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Coherent Sheaves on Configuration Schemes[1]

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We introduce and study *configuration schemes*, which are obtained by "glueing" usual schemes along closed embeddings. The category of coherent sheaves on a configuration scheme is investigated. *Smooth* configuration schemes provide alternative "resolutions of singularities" of usual singular schemes. We consider in detail the case when the singular scheme is a union of hyperplanes in a projective space.

1. INTRODUCTION

We introduce and study *configuration schemes*, or rather, coherent sheaves on them. Namely, let S be a finite poset considered as a category; i.e., for $\alpha, \beta \in S$ the set of morphisms $\mathrm{Mor}(\alpha, \beta)$ contains a unique element if $\alpha \geq \beta$ and is empty otherwise. Consider a functor X from the opposite category S^{opp} to the category of schemes. Thus, in particular, for each $\alpha \in S$ we are given a scheme $X(\alpha)$ with a morphism $f_{\beta\alpha} : X(\beta) \to X(\alpha)$ if $\beta \leq \alpha$. We require $f_{\beta\alpha}$'s to be *closed embeddings* and then call this data a configuration scheme X/S. (One could relax this condition by requiring

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only that $f_{\beta\alpha}$'s be affine morphisms.) We say that X/S is smooth if each $X(\alpha)$ is a smooth scheme.

A *coherent sheaf* F on X/S is a collection of coherent sheaves $\{F_\alpha \in \mathrm{coh}(X)(\alpha))\}$ with morphisms $\sigma_{\alpha\beta} : f^*_{\beta\alpha}F_\alpha \to F_\beta$ satisfying an obvious compatibility condition. The category $\mathrm{coh}(X/S)$ of coherent sheaves on X/S is an abelian category. If X/S is smooth, the category $\mathrm{coh}(X/S)$ has finite cohomological dimension (which is usually twice what one expects in the coherent theory).

Configuration schemes appear naturally in the following situation. Let X_0 be a (singular) reducible scheme with smooth components, which intersect along smooth subschemes. Let S be the poset indexing all nonempty intersections of these components, and for $\alpha \in S$ let $X(\alpha)$ be the corresponding intersection itself. We get a smooth configuration scheme X/S with the obvious inverse image functor (or "localization" functor)

$$\mathscr{L} : \mathrm{coh}(X_0) \to \mathrm{coh}(X/S).$$

The configuration scheme X/S can be considered as a (nontraditional) resolution of singularities of X_0, which may behave better than the usual resolution $\widetilde{X}_0 \to X_0$. For example, one can reconstruct the scheme X_0 from the configuration scheme X/S, whereas $\widetilde{X}_0$ does not "remember" that it is a resolution of X_0. In the last section of this paper we give examples of some interesting auto-equivalences of the derived category $D^b(\mathrm{coh}(X_0))$ and show that they "extend" to autoequivalences of $D^b(\mathrm{coh}(X/S))$ (whereas they do not extend to $D^b(\mathrm{coh}(\widetilde{X}_0))$).

The derived categories $D^b(\mathrm{coh}(X/S))$ for smooth configuration schemes X/S tend to have similar properties to those of the derived categories $D^b(\mathrm{coh}(Y))$ for smooth schemes Y. For example, if each $X(\alpha)$ is a (smooth) projective variety, then every cohomological functor

$$D^b(\mathrm{coh}(X/S)) \to \mathrm{Vect}$$

is representable. In particular, $D^b(\mathrm{coh}(X/S))$ has a Serre functor [BK].

We work out in some detail the case when X_0 is the union of hyperplanes in general position in a projective space. We call the corresponding configuration scheme X/S the *hyperplane configuration scheme*. We prove that the derived localization functor $\mathbb{L}\mathscr{L} : D^b(\mathrm{coh}(X_0)) \to D^-(\mathrm{coh}(X/S))$ is fully faithful when restricted to the full subcategory of perfect complexes in $\mathrm{coh}(X_0)$ (Theorem 4.5), which again shows the intimate relation of the singular scheme X_0 and its "desingularization" X/S. Also we find a strong exceptional collection E in $D^b(\mathrm{coh}(X/S))$ [Bo] which implies that the category $D^b(\mathrm{coh}(X/S)$ is equivalent to the derived category $D^b((A - \mathrm{mod})_f)$ of finite right *A-modules* for a finite dimensional algebra A. This is again in analogy with $D^b(\mathrm{coh}(Y))$ for certain smooth projective schemes Y (see [Bo]).

Finally I want to mention that some configuration schemes X/S (or rather categories $\mathrm{coh}(X/S)$) are intimately related to categories of *constructible* sheaves on algebraic varieties. In the next paper I will use configuration schemes to construct a mirror-type correspondence between coherent and constructible sheaves.

2. CONFIGURATION CATEGORIES

1. *Poset Categories and Configuration Categories.* Let S be a poset, which we consider as a category. That is, for $\alpha, \beta \in S$ the set $\mathrm{Mor}(\alpha, \beta)$ consists of a single element if $\alpha \geq \beta$ and is empty otherwise. Let $\mathscr{C}$ be a functor from S to the category *Cat* of all categories. That is, for each $\alpha \in S$ we have a category $\mathscr{C}(\alpha)$ and if $\alpha \geq \beta$ we have a functor $\phi_{\alpha\beta} : \mathscr{C}(\alpha) \to \mathscr{C}(\beta)$ such that $\phi_{\alpha\gamma} = \phi_{\beta\gamma}\phi_{\alpha\beta}$ in case $\alpha \geq \beta \geq \gamma$.

The pair $(S, \mathscr{C})$ defines a new category $\mathscr{M}(S, \mathscr{C})$ which may be considered as a glueing of categories $\mathscr{C}(\alpha)$ along the functors $\phi_{\alpha\beta}$. Namely, objects of $\mathscr{M}(S, \mathscr{C})$ are collections

$$F = (F_\alpha \in \mathscr{C}(\alpha))_{\alpha \in S}$$

together with morphisms

$$\sigma_{\alpha\beta}(F) = \sigma_{\alpha\beta} : \phi_{\alpha\beta}(F_\alpha) \to F_\beta$$

such that $\sigma_{\alpha\gamma} = \sigma_{\beta\gamma} \cdot \phi_{\beta\gamma}(\sigma_{\alpha\beta})$. Morphisms $f \in \mathrm{Mor}(F, G)$ are collections $f = \{f_\alpha \in \mathrm{Mor}_{\mathscr{C}(\alpha)}(F_\alpha, G_\alpha)\}$, which are compatible with the structure morphisms $\sigma_{\alpha\beta}$.

DEFINITION 2.1. We call $\mathscr{M}(S, \mathscr{C})$ a *poset category* over S. We say that $\mathscr{M}(S, \mathscr{C})$ is an *abelian poset category* if each $\mathscr{C}(\alpha)$ is an abelian category and the functors $\phi_{\alpha\beta}$ are *right exact*.

LEMMA 2.2. *An abelian poset category $\mathscr{M}(S, \mathscr{C})$ is an abelian category.*

Proof. Indeed, given two objects F, G in $\mathscr{M}(S, \mathscr{C})$ and a morphism $f : F \to G$, define $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ componentwise. Namely, put $\mathrm{Ker}(f)_\alpha := \mathrm{Ker}(f_\alpha)$, $\mathrm{Coker}(f)_\alpha := \mathrm{Coker}(f_\alpha)$. Note that $\mathrm{Coker}(f)$ is a well defined object in $\mathscr{M}(S, \mathscr{C})$ since the functors $\phi_{\alpha\beta}$ are right exact. ∎

In this paper we will only be interested in abelian poset categories. For simplicity we only consider $\mathbb{C}$-linear abelian categories.

DEFINITION 2.3. An abelian poset category $\mathscr{M}(S, \mathscr{C})$ is called a *configuration category* if the set S is finite and each functor $\phi_{\alpha\beta}$ has a right adjoint $\psi_{\beta\alpha}$, which is exact.

EXAMPLE 2.4. Our main example of an abelian poset category comes from geometry. Namely, let $X : S^{\mathrm{opp}} \to \mathrm{Sch}/\mathbb{C}$ be a functor from the opposite category S^{opp} to the category of schemes (say, of finite type over $\mathbb{C}$). We call this data a *poset of schemes* and denote it X/S. Then for $\alpha \geq \beta$ in S we have a morphism of schemes $f_{\beta\alpha} : X(\beta) \to X(\alpha)$. This induces the inverse image functor $\phi_{\alpha\beta} := f^*_{\beta\alpha} : \mathrm{coh}(X(\alpha)) \to \mathrm{coh}(X(\beta))$ between the categories of coherent sheaves. This functor is right exact. Thus we obtain a functor $\mathscr{C} : S \to Cat$, such that $\mathscr{C}(\alpha) = \mathrm{coh}(X(\alpha))$ and $\phi_{\alpha\beta}$ as above, which gives rise to the corresponding abelian poset category $\mathscr{M}(S, \mathscr{C})$. An object F of $\mathscr{M}(S, \mathscr{C})$ is a collection of coherent sheaves $F = \{F_\alpha \in \mathrm{coh}(X(\alpha))\}$ with morphisms $\sigma_{\alpha\beta} : f^*_{\beta\alpha} F_\alpha \to F_\beta$, satisfying the compatibility condition.

DEFINITION 2.5. In the notation of the previous example assume that the set S is finite and all morphisms $f_{\beta\alpha}$ are closed embeddings. Then we call the poset of schemes X/S a *configuration scheme*.

DEFINITION 2.6. Let X/S be a configuration scheme. The direct image functor $\psi_{\beta\alpha} := f_{\beta\alpha *} : \mathrm{coh}(X(\beta)) \to \mathrm{coh}(X(\alpha))$ is the right adjoint to $\phi_{\alpha\beta}$ and it is exact. Thus $\mathscr{M}(S, \mathscr{C})$ becomes a configuration category, which is called the *category of coherent sheaves on* X/S and denoted $\mathrm{coh}(X/S)$. Similarly, we define the *category* $\mathrm{qcoh}(X/S)$ *of quasicoherent sheaves on* X/S. Clearly, $\mathrm{coh}(X/S)$ is a full subcategory of $\mathrm{qcoh}(X/S)$. We say that X/S is a *smooth* configuration scheme, if all schemes $X(\alpha)$ are smooth.

Remark 2.7. Note that we could weaken the condition that the $f_{\alpha\beta}$ are closed embeddings and require only that they be affine morphisms. Then the category $\mathrm{coh}(X/S)$ is still a configuration category.

2. *Motivation.* Our main interest lies in the categories $\mathrm{coh}(X/S)$ and $D^b(\mathrm{coh}(X/S))$ for smooth configuration schemes X/S. Such schemes arise from degenerations of families of smooth schemes. (More precisely, this happens when the singular fiber has smooth components which intersect along smooth subschemes.) We will show that for a smooth configuration scheme X/S the category $\mathrm{coh}(X/S)$ inherits many of the good properties of the categories $\mathrm{coh}(X(\alpha))$. For example, $\mathrm{coh}(X/S)$ has finite cohomological dimension (Corollary 3.5). Consider the bounded derived category $D^b(\mathrm{coh}(X/S))$. If each (smooth) scheme $X(\alpha)$ is projective, then every covariant or contravariant cohomological functor from $D^b(\mathrm{coh}(X/S))$ to Vect is representable (Theorem 3.14). In particular there exists a Serre functor in $D^b(\mathrm{coh}(X/S))$ (Theorem 3.18).

3. *Operations in Configuration Categories.* Let $\mathscr{M} = \mathscr{M}(S, \mathscr{C})$ be an abelian poset category. For $F \in \mathscr{M}$ define its support $\mathrm{Supp}(F) = \{\alpha \in S | F_\alpha \neq 0\}$.

Define a topology on S by taking as a basis of open sets the subsets $U_\alpha = \{\beta \in S | \beta \leq \alpha\}$.

Note that $Z_\alpha = \{\gamma \in S | \gamma \geq \alpha\}$ is a closed subset in S.

Let $U \subset S$ be open and $Z = S - U$ the complementary closed. Let $\mathscr{M}_U$ (resp. $\mathscr{M}_Z$) be the full subcategory of $\mathscr{M}$ consisting of objects F with support in U (resp. in Z). For every object F in $\mathscr{M}$ there is a natural short exact sequence

$$0 \to F_U \to F \to F_Z \to 0,$$

where $F_U \in \mathscr{M}_U, F_Z \in \mathscr{M}_Z$. Indeed, take

$$(F_U)_\alpha = \begin{cases} F_\alpha, & \text{if } \alpha \in U, \\ 0, & \text{if } \alpha \in Z. \end{cases}$$

$$(F_Z)_\alpha = \begin{cases} F_\alpha, & \text{if } \alpha \in Z, \\ 0, & \text{if } \alpha \in U. \end{cases}$$

We may consider U (resp. Z) as a subcategory of S and restrict the functor $\mathscr{C}$ to U (resp. Z). This gives abelian poset categories $\mathscr{M}(U, \mathscr{C}|_U)$ and $\mathscr{M}(Z, \mathscr{C}|_Z)$.

Denote by $j : U \hookrightarrow S$ and $i : Z \hookrightarrow S$ the inclusions. We get the obvious restriction functors

$$j^* = j^! : \mathscr{M} \to \mathscr{M}(U, \mathscr{C}|_U), \qquad i^* : \mathscr{M} \to \mathscr{M}(Z, \mathscr{C}|_Z).$$

Clearly these functors are exact. The functor j^* has an exact left adjoint $j_! : \mathscr{M}(U, \mathscr{C}|_U) \to \mathscr{M}$ ("extension by zero"). Its image is the subcategory $\mathscr{M}_U$. The functor i^* has an exact right adjoint $i_* = i_! : \mathscr{M}(Z, \mathscr{C}|_Z) \to \mathscr{M}$ (also "extension by zero"). Its image is the subcategory $\mathscr{M}_Z$. It follows that j^* and i_* preserve injectives (as right adjoints to exact functors). We have $j^* j_! = Id, i^* i_* = Id$.

Note that the short exact sequence above is just

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0,$$

where the two middle arrows are the adjunction maps.

For $\alpha \in S$ denote by $j_\alpha : \{\alpha\} \hookrightarrow S$ the inclusion. The inverse image functor $j_\alpha^* : \mathscr{M} \to \mathscr{M}_{\{\alpha\}} = \mathscr{C}(\alpha), F \mapsto F_\alpha$ has a right-exact left adjoint $j_{\alpha+}$ defined as

$$(j_{\alpha+} P)_\beta = \begin{cases} \phi_{\alpha\beta} P, & \text{if } \beta \leq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $P \in \mathscr{C}(\alpha)$, $\mathrm{Supp} j_{\alpha+} P \subset U_\alpha$.

We also consider the "extension by zero" functor $j_{\alpha!} : \mathscr{C}(\alpha) \to \mathscr{M}$ defined by

$$j_{\alpha!}(P)_\beta = \begin{cases} P, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 2.8. *Assume that $\mathcal{M}(S, \mathcal{C})$ is a configuration category. Then the functor $j_\alpha^* : \mathcal{M}(S, \mathcal{C}) \to \mathcal{C}(\alpha)$ has a right adjoint $j_{\alpha*}$. The functor $j_{\alpha*}$ is exact and preserves injectives. For $P \in \mathcal{C}(\alpha)$, $\mathrm{Supp}(j_{\alpha*}P) \subset Z_\alpha$.*

Proof. Given $P \in \mathcal{C}(\alpha)$ we set

$$j_{\alpha*}(P)_\gamma = \begin{cases} \psi_{\alpha\gamma}(P), & \text{if } \gamma \geq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

and the structure map

$$\sigma_{\gamma\delta} : \phi_{\gamma\delta}(j_{\alpha*}P)_\gamma \to (j_{\alpha*}P)_\delta$$

is the adjunction map

$$\phi_{\gamma\delta}\psi_{\alpha\gamma}P = \phi_{\gamma\delta}\psi_{\delta\gamma}\psi_{\alpha\delta}P \to \psi_{\alpha\delta}P$$

if $\alpha \leq \delta \leq \gamma$ and $\sigma_{\gamma\delta} = 0$ otherwise.

It is clear that $j_{\alpha*}$ is exact (since the ψ's are such) and that $\mathrm{Supp}\,(j_{\alpha*}P) \subset Z_\alpha$.

Let us prove that $j_{\alpha*}$ is the right adjoint to j_α^*.

Let $P \in \mathcal{C}(\alpha)$ and $M \in \mathcal{M}(S, \mathcal{C})$. Given $f_\alpha \in \mathrm{Hom}(M_\alpha, P)$ for each $\gamma \geq \alpha$ we obtain a map $f_\alpha \cdot \sigma_{\gamma\alpha}(M) : \phi_{\gamma\alpha}M_\gamma \to P$ and hence by adjunction $f_\gamma : M_\gamma \to \psi_{\alpha\gamma}P = (j_{\alpha*}P)_\gamma$. The collection $f = \{f_\gamma\}$ is a morphism $f : M \to j_{\alpha*}P$. It remains to show that the constructed map

$$\mathrm{Hom}(M_\alpha, P) \to \mathrm{Hom}(M, j_{\alpha*}P)$$

is surjective or, equivalently, that the restriction map

$$\mathrm{Hom}(M, j_{\alpha*}P) \to \mathrm{Hom}(M_\alpha, P), \qquad f \mapsto f_\alpha$$

in injective.

Assume that $0 \neq f \in \mathrm{Hom}(M, j_{\alpha*}P)$, i.e., $f_\gamma \neq 0$ for some $\gamma \geq \alpha$. By definition we have the commutative diagram

$$\begin{array}{ccc} \phi_{\gamma\alpha}M_\gamma & \xrightarrow{\phi_{\gamma\alpha}(f_\gamma)} & \phi_{\gamma\alpha}\psi_{\alpha\gamma}P \\ {\scriptstyle \sigma_{\gamma\alpha}(M)}\downarrow & & \downarrow{\scriptstyle \epsilon_P} \\ M_\alpha & \xrightarrow{f_\alpha} & P, \end{array}$$

where ϵ_P is the adjunction morphism. Note that $\epsilon_P\phi_{\gamma\alpha}(f_\gamma) : \phi_{\gamma\alpha}M_\gamma \to P$ is the morphism, which corresponds to $f_\gamma : M_\gamma \to \psi_{\alpha\gamma}P$ by the adjunction property. Hence $\epsilon_P\phi_{\gamma\alpha}(f_\gamma) \neq 0$. Therefore $f_\alpha \neq 0$. This shows the injectivity of the restriction map $f \mapsto f_\alpha$ and proves that $j_{\alpha*}$ is the right adjoint to j_α^*. Finally, $j_{\alpha*}$ preserves injectives being the right adjoint to an exact functor. ∎

4. *Cohomological Dimension of Configuration Categories.*

LEMMA 2.9. *Let $\mathcal{M} = \mathcal{M}(S, \mathcal{C})$ be a configuration category. Assume that each category $\mathcal{C}(\alpha)$ has enough injectives. Then so does $\mathcal{M}$.*

Proof. We will prove the assertion by induction on the cardinality of S. Let $\beta \in S$ be a smallest element. Put $U = U_\beta = \{\beta\}$, $Z = S - U$. Let $j = j_\beta : U \hookrightarrow S$ and $i : Z \hookrightarrow S$ be the corresponding open and closed embedding.

For F in $\mathcal{M}$ we need to find a monomorphism $F \to I$, where I is an injective object in $\mathcal{M}$. It suffices to do so for $F_U = j_! j^* F$ and $F_Z = i_* i^* F$. Let $j^* F \to I_1$, $i^* F \to I_2$ be similar monomorphisms in categories $\mathcal{M}(U, \mathcal{C}|_U) = \mathcal{C}(\beta)$ and $\mathcal{M}(Z, \mathcal{C}|_Z)$ (this is possible by the induction assumption). Then

$$j_* j^* F \to j_* I_1, \qquad i_* i^* F \to i_* I_2$$

are monomorphisms in $\mathcal{M}$ with $j_* I_1$ and $i_* I_1$ being injective. It remains to note the obvious monomorphism $j_! j^* F \to j_* j^* F$. ∎

PROPOSITION 2.10. *Let $\mathcal{M} = \mathcal{M}(S, \mathcal{C})$ be a configuration category. Assume that each category $\mathcal{C}(\alpha)$ has enough injectives and has finite cohomological dimension. Then $\mathcal{M}$ also has finite cohomological dimension.*

Proof. We argue by induction on the cardinality of S as in the proof of the last lemma. In fact we will use the notation of that proof.

Given F in $\mathcal{M}$ we need to find a finite injective resolution of F. It suffices to do so for $j_! j^* F$ and $i_* i^* F$. Let $j^* F \to I_1^\bullet$, $i^* F \to I_2^\bullet$ be such resolutions in categories $\mathcal{M}(U, \mathcal{C}|_U)$ and $\mathcal{M}(Z, \mathcal{C}|_Z)$, respectively. Then $j_* j^* F \to j_* I_1^\bullet$, $i_* i^* F \to i_* I_2^\bullet$ will be injective resolutions in $\mathcal{M}$. Consider the short exact sequence

$$0 \to j_! j^* F \to j_* j^* F \to G \to 0.$$

Then $\mathrm{Supp}(G) \subset Z$ and so by induction $G = i_* i^* G$ has a finite injective resolution in $\mathcal{M}$. Therefore the same is true for $j_! j^* F$. ∎

5. *Derived Categories of Configuration Categories.* Let $\mathcal{M} = \mathcal{M}(S, \mathcal{C})$ be a configuration category. Denote by $D^b(\mathcal{M})$ its bounded derived category.

Let $U \overset{j}{\hookrightarrow} S \overset{i}{\hookleftarrow} Z$ be embeddings of an open U and a complementary closed Z. The exact functors $j^*, j_!, i^*, i_*$ extend trivially to corresponding functors between derived categories $D^b(\mathcal{M})$, $D^b(\mathcal{M}(U, \mathcal{C}|_U))$, $D^b(\mathcal{M}(Z, \mathcal{C}|_Z))$. If $j_\alpha : \{\alpha\} \hookrightarrow S$ is the inclusion, then the same is true for the exact functor $j_{\alpha *}$. When this causes no confusion we will denote the derived functor again by the same symbol; i.e., we will write $j_!$ for $\mathbb{R} j_!$, etc.

Recall that functors $j_!$ and i_* identify categories $\mathcal{M}(U, \mathcal{C}|_U)$ and $\mathcal{M}(Z, \mathcal{C}|_Z)$ with $\mathcal{M}_U$ and $\mathcal{M}_Z$, respectively. Denote by $D^b_U(\mathcal{M})$ and $D^b_Z(\mathcal{M})$

the full subcategories of $D^b(\mathscr{M})$ consisting of complexes with cohomology objects in $\mathscr{M}_U$ and $\mathscr{M}_Z$, respectively.

LEMMA 2.11. *Assume that all categories $\mathscr{C}(\alpha)$ have enough injectives. Then in the above notation the functors $i_* : D^b(\mathscr{M}(Z, \mathscr{C}|_Z)) \to D^b(\mathscr{M})$ and $j_! : D^b(\mathscr{M}(U, \mathscr{C}|_U)) \to D^b(\mathscr{M})$ are fully faithful. The essential images of these functors are the full subcategories $D^b_Z(\mathscr{M})$ and $D^b_U(\mathscr{M})$, respectively.*

Proof. We need to prove that functors i_* and $j_!$ preserve Ext's between objects.

Let $F, G \in \mathscr{M}(Z, \mathscr{C}|_Z)$ and let $G \to I^\bullet$ be an injective resolution (it exists by Lemma 2.9). Then $i_*G \to i_*I^\bullet$ is still an injective resolution. We have

$$\begin{aligned}\mathrm{Ext}^k(i_*F, i_*G) &= H^k\mathrm{Hom}^\bullet(i_*F, i_*I^\bullet)\\ &= H^k\mathrm{Hom}^\bullet(i^*i_*F, I^\bullet)\\ &= H^k\mathrm{Hom}^\bullet(F, I^\bullet)\\ &= \mathrm{Ext}^k(F, G).\end{aligned}$$

Thus i^* preserves Ext's.

Let $F, G \in \mathscr{M}(U, \mathscr{C}|_U)$. Let $j_!G \to J^\bullet$ be an injective resolution. Then $G = j^*j_!G \to j^*J^\bullet$ is still an injective resolution. We have

$$\begin{aligned}\mathrm{Ext}^k(j_!F, j_!G) &= H^k\mathrm{Hom}^\bullet(j_!F, J^\bullet)\\ &= H^k\mathrm{Hom}^\bullet(F, j^*J^\bullet)\\ &= \mathrm{Ext}^k(F, G).\end{aligned}$$

∎

We immediately obtain the following corollary

COROLLARY 2.12. *Assume that all categories $\mathscr{C}(\alpha)$ have enough injectives. Then the categories $D^b(\mathscr{M}_U)$ and $D^b(\mathscr{M}_Z)$ are naturally equivalent to $D^b_U(\mathscr{M})$ and $D^b_Z(\mathscr{M})$, respectively.*

COROLLARY 2.13. *Assume that each category $\mathscr{C}(\beta)$ has enough injectives. Fix $\alpha \in S$. Let $i : \{\alpha\} \hookrightarrow U_\alpha$ and $j : U_\alpha \hookrightarrow S$ be the closed and the open embeddings, respectively. Then the composition functor*

$$j_!i_* : D^b(\mathscr{C}(\alpha)) \to D^b(\mathscr{M}(S, \mathscr{C}))$$

is fully faithful. In particular, the derived category $D^b(\mathscr{C}(\alpha))$ is naturally a full subcategory of $D^b(\mathscr{M}(S, \mathscr{C}))$.

Proof. Indeed, by Lemma 2.11 above the functors

$$i_* : D^b(\mathscr{C}(\alpha)) \to D^b(\mathscr{M}(U_\alpha, \mathscr{C}|_{U_\alpha}))$$

and

$$j_! : D^b(\mathscr{M}(U_\alpha, \mathscr{C}|_{U\alpha})) \to D^b(\mathscr{M}(S, \mathscr{C}))$$

are fully faithful. So is their composition. ∎

Recall the following definitions from [BK].

DEFINITION 2.14. Let $\mathscr{A}$ be an additive category, and let $\mathscr{B} \subset \mathscr{A}$ be a full subcategory. A *right orthogonal* to $\mathscr{B}$ in $\mathscr{A}$ is a full subcategory $\mathscr{B}^\perp \subset \mathscr{A}$ consisting of all objects C such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathscr{B}$.

DEFINITION 2.15. Let $\mathscr{A}$ be a triangulated category, and let $\mathscr{B} \subset \mathscr{A}$ be a full triangulated subcategory. We say that $\mathscr{B}$ is *right-admissible* if for each $X \in \mathscr{A}$ there exists an exact triangle $B \to X \to C$ with $B \in \mathscr{B}$, $C \in \mathscr{B}^\perp$.

LEMMA 2.16. (a) *In the category $\mathscr{M}$ we have $\mathscr{M}_U^\perp = \mathscr{M}_Z$.*

(b) *Assume that all categories $\mathscr{C}(\alpha)$ have enough injectives (so that, in particular, $D^b(\mathscr{M}_U)$ and $D^b(\mathscr{M}_Z)$ are full subcategories of $D^b(\mathscr{M})$). Then*

(i) *in $D^b(\mathscr{M})$ we have $D^b(\mathscr{M}_U)^\perp = D^b(\mathscr{M}_Z)$,*

(ii) *the subcategory $D^b(\mathscr{M}_U) \subset D^b(\mathscr{M})$ is right-admissible.*

Proof. (a) Given $F \in \mathscr{M}_U$, $G \in \mathscr{M}$ we have $\mathrm{Hom}(F, G) = \mathrm{Hom}(F, G_U)$. Hence $\mathrm{Hom}(F, G) = 0$ for all F iff $G_U = 0$ or, equivalently, $G \in \mathscr{M}_Z$.

(b) (i) Let $G^\bullet \in D^b(\mathscr{M})$. If $G^\bullet \in D^b(\mathscr{M}_U)^\perp$, then $G_U^\bullet$ is acyclic, i.e., $G^\bullet \in D^b_Z(\mathscr{M})$. Vice versa, and injective complex $G^\bullet$ in $D^b(\mathscr{M}_Z)$ stays injective in $D^b(\mathscr{M})$. Thus $D^b(\mathscr{M}_Z) \subset D^b(\mathscr{M}_U)^\perp$.

(ii) Given $X^\bullet \in D^b(\mathscr{M})$ the required exact triangle is $X_U^\bullet \to X^\bullet \to X_Z^\bullet$. ∎

3. CATEGORIES $\mathrm{coh}(X/S)$ AND $\mathrm{qcoh}(X/S)$ FOR SMOOTH CONFIGURATION SCHEMES X/S

We now turn to the subject of our primary interest: the category $\mathrm{coh}(X/S)$ for a smooth configuration scheme X/S. This category does not have enough injectives, so we will study $\mathrm{coh}(X/S)$ (as usual) by embedding it in a larger category $\mathrm{qcoh}(X/S)$.

So let X/S be a smooth configuration scheme.

LEMMA 3.1. *The category* $\mathrm{qcoh}(X/S)$ *has enough injectives.*

Proof. This is a direct consequence of the general Lemma 2.9. ∎

COROLLARY 3.2. *The category* $\mathrm{qcoh}(X/S)$ *has finite cohomological dimension*.

Proof. This follows from Lemma 3.1, Proposition 2.10, and the well known fact that the category $\mathrm{qcoh}(Y)$ has finite cohomological dimension if Y is smooth. ∎

Consider the bounded derived categories $D^b(\mathrm{coh}(X/S))$ and $D^b(\mathrm{qcoh}(X/S))$. Let $D^b_{\mathrm{coh}}(\mathrm{qcoh}(X/S)) \subset D^b(\mathrm{qcoh}(X/S))$ be the full subcategory consisting of complexes with cohomologies in $\mathrm{coh}(X/S)$. The next proposition is the analogue of a similar well known result for usual schemes.

PROPOSITION 3.3. *The categories* $D^b(\mathrm{coh}(X/S))$ *and* $D^b_{\mathrm{coh}}(\mathrm{qcoh}(X/S))$ *are equivalent*.

LEMMA 3.4. *Let* $f : F \to T$ *be an epimorphism, where* $F \in \mathrm{qcoh}(X/S)$ *and* $T \in \mathrm{coh}(X/S)$. *There exists a subobject* $G \subset F$ *such that* $G \in \mathrm{coh}(X/S)$ *and* $f|_G : G \to T$ *is still an epimorphism*.

Proof. The lemma is well known for usual noetherian schemes. For each $\alpha \in S$ choose a coherent subsheaf $P_\alpha \subset F_\alpha$ such that $f_\alpha|_{P_\alpha} : P_\alpha \to T_\alpha$ is an epimorphism. The inclusion $P_\alpha \hookrightarrow F_\alpha$ defines the corresponding morphism $t_\alpha : i_{\alpha+}P_\alpha \to F$. Put

$$P := \bigoplus_\alpha i_{\alpha+}P_\alpha \in \mathrm{coh}(X/S), \qquad t = \oplus t_\alpha : P \to F.$$

Then the composition $f \cdot t : P \to T$ is an epimorphism and we can take $G = Im(t)$. ∎

Proof (of Proposition). The proposition follows from the above lemma by standard homological algebra as (for example) in [ST, pp. 12, 13]. ∎

COROLLARY 3.5. *The category* $\mathrm{coh}(X/S)$ *has finite cohomological dimension*.

Proof. This follows from Corollary 3.2 and Proposition 3.3. ∎

Let $U \overset{j}{\hookrightarrow} S \overset{i}{\hookleftarrow} Z$ be embeddings of an open and a complementary closed subset. Consider the functors

$$j_! : D^b(\mathrm{coh}(X/U)) \to D^b(\mathrm{coh}(X/S)),$$
$$i_* : D^b(\mathrm{coh}(X/Z)) \to D^b(\mathrm{coh}(X/S)).$$

PROPOSITION 3.6. (i) *The functors* $j_!$ *and* i_* *are fully faithful. Thus* $D^b(\mathrm{coh}(X/U))$ *and* $D^b(\mathrm{coh}(X/Z))$ *can be considered as full triangulated subcategories of* $D^b(\mathrm{coh}(X/S))$.

(ii) $D^b(\mathrm{coh}(X/Z))$ *is the right orthogonal to* $D^b(\mathrm{coh}(X/U))$ *in* $D^b(\mathrm{coh}(X/S))$.

(iii) $D^b(\mathrm{coh}(X/U))$ *is right-admissible in* $D^b(\mathrm{coh}(X/S))$.

Proof. (i) By Lemma 2.11 the analogous fact holds for the categories $D^b(\mathrm{qcoh})$ instead of $D^b(\mathrm{coh})$. Hence it also holds for the categories $D^b_{\mathrm{coh}}(\mathrm{qcoh})$. Therefore it holds for $D^b(\mathrm{coh})$ by the above proposition.

(ii) By Lemma 2.16(b)(i) the analogous fact holds for the categories $D^b(\mathrm{qcoh})$ instead of $D^b(\mathrm{coh})$. Actually, the same proof shows that the assertion is true for categories $D^b_{\mathrm{coh}}(\mathrm{qcoh})$. Hence by Proposition 3.3 it is also true for $D^b(\mathrm{coh})$.

(iii) The required exact triangle is $F_U \to F \to F_Z$. ∎

COROLLARY 3.7. *Fix* $\alpha \in S$. *Let* $i : \{\alpha\} \hookrightarrow U_\alpha$ *and* $j : U_\alpha \hookrightarrow S$ *be closed and open embeddings, respectively. Then the composition of functors*

$$j_! i_* : D^b(\mathrm{coh}(X(\alpha))) \to D^b(\mathrm{coh}(X/S))$$

is fully faithful. In particular, the derived category $D^b(\mathrm{coh}(X(\alpha)))$ *is naturally a full subcategory of* $D^b(\mathrm{coh}(X/S))$.

Proof. By Proposition 3.6 above the functors $j_!$ and i_* are fully faithful. So is their composition. ∎

Recall the following definitions from [BK].

DEFINITION 3.8. A triangulated category $\mathcal{A}$ is of *finite type* if for all $B, C \in \mathcal{A}$ the space $\mathrm{Hom}(B, C)$ is finite dimensional and $\mathrm{Ext}^i(B, C) = 0$ for $|i| \gg 0$.

DEFINITION 3.9. Let $\mathcal{A}$ be a triangulated category and $h : \mathcal{A} \to \mathrm{Vect}$ be a contravariant cohomological functor. We say that h is of *finite type* if for all $E \in \mathcal{A}$ the vector spaces $h^i(E) = h(E[-i])$ are finite dimensional and are equal to zero for almost all i. Similarly, for covariant cohomological functors.

DEFINITION 3.10. Let $\mathcal{A}$ be a triangulated category of finite type. We say that $\mathcal{A}$ is *right-saturated* if every contravariant cohomological functor $h : \mathcal{A} \to \mathrm{Vect}$ of finite type is representable, i.e., $h(\cdot) = \mathrm{Hom}(\cdot, X)$ for some $X \in \mathcal{A}$. Similarly we define left-saturatedness using covariant cohomological functors. The category $\mathcal{A}$ is *saturated* if it is left- and right-saturated.

One of the main results in [BK] is the following theorem.

THEOREM 3.11. *Let* X *be a smooth projective variety. Then the category* $D^b(\mathrm{coh}(X))$ *is saturated.*

The technique developed in [BK] allows us to extend the above theorem to the category $D^b(\mathrm{coh}(X/S))$ for a smooth configuration scheme with projective $X(\alpha)$'s.

LEMMA 3.12. *Assume that each $X(\alpha)$ is a (smooth) projective variety. Then the category $D^b(\mathrm{coh}(X/S))$ is of finite type.*

Proof. Put $\mathscr{M} = \mathrm{coh}(X/S)$. Let $\beta \in S$ be a smallest element, $U = U_\beta = \{\beta\}$ the corresponding open subset, and $Z = S - U$ the complementary closed. Given $F, G \in \mathscr{M}$ consider the short exact sequences in $\mathscr{M}$.

$$0 \to F_U \to F \to F_Z \to 0, \qquad 0 \to G_U \to G \to G_Z \to 0.$$

To show that $\oplus_k \dim \mathrm{Ext}^k(F, G) < \infty$ it suffices to prove the same for the other four members of the above exact sequences. By induction on the cardinality of S (and using Proposition 3.6(i)) we may assume that

$$\bigoplus_k \dim \mathrm{Ext}^k(F_U, G_U) < \infty, \qquad \bigoplus_k \dim \mathrm{Ext}^k(F_Z, G_Z) < \infty.$$

Also by Proposition 3.6(ii), $\mathrm{Ext}^\bullet(F_U, G_Z) = 0$. It remains to consider $\mathrm{Ext}^k(F_Z, G_U)$.

Denote by $j : U \hookrightarrow S$ the open embedding. Consider the short exact sequence

$$0 \to G_U \to j_* j^* G_U \to P \to 0.$$

Then $P \in \mathscr{M}_Z$ and by induction $\oplus_k \dim \mathrm{Ext}^k(F_Z, P) < \infty$. We claim that $\mathrm{Ext}^\bullet(F_Z, j_* j^* G_U) = 0$. Indeed, let $j^* G_U \to I^\bullet$ be an injective resolution (in $\mathrm{qcoh}(X(\beta))$). Then $j_* j^* G_U \to j_* I^\bullet$ remains an injective resolution (Lemma 2.8), so that $\mathrm{Ext}^\bullet(F_Z, j_* j^* G_U) = H^\bullet \mathrm{Hom}^\bullet(F_Z, j_* I^\bullet)$. But for any $T \in \mathrm{qcoh}(X(\beta))$

$$\mathrm{Hom}(F_Z, j_* T) = \mathrm{Hom}(j^* F_Z, T) = \mathrm{Hom}(0, T) = 0.$$

Hence also $\oplus_k \dim \mathrm{Ext}^k(F_Z, G_U) < \infty$. ∎

The next theorem is again from [BK].

THEOREM 3.13. *Let $\mathscr{A}$ be a triangulated category of finite type, and $\mathscr{B} \subset \mathscr{A}$ a right-admissible triangulated subcategory, $\mathscr{C} = \mathscr{B}^\perp$. Assume that $\mathscr{B}$ and $\mathscr{C}$ are right- (resp. left-) saturated. Then so is $\mathscr{A}$.*

We have the following counterpart of Theorem 3.11.

THEOREM 3.14. *Assume that each $X(\alpha)$ is a (smooth) projective variety. Then the category $D^b(\mathrm{coh}(X/S))$ is saturated.*

Proof. We will use Theorem 3.11 as a basis for induction on the cardinality of S. By Lemma 3.12 the category $D^b(\mathrm{coh}(X/S))$ is of finite type. Let $U \subset S$ be open and $Z = S - U$. By Proposition 3.6 the full subcategory $D^b(\mathrm{coh}(X/U)) \subset D^b(\mathrm{coh}(X/S))$ is right-admissible and $D^b(\mathrm{coh}(X/U))^{\perp} = D^b(\mathrm{coh}(X/Z))$. By the induction hypothesis the categories $D^b(\mathrm{coh}(X/U))$ and $D^b(\mathrm{coh}(X/Z))$ are saturated. Hence by Theorem 3.13 so is $D^b(\mathrm{coh}(X/S))$. ∎

1. *The Serre Functor.*

DEFINITION 3.15 [BK]. Let $\mathcal{A}$ be a tringulated category with finite dimensional Hom's. Let $F : \mathcal{A} \to \mathcal{A}$ be a covariant auto-equivalence which commutes with the translation functor [1]. We call F a *Serre functor* if there exist bifunctorial isomorphisms

$$\phi_{E,G} : \mathrm{Hom}(E, G) \to \mathrm{Hom}(G, F(E))^*$$

(for $E, G \in \mathcal{A}$) with the property that the composition

$$\left(\phi^{-1}_{F(E),F(G)}\right)^* \cdot \phi_{E,G} : \mathrm{Hom}(E,G) \to \mathrm{Hom}(G,F(E))^* \to \mathrm{Hom}(F(E),F(G))$$

coincides with the isomorphism induced by F.

In fact the compatibility condition in the above definition is superfluous, as was proved in [RVdB].

EXAMPLE 3.16. Let X be a smooth projective variety, $n = \dim X$, $\mathcal{A} = D^b(\mathrm{coh}(X))$, and $K_X = \Omega^n_X$ the canonical sheaf. Then $F(\cdot) = (\cdot) \otimes K_X[n]$ is a Serre functor in $\mathcal{A}$.

PROPOSITION 3.17 [BK]. *Let $\mathcal{A}$ be a triangulated category of finite type.*

(i) *A serre functor $F : \mathcal{A} \to \mathcal{A}$ is exact; i.e., it takes exact triangles to exact triangles.*

(ii) *Any two Serre functors in $\mathcal{A}$ are canonically isomorphic.*

(iii) *If $\mathcal{A}$ is saturated, then it has a Serre functor.*

We immediately obtain the following theorem.

THEOREM 3.18. *Assume that each $X(\alpha)$ is a (smooth) projective variety. Then the category $D^b(\mathrm{coh}(X/S))$ has a Serre functor.*

Proof. Follows from Theorem 3.14 and Proposition 3.17. ∎

2. Cohomology of Coherent Sheaves on Configuration Schemes. Let X/S be a configuration scheme.

DEFINITION 3.19. Consider the object $G \in \mathrm{coh}(X/S)$ such that $G_\alpha = \mathcal{O}_{X(\alpha)}$ and for $\beta \leq \alpha$ the map $\sigma_{\alpha\beta} : \phi_{\alpha\beta}(G_\alpha) \to G_\beta$ is the identity. We call this object the *structure sheaf of X/S* and denote it $\mathcal{O}_{X/S}$.

DEFINITION 3.20. Let $F \in \mathrm{coh}(X/S)$. We define the cohomology of F as

$$H^i(F) = H^i(X/S, F) := \mathrm{Ext}^i(\mathcal{O}_{X/S}, F).$$

Note that any $G \in \mathrm{coh}(X/S)$ has a finite left resolution by sheaves which are finite direct sums of G_{U_α}'s for various $\alpha \in S$:

$$0 \to \oplus G_{U_\beta} \to \cdots \to \oplus G_{U_\alpha} \to G \to 0.$$

Hence, in particular, $\mathcal{O}_{X/S}$ has such a resolution. Thus to compute $H^i(X/S, F)$ we can use a spectral sequence with the E_1 term equal to a direct sum of vector spaces $H^i(X/U_\alpha, F)$ for various $\alpha \in S, i \geq 0$.

The next lemma shows how to compute the cohomology $H^\bullet(X/U_\alpha, F)$.

LEMMA 3.21. *Assume that $\alpha \in S$ is the unique maximal element, i.e., $S = U_\alpha$. Then for any $F \in \mathrm{coh}(X/S)$ there is a natural isomorphism*

$$H^\bullet(X/S, F) \simeq H^\bullet(X(\alpha), F_\alpha).$$

Proof. Let $j_\alpha : \{\alpha\} \hookrightarrow S$ be the embedding. Note that $\mathcal{O}_{X/S} = j_{\alpha+}\mathcal{O}_{X(\alpha)}$. Hence for $F \in \mathrm{coh}(X/S)$

$$\mathrm{Hom}(\mathcal{O}_{X/S}, F) \simeq \mathrm{Hom}(\mathcal{O}_{X(\alpha)}, F_\alpha),$$

which proves the lemma for H^0. To complete the proof of the lemma it suffices to prove the following claim.

CLAIM. *Every $F \in \mathrm{coh}(X/S)$ can be embedded in an injective object $I \in \mathrm{qcoh}(X/S)$ such that I_α is acyclic for the functor* $\mathrm{Hom}(\mathcal{O}_{X(\alpha)}, \cdot)$.

Proof of the Claim. For $\beta \in S$ let $j_\beta : \{\beta\} \hookrightarrow S$ be the inclusion. For each β choose a monomorphism $F_\beta \to I_\beta$, where $I_\beta \in \mathrm{qcoh}(X(\beta))$ is an injective object. By adjunction this defines a map

$$t^\beta : F \to j_{\beta*}I_\beta,$$

where the object $j_{\beta*}I_\beta \in \mathrm{qcoh}(X/S)$ is injective (Lemma 2.8). Put

$$I := \oplus_{\beta \in S} j_{\beta*}I_\beta$$

with the embedding

$$t = \oplus t^\beta : F \hookrightarrow I.$$

We claim that for each β the sheaf $(j_{\beta*}I_\beta)_\alpha \in \mathrm{qcoh}(X(\alpha))$ is acyclic for $\mathrm{Hom}(\mathcal{O}_{X(\alpha)}, \cdot)$ (and hence I is such). Indeed, it is equal to $\psi_{\beta\alpha}(I_\beta) = f_{\beta\alpha*}(I_\beta)$ for the closed embedding $f_{\beta\alpha} : X(\beta) \hookrightarrow X(\alpha)$, and $H^\bullet(X(\beta), I_\beta) = H^\bullet(X(\alpha), f_{\beta\alpha*}I_\beta)$. This proves the claim and the lemma. ∎

We immediately obtain the following corollary.

COROLLARY 3.22. *Assume that $\alpha \in S$ is the unique maximal element, i.e., $S = U_\alpha$. Let $F \in \mathrm{coh}(X/S)$ be such that $F_\alpha = 0$. Then $H^\bullet(X/S, F) = 0$.*

4. HYPERPLANE CONFIGURATION SCHEMES

1. *Definition and Cohomological Dimension of Hyperplane Configuration Schemes.* We want to consider in detail the simplest type of smooth configuration schemes, namely, the ones defined by d hyperplanes in general position in $\mathbb{P}^{n+1}$.

Let $L_1, L_2, \ldots, L_d$ be hyperplanes in general position in $\mathbb{P}^{n+1}$. Let S be the poset of nonempty subsets of $\{1, \ldots, d\}$ of cardinality $\leq n+1$. (For $\alpha, \beta \in S$ we put $\beta \leq \alpha$ iff $\beta \subseteq \alpha$.) For $\alpha = \{i_1, \ldots, i_{j+1}\} \in S$ define

$$X(\alpha) := L_{i_1} \cap \cdots \cap L_{i_{j+1}} \simeq \mathbb{P}^{n-j}.$$

Put $d(\alpha) = \dim X(\alpha) = n - j$. We will also denote $\mathbb{P}_\alpha = X(\alpha) \simeq \mathbb{P}^{d(\alpha)}$. If $\alpha \geq \beta$ we have the natural closed embedding $f_{\beta\alpha} : \mathbb{P}_\beta \to \mathbb{P}_\alpha$. This defines a functor $X : S^{\mathrm{opp}} \to \mathrm{Sch}/\mathscr{C}$, which is a smooth configuration scheme.

DEFINITION 4.1. We call a smooth configuration scheme X/S as above *a hyperplane configuration scheme (of type* (n, d)*)*.

By the general Corollary 3.5 we know that the cohomological dimension of the category $\mathrm{coh}(X/S)$ is finite. The next proposition is a more precise result.

PROPOSITION 4.2. *Let X/S be a hyperplane configuration scheme of type (n, d). Then the cohomological dimension of* $\mathrm{coh}(X/S)$ *is* $2n$ *if* $d \geq n$ *and* $n + d$ *if* $d < n$; *i.e., it is equal to* $\min\{2n, n + d\}$.

Proof. First we prove the upper bound on the cohomological dimension. For each $k \geq 0$ denote by $Z_k \subset S$ the closed subset

$$Z_k := \{\alpha \in S | d(\alpha) \geq k\}.$$

Thus $Z_n \subset Z_{n-1} \subset \ldots$.

Let $F, G \in \mathrm{coh}(X/S)$. Assume that $\mathrm{Supp}(G) \subset Z_k$. It suffices to prove that $\mathrm{Ext}^i(F, G) = 0$ for $i > 2n - k$. We will prove it by descending induction on k.

If $k = n$, then

$$\mathrm{Ext}^i(F, G) = \oplus_j \mathrm{Ext}^i(F|_{L_j}, G|_{L_j}).$$

Thus $\mathrm{Ext}^i(F, G) = 0$ for $i > n$. Assume the assertion is proved for k and $\mathrm{Supp} G \subset Z_{k-1}$. Put $U = Z_{k-1} - Z_k$. Consider the short exact sequence

$$0 \to G_U \to G \to G_{Z_k} \to 0.$$

It suffices to prove that $\mathrm{Ext}^i(F, G_U) = 0$ for $i > 2n - k + 1$. We may assume that $S = Z_{k-1}$. Note that

$$G_U = \oplus_{\alpha \in U} G_\alpha.$$

Fix one $\alpha \in U$. This is a smallest element in S. Let $j_\alpha : \{\alpha\} \hookrightarrow S$ denote the open embedding. We have the short exact sequence of sheaves on S

$$0 \to G_\alpha \to j_{\alpha *} j_\alpha^* G_\alpha \to (j_{\alpha *} j_\alpha^* G_\alpha)/G_\alpha \to 0.$$

Since $j_{\alpha *}$ is exact and preserves injectives, the object $j_{\alpha *} j_\alpha^* G_\alpha$ has an injective resolution of length $\leq k-1$. On the other hand $\mathrm{Supp}((j_{\alpha *} j_\alpha^* G_\alpha)/G_\alpha) \subset Z_k$. Thus by induction

$$\mathrm{Ext}^i(F, (j_{\alpha *} j_\alpha^* G_\alpha)/G_\alpha) = 0$$

for $i > 2n-k$. Hence $\mathrm{Ext}^i(F, G_\alpha) = 0$ for $i > 2n-k+1$. This completes the induction step and proves the upper bound on the cohomological dimension.

To show that the bound is actually achieved choose an element $\beta \in S$. Put $m := \dim \mathbb{P}_\beta$. Choose a point $p \in \mathbb{P}_\beta$. Since $\mathbb{P}_\beta \subset \mathbb{P}_\gamma$ for $\beta \leq \gamma$, the point p also belongs to all $\mathbb{P}_\gamma$ for $\gamma \geq \beta$. We denote the corresponding sky-scraper sheaf $\mathbb{C}_p^\gamma \in \mathrm{coh}(\mathbb{P}_\gamma)$. For $\gamma \in S$ denote the embedding $j_\gamma : \{\gamma\} \hookrightarrow S$. Let $\alpha \geq \beta$ be a maximal element. It suffices to prove that

$$\mathrm{Ext}^{2n-m}(j_{\alpha *}\mathbb{C}_p^\alpha, j_{\beta !}\mathbb{C}_p^\beta) \neq 0.$$

In case $m = n$ (i.e., $\alpha = \beta$) we have

$$\mathrm{Ext}^n(j_{\alpha *}\mathbb{C}_p^\alpha, j_{\alpha !}\mathbb{C}_p^\alpha) = \mathrm{Ext}^n_{\mathrm{coh}(\mathbb{P}_\alpha)}(\mathbb{C}_p, \mathbb{C}_p) \neq 0.$$

Now we argue by descending induction on m. The sheaf $j_{\beta !}\mathbb{C}_p^\beta$ has a resolution

$$0 \to j_{\beta !}\mathbb{C}_p^\beta \to j_{\beta *}\mathbb{C}_p^\beta \to \bigoplus_{\substack{\gamma \geq \beta \\ d(\gamma) = d(\beta)+1}} j_{\gamma *}\mathbb{C}_p^\gamma \to \cdots \to \bigoplus_{\substack{\delta \geq \beta \\ d(\delta) = n}} j_{\delta *}\mathbb{C}_p^\delta \to 0.$$

So it suffices to prove that $\mathrm{Ext}^\bullet(j_{\alpha *}\mathbb{C}_p^\alpha, j_{\gamma *}\mathbb{C}_p^\gamma) = 0$ for all $\gamma < \alpha$. But this follows because $j_{\gamma *}$ is exact and preserves injectives and $\mathrm{Hom}(j_{\alpha *}\mathbb{C}_p^\alpha, j_{\gamma *}T) = 0$ for all $T \in \mathrm{qcoh}(\mathbb{P}_\gamma)$. This proves the proposition. ∎

2. The Čech Complex. Let X/S be a hyperplane configuration scheme of type (n, d). Consider the open covering of S by the (irreducible) sets U_α, $d(\alpha) = n$. Given $F \in D^b(\mathrm{coh}(X/S))$ we have the corresponding Čech resolution $\check{C}^\bullet(F) \to F$ with

$$\check{C}^\bullet(F) := 0 \to \bigoplus_{d(\gamma)=0} F_{U_\gamma} \to \cdots \bigoplus_{d(\beta)=n-1} F_{U_\beta} \to \bigoplus_{d(\alpha)=n} F_{U_\alpha} \to 0,$$

where $F_{U_\delta} := j_{\delta !} j_\delta^* F$ for the open embedding $j_\delta : U_\delta \hookrightarrow S$. The differentials in this complex are the sums of the natural embeddings $F_{U_\beta} \hookrightarrow F_{U_\alpha}$ taken with $\pm$ sign. Notice that for each $\alpha \in S$ there is exactly one summand F_{U_α} in $\check{C}^\bullet(F)$.

In case $F = \mathcal{O}_{X/S}$ we will denote $\check{C}^\bullet = \check{C}^\bullet(\mathcal{O}_{X/S})$. Note that for each $\alpha \in S$, $j \geq 0$, $i > 0$,

$$\mathrm{Ext}^i(\mathcal{O}_{U_\alpha}, \mathcal{O}_{X/S}(j)) = \mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}_\alpha}, \mathcal{O}_{\mathbb{P}_\alpha}(j)) = 0.$$

Hence the complex $\mathrm{Hom}^\bullet(\check{C}^\bullet, \mathcal{O}_{X/S}(j))$ represents $\mathbb{R}\mathrm{Hom}^\bullet(\mathcal{O}_{X/S}, \mathcal{O}_{X/S}(j))$.

EXAMPLE 4.3. Let X/S be a *Calabi–Yau* hyperplane configuration scheme, i.e., $d = n+2$. Then the poset S is isomorphic to the poset of simplices in the standard triangulation of the boundary of an $(n+1)$-simplex. Thus the complex $\mathrm{Hom}^\bullet(\check{C}^\bullet, \mathcal{O}_{X/S})$ is isomorphic to the (shifted by n to the right) simplicial complex of the boundary of an $(n+1)$-simplex.

3. *Localization Theorem.* Let X_0 be the reduced scheme, which is the union $X_0 = L_1 \cup \cdots \cup L_d \subset \mathbb{P}^{n+1}$. We have the obvious localization functor

$$\mathcal{L} : \mathrm{coh}(X_0) \to \mathrm{coh}(X/S).$$

Namely, for $\alpha \in S$ denote the closed embedding $t_\alpha : \mathbb{P}_\alpha \to X_0$. Then given $F \in \mathrm{coh}(X_0)$ we define $\mathcal{L}(F)_\alpha := t_\alpha^* F \in \mathrm{coh}(\mathbb{P}_\alpha)$. The functor $\mathcal{L}$ is right exact, hence induces its left derived

$$\mathbb{L}\mathcal{L} : D^b(\mathrm{coh}(X_0)) \to D^-(\mathrm{coh}(X/S)).$$

We will denote $\mathbb{L}\mathcal{L}(\mathcal{O}_{X_0}(i))$ simply by $\mathcal{O}_{X/S}(i)$. The endofunctor

$$\mathcal{O}_{X/S}(i) \underset{\mathcal{O}_{X/S}}{\bigotimes} (\cdot) : \mathrm{coh}(X/S) \to \mathrm{coh}(X/S)$$

is an auto-equivalence.

DEFINITION 4.4. A *perfect complex* in $D^b(\mathrm{coh}(X_0))$ is a finite complex consisting of locally free sheaves. Denote by $\mathrm{Perf}(X_0)$ the full subcategory of $D^b(\mathrm{coh}(X_0))$ consisting of perfect complexes.

Note that for a perfect complex $P^\bullet \in \mathrm{Perf}(X_0)$ we have $\mathcal{L}(P^\bullet) = \mathbb{L}\mathcal{L}(P^\bullet) \in D^b(\mathrm{coh}(X/S))$.

THEOREM 4.5. *The functor* $\mathcal{L} : \mathrm{Perf}(X_0) \to D^b(\mathrm{coh}(X/S))$ *is fully faithful.*

We need a few lemmas.

LEMMA 4.6. *For* $i \geq 0$ *the spaces* $H^\bullet(X_0, \mathcal{O}_{X_0}(i))$ *and* $H^\bullet(X/S, \mathcal{O}_{X/S}(i))$ *are isomorphic.*

Proof. The scheme X_0 has a covering by closed subschemes

$$X_0 = \bigcup_{d(\alpha)=n} \mathbb{P}_\alpha.$$

Consider the corresponding Čech resolution $\mathcal{O}_{X_0}(i) \to J^\bullet$ in $\mathrm{coh}(X_0)$, where

$$J^\bullet = \bigoplus_{d(\alpha)=n} \mathcal{O}_{\mathbb{P}_\alpha}(i) \to \bigoplus_{d(\beta)=n-1} \mathcal{O}_{\mathbb{P}_\beta}(i) \to \dots.$$

(The fact that $\mathcal{O}_{X_0}(i) \to J^\bullet$ is a quasi-isomorphism can be easily checked by a local computation, since the hyperplanes L_i are in general position.) Note that each sheaf $\mathcal{O}_{\mathbb{P}_\gamma}(i)$ is acyclic for $\mathrm{Hom}(\mathcal{O}_{X_0}, \cdot)$ (since $i \geq 0$), hence the complex $\mathrm{Hom}^\bullet(\mathcal{O}_{X_0}, J^\bullet)$ represents $\mathbb{R}\,\mathrm{Hom}^\bullet(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}(i))$.

On the other hand $\mathbb{R}\,\mathrm{Hom}^\bullet(\mathcal{O}_{X/S}, \mathcal{O}_{X/S}(i))$ is represented by the complex $\mathrm{Hom}^\bullet(\check{C}^\bullet, \mathcal{O}_{X/S}(i))$, which is naturally isomorphic to $\mathrm{Hom}^\bullet(\mathcal{O}_{X_0}, J^\bullet)$. This proves the lemma. ∎

COROLLARY 4.7. *We have* $H^j(X/S, \mathcal{O}_{X/S}(i)) = 0$ *for* $j > 0$, $i > 0$.

Proof. This follows from the last lemma and the vanishing of the corresponding cohomology groups on X_0. ∎

LEMMA 4.8. *For all* $j \geq i$ *the map*

$$\mathcal{L} : \mathrm{Hom}(\mathcal{O}_{X_0}(i), \mathcal{O}_{X_0}(j)) \to \mathrm{Hom}(\mathcal{O}_{X/S}(i), \mathcal{O}_{X/S}(j))$$

is an isomorphism.

Proof. Since $\mathcal{L}$ commutes with tensoring by $\mathcal{O}_{X_0}(i)$ (resp. $\mathcal{O}_{X/S}(i)$), it suffices to show that for all $j \geq 0$ the map

$$\mathcal{L} : H^0(X_0, \mathcal{O}_{X_0}(j)) \to H^0(X/S, \mathcal{O}_{X/S}(j))$$

is an isomorphism. By Lemma 4.6 the two spaces have the same dimension. Also it is clear that the map is injective. Hence it is an isomorphism. ∎

LEMMA 4.9. *For any* $F \in \mathrm{coh}(X_0)$ *and any* i *the map*

$$\mathbb{L}\mathcal{L} : \mathrm{Ext}^\bullet(F, \mathcal{O}_{X_0}(i)) \to \mathrm{Ext}^\bullet(\mathbb{L}\mathcal{L}(F), \mathcal{O}_{X/S}(i))$$

is an isomorphism.

Proof. We can find a left resolution $P^\bullet \to F$, where elements of $P^\bullet$ are direct sums of sheaves $\mathcal{O}_{X_0}(-j_k)$ with $j_k > -i$. Then $\mathrm{Hom}^\bullet(P^\bullet, \mathcal{O}_{X_0}(i))$ represents $\mathbb{R}\,\mathrm{Hom}^\bullet(F, \mathcal{O}_{X_0}(i))$ and by Corollary 4.7, $\mathrm{Hom}^\bullet(\mathcal{L}(P^\bullet), \mathcal{O}_{X/S}(i))$ represents $\mathbb{R}\,\mathrm{Hom}^\bullet(\mathbb{L}\mathcal{L}(F), \mathcal{O}_{X/S}(i))$. But the functor $\mathcal{L}$ establishes an isomorphism of complexes $\mathrm{Hom}^\bullet(P^\bullet, \mathcal{O}_{X_0}(i))$ and $\mathrm{Hom}^\bullet(\mathcal{L}(P^\bullet), \mathcal{O}_{X/S}(i))$ by Lemma 4.8. ∎

Proof (of Theorem). It follows from the last lemma that the functor $\mathbb{L}\mathcal{L}$ is fully faithful on the triangulated subcategory of $D^b(\mathrm{coh}(X_0))$, generated by line bundles $\mathcal{O}_{X_0}(i)$. To prove the theorem it suffices to show that any locally free sheaf F on X_0 is a direct summand (in $D^b(\mathrm{coh}(X_0))$) of a finite complex $P^\bullet$ which consists of direct sums of line bundles $\mathcal{O}_{X_0}(i)$. For any $s \geq 0$ we can find an exact complex in $\mathrm{coh}(X_0)$

$$0 \to T \to P^{-s} \to \cdots \to P^{-1} \to P^0 \to F \to 0,$$

where each P^{-j} is a direct sum of line bundles $\mathcal{O}_{X_0}(i)$. It represents an element in $\mathrm{Ext}^{s+1}(F, T)$. Note that the functor $\mathrm{Hom}(F, (\cdot))$ is isomorphic to $\Gamma(X_0, (\cdot) \otimes_{\mathcal{O}_{X_0}} F^*)$. Thus $\mathrm{Ext}^{s+1}(F, T) = 0$ for $s \gg 0$ and so $P^\bullet \simeq F \oplus T[s]$. ∎

4. *Exceptional Collections and Representations of Quivers.* For a hyperplane configuration scheme X/S we will prove that the derived category $D^b(\mathrm{coh}(X/S))$ is equivalent to the derived category of finite dimensional modules over a certain finite dimensional algebra A. Let us first recall some basic definitions (see, for example, [Bo]).

Let $\mathcal{A}$ be a triangulated category.

DEFINITION 4.10. An object E in $\mathcal{A}$ is called *exceptional* if $\mathrm{Ext}^i(E, E) = 0$ for $i \neq 0$, $\mathrm{Hom}(E, E) = \mathbb{C}$.

DEFINITION 4.11. An ordered collection $(E_0, \ldots, E_m)$ of exceptional objects is called an *exceptional collection* in $\mathcal{A}$ if $\mathrm{Ext}^i(E_j, E_k) = 0$ for all i when $j > k$.

DEFINITION 4.12. An exceptional collection $(E_0, \ldots, E_m)$ in $\mathcal{A}$ satisfying the condition $\mathrm{Ext}^i(E_j, E_k) = 0$ for all j and k if $i \neq 0$ is called a *strong exceptional collection*.

Given a strong exceptional collection $(E_0, \ldots, E_m)$ in $\mathcal{A}$ put $E = \oplus E_i$ and $A = \mathrm{Hom}(E, E)$. Then A is a finite-dimensional algebra. It is actually the *path algebra* of a certain quiver with relations which contains $n + 1$ vertices (see [Bo] for the discussion of this topic). The following theorem is proved in [Bo].

THEOREM 4.13. *Let Y be a smooth projective variety. Assume that the derived category $\mathcal{A} = D^b(\mathrm{coh}(Y))$ is generated (as a triangulated category) by a strong exceptional collection $(E_0, \ldots, E_m)$. Then $\mathcal{A}$ is equivalent to the derived category $D^b(A - \mathrm{mod})$ of right finite-dimensional A-modules.*

Remark 4.14. Strong exceptional collections as in the above theorem are known for projective spaces, quadrics, Grassmannians, and flag varieties.

Remark 4.15. Let $p_k : E \to E_k$ be the projection on the kth summand. Consider the projective right A-module $P_k := p_k A$. We have the natural isomorphism

$$\mathrm{Hom}_A(P_k, P_l) = \mathrm{Hom}(E_k, E_l).$$

Here we provide a counterpart of the above theorem for hyperplane configuration schemes. Namely, let X/S be a hyperplane configuration scheme of type (n, d). For $\alpha \in S$ put

$$E_{\alpha,i} := \mathcal{O}_{U_\alpha}(i) \in \mathrm{coh}(X/S).$$

Consider the collection

$$\mathcal{E} = \{E_{\alpha,i} | \alpha \in S, 0 \le i \le \dim \mathbb{P}_\alpha\}.$$

It has a natural lexicographical partial order, i.e., $E_{\alpha,i} > E_{\beta,j}$ iff $\alpha > \beta$, or $\alpha = \beta$ and $i > j$. Again we put

$$E = \oplus_{E_{\alpha,i} \in \mathcal{E}} E_{\alpha,i}, \qquad A = \mathrm{Hom}(E, E).$$

THEOREM 4.16. *Let X/S be a hyperplane configuration scheme. Put $\mathcal{A} = D^b(\mathrm{coh}(X/S))$. Consider the collection $\mathcal{E}$ as above with any linear ordering which is compatible with its natural partial order. Then*

(i) *$\mathcal{E}$ is a strong exceptional collection in $\mathcal{A}$;*

(ii) *$\mathcal{E}$ generates the triangulated category $\mathcal{A}$;*

(iii) *the category $\mathcal{A}$ is equivalent to the derived category $D^b(A - \mathrm{mod})$ of finite dimensional right A-modules.*

Proof. (i) Let $E_{\alpha,i}, E_{\beta,j} \in \mathcal{E}$. It follows from Lemma 3.21 that

$$\mathrm{Ext}^\bullet(E_{\alpha,i}, E_{\beta,j}) = H^\bullet(\mathbb{P}_\alpha, (E_{\beta,j-i})_\alpha).$$

Thus $\mathrm{Ext}^\bullet(E_{\alpha,i}, E_{\beta,j}) = 0$ unless $\beta \ge \alpha$. Assume that $\beta \ge \alpha$. Then by the same formula

$$\mathrm{Ext}^\bullet(E_{\alpha,i}, E_{\beta,j}) = H^\bullet(\mathbb{P}_\alpha, \mathcal{O}_{\mathbb{P}_\alpha}(j - i)).$$

If $j < i$, then again $\mathrm{Ext}^\bullet = 0$ (because $i - j \le \dim(\mathbb{P}_\alpha)$). If $j \ge i$ then only Ext^0 is nonzero. Thus $\mathcal{E}$ is a strong exceptional collection.

(ii) We will use the following theorem of A. Beilinson: for a projective space $\mathbb{P}^k$ the line bundles $\mathcal{O}_{\mathbb{P}^k}, \mathcal{O}_{\mathbb{P}^k}, (1), \ldots, \mathcal{O}_{\mathbb{P}^k}(k)$ generate the category $D^b(\mathrm{coh}(\mathbb{P}^k))$.

For $\alpha \in S$ denote by $i_\alpha : \{\alpha\} \hookrightarrow U_\alpha$ and $j_\alpha : U_\alpha \hookrightarrow S$ the closed and open embeddings, respectively. Recall that the functor

$$j_{\alpha!} i_{\alpha *} : D^b(\mathrm{coh}(\mathbb{P}_\alpha)) \to D^b(\mathrm{coh}(X/S))$$

is fully faithful (Corollary 3.7). The images of these functors generate the triangulated category $D^b(\mathrm{coh}(X/S))$. So it suffices to prove that the collection $\mathscr{E}$ generates each subcategory $j_{\alpha!}j_{\alpha*}D^b(\mathrm{coh}(\mathbb{P}_\alpha))$. We will prove this assertion by induction on α (using the partial order of S).

If α is a smallest element the assertion follows from the above mentioned theorem of Beilinson about $D^b(\mathrm{coh}(\mathbb{P}^k))$. Suppose we proved the assertion for each $\beta \in S$ such that $\beta < \alpha$. Put $V = U_\alpha - \{\alpha\}$. For $E_{\alpha,i} \in \mathscr{E}$ consider the exact sequence

$$0 \to (E_{\alpha,i})_V \to E_{\alpha,i} \to j_{\alpha!}i_{\alpha*}\mathscr{O}_{\mathbb{P}_\alpha}(i) \to 0.$$

By our assumption $(E_{\alpha,i})_V$ is contained in the subcategory of $D^b(\mathrm{coh}(X/S))$ generated by $\mathscr{E}$. So the same is true for $j_{\alpha!}i_{\alpha*}\mathscr{O}_{\mathbb{P}_\alpha}(i)$. Now the assertion for α follows from the same fact about $\mathbb{P}^k$ mentioned in the beginning.

(iii) The proof is essentially the same as that of the corresponding Theorem 6.2 in [Bo]. We present it for completeness.

Let $A - \mathrm{mod}$ denote the abelian category of right A-modules, and let $(A - \mathrm{mod})_f \subset A - \mathrm{mod}$ denote the full subcategory of finitely generated A-modules. Let $D^b(A - \mathrm{mod})$, $D^b((A - \mathrm{mod})_f)$ be the corresponding bounded derived categories. Denote by $D^b_f(A - \mathrm{mod}) \subset D^b(A - \mathrm{mod})$ the full subcategory consisting of complexes with cohomologies in $(A - \mathrm{mod})_f$. The natural functor $D^b((A - \mathrm{mod})_f) \to D^b_f(A - \mathrm{mod})$ is an equivalence. We are going to construct an equivalence

$$\theta : D^b(\mathrm{coh}(X/S)) \xrightarrow{\sim} D^b_f(A - \mathrm{mod}).$$

For each $F \in D^b(\mathrm{coh}(X/S))$ choose an injective resolution $F \to I^\bullet(F)$. Put $\theta(F) := \mathrm{Hom}^\bullet(E, I^\bullet(F))$. The algebra A acts on this complex from the right preserving each component and commuting with the differential. Thus $\theta(F)$ is a complex of right $A-$modules. The cohomologies $H^i(\theta(F))$ are finite dimensional vector spaces and are zero for $|i| \gg 0$. Thus $\theta(F)$ is an object in $D^b_f(A - \mathrm{mod})$ and we obtain a functor

$$\theta : D^b(\mathrm{coh}(X/S)) \to D^b_f(A - \mathrm{mod}).$$

Let us prove that θ is an equivalence.

Denote by $p_{\alpha,i} : E \to E_{\alpha,i}$ the projection. Let $p_{\alpha,i} := p_{\alpha,i}A$ be the corresponding projective right A-module.

Note that $H^j(\theta(E_{\alpha,i})) = 0$ for $j \neq 0$ and $H^0(\theta(E_{\alpha,i})) = P_{\alpha,i}$. The collections $\mathscr{E} = \{E_{\alpha,i}\}$ and $P_{\alpha,i}$ generate the categories $D^b(\mathrm{coh}(X/S))$ and $D^b_f(A - \mathrm{mod})$, respectively. Thus it suffices to prove that the map

$$\theta : \mathrm{Ext}^\bullet(E_{\alpha,i}, E_{\beta,j}) \to \mathrm{Ext}^\bullet(P_{\alpha,i}, P_{\beta,j})$$

is an isomorphism. But the higher Ext's on both sides vanish and

$$\theta : \mathrm{Hom}(E_{\alpha,i}, E_{\beta,j}) \to \mathrm{Hom}_A(P_{\alpha,i}, P_{\beta,j})$$

is an isomorphism by Remark 4.15 above. ∎

5. *Spherical Objects and Twist Functors.* We recall some facts about spherical objects and the corresponding twist functors following [ST]. In the next subsection we will apply these results to Calabi-Yau hyperplane configuration schemes.

Let $\mathscr{C}$ be an abelian category and $\mathscr{C}' \subset \mathscr{C}$ be a full abelian subcategory with the following properties

(C1) $\mathscr{C}'$ is a Serre subcategory of $\mathscr{C}$;

(C2) $\mathscr{C}$ contains infinite direct sums and products;

(C3) $\mathscr{C}$ has enough injectives, and any direct sum of injectives is again an injective;

(C4) for any epimorphism $f : A \twoheadrightarrow A'$. with $A \in \mathscr{C}$ and $A' \in \mathscr{C}'$, there is a $B' \in \mathscr{C}'$ and $g : B' \to A$ such that fg is an epimorphism.

LEMMA 4.17. *Let X be a Noetherian scheme over a field k and $C = \mathrm{qcoh}(X), \mathscr{C}' = \mathrm{coh}(X)$. Then properties* (C1)–(C4) *are satisfied.*

Proof. See [ST]. ∎

DEFINITION 4.18. Let $K^+(\mathscr{C})$ be the homotopy category of bounded below complexes of objects in $\mathscr{C}$. Denote by $\mathscr{K} \subset K^+(\mathscr{C})$ the full subcategory whose objects are complexes $C^\bullet$ of injectives in $\mathscr{C}$ which satisfy $H^i(C^\bullet) \in \mathscr{C}'$ for all i and $H^i(C^\bullet) = 0$ for all $i \gg 0$.

The following lemma is well known (also proved in [ST]).

LEMMA 4.19. *The obvious functors*

$$D^b(\mathscr{C}') \to D^b_{\mathscr{C}'}(\mathscr{C}) \leftarrow \mathscr{K}$$

are equivalences. Thus in particular $D^b(\mathscr{C}') \simeq \mathscr{K}$.

Let us define some operations in the category $\mathscr{K}$.

DEFINITION 4.20. Let $E \in \mathscr{K}$ be an object satisfying the following finiteness conditions

(K1) E is a bounded complex,

(K2) for any $F \in \mathscr{K}$

$$\bigoplus_i \dim \mathrm{Ext}^i(E, F) < \infty, \qquad \bigoplus_i \dim \mathrm{Ext}^i(F, E) < \infty.$$

Then one defines the *twist functor* $T_E : \mathcal{K} \to \mathcal{K}$ by

$$T_E(F) := \mathrm{Cone}(\mathrm{Hom}^\bullet(E, F) \otimes E \xrightarrow{ev} F),$$

where ev is the obvious evaluation map.

Remarks. (1) T_E is an exact functor; i.e., it takes exact triangles to exact triangles.

(2) If $E, E_1 \in \mathcal{K}$ satisfying (K1), (K2) are isomorphic, then the corresponding functors T_E, T_{E_1} are isomorphic.

DEFINITION 4.21. An object $E \in \mathcal{K}$ is called *n-spherical* for some $n > 0$ if it satisfies (K1), (K2) above and in addition

(K3) $\mathrm{Ext}^i(E, E)$ is equal to $\mathbb{C}$ for $i = 0, n$ and zero otherwise;

(K4) the composition

$$\mathrm{Ext}^j(E, F) \times \mathrm{Ext}^{n-j}(F, E) \to \mathrm{Ext}^n(E, E) \simeq \mathbb{C}$$

is a nondegenerate pairing for all $F \in \mathcal{K}$, $j \in \mathbb{Z}$.

PROPOSITION 4.22. *If E is n-spherical for some $n > 0$, then the functor $T_E : \mathcal{K} \to \mathcal{K}$ is an equivalence.*

Proof. See [ST]. ∎

Using the equivalence $D^b(\mathscr{C}') \simeq \mathcal{K}$ we obtain the following corollary.

COROLLARY 4.23. *In the setup of the last proposition the n-spherical object E induces an exact auto-equivalence of $D^b(\mathscr{C}')$ which we also denote T_E.*

EXAMPLE 4.24. The standard example of an n-spherical object in $D^b(\mathrm{coh}(X))$ is $\mathcal{O}_X$ for a smooth n-dimensional Calabi–Yau variety X. Indeed, by definition $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{O}_X)$ is $\mathbb{C}$ for $i = 0, n$ and zero otherwise. Also by definition $\mathcal{O}_X \simeq \omega_X$ and hence by Serre duality the natural pairing

$$\mathrm{Ext}^i(\mathcal{O}_X, F) \times \mathrm{Ext}^{n-i}(F, \mathcal{O}_X) \to \mathrm{Ext}^n(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathbb{C}$$

is perfect for all $F \in D^b(\mathrm{coh}(X))$ and all i. Thus $\mathcal{O}_X$ is indeed n-spherical. Examples of other spherical objects in $D^b(\mathrm{coh}(X))$ are discussed in [ST].

6. *Calabi–Yau Hyperplane Configuration Schemes.* Let X/S be a Calabi–Yau hyperplane configuration scheme; i.e., it is of type $(n, n+2)$ for some $n > 0$. Let X_0, as usual, be the corresponding singular subscheme of $\mathbb{P}^{n+1}$ which is the union of the $n+2$ hyperplanes. The object $\mathcal{O}_{X_0} \in D^b(\mathrm{coh}(X_0))$ is the dualizing complex on X_0. That is, the pairing

$$\mathrm{Ext}^i(\mathcal{O}_{X_0}, F) \times \mathrm{Ext}^{n-i}(F, \mathcal{O}_{X_0}) \to \mathrm{Ext}^n(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}) \simeq \mathbb{C}$$

is perfect for all $F \in D^b(\mathrm{coh}(X_0))$ and all i. Thus for all $F \in \mathrm{coh}(X_0)$ we have $\mathrm{Ext}^{n+1}(F, \mathcal{O}_{X_0}) = 0$. Therefore the same is true for all $F \in \mathrm{qcoh}(X_0)$. It follows that $\mathcal{O}_{X_0}$ has finite injective dimension in $\mathrm{qcoh}(X_0)$. So there exists a finite injective resolution $\mathcal{O}_{X_0} \to E$ in $\mathrm{qcoh}(X_0)$ such that $E \in \mathcal{K}$ satisfies the properties (K1)–(K4) above. Hence the corresponding twist functor $T_{\mathcal{O}_{X_0}} = T_E : D^b(\mathrm{coh}(X_0)) \to D^b(\mathrm{coh}(X_0))$ is an equivalence. It follows that $\mathcal{O}_{X_0}(j)$ is also n-spherical for all $j \in \mathbb{Z}$. Hence for each j we obtain the corresponding auto-equivalence of $D^b(\mathrm{coh}(X_0))$. We are going to show that these auto-equivalences survive when we pass from X_0 to the configuration scheme X/S.

First we need a lemma.

LEMMA 4.25. *The categories* $\mathcal{C} = \mathrm{qcoh}(X/S)$ *and* $\mathcal{C}' = \mathrm{coh}(X/S)$ *satisfy the assumptions* (C1)–(C4) *of the last subsection*.

Proof. Assumption (C1) is obvious and (C4) was proved in Lemma 3.4 above. Let us prove (C2).

Let $F^i \in \mathrm{qcoh}(X/S)$ for $i \in I$. Let us define the object $F = \oplus F^i \in \mathrm{qcoh}(X/S)$. Namely, for $\alpha \in S$ put $F_\alpha = \oplus F^i_\alpha$. Then for $f_{\beta\alpha} : X(\beta) \to X(\alpha)$ we have

$$f^*_{\beta\alpha}\Big(\bigoplus F^i_\alpha\Big) = \bigoplus f^*_{\beta\alpha} F^i_\alpha.$$

Hence we define the morphism $\sigma_{\alpha\beta}(F) : f^*_{\beta\alpha} F_\alpha \to F_\beta$ to be the direct sum of morphisms $\sigma_{\alpha\beta}(F^i)$.

The definition of $G = \Pi F^i \in \mathrm{qcoh}(X/S)$ is similar. Namely, for $\alpha \in S$ put $G_\alpha = \Pi F^i_\alpha$. Then for $f_{\beta\alpha} : X(\beta) \to X(\alpha)$ we have a canonical morphism

$$m_{\alpha\beta} : f^*_{\beta\alpha}\Big(\prod F^i_\alpha\Big) \to \prod f^*_{\beta\alpha} F^i_\alpha.$$

Hence we define the morphism $\sigma_{\alpha\beta}(G) : f^*_{\beta\alpha} G_\alpha \to G_\beta$ as the composition of $m_{\beta\alpha}$ with the product of morphisms $\sigma_{\alpha\beta}(F^i)$.

One checks directly that F and G satisfy the universal properties of the direct sum and the direct product, respectively.

Let us prove (C3). Also by Lemma 3.1 the category $\mathrm{qcoh}(X/S)$ has enough injectives. It remains to prove that a direct sum of injectives in $\mathrm{qcoh}(X/S)$ is again an injective. Note that the category $\mathrm{qcoh}(X/S)$ is locally noetherian; i.e., it has a family of noetherian generators (any object in $\mathrm{coh}(X/S)$ is noetherian). Hence a direct sum of injectives in $\mathrm{qcoh}(X/S)$ is an injective by Corollary 6.50 in [BD]. ∎

Remark 4.26. The assertion of the lemma is true for arbitrary configuration schemes (the proof is the same).

THEOREM 4.27. *Let* X/S *be a Calabi–Yau hyperplane configuration scheme of type* $(n, n+2)$. *Then for any* t *the line bundle* $\mathcal{O}_{X/S}(t)$ *is an* n*-spherical object in* $D^b(\mathrm{coh}(X/S))$.

Proof. We may assume that $t = 0$.

The finiteness conditions (K1), (K2) are satisfied for any $E \in D^b(\mathrm{coh}(X/S))$ by Corollary 3.2 and Lemma 3.12. Also by Theorem 4.5, $\mathrm{Ext}^i(\mathcal{O}_{X/S}, \mathcal{O}_{X/S}) = \mathrm{Ext}^i(\mathcal{O}_{X_0}, \mathcal{O}_{X_0})$. So we only need to prove that for all $F \in D^b(\mathrm{coh}(X/S))$ the natural pairing

$$\mathrm{Ext}^i(\mathcal{O}_{X/S}, F) \times \mathrm{Ext}^{n-i}(F, \mathcal{O}_{X/S}) \to \mathrm{Ext}^n(\mathcal{O}_{X/S}, \mathcal{O}_{X/S}) \simeq \mathbb{C}$$

is perfect.

Step 1. We can find a left resolution $P^\bullet \to F$, where $P^\bullet$ consists of finite direct sums of sheaves $\mathcal{O}_{U_\alpha}(j)$ for various $\alpha \in S$, $j < 0$. Thus we may assume that $F = \mathcal{O}_{U_\alpha}(j)$ for some $\alpha \in S$ and $j < 0$.

Step 2. Note that

$$\mathrm{Ext}^k(\mathcal{O}_{U_\alpha}(j), \mathcal{O}_{X/S}) = \mathrm{Ext}^k(\mathcal{O}_{U_\alpha}(j), \mathcal{O}_{U_\alpha}) = \mathrm{Ext}^k(\mathcal{O}_{\mathbb{P}_\alpha}(j), \mathcal{O}_{\mathbb{P}_\alpha}) = 0,$$

unless $k = 0$.

LEMMA 4.28. $\mathrm{Ext}^i(\mathcal{O}_{X/S}, \mathcal{O}_{U_\alpha}(j)) = 0$ *if* $i \neq n$.

Proof of Lemma. Consider the Čech resolution $\check{C}^\bullet \to \mathcal{O}_{X/S}$ as defined above.

By Lemma 3.21 we have

$$\mathrm{Ext}^p(\mathcal{O}_{U_\gamma}, \mathcal{O}_{U_\alpha}(j)) = \begin{cases} 0, & \text{if } \gamma \not\leq \alpha, \\ \mathrm{Ext}^p(\mathcal{O}_{\mathbb{P}_\gamma}, \mathcal{O}_{\mathbb{P}_\gamma}(j)), & \text{otherwise.} \end{cases}$$

Since $j < 0$, $\mathrm{Ext}^p(\mathcal{O}_{\mathbb{P}_\gamma}, \mathcal{O}_{\mathbb{P}_\gamma}(j)) = 0$, unless $p = d(\gamma)$. So the spectral sequence argument applied to the naturally filtered complex $\mathbb{R}\mathrm{Hom}^\bullet(\check{C}^\bullet, \mathcal{O}_{U_\alpha}(j))$ yields $\mathrm{Ext}^i(\mathcal{O}_{X/S}, \mathcal{O}_{U_\alpha}(j)) = 0$ for $i \neq n$. This proves the lemma.

Therefore, it suffices to show that the pairing

$$\mathrm{Ext}^n(\mathcal{O}_{X/S}, \mathcal{O}_{U_\alpha}(j)) \times \mathrm{Hom}(\mathcal{O}_{U_\alpha}(j), \mathcal{O}_{X/S}) \to \mathrm{Ext}^n(\mathcal{O}_{X/S}, \mathcal{O}_{X/S}) \tag{1}$$

is perfect.

Step 3. Let $\check{C}^\bullet_\alpha \subset \check{C}^\bullet$ be the subcomplex consisting of direct sums of objects $\mathcal{O}_{U_\beta}$ with $\beta \leq \alpha$.

LEMMA 4.29. *The inclusion* $\check{C}^\bullet_\alpha \hookrightarrow \check{C}^\bullet$ *induces isomorphisms*

$$\mathrm{Ext}^n(\check{C}^\bullet, \mathcal{O}_{U_\alpha}(j)) \xrightarrow{\sim} \mathrm{Ext}^n(\check{C}^\bullet_\alpha, \mathcal{O}_{U_\alpha}(j)),$$

and

$$\mathrm{Ext}^n(\check{C}^\bullet, \mathcal{O}_{X/S}) \xrightarrow{\sim} \mathrm{Ext}^n(\check{C}^\bullet_\alpha, \mathcal{O}_{X/S}).$$

Proof of Lemma. The first one follows from Lemma 3.21. Let us prove the second one. The complex $\mathrm{Hom}^\bullet(\check{C}^\bullet, \mathcal{O}_{X/S})$ is isomorphic to the (shifted by n to the right) chain complex of the boundary of an $(n+1)$-simplex with the standard triangulation. Its quotient $\mathrm{Hom}^\bullet(\check{C}^\bullet_\alpha, \mathcal{O}_{X/S})$ corresponds to the (shifted) chain complex of a subsimplex. Hence the projection $\mathrm{Hom}^\bullet(\check{C}^\bullet, \mathcal{O}_{X/S}) \to \mathrm{Hom}^\bullet(\check{C}^\bullet_\alpha, \mathcal{O}_{X/S})$ induces an isomorphism on the top cohomology. This proves the lemma.

It follows that the pairing (1) is isomorphic to

$$\mathrm{Ext}^n(\check{C}^\bullet_\alpha, \mathcal{O}_{U_\alpha}(j)) \times \mathrm{Hom}(\mathcal{O}_{U_\alpha}(j), \mathcal{O}_{X/S}) \to \mathrm{Ext}^n(\check{C}^\bullet_\alpha, \mathcal{O}_{X/S}).$$

Now again by Lemma 3.21 we can replace $\mathcal{O}_{X/S}$ by $\mathcal{O}_{U_\alpha}$ to get the isomorphic pairing

$$\mathrm{Ext}^n(\check{C}^\bullet_\alpha, \mathcal{O}_{U_\alpha}(j)) \times \mathrm{Hom}(\mathcal{O}_{U_\alpha}(j), \mathcal{O}_{U_\alpha}) \to \mathrm{Ext}^n(\check{C}^\bullet_\alpha, \mathcal{O}_{U_\alpha}).$$

Let $j_\alpha : \{\alpha\} \hookrightarrow U_\alpha$ be the closed embedding. Note that $\mathcal{O}_{U_\alpha} = j_{\alpha+}\mathcal{O}_{\mathbb{P}_\alpha}$, and $\mathcal{O}_{U_\alpha}(j) = j_{\alpha+}\mathcal{O}_{\mathbb{P}_\alpha}(j)$. Thus the last pairing is

$$\mathrm{Ext}^n(\check{C}^\bullet_\alpha, j_{\alpha+}\mathcal{O}_{\mathbb{P}_\alpha}(j)) \times \mathrm{Hom}(j_{\alpha+}\mathcal{O}_{\mathbb{P}_\alpha}(j), j_{\alpha+}\mathcal{O}_{\mathbb{P}_\alpha}) \to \mathrm{Ext}^n(\check{C}^\bullet_\alpha, j_{\alpha+}\mathcal{O}_{\mathbb{P}_\alpha}). \tag{2}$$

Step 4.

LEMMA 4.30. *Let $F \in D^b(\mathrm{coh}(\mathbb{P}_\alpha))$ be a finite complex of locally free sheaves. Then for $i \gg 0$ we have a natural quasi-isomorphism of complexes*

$$\mathrm{Hom}^\bullet(\mathcal{O}_{\mathbb{P}_\alpha}(-i), F(-d(\alpha)-1)) \xrightarrow{\sim} \mathrm{Hom}^\bullet(\check{C}^\bullet_\alpha(-i), j_{\alpha+}F)[n-d(\alpha)],$$

where the first Hom is computed in the category $\mathrm{coh}(\mathbb{P}_\alpha)$ *and the second in* $\mathrm{coh}(X/S)$ (*or in* $\mathrm{coh}(X/U_\alpha)$).

Proof of Lemma. Let $\beta_1, \dots, \beta_{d(\alpha)+1} \in S$ be all elements with the property $\beta_i < \alpha$, $d(\beta_i) = d(\alpha) - 1$. Then for each i, $\mathbb{P}_{\beta_i}$ is a hyperplane in $\mathbb{P}_\alpha$ (more precisely, $f_{\beta_i\alpha}(\mathbb{P}_{\beta_i})$ is such). Put $Y = \cup\mathbb{P}_{\beta_i} \subset \mathbb{P}_\alpha$. We have the exact sequence of complexes on $\mathbb{P}_\alpha$

$$0 \to F(-d(\alpha)-1) \to F \to F_Y \to 0.$$

Let $0 \to F_Y \to J^\bullet(F_Y) \to 0$ be the Čech resolution (as in the proof of Lemma 4.6) of F_Y corresponding to the closed covering of Y by $\mathbb{P}_{\beta_i}$'s, i.e.,

$$J^\bullet(F_Y) = 0 \to \bigoplus_i F_{\mathbb{P}_{\beta_i}} \to \bigoplus_{i<j} F_{\mathbb{P}_{\beta_i} \cap \mathbb{P}_{\beta_j}} \to \dots.$$

Denote by $J_F^\bullet$ the complex $0 \to F \to J^\bullet(F_Y) \to 0$. Then $J_F^\bullet$ is quasi-isomorphic to the complex $F(-d(\alpha)-1)$. Thus for $i \gg 0$ the natural map of complexes

$$\mathrm{Hom}^\bullet(\mathscr{O}_{\mathbb{P}_\alpha}(-i), F(-d(\alpha)-1)) \to \mathrm{Hom}^\bullet(\mathscr{O}_{\mathbb{P}_\alpha}(-i), J_F^\bullet)$$

is a quasi-isomorphism. It remains to notice the natural isomorphism of complexes

$$\mathrm{Hom}^\bullet_{\mathrm{coh}(\mathbb{P}_\alpha)}(\mathscr{O}_{\mathbb{P}_\alpha}(-i), J_F^\bullet) = \mathrm{Hom}^\bullet_{\mathrm{coh}(X/S)}(\check{C}_\alpha^\bullet(-i), j_{\alpha+}F)[n-d(\alpha)].$$

This proves the lemma.

Step 5. Let $F \in D^b(\mathrm{coh}(\mathbb{P}_\alpha))$ be as in the above lemma. We may take a left resolution $P^\bullet \to \mathscr{O}_{\mathbb{P}_\alpha}$ consisting of finite sums $\oplus \mathscr{O}_{\mathbb{P}_\alpha}(-i)$ for $i \gg 0$. This induces a quasi-isomorphism of Čech resolutions $\check{C}_\alpha^\bullet(P^\bullet) \xrightarrow{\sim} \check{C}_\alpha^\bullet$. We have

$$\mathrm{Hom}^\bullet(P^\bullet, F(-d(\alpha)-1)) = \mathbb{R}\mathrm{Hom}^\bullet(\mathscr{O}_{\mathbb{P}_\alpha}, F(-d(\alpha)-1)),$$

$$\mathrm{Hom}^\bullet(\check{C}_\alpha^\bullet(P^\bullet), j_{\alpha+}(F)) = \mathbb{R}\,\mathrm{Hom}^\bullet(\check{C}_\alpha^\bullet, j_{\alpha+}(F)).$$

Thus the above lemma implies a natural quasi-isomorphism of complexes

$$\mathbb{R}\,\mathrm{Hom}^\bullet(\mathscr{O}_{\mathbb{P}_\alpha}, F(-d(\alpha)-1)) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}^\bullet(\check{C}_\alpha^\bullet, j_{\alpha+}F)[n-d(\alpha)].$$

This shows that the pairing (2) above is isomorphic to the pairing

$$\mathrm{Ext}^{d(\alpha)}(\mathscr{O}_{\mathbb{P}_\alpha}, \mathscr{O}_{\mathbb{P}_\alpha}(j-d(\alpha)-1)) \times \mathrm{Hom}(\mathscr{O}_{\mathbb{P}_\alpha}(j-d(\alpha)-1), \mathscr{O}_{\mathbb{P}_\alpha}(-d(\alpha)-1)) \to \mathrm{Ext}^{d(\alpha)}(\mathscr{O}_{\mathbb{P}_\alpha}, \mathscr{O}_{\mathbb{P}_\alpha}(-d(\alpha)-1)),$$

where everything is computed in $\mathrm{coh}(\mathbb{P}_\alpha)$. The last pairing is perfect by the Serre duality on $\mathbb{P}_\alpha$. This proves the theorem. ∎

COROLLARY 4.31. *Let X/S be a Calabi–Yau hyperplane configuration scheme of type $(n, n+2)$. Then for any $t \in \mathbb{Z}$ the twist functor $T_{\mathscr{O}_{X/S}(t)}$ is an autoequivalence of $D^b(\mathrm{coh}(X/S))$.*

Proof. This follows immediately from the last theorem, Lemma 4.25, and Corollary 4.23. ∎

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