$ is dense in $\eta(A)$. Thus, for any $\varepsilon > 0$, if $(y_1, \ldots, y_k) \in \eta(A)$, there exists a $(z_1, \ldots, z_k) \in \eta(A')$ such that $|z_i - y_i| < \varepsilon$, $i = 1, \ldots, k$.

**Corollary.** Suppose $y_1, \ldots, y_k$ are real numbers that satisfy a system of homogeneous linear equations having rational coefficients. Then for any $\varepsilon > 0$, there exist rational numbers $z_1, \ldots, z_k$ which satisfy the same system of equations and $|z_i - y_i| < \varepsilon$, $i = 1, \ldots, k$. 
THEOREM 6. If $f$ and $g$ are summable simple functions, then

$$\int f \, d\mu + \int g \, d\mu = \int (f + g) \, d\mu.$$ 

Proof. In the proof of Lemma 4 it was not assumed that all the $R_i$, $C_j$, and $D_t$ were nonempty. Thus the result holds if $f$ and $g$ have any nonnegative integer values. If $a$ is constant, it is clear that

$$\int (a + f) \, d\mu = \int a \, d\mu + \int f \, d\mu.$$ 

Hence, the result holds if $f$ and $g$ have any integer values. Now suppose $f$ and $g$ have rational values. Then there exists a constant $b \neq 0$ such that $bf$ and $bg$ have integer values. Then

$$\int f \, d\mu + \int g \, d\mu = \frac{1}{b} \left( \int bf \, d\mu + \int bg \, d\mu \right) = \frac{1}{b} \int (f + g) \, d\mu = \int (f + g) \, d\mu.$$ 

Hence, the result holds if $f$ and $g$ have rational values. Finally, suppose that $f$ and $g$ have arbitrary distinct real values $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ (there is no loss of generality in assuming that they have the same number of values).

There may be indices $i \neq k$, $j \neq m$ such that

$$\alpha_i + \beta_j - \alpha_k - \beta_m = 0.$$ 

Let $S$ be the system of all such equations. Choose an $\varepsilon > 0$ such that

$$\varepsilon < |\alpha_i - \alpha_j|, |\beta_j - \beta_j|, |\alpha_i + \beta_j - \alpha_k - \beta_m|$$

for all $i, j, k, m$ for which the terms on the right of the inequality are non-zero. Applying the previous corollary, there exist rationals $p_i, q_i, i = 1, \ldots, n$, which satisfy $S$ and $|\alpha_i - p_i|, |\beta_j - q_j| < \varepsilon/4, i = 1, \ldots, n$. If $\alpha_i + \beta_j - \alpha_k - \beta_m \neq 0$, then $p_i + q_j - p_k - q_m \neq 0$. Indeed, if $p_i + q_j - p_k - q_m = 0$, then

$$|\alpha_i + \beta_j - \alpha_k - \beta_m| = |(\alpha_i - p_i) + (\beta_j - q_j) - (\alpha_k - p_k) - (\beta_m - q_m)|$$

$$\leq |\alpha_i - p_i| + |\beta_j - q_j| + |\alpha_k - p_k| + |\beta_m - q_m| < \varepsilon.$$ 

This gives a contradiction.

Define $f_1, g_1$ by $f_1(w) = p_i$ if $f(w) = \alpha_i$ and $g_1(w) = q_i$ if $g(w) = \beta_i$. Because of the above inequalities, the $p_i$'s are distinct and so are the $q_i$'s. It follows that

$$\{f = \alpha_i\} = \{f_1 = p_i\}, \quad \{g = \beta_i\} = \{g_1 = q_i\}, i = 1, \ldots, n.$$
Our work in the previous paragraph shows that
\[ \{ f + g = \alpha_i + \beta_j \} = \{ f_1 + g_1 = p_i + q_j \}. \]

Since \( f_i \) and \( g_1 \) are summable simple functions with rational values, we have
\[ \sum p_i \mu(f_i = p_i) + \sum q_i \mu(g_1 = q_i) = \sum (p_i + q_j) \mu(f_i + g_1 = p_i + q_j), \]
where the sum on the right of the equality is over the distinct \( p_i + q_j \). We then obtain
\[
\begin{align*}
\left| \sum \alpha_i \mu(f = \alpha_i) + \sum \beta_i \mu(g = \beta_i) - \sum (\alpha_i + \beta_j) \mu(f + g = \alpha_i + \beta_j) \right| & \\
\leq & \sum |\alpha_i - p_i| \mu(f = \alpha_i) + \sum |\beta_i - q_j| \mu(g = \beta_i) \\
& + \sum |\alpha_i + \beta_j - p_i - q_j| \mu(f + g = \alpha_i + \beta_j) \\
& \leq \varepsilon \mu(\Omega).
\end{align*}
\]

Letting \( \varepsilon \to 0 \) completes the proof. 

4. Example

We have shown that the generalized integral is additive on two summable simple functions. What about three summable simple functions? If \( f_1, f_2, \) and \( f_3 \) are summable simple functions and \( f_1 + f_2 + f_3 \) is measurable, then is
\[ \int f_1 \, d\mu + \int f_2 \, d\mu + \int f_3 \, d\mu + \int (f_1 + f_2 + f_3) \, d\mu ? \]

The following example shows that in general the answer is negative.

Let \( \Omega = \{ 1, 2, 3, 4, 5, 6 \} \) and let \( \mathcal{C} \) be the \( \sigma \)-class consisting of the subsets \( \emptyset, \Omega, \{ 1, 2, 3 \}, \{ 4, 5, 6 \}, \{ 1, 2, 4 \}, \{ 3, 5, 6 \}, \{ 1, 3, 6 \}, \{ 2, 4, 5 \}, \{ 1, 4, 6 \}, \{ 2, 3, 5 \} \). Define the measure \( \mu \) on \( \mathcal{C} \) by
\[
\mu(\Omega) = \mu(\{ 1, 2, 3 \}) = \mu(\{ 1, 2, 4 \}) = \mu(\{ 1, 3, 6 \}) = \mu(\{ 2, 3, 5 \}) = 1,
\]
and \( \mu \) is zero on the other sets. Let \( f_1, f_2, \) and \( f_3 \) be the characteristic functions of \( \{ 1, 4, 6 \}, \{ 3, 5, 6 \}, \) and \( \{ 2, 4, 5 \} \), respectively. Then \( f_1, f_2, f_3 \)
are measurable functions. Also, \( f = f_1 + f_2 + f_3 \) is measurable, since \( f^{-1}(\{1\}) = \{1, 2, 3\} \), \( f^{-1}(\{2\}) = \{4, 5, 6\} \). Now

\[
\int f_1 \, d\mu + \int f_2 \, d\mu + \int f_3 \, d\mu = 0 \neq 1 = \int (f_1 + f_2 + f_3) \, d\mu.
\]

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