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Scalar Levin-type sequence transformations

Herbert H.H. Homeier [*,1]

Institut für Physikalische und Theoretische Chemie, Universität Regensburg, D-93040 Regensburg, Germany

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Abstract

Sequence transformations are important tools for the convergence acceleration of slowly convergent scalar sequences or series and for the summation of divergent series. The basic idea is to construct from a given sequence $\{\{s_n\}\}$ a new sequence $\{\{s'_n\}\} = \mathscr{T}(\{\{s_n\}\})$ where each s'_n depends on a finite number of elements $s_{n_1},\dots,s_{n_m}$. Often, the s_n are the partial sums of an infinite series. The aim is to find transformations such that $\{\{s'_n\}\}$ converges faster than (or sums) $\{\{s_n\}\}$. Transformations $\mathscr{T}(\{\{s_n\}\},\{\{\omega_n\}\})$ that depend not only on the sequence elements or partial sums s_n but also on an auxiliary sequence of the so-called remainder estimates ω_n are of Levin-type if they are linear in the s_n, and nonlinear in the ω_n. Such remainder estimates provide an easy-to-use possibility to use asymptotic information on the problem sequence for the construction of highly efficient sequence transformations. As shown first by Levin, it is possible to obtain such asymptotic information easily for large classes of sequences in such a way that the ω_n are simple functions of a few sequence elements s_n. Then, nonlinear sequence transformations are obtained. Special cases of such Levin-type transformations belong to the most powerful currently known extrapolation methods for scalar sequences and series. Here, we review known Levin-type sequence transformations and put them in a common theoretical framework. It is discussed how such transformations may be constructed by either a model sequence approach or by iteration of simple transformations. As illustration, two new sequence transformations are derived. Common properties and results on convergence acceleration and stability are given. For important special cases, extensions of the general results are presented. Also, guidelines for the application of Levin-type sequence transformations are discussed, and a few numerical examples are given. © 2000 Elsevier Science B.V. All rights reserved.

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[*] Fax: +49-941-943-4719.
E-mail address: herbert.homeier@na-net.ornl.gov (H.H.H. Homeier)
[1] WWW: http://www.chemie.uni-regensburg.de/ ~hoh05008

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1. Introduction

In applied mathematics and the numerate sciences, extrapolation methods are often used for the convergence acceleration of slowly convergent sequences or series and for the summation of divergent series. For an introduction to such methods, and also further information that cannot be covered here, see the books of Brezinski and Redivo Zaglia [14] and Wimp [102] and the work of Weniger [84,88] and Homeier [40], but also the books of Baker [3], Baker and Graves-Morris [5], Brezinski [7,8,10–12], Graves-Morris [24,25], Graves-Morris, Saff and Varga [26], Khovanskii [52], Lorentzen and Waadeland [56], Nikishin and Sorokin [62], Petrushev and Popov [66], Ross [67], Saff and Varga [68], Wall [83], Werner and Buenger [101] and Wuytack [103].

For the discussion of extrapolation methods, one considers a sequence $\{\{s_n\}\} = \{\{s_0, s_1, \dots\}\}$ with elements s_n or the terms $a_n = s_n - s_{n-1}$ of a series $\sum_{j=0}^{\infty} a_j$ with partial sums $s_n = \sum_{j=0}^{n} a_j$ for large n. A common approach is to rewrite s_n as

$$s_n = s + R_n, \tag{1}$$

where s is the limit (or antilimit in the case of divergence) and R_n is the remainder or tail. The aim then is to find a new sequence $\{\{s'_n\}\}$ such that

$$s'_n = s + R'_n, \qquad R'_n/R_n \to 0 \text{ for } n \to \infty. \tag{2}$$

Thus, the sequence $\{\{s'_n\}\}$ converges faster to the limit s (or diverges less violently) than $\{\{s_n\}\}$.

To find the sequence $\{\{s'_n\}\}$, i.e., to construct a sequence transformation $\{\{s'_n\}\} = \mathscr{T}(\{\{s_n\}\})$, one needs asymptotic information about the s_n or the terms a_n for large n, and hence about the R_n. This information then allows to eliminate the remainder at least asymptotically, for instance by substracting the dominant part of the remainder. Either such information is obtained by a careful mathematical analysis of the behavior of the s_n and/or a_n, or it has to be extracted numerically from the values of a finite number of the s_n and/or a_n by some method that ideally can be proven to work for a large class of problems.

Suppose that one knows quantities ω_n such that $R_n/\omega_n = \mathrm{O}(1)$ for $n \to \infty$, for instance

$$\lim_{n\to\infty} R_n/\omega_n = c \neq 0, \tag{3}$$

where c is a constant. Such quantities are called remainder estimates. Quite often, such remainder estimates can be found with relatively low effort but the exact value of c is often quite hard to calculate. Then, it is rather natural to rewrite the rest as $R_n = \omega_n \mu_n$ where $\mu_n \to c$. The problem is how to describe or model the μ_n. Suppose that one has a system of known functions $\psi_j(n)$ such that $\psi_0(n) = 1$ and $\psi_{j+1} = \mathrm{o}(\psi_j(n))$ for $j \in \mathbb{N}_0$. An example of such a system is $\psi_j(n) = (n+\beta)^{-j}$ for some $\beta \in \mathbb{R}_+$. Then, one may model μ_n as a linear combination of the $\psi_j(n)$ according to

$$\mu_n \sim \sum_{j=0}^{\infty} c_j \psi_j(n) \qquad \text{for } n \to \infty, \tag{4}$$

whence the problem sequence is modelled according to

$$s_n \sim s + \omega_n \sum_{j=0}^{\infty} c_j \psi_j(n). \tag{5}$$

The idea now is to eliminate the leading terms of the remainder with the unknown constants c_j up to $j = k - 1$, say. Thus, one uses a model sequence with elements

$$\sigma_m = \sigma + \omega_m \sum_{j=0}^{k-1} c_j \psi_j(m), \qquad m \in \mathbb{N}_0 \tag{6}$$

and calculates σ exactly by solving the system of $k+1$ equations resulting for $m = n, n+1, \ldots, n+k$ for the unknowns σ and c_j, $j = 0, \ldots, k-1$. The solution for σ is a ratio of determinants (see below) and may be denoted symbolically as

$$\sigma = T(\sigma_n, \ldots, \sigma_{n+k}; \omega_n, \ldots, \omega_{n+k}; \psi_j(n), \ldots, \psi_j(n+k)). \tag{7}$$

The resulting sequence transformation is

$$\mathscr{T}(\{\{s_n\}\}, \{\{\omega_n\}\}) = \{\{\mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\})\}\} \tag{8}$$

with

$$\mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}) = T(s_n, \ldots, s_{n+k}; \omega_n, \ldots, \omega_{n+k}; \psi_j(n), \ldots, \psi_j(n+k)). \tag{9}$$

It eliminates the leading terms of the asymptotic expansion (5). The model sequences (6) are in the kernel of the sequence transformation $\mathscr{T}$, defined as the set of all sequences such that $\mathscr{T}$ reproduces their (anti)limit exactly.

A somewhat more general approach is based on model sequences of the form

$$\sigma_n = \sigma + \sum_{j=1}^{k} c_j g_j(n), \qquad n \in \mathbb{N}_0, \; k \in \mathbb{N}. \tag{10}$$

Virtually all known sequence transformations can be derived using such model sequences. This leads to the **E** algorithm as described below in Section 3.1. Also, some further important examples of sequence transformations are described in Section 3.

However, the introduction of remainder estimates proved to be an important theoretical step since it allows to make use of asymptotic information of the remainder easily. The most prominent of the resulting sequence transformations $\mathscr{T}(\{\{s_n\}\}, \{\{\omega_n\}\})$ is the Levin transformation [53] that corresponds to the asymptotic system of functions given by $\psi_j(n) = (n+\beta)^{-j}$, and thus, to Poincare-type expansions of the μ_n. But also other systems are of importance, like $\psi_j(n) = 1/(n+\beta)_j$ leading to factorial series, or $\psi_j(n) = t_n^j$ corresponding to Taylor expansions of t-dependent functions at the abscissae t_n that tend to zero for large n. The question which asymptotic system is best, cannot be decided generally. The answer to this question depends on the extrapolation problem. To obtain efficient extrapolation procedures for large classes of problems requires to use various asymptotic systems, and thus, a larger number of different sequence transformations. Also, different choices of ω_n lead to different variants of such transformations. Levin [53] has pioneered this question and introduced three variants that are both simple and rather successful for large classes of problems. These variants and some further ones will be discussed. The question which variant is best, also cannot be decided generally. There are, however, a number of results that favor certain variants for certain problems. For example, for Stieltjes series, the choice $\omega_n = a_{n+1}$ can be theoretically justified (see Appendix A).

Thus, we will focus on sequence transformations that involve an auxiliary sequence $\{\{\omega_n\}\}$. To be more specific, we consider transformations of the form $\mathscr{T}(\{\{s_n\}\},\{\{\omega_n\}\}) = \{\{\mathscr{T}_n^{(k)}\}\}$ with

$$\mathscr{T}_n^{(k)} = \frac{\sum_{j=0}^{k} \lambda_{n,j}^{(k)} s_{n+j}/\omega_{n+j}}{\sum_{j=0}^{k} \lambda_{n,j}^{(k)}/\omega_{n+j}}. \tag{11}$$

This will be called a Levin-type transformations. The known sequence transformations that involve remainder estimates, for instance the $\mathscr{C}$, $\mathscr{S}$, and $\mathscr{M}$ transformations of Weniger [84], the W algorithm of Sidi [73], and the $\mathscr{J}$ transformation of Homeier with its many special cases like the important ${}_p\mathbf{J}$ transformations [35,36,38–40,46], are all of this type. Interestingly, also the $\mathscr{H}$, $\mathscr{I}$, and $\mathscr{K}$ transformations of Homeier [34,35,37,40–44] for the extrapolation of orthogonal expansions are of this type although the ω_n in some sense cease to be remainder estimates as defined in Eq. (3).

The Levin transformation was also generalized in a different way by Levin and Sidi [54] who introduced the $d^{(m)}$ transformations. This is an important class of transformations that would deserve a thorough review itself. This, however, is outside the scope of the present review. We collect some important facts regarding this class of transformations in Section 3.2.

Levin-type transformations as defined in Eq. (11) have been used for the solution of a large variety of problems. For instance, Levin-type sequence transformations have been applied for the convergence acceleration of infinite series representations of molecular integrals [28,29, 33,65,82,98–100], for the calculation of the lineshape of spectral holes [49], for the extrapolation of cluster- and crystal-orbital calculations of one-dimensional polymer chains to infinite chain length [16,88,97], for the calculation of special functions [28,40,82,88,89,94,100], for the summation of divergent and acceleration of convergent quantum mechanical perturbation series [17,18,27,85,90–93,95,96], for the evaluation of semiinfinite integrals with oscillating integrands and Sommerfeld integral tails [60,61,75,81], and for the convergence acceleration of multipolar and orthogonal expansions and Fourier series [34,35,37,40–45,63,77,80]. This list is clearly not complete but sufficient to demonstrate the possibility of successful application of these transformations.

The outline of this survey is as follows: After listing some definitions and notations, we discuss some basic sequence transformations in order to provide some background information. Then, special definitions relevant for Levin-type sequence transformations are given, including variants obtained by choosing specific remainder estimates ω_n. After this, important examples of Levin-type sequence transformations are introduced. In Section 5, we will discuss approaches for the construction of Levin-type sequence transformations, including model sequences, kernels and annihilation operators, and also the concept of hierarchical consistency. In Section 6, we derive basic properties, those of limiting transformations and discuss the application to power series. In Section 7, results on convergence acceleration are presented, while in Section 8, results on the numerical stability of the transformations are provided. Finally, we discuss guidelines for the application of the transformations and some numerical examples in Section 9.

2. Definitions and notations

2.1. General definitions

2.1.1. Sets

Natural numbers:

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{12}$$

Integer numbers:

$$\mathbb{Z} = \mathbb{N} \cup \{0, -1, -2, -3, \ldots\}. \tag{13}$$

Real numbers and vectors:

$$\begin{aligned} &\mathbb{R} = \{x : x \text{ real}\}, \\ &\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\} \\ &\mathbb{R}^n = \{(x_1, \ldots, x_n) \,|\, x_j \in \mathbb{R}, j = 1, \ldots, n\}. \end{aligned} \tag{14}$$

Complex numbers:

$$\begin{aligned} &\mathbb{C} = \{z = x + \mathrm{i}y : x \in \mathbb{R}, y \in \mathbb{R},\ \mathrm{i}^2 = -1\}, \\ &\mathbb{C}^n = \{(z_1, \ldots, z_n) \,|\, z_j \in \mathbb{C}, j = 1, \ldots, n\}. \end{aligned} \tag{15}$$

For $z = x + \mathrm{i}y$, real and imaginary parts are denoted as $x = \Re(z)$, $y = \Im(z)$. We use $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

Vectors with nonvanishing components:

$$\mathbb{F}^n = \{(z_1, \ldots, z_n) \,|\, z_j \in \mathbb{C}, z_j \neq 0,\ j = 1, \ldots, n\}. \tag{16}$$

Polynomials:

$$\mathbb{P}^k = \left\{ P : z \mapsto \sum_{j=0}^{k} c_j z^j \,|\, z \in \mathbb{C}, (c_0, \ldots, c_k) \in \mathbb{K}^{k+1} \right\}. \tag{17}$$

Sequences:

$$\mathbb{S}^{\mathbb{K}} = \{\{\{s_0, s_1, \ldots, s_n, \ldots\}\} \,|\, s_n \in \mathbb{K},\ n \in \mathbb{N}_0\}. \tag{18}$$

Sequences with nonvanishing terms:

$$\mathbb{O}^{\mathbb{K}} = \{\{\{s_0, s_1, \ldots, s_n, \ldots\}\} \,|\, s_n \neq 0,\ s_n \in \mathbb{K},\ n \in \mathbb{N}_0\}. \tag{19}$$

2.1.2. Special functions and symbols

Gamma function [58, p. 1]:

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\, \mathrm{d}t \quad (z \in \mathbb{R}_+). \tag{20}$$

Factorial:

$$n! = \Gamma(n+1) = \prod_{j=1}^{n} j. \tag{21}$$

Pochhammer symbol [58, p. 2]:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{j=1}^{n}(a+j-1). \tag{22}$$

Binomial coefficients [1, p. 256, Eq. (6.1.21)]:

$$\binom{z}{w} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}. \tag{23}$$

Entier function:

$$[\![x]\!] = \max\{j \in \mathbb{Z}\colon j \leqslant x,\ x \in \mathbb{R}\}. \tag{24}$$

2.2. Sequences, series and operators

2.2.1. Sequences and series

For Stieltjes series see Appendix A.

Scalar sequences with elements s_n, tail R_n, and limit s:

$$\{\{s_n\}\} = \{\{s_n\}\}_{n=0}^{\infty} = \{\{s_0, s_1, s_2, \ldots\}\} \in \mathbb{S}^{\mathbb{K}}, \quad R_n = s_n - s, \quad \lim_{n\to\infty} s_n = s. \tag{25}$$

If the sequence is not convergent but summable to s, s is called the antilimit. The nth element s_n of a sequence $\sigma = \{\{s_n\}\} \in \mathbb{S}^{\mathbb{K}}$ is also denoted by $\langle\sigma\rangle_n$. A sequence is called a *constant sequence*, if all elements are constant, i.e., if there is a $c \in \mathbb{K}$ such that $s_n = c$ for all $n \in \mathbb{N}_0$, in which case it is denoted by $\{\{c\}\}$. The constant sequence $\{\{0\}\}$ is called the *zero sequence*.

Scalar series with terms $a_j \in \mathbb{K}$, partial sums s_n, tail R_n, and limit/antilimit s:

$$s = \sum_{j=0}^{\infty} a_j, \quad s_n = \sum_{j=0}^{n} a_j, \quad R_n = -\sum_{j=n+1}^{\infty} a_j = s_n - s. \tag{26}$$

We say that $\hat{a}_n$ are Kummer-related to the a_n with limit or antilimit $\hat{s}$ if $\hat{a}_n = \triangle \hat{s}_{n-1}$ satisfy $a_n \sim \hat{a}_n$ for $n \to \infty$ and $\hat{s}$ is the limit (or antilimit) of $\hat{s}_n = \sum_{j=0}^{n} \hat{a}_j$.

Scalar power series in $z \in \mathbb{C}$ with coefficients $c_j \in \mathbb{K}$, partial sums $f_n(z)$, tail $R_n(z)$, and limit/antilimit $f(z)$:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j, \quad f_n(z) = \sum_{j=0}^{n} c_j z^j, \quad R_n(z) = \sum_{j=n+1}^{\infty} c_j z^j = f(z) - f_n(z). \tag{27}$$

2.2.2. Types of convergence

Sequences $\{\{s_n\}\}$ satisfying the equation

$$\lim_{n\to\infty} (s_{n+1} - s)/(s_n - s) = \rho \tag{28}$$

are called *linearly convergent* if $0 < |\rho| < 1$, *logarithmically convergent* for $\rho = 1$ and *hyperlinearly convergent* for $\rho = 0$. For $|\rho| > 1$, the sequence diverges.

A sequence $\{\{u_n\}\}$ accelerates a sequence $\{\{v_n\}\}$ to s if

$$\lim_{n\to\infty} (u_n - s)/(v_n - s) = 0. \tag{29}$$

If $\{\{v_n\}\}$ converges to s then we also say that $\{\{u_n\}\}$ converges faster than $\{\{v_n\}\}$.

A sequence $\{\{u_n\}\}$ accelerates a sequence $\{\{v_n\}\}$ to s with order $\alpha > 0$ if

$$(u_n - s)/(v_n - s) = O(n^{-\alpha}). \tag{30}$$

If $\{\{v_n\}\}$ converges to s then we also say that $\{\{u_n\}\}$ converges faster than $\{\{v_n\}\}$ with order α.

2.2.3. Operators

Annihilation operator: An operator $\mathscr{A}$: $\mathbb{S}^{\mathbb{K}} \to \mathbb{K}$ is called an annihilation operator for a given sequence $\{\{\tau_n\}\}$ if it satisfies

$$\begin{aligned}&\mathscr{A}(\{\{s_n + zt_n\}\}) = \mathscr{A}(\{\{s_n\}\}) + z\mathscr{A}(\{\{t_n\}\}) \quad \text{for all } \{\{s_n\}\} \in \mathbb{S}^{\mathbb{K}},\ \{\{t_n\}\} \in \mathbb{S}^{\mathbb{K}}, z \in \mathbb{K},\\ &\mathscr{A}(\{\{\tau_n\}\}) = 0.\end{aligned} \tag{31}$$

Forward difference operator.

$$\begin{aligned}&\triangle_m g(m) = g(m+1) - g(m),\ \ \triangle_m g_m = g_{m+1} - g_m,\\ &\triangle_m^k = \triangle_m \triangle_m^{k-1},\\ &\triangle = \triangle_n,\\ &\triangle^k g_n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g_{n+j}.\end{aligned} \tag{32}$$

Generalized difference operator $\nabla_n^{(k)}$ for given quantities $\delta_n^{(k)} \neq 0$:

$$\nabla_n^{(k)} = (\delta_n^{(k)})^{-1} \triangle . \tag{33}$$

Generalized difference operator $\tilde{\nabla}_n^{(k)}$ for given quantities $\zeta_n^{(k)} \neq 0$:

$$\tilde{\nabla}_n^{(k)} = (\zeta_n^{(k)})^{-1} \triangle^2 . \tag{34}$$

Generalized difference operator $\triangledown_n^{(k)}[\alpha]$ for given quantities $\Delta_n^{(k)} \neq 0$:

$$\triangledown_n^{(k)}\,[\alpha] f_n = (\Delta_n^{(k)})^{-1} (f_{n+2} - 2\cos\alpha f_{n+1} + f_n). \tag{35}$$

Generalized difference operator $\partial_n^{(k)}[\zeta]$ for given quantities $\tilde{\Delta}_n^{(k)} \neq 0$:

$$\partial_n^{(k)}[\zeta] f_n = (\tilde{\Delta}_n^{(k)})^{-1} (\zeta_{n+k}^{(2)} f_{n+2} + \zeta_{n+k}^{(1)} f_{n+1} + \zeta_{n+k}^{(0)} f_n). \tag{36}$$

Weighted difference operators for given $P^{(k-1)} \in \mathbb{P}^{k-1}$:

$$\mathscr{W}_n^{(k)} = \mathscr{W}_n^{(k)}[P^{(k-1)}] = \triangle^{(k)} P^{(k-1)}(n). \tag{37}$$

Polynomial operators $\mathscr{P}$ for given $P^{(k)} \in \mathbb{P}^{(k)}$: Let $P^{(k)}(x) = \sum_{j=0}^{k} p_j^{(k)} x^j$. Then put

$$\mathscr{P}[P^{(k)}] g_n = \sum_{j=0}^{k} p_j^{(k)} g_{n+j}. \tag{38}$$

Divided difference operator. For given $\{\{x_n\}\}$ and $k, n \in \mathbb{N}_0$, put

$$\square_n^{(k)}[\{\{x_n\}\}](f(x)) = \square_n^{(k)}(f(x)) = f[x_n, \ldots, x_{n+k}] = \sum_{j=0}^{k} f(x_{n+j}) \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{1}{x_{n+j} - x_{n+i}},$$

$$\Box_n^{(k)}[\{\{x_n\}\}]g_n = \Box_n^{(k)}g_n = \sum_{j=0}^{k} g_{n+j} \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{1}{x_{n+j}-x_{n+i}}. \tag{39}$$

3. Some basic sequence transformations

3.1. E Algorithm

Putting for sequences $\{\{y_n\}\}$ and $\{\{g_j(n)\}\}$, $j=1,\ldots,k$

$$E_n^{(k)}[\{\{y_n\}\};\{\{g_j(n)\}\}] = \begin{vmatrix} y_n & \cdots & y_{n+k} \\ g_1(n) & \cdots & g_1(n+k) \\ \vdots & \ddots & \vdots \\ g_k(n) & \cdots & g_k(n+k) \end{vmatrix}, \tag{40}$$

one may define the sequence transformation

$$\mathbf{E}_n^{(k)}(\{\{s_n\}\}) = \frac{E_n^{(k)}[\{\{s_n\}\};\{\{g_j(n)\}\}]}{E_n^{(k)}[\{\{1\}\};\{\{g_j(n)\}\}]}. \tag{41}$$

As is plain using Cramer's rule, we have $\mathbf{E}_n^{(k)}(\{\{\sigma_n\}\}) = \sigma$ if the σ_n satisfy Eq. (10). Thus, the sequence transformation yields the limit σ exactly for model sequences (10).

The sequence transformation $\mathbf{E}$ is known as the $\mathbf{E}$ algorithm or also as Brezinski–Håvie–Protocol [102, Section 10] after two of its main investigators, Håvie [32] and Brezinski [9]. A good introduction to this transformation is also given in the book of Brezinski and Redivo Zaglia [14, Section 2.1], cf. also Ref. [15].

Numerically, the computation of the $\mathbf{E}_n^{(k)}(\{\{s_n\}\})$ can be performed recursively using either the algorithm of Brezinski [14, p. 58f]

$$\begin{aligned}
&\mathbf{E}_n^{(0)}(\{\{s_n\}\}) = s_n, \quad g_{0,i}^{(n)} = g_i(n), \quad n\in\mathbb{N}_0,\ i\in\mathbb{N},\\
&\mathbf{E}_n^{(k)}(\{\{s_n\}\}) = \mathbf{E}_n^{(k-1)}(\{\{s_n\}\}) - \frac{\mathbf{E}_{(n+1)}^{(k-1)}(\{\{s_n\}\}) - \mathbf{E}_n^{(k-1)}(\{\{s_n\}\})}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}} g_{k-1,k}^{(n)},\\
&g_{k,i}^{(n)} = g_{k-1,i}^{(n)} - \frac{g_{k-1,i}^{(n+1)} - g_{k-1,i}^{(n)}}{g_{k-1,k}^{(n+1)} - g_{k-,1,k}^{(n)}} g_{k-,1,k}^{(n)}, \quad i = k+1, k+2, \ldots
\end{aligned} \tag{42}$$

or the algorithm of Ford and Sidi [22] that requires additionally the quantities $g_{k+1}(n+j)$, $j=0,\ldots,k$ for the computation of $\mathbf{E}_n^{(k)}(\{\{s_n\}\})$. The algorithm of Ford and Sidi involves the quantities

$$\Psi_{k,n}(u) = \frac{E_n^{(k)}[\{\{u_n\}\};\{\{g_j(n)\}\}]}{E_n^{(k)}[\{\{g_{k+1}(n)\}\};\{\{g_j(n)\}\}]} \tag{43}$$

for any sequence $\{\{u_0,u_1,\ldots\}\}$, where the $g_i(n)$ are not changed even if they depend on the u_n and the u_n are changed. Then we have

$$\mathbf{E}_n^{(k)}(\{\{s_n\}\}) = \frac{\Psi_k^{(n)}(s)}{\Psi_k^{(n)}(1)} \tag{44}$$

and the Ψ are calculated recursively via

$$\Psi_{k,n}(u) = \frac{\Psi_{k-1,n+1}(u) - \Psi_{k-1,n}(u)}{\Psi_{k-1,n+1}(g_{k+1}) - \Psi_{k-1,n}(g_{k+1})}. \tag{45}$$

Of course, for $g_j(n) = \omega_n \psi_{j-1}(n)$, i.e., in the context of sequences modelled via expansion (5), the **E** algorithm may be used to obtain an explicit representation for any Levin-type sequence transformation of the form (cf. Eq. (9))

$$\mathscr{T}_n^{(k)} = T(s_n, \ldots, s_{n+k}; \omega_n, \ldots, \omega_{n+k}; \psi_j(n), \ldots, \psi_j(n+k)) \tag{46}$$

as ratio of two determinants

$$\mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}) = \frac{E_n^{(k)}[\{\{s_n/\omega_n\}\}; \{\{\psi_{j-1}(n)\}\}]}{E_n^{(k)}[\{\{1/\omega_n\}\}; \{\{\psi_{j-1}(n)\}\}]}. \tag{47}$$

This follows from the identity [14]

$$\frac{E_n^{(k)}[\{\{s_n\}\}; \{\{\omega_n \psi_{j-1}(n)\}\}]}{E_n^{(k)}[\{\{1\}\}; \{\{\omega_n \psi_{j-1}(n)\}\}]} = \frac{E_n^{(k)}[\{\{s_n/\omega_n\}\}; \{\{\psi_{j-1}(n)\}\}]}{E_n^{(k)}[\{\{1/\omega_n\}\}; \{\{\psi_{j-1}(n)\}\}]}, \tag{48}$$

that is an easy consequence of usual algebraic manipulations of determinants.

3.2. The $d^{(m)}$ transformations

As noted in the introduction, the $d^{(m)}$ transformations were introduced by Levin and Sidi [54] as a generalization of the u variant of the Levin transformation [53]. We describe a slightly modified variant of the $d^{(m)}$ transformations [77]:

Let s_r, $r = 0, 1, \ldots$ be a real or complex sequence with limit or antilimit s and terms $a_0 = s_0$ and $a_r = s_r - s_{r-1}$, $r = 1, 2, \ldots$ such that $s_r = \sum_{r=0}^{r} a_j$, $r = 0, 1, \ldots$. For given $m \in \mathbb{N}$ and $\xi_l \in \mathbb{N}_0$ with $l \in \mathbb{N}_0$ and $0 \leqslant \xi_0 < \xi_1 < \xi_2 < \cdots$ and $v = (n_1, \ldots, n_m)$ with $n_j \in \mathbb{N}_0$ the $d^{(m)}$ transformation yields a table of approximations $s_v^{(m,j)}$ for the (anti-)limit s as solution of the linear system of equations

$$s_{\xi_l} = s_v^{(m,j)} + \sum_{k=1}^{m} (\xi_l + \alpha)^k [\Delta^{k-1} a_{\xi_l}] \sum_{i=0}^{n_k} \frac{\bar{\beta}_{ki}}{(\xi_l + \alpha)^i}, \quad j \leqslant l \leqslant j + N \tag{49}$$

with $\alpha > 0$, $N = \sum_{k=1}^{m} n_k$ and the $N+1$ unknowns $s_v^{(m,j)}$ and $\bar{\beta}_{k\,i}$. The $[\Delta^k a_j]$ are defined via $[\Delta^0 a_j] = a_j$ and $[\Delta^k a_j] = [\Delta^{k-1} a_{j+1}] - [\Delta^{k-1} a_j]$, $k = 1, 2, \ldots$. In most cases, all n_k are chosen equal and one puts $v = (n, n, \ldots, n)$. Apart from the value of α, only the input of m and of ξ_ℓ is required from the user. As transformed sequence, often one chooses the elements $s_{(n,\ldots,n)}^{(m,0)}$ for $n = 0, 1, \ldots$. The u variant of the Levin transformation is obtained for $m = 1$, $\alpha = \beta$ and $\xi_l = l$. The definition above differs slightly from the original one [54] and was given in Ref. [22] with $\alpha = 1$.

Ford and Sidi have shown, how these transformations can be calculated recursively with the $\mathbf{W}^{(m)}$ algorithms [22]. The $d^{(m)}$ transformations are the best known special cases of the *generalised Richardson Extrapolation process* (GREP) as defined by Sidi [72,73,78].

The $d^{(m)}$ transformations are derived by asymptotic analysis of the remainders $s_r - s$ for $r \to \infty$ for the family $\tilde{B}^{(m)}$ of sequences $\{\{a_r\}\}$ as defined in Ref. [54]. For such sequences, the a_r satisfy a difference equation of order m of the form

$$a_r = \sum_{k=1}^{m} p_k(r) \Delta^k a_r. \tag{50}$$

The $p_k(r)$ satisfy the asymptotic relation

$$p_k(r) \sim r^{i_k} \sum_{\ell=0}^{\infty} \frac{p_{k\ell}}{r^{\ell}} \quad \text{for } r \to \infty. \tag{51}$$

The i_k are integers satisfying $i_k \leqslant k$ for $k = 1, \ldots, m$. This family of sequences is very large. But still, Levin and Sidi could prove [54, Theorem 2] that under mild additional assumptions, the remainders for such sequences satisfy

$$s_r - s \sim \sum_{k=1}^{m} r^{j_k} (\Delta^{k-1} a_r) \sum_{\ell=0}^{\infty} \frac{\beta_{k\ell}}{r^{\ell}} \quad \text{for } r \to \infty. \tag{52}$$

The j_k are integers satisfying $j_k \leqslant k$ for $k = 1, \ldots, m$. A corresponding result for $m = 1$ was proven by Sidi [71, Theorem 6.1].

System (49) now is obtained by truncation of the expansions at $\ell = n_n$, evaluation at $r = \xi_l$, and some further obvious substitutions.

The introduction of suitable ξ_l was shown to improve the accuracy and stability in difficult situations considerably [77].

3.3. Shanks transformation and epsilon algorithm

An important special case of the **E** algorithm is the choice $g_j(n) = \triangle s_{n+j-1}$ leading to the Shanks transformation [70]

$$e_k(s_n) = \frac{E_n^{(k)}[\{\{s_n\}\}; \{\{\triangle s_{n+j-1}\}\}]}{E_n^{(k)}[\{\{1\}\}; \{\{\triangle s_{n+j-1}\}\}]}. \tag{53}$$

Instead of using one of the recursive schemes for the **E** algorithms, the Shanks transformation may be implemented using the epsilon algorithm [104] that is defined by the recursive scheme

$$\begin{aligned} &\varepsilon_{-1}^{(n)} = 0, \quad \varepsilon_0^{(n)} = s_n, \\ &\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + 1/[\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}]. \end{aligned} \tag{54}$$

The relations

$$\varepsilon_{2k}^{(n)} = e_k(s_n), \quad \varepsilon_{2k+1}^{(n)} = 1/e_k(\triangle s_n) \tag{55}$$

hold and show that the elements $\varepsilon_{2k+1}^{(n)}$ are only auxiliary quantities.

The kernel of the Shanks transformation e_k is given by sequences of the form

$$s_n = s + \sum_{j=0}^{k-1} c_j \triangle s_{n+j}. \tag{56}$$

See also [14, Theorem 2.18].

Additionally, one can use the Shanks transformation – and hence the epsilon algorithm – to compute the upper-half of the Padé table according to [70,104]

$$e_k(f_n(z)) = [n + k/k]_f(z) \quad (k \geqslant 0, n \geqslant 0), \tag{57}$$

where

$$f_n(z) = \sum_{j=0}^{n} c_j z^j \tag{58}$$

are the partial sums of a power series of a function $f(z)$. Padé approximants of $f(z)$ are rational functions in z given as ratio of two polynomials $p_\ell \in \mathbb{P}^{(\ell)}$ and $q_m \in \mathbb{P}^{(m)}$ according to

$$[\ell/m]_f(z) = p_\ell(z)/q_m(z), \tag{59}$$

where the Taylor series of f and $[\ell/m]_f$ are identical to the highest possible power of z, i.e.,

$$f(z) - p_\ell(z)/q_m(z) = \mathrm{O}(z^{\ell+m+1}). \tag{60}$$

Methods for the extrapolation of power series will be treated later.

3.4. Aitken process

The special case $\varepsilon_2^{(n)} = e_1(s_n)$ is identical to the famous Δ^2 method of Aitken [2]

$$s_n^{(1)} = s_n - \frac{(s_{n+1} - s_n)^2}{s_{n+2} - 2s_{n+1} + s_n} \tag{61}$$

with kernel

$$s_n = s + c\,(s_{n+1} - s_n), \quad n \in \mathbb{N}_0. \tag{62}$$

Iteration of the Δ^2 method yields the iterated Aitken process [14,84,102]

$$\begin{aligned} \mathbf{A}_n^{(0)} &= s_n, \\ \mathbf{A}_n^{(k+1)} &= \mathbf{A}_n^{(k)} - \frac{(\mathbf{A}_{n+1}^{(k)} - \mathbf{A}_n^{(k)})^2}{\mathbf{A}_{n+2}^{(k)} - 2\mathbf{A}_{n+1}^{(k)} + \mathbf{A}_n^{(k)}}. \end{aligned} \tag{63}$$

The iterated Aitken process and the epsilon algorithm accelerate linear convergence and can sometimes be applied successfully for the summation of alternating divergent series.

3.5. Overholt process

The Overholt process is defined by the recursive scheme [64]

$$\begin{aligned} V_n^{(0)}(\{\{s_n\}\}) &= s_n, \\ V_n^{(k)}(\{\{s_n\}\}) &= \frac{(\triangle s_{n+k-1})^k V_{n+1}^{(k-1)}(\{\{s_n\}\}) - (\triangle s_{n+k})^k V_n^{(k-1)}(\{\{s_n\}\})}{(\triangle s_{n+k-1})^k - (\triangle s_{n+k})^k} \end{aligned} \tag{64}$$

for $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. It is important for the convergence acceleration of fixed point iterations.

4. Levin-type sequence transformations

4.1. Definitions for Levin-type transformations

A set $\Lambda^{(k)} = \{\lambda_{n,j}^{(k)} \in \mathbb{K} \,|\, n \in \mathbb{N}_0, 0 \leqslant j \leqslant k\}$ is called a *coefficient set of order* k with $k \in \mathbb{N}$ if $\lambda_{n,k}^{(k)} \neq 0$ for all $n \in \mathbb{N}_0$. Also, $\Lambda = \{\Lambda^{(k)} \,|\, k \in \mathbb{N}\}$ is called *coefficient set*. Two coefficient sets

$\Lambda=\{\{\lambda_{n,j}^{(k)}\}\}$ and $\hat{\Lambda}=\{\{\hat{\lambda}_{n,j}^{(k)}\}\}$ are called *equivalent*, if for all n and k, there is a constant $c_n^{(k)}\neq 0$ such that $\hat{\lambda}_{n,j}^{(k)}=c_n^{(k)}\lambda_{n,j}^{(k)}$ for all j with $0\leqslant j\leqslant k$.

For each coefficient set $\Lambda^{(k)}=\{\lambda_{n,j}^{(k)}|n\in\mathbb{N}_0, 0\leqslant j\leqslant k\}$ of order k, one may define a *Levin-type sequence transformation of order* k by

$$\begin{aligned}\mathscr{T}[\Lambda^{(k)}]&:\mathbb{S}^{\mathbb{K}}\times\mathbb{Y}^{(k)}\to\mathbb{S}^{\mathbb{K}}\\ &:(\{\{s_n\}\},\{\{\omega_n\}\})\mapsto\{\{s_n'\}\}=\mathscr{T}[\Lambda^{(k)}](\{\{s_n\}\},\{\{\omega_n\}\})\end{aligned}\tag{65}$$

with

$$s_n'=\mathscr{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\})=\frac{\sum_{j=0}^{k}\lambda_{n,j}^{(k)}s_{n+j}/\omega_{n+j}}{\sum_{j=0}^{k}\lambda_{n,j}^{(k)}/\omega_{n+j}}\tag{66}$$

and

$$\mathbb{Y}^{(k)}=\left\{\{\{\omega_n\}\}\in\mathbb{O}^{\mathbb{K}}\colon\ \sum_{j=0}^{k}\lambda_{n,j}^{(k)}/\omega_{n+j}\neq 0 \text{ for all } n\in\mathbb{N}_0\right\}.\tag{67}$$

We call $\mathscr{T}[\Lambda]=\{\mathscr{T}[\Lambda^{(k)}]|\ k\in\mathbb{N}\}$ the Levin-type sequence transformation corresponding to the coefficient set $\Lambda=\{\Lambda^{(k)}|\ k\in\mathbb{N}\}$. We write $\mathscr{T}^{(k)}$ and $\mathscr{T}$ instead of $\mathscr{T}[\Lambda^{(k)}]$ and $\mathscr{T}[\Lambda]$, respectively, whenever the coefficients $\lambda_{n,j}^{(k)}$ are clear from the context. Also, if two coefficient sets Λ and $\hat{\Lambda}$ are equivalent, they give rise to the same sequence transformation, i.e., $\mathscr{T}[\Lambda]=\mathscr{T}[\hat{\Lambda}]$, since

$$\frac{\sum_{j=0}^{k}\hat{\lambda}_{n,j}^{(k)}s_{n+j}/\omega_{n+j}}{\sum_{j=0}^{k}\hat{\lambda}_{n,j}^{(k)}/\omega_{n+j}}=\frac{\sum_{j=0}^{k}\lambda_{n,j}^{(k)}s_{n+j}/\omega_{n+j}}{\sum_{j=0}^{k}\lambda_{n,j}^{(k)}/\omega_{n+j}}\quad\text{for } \hat{\lambda}_{n,j}^{(k)}=c_n^{(k)}\lambda_n^{(k)}\tag{68}$$

with arbitrary $c_n^{(k)}\neq 0$.

The number $\mathscr{T}_n^{(k)}$ are often arranged in a two-dimensional table

$$\begin{array}{cccc}\mathscr{T}_0^{(0)} & \mathscr{T}_0^{(1)} & \mathscr{T}_0^{(2)} & \cdots\\ \mathscr{T}_1^{(0)} & \mathscr{T}_1^{(1)} & \mathscr{T}_1^{(2)} & \cdots\\ \mathscr{T}_2^{(0)} & \mathscr{T}_2^{(1)} & \mathscr{T}_2^{(2)} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{array}\tag{69}$$

that is called the $\mathscr{T}$ table. The transformations $\mathscr{T}^{(k)}$ thus correspond to columns, i.e., to following vertical paths in the table. The numerators and denominators such that $\mathscr{T}_n^{(k)}=N_n^{(k)}/D_n^{(k)}$ also are often arranged in analogous N and D tables.

Note that for fixed N, one may also define a transformation

$$\mathscr{T}_N:\{\{s_{n+N}\}\}\mapsto\{\{\mathscr{T}_N^{(k)}\}\}_{k=0}^{\infty}.\tag{70}$$

This corresponds to horizontal paths in the $\mathscr{T}$ table. These are sometimes called diagonals, because rearranging the table in such a way that elements with constant values of $n+k$ are members of the same row, $\mathscr{T}_N^{(k)}$ for fixed N correspond to diagonals of the rearranged table.

For a given coefficient set Λ define the *moduli* by

$$\mu_n^{(k)}=\max_{0\leqslant j\leqslant k}\{|\lambda_{n,j}^{(k)}|\}\tag{71}$$

and the *characteristic polynomials* by

$$\Pi_n^{(k)} \in \mathbb{P}^{(k)}:\ \Pi_n^{(k)}(z) = \sum_{j=0}^{k} \lambda_{n,j}^{(k)} z^j \tag{72}$$

for $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$.

Then, $\mathscr{T}[\Lambda]$ is said to be *in normalized form* if $\mu_n^{(k)} = 1$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Is is said to be *in subnormalized form* if for all $k \in \mathbb{N}$ there is a constant $\tilde{\mu}^{(k)}$ such that $\mu_n^{(k)} \leqslant \tilde{\mu}^{(k)}$ for all $n \in \mathbb{N}_0$.

Any Levin-type sequence transformation $\mathscr{T}[\Lambda]$ can rewritten in normalized form. To see this, use

$$c_n^{(k)} = 1/\mu_n^{(k)} \tag{73}$$

in Eq. (68). Similarly, each Levin-type sequence transformation can be rewritten in (many different) subnormalized forms.

A Levin-type sequence transformation of order k is said to be *convex* if $\Pi_n^{(k)}(1) = 0$ for all n in $\mathbb{N}_0$. Equivalently, it is convex if $\{\{1\}\} \notin \mathbb{Y}^{(k)}$, i.e., if the transformation vanishes for $\{\{s_n\}\} = \{\{c\omega_n\}\}$, $c \in \mathbb{K}$. Also, $\mathscr{T}[\Lambda]$ is called convex, if $\mathscr{T}[\Lambda^{(k)}]$ is convex for all $k \in \mathbb{N}$. We will see that this property is important for ensuring convergence acceleration for linearly convergent sequences.

A given Levin-type transformation $\mathscr{T}$ can also be rewritten as

$$\mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}) = \sum_{j=0}^{k} \gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n)\, s_{n+j}, \quad \boldsymbol{\omega}_n = (\omega_n, \ldots, \omega_{n+k}) \tag{74}$$

with

$$\gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n) = \frac{\lambda_{n,j}^{(k)}}{\omega_{n+j}} \left[\sum_{j'=0}^{k} \frac{\lambda_{n,j'}^{(k)}}{\omega_{n+j'}} \right]^{-1}, \quad \sum_{j=0}^{k} \gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n) = 1. \tag{75}$$

Then, one may define *stability indices* by

$$\boldsymbol{\Gamma}_n^{(k)}(\mathscr{T}) = \sum_{j=0}^{k} |\gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n)| \geqslant 1. \tag{76}$$

Note that any sequence transformation $\mathscr{Q}$

$$\mathscr{Q}_n^{(k)} = \sum_{j=0}^{k} q_{n,j}^{(k)} s_{n+j} \tag{77}$$

with

$$\sum_{j=0}^{k} q_{n,j}^{(k)} = 1 \tag{78}$$

can formally be rewritten as a Levin-type sequence transformation according to $\mathscr{Q}_n^{(k)} = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\})$ with coefficients $\lambda_{n,j}^{(k)} = \omega_{n+j} q_{n,j}^{(k)} \rho_n^{(k)}$ where the validity of Eq. (78) requires to set

$$\rho_n^{(k)} = \sum_{j=0}^{k} \lambda_{n,j}^{(k)}/\omega_{n+j}. \tag{79}$$

If for given $k \in \mathbb{N}$ and for a transformation $\mathscr{T}[\Lambda^{(k)}]$ the following limits exist and have the values:

$$\lim_{n\to\infty} \lambda_{n,j}^{(k)} = \mathring{\lambda}_j^{(k)} \tag{80}$$

for all $0 \leqslant j \leqslant k$, and if $\mathring{\Lambda}^{(k)}$ is a coefficient set of order k which means that at least the limit $\mathring{\lambda}_k^{(k)}$ does not vanish, then a limiting transformation $\mathring{\mathscr{T}}[\mathring{\Lambda}^{(k)}]$ exists where $\mathring{\Lambda}^{(k)} = \{\mathring{\lambda}_j^{(k)}\}$. More explicitly, we have

$$\begin{aligned} \mathring{\mathscr{T}}[\Lambda^{(k)}] &: \mathbb{S}^{\mathbb{K}} \times \mathring{\mathbb{Y}}^{(k)} \to \mathbb{S}^{\mathbb{K}} \\ &: (\{\{s_n\}\}, \{\{\omega_n\}\}) \mapsto \{\{s_n'\}\} \end{aligned} \tag{81}$$

with

$$s_n' = \mathring{\mathscr{T}}^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}) = \frac{\sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} s_{n+j}/\omega_{n+j}}{\sum_{j=0}^{k} \mathring{\lambda}_j^{(k)}/\omega_{n+j}} \tag{82}$$

and

$$\mathring{\mathbb{Y}}^{(k)} = \left\{ \{\{\omega_n\}\} \in \mathbb{O}^{\mathbb{K}} : \sum_{j=0}^{k} \mathring{\lambda}_j^{(k)}/\omega_{n+j} \neq 0 \quad \text{for all } n \in \mathbb{N}_0 \right\}. \tag{83}$$

Obviously, this limiting transformation itself is a Levin-type sequence transformation and automatically is given in subnormalized form.

4.1.1. Variants of Levin-type transformations

For the following, assume that $\beta > 0$ is an arbitrary constant, $a_n = \triangle s_{n-1}$, and $\hat{a}_n$ are Kummer-related to the a_n with limit or antilimit $\hat{s}$ (cf. Section 2.2.1).

A *variant* of a Levin-type sequence transformation $\mathscr{T}$ is obtained by a particular choice ω_n. For $\omega_n = f_n(\{\{s_n\}\})$, the transformation $\mathscr{T}$ is nonlinear in the s_n. In particular, we have [50,53,79]:

t Variant:

$${}^t\omega_n = \triangle s_{n-1} = a_n : {}^t\mathscr{T}_n^{(k)}(\{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{{}^t\omega_n\}\}). \tag{84}$$

u Variant:

$${}^u\omega_n = (n+\beta) \triangle s_{n-1} = (n+\beta)a_n : {}^u\mathscr{T}_n^{(k)}(\beta, \{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{{}^u\omega_n\}\}). \tag{85}$$

v Variant:

$${}^v\omega_n = -\frac{\triangle s_{n-1} \triangle s_n}{\triangle^2 s_{n-1}} = \frac{a_n a_{n+1}}{a_n - a_{n+1}} : {}^v\mathscr{T}_n^{(k)}(\{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{{}^v\omega_n\}\}). \tag{86}$$

$\tilde{t}$ Variant:

$${}^{\tilde{t}}\omega_n = \triangle s_n = a_{n+1} : {}^{\tilde{t}}\mathscr{T}_n^{(k)}(\{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{{}^{\tilde{t}}\omega_n\}\}). \tag{87}$$

lt Variant:

$${}^{lt}\omega_n = \hat{a}_n : {}^{lt}\mathscr{T}_n^{(k)}(\{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{{}^{lt}\omega_n\}\}). \tag{88}$$

lu Variant:

$$^{lu}\omega_n = (n+\beta)\hat{a}_n\colon\ ^{lu}\mathscr{T}_n^{(k)}(\beta, \{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{^{lu}\omega_n\}\}). \tag{89}$$

lv Variant:

$$^{lv}\omega_n = \frac{\hat{a}_n\hat{a}_{n+1}}{\hat{a}_n - \hat{a}_{n+1}}\colon\ ^{lv}\mathscr{T}_n^{(k)}(\{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{^{lv}\omega_n\}\}). \tag{90}$$

$l\tilde{t}$ Variant:

$$^{l\tilde{t}}\omega_n = \hat{a}_{n+1}\colon\ ^{l\tilde{t}}\mathscr{T}_n^{(k)}(\{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{^{l\tilde{t}}\omega_n\}\}). \tag{91}$$

K Variant:

$$^{K}\omega_n = \hat{s}_n - \hat{s}\colon\ ^{K}\mathscr{T}_n^{(k)}(\{\{s_n\}\}) = \mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{^{K}\omega_n\}\}). \tag{92}$$

The K variant of a Levin-type transformation $\mathscr{T}$ is linear in the s_n. This holds also for the lt, lu, lv and $l\tilde{t}$ variants.

4.2. Important examples of Levin-type sequence transformations

In this section, we present important Levin-type sequence transformations. For each transformation, we give the definition, recursive algorithms and some background information.

4.2.1. $\mathscr{J}$ transformation

The $\mathscr{J}$ transformation was derived and studied by Homeier [35,36,38–40,46]. Although the $\mathscr{J}$ transformation was derived by hierarchically consistent iteration of the simple transformation

$$s'_n = s_{n+1} - \omega_{n+1}\frac{\triangle s_n}{\triangle \omega_n}, \tag{93}$$

it was possible to derive an explicit formula for its kernel as is discussed later. It may be defined via the recursive scheme

$$\begin{aligned}
&N_n^{(0)} = s_n/\omega_n, \quad D_n^{(0)} = 1/\omega_n,\\
&N_n^{(k)} = \nabla_n^{(k-1)} N_n^{(k-1)}, \quad D_n^{(k)} = \nabla_n^{(k-1)} D_n^{(k-1)},\\
&\mathscr{J}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\delta_n^{(k)}\}) = N_n^{(k)}/D_n^{(k)},
\end{aligned} \tag{94}$$

where the generalized difference operator defined in Eq. (33) involves quantities $\delta_n^{(k)} \neq 0$ for $k \in \mathbb{N}_0$. Special cases of the $\mathscr{J}$ transformation result from corresponding choices of the $\delta_n^{(k)}$. These are summarized in Table 1.

Using generalized difference operators $\nabla_n^{(k)}$, we also have the representation [36, Eq. (38)]

$$\mathscr{J}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\{\delta_n^{(k)}\}\}) = \frac{\nabla_n^{(k-1)}\nabla_n^{(k-2)}\dots\nabla_n^{(0)}[s_n/\omega_n]}{\nabla_n^{(k-1)}\nabla_n^{(k-2)}\dots\nabla_n^{(0)}[1/\omega_n]}. \tag{95}$$

The $\mathscr{J}$ transformation may also be computed using the alternative recursive schemes [36,46]

$$\begin{aligned}
&\hat{D}_n^{(0)} = 1/\omega_n, \qquad \hat{N}_n^{(0)} = s_n/\omega_n,\\
&\hat{D}_n^{(k)} = \Phi_n^{(k-1)}\hat{D}_{n+1}^{(k-1)} - \hat{D}_n^{(k-1)}, \quad k \in \mathbb{N},\\
&\hat{N}_n^{(k)} = \Phi_n^{(k-1)}\hat{N}_{n+1}^{(k-1)} - \hat{N}_n^{(k-1)}, \quad k \in \mathbb{N},
\end{aligned} \tag{96}$$

Table 1
Special cases of the $\mathscr{J}$ transformation[a]

Case	$\psi_j(n)$[b]	$\delta_n^{(k)}$[c]
Drummond transformation $\mathscr{D}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\})$	n^j	1
Homeier $\mathscr{I}$ transformation $\mathscr{I}_n^{(k)}(\alpha,\{\{s_n\}\},\{\{\omega_n\}\},\{\Delta_n^{(k)}\})$ $=\mathscr{J}_n^{(2k)}(\{\{s_n\}\},\{\{e^{-i\alpha n}\omega_n\}\},\{\delta_n^{(k)}\})$	Eq. (231)	$\delta_n^{(2\ell)}=\exp(2\mathrm{i}\alpha n)$, $\delta_n^{(2\ell+1)}=\exp(-2\mathrm{i}\alpha n)\Delta_n^{(\ell)}$
Homeier $\mathscr{F}$ transformation $\mathscr{F}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\{\{x_n\}\})$	$1/(x_n)_j$	$\frac{x_{n+k+1}-x_n}{x_n+k-1}\prod_{j=0}^{n-1}\frac{(x_j+k)(x_{j+k+1}+k-1)}{(x_j+k-1)(x_{j+k+2}+k)}$
Homeier ${}_p\mathbf{J}$ transformation ${}_p\mathbf{J}_n^{(k)}(\beta,\{\{s_n\}\},\{\{\omega_n\}\})$	Eq. (231)	$\frac{1}{(n+\beta+(p-1)k)_2}$
Levin transformation $\mathscr{L}_n^{(k)}(\beta,\{\{s_n\}\},\{\{\omega_n\}\})$	$(n+\beta)^{-j}$	$\frac{1}{(n+\beta)(n+\beta+k+1)}$
generalized $\mathscr{L}$ transformation $\mathscr{L}_n^{(k)}(\alpha,\beta,\{\{s_n\}\},\{\{\omega_n\}\})$	$(n+\beta)^{-j\alpha}$	$\frac{(n+\beta+k+1)^\alpha-(n+\beta)^\alpha}{(n+\beta)^\alpha(n+\beta+k+1)^\alpha}$
Levin-Sidi $d^{(1)}$ transformation [22,54,77] $(d^{(1)})_n^{(k)}(\alpha,\{\{s_n\}\})$	$(R_n+\alpha)^{-j}$	$\frac{1}{R_{n+k+1}+\alpha}-\frac{1}{R_n+\alpha}$
Mosig–Michalski algorithm [60,61] $M_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\{\{x_n\}\})$	Eq. (231)	$\frac{1}{x_n^2}\left(1-\frac{\omega_n x_{n+1}^{2k}}{\omega_{n+1}x_n^{2k}}\right)$
Sidi W algorithm (GREP$^{(1)}$) [73,77,78] $W_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\{\{t_n\}\})$	t_n^j	$t_{n+k+1}-t_n$
Weniger $\mathscr{C}$ transformation [87] $\mathscr{C}_n^{(k)}(\gamma,\beta/\gamma,\{\{s_n\}\},\{\{\omega_n\}\})$	$\frac{1}{(\gamma n+\beta)_j}$	$\frac{(n+1+(\beta+k-1)/\gamma)_k}{(n+(\beta+k)/\gamma)_{k+2}}$
Weniger $\mathscr{M}$ transformation $\mathscr{M}_n^{(k)}(\xi,\{\{s_n\}\},\{\{\omega_n\}\})$	$\frac{1}{(-n-\xi)_j}$	$\frac{(n+1+\xi-(k-1))_k}{(n+\xi-k)_{k+2}}$
Weniger $\mathscr{S}$ transformation $\mathscr{S}_n^{(k)}(\beta,\{\{s_n\}\},\{\{\omega_n\}\})$	$1/(n+\beta)_j$	$\frac{1}{(n+\beta+2k)_2}$
Iterated Aitken process [2,84] $\mathbf{A}_n^{(k)}(\{\{s_n\}\})$ $=\mathscr{J}_n^{(k)}(\{\{s_n\}\},\{\{\triangle s_n\}\},\{\delta_n^{(k)}\})$	Eq. (231)	$\frac{(\triangle\mathbf{A}_n^{(k+1)}(\{\{s_n\}\}))(\triangle^2\mathbf{A}_n^{(k)})(\{\{s_n\}\})}{(\triangle\mathbf{A}_n^{(k)}(\{\{s_n\}\}))(\triangle\mathbf{A}_{n+1}^{(k)}(\{\{s_n\}\}))}$
Overholt process [64] $V_n^{(k)}(\{\{s_n\}\})$ $=\mathscr{J}_n^{(k)}(\{\{s_n\}\},\{\{\triangle s_n\}\},\{\delta_n^{(k)}\})$	Eq. (231)	$\frac{(\Delta s_{n+k+1})\Delta[(\Delta s_{n+k})^{k+1}]}{(\Delta s_{n+k})^{k+1}}$

[a]Refs. [36,38,40].
[b]For the definition of the $\psi_{j,n}$ see Eq. (5).
[c]Factors independent of n are irrelevant.

$$\mathscr{J}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\{\delta_n^{(k)}\}) = \frac{\hat{N}_n^{(k)}}{\hat{D}_n^{(k)}}$$

with

$$\Phi_n^{(0)} = 1, \qquad \Phi_n^{(k)} = \frac{\delta_n^{(0)}\delta_n^{(1)}\cdots\delta_n^{(k-1)}}{\delta_{n+1}^{(0)}\delta_{n+1}^{(1)}\cdots\delta_{n+1}^{(k-1)}}, \quad k \in \mathbb{N} \tag{97}$$

and

$$\begin{aligned}
&\tilde{D}_n^{(0)} = 1/\omega_n, \quad \tilde{N}_n^{(0)} = s_n/\omega_n,\\
&\tilde{D}_n^{(k)} = \tilde{D}_{n+1}^{(k-1)} - \Psi_n^{(k-1)}\tilde{D}_n^{(k-1)}, \quad k \in \mathbb{N}\,,\\
&\tilde{N}_n^{(k)} = \tilde{N}_{n+1}^{(k-1)} - \Psi_n^{(k-1)}\tilde{N}_n^{(k-1)}, \quad k \in \mathbb{N},
\end{aligned} \tag{98}$$

$$\mathscr{J}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\{\delta_n^{(k)}\}) = \frac{\tilde{N}_n^{(k)}}{\tilde{D}_n^{(k)}}$$

with

$$\Psi_n^{(0)} = 1, \qquad \Psi_n^{(k)} = \frac{\delta_{n+k}^{(0)}\delta_{n+k-1}^{(1)}\cdots\delta_{n+1}^{(k-1)}}{\delta_{n+k-1}^{(0)}\delta_{n+k-2}^{(1)}\cdots\delta_n^{(k-1)}}, \quad k \in \mathbb{N}. \tag{99}$$

The quantities $\Psi_n^{(k)}$ should not be mixed up with the $\Psi_{k,n}(u)$ as defined in Eq. (43).

As shown in [46], the coefficients for the algorithm (96) that are defined via $\hat{D}_n^{(k)} = \sum_{j=0}^{k} \lambda_{n,j}^{(k)}/\omega_{n+j}$, satisfy the recursion

$$\lambda_{n,j}^{(k+1)} = \Phi_n^{(k)}\lambda_{n+1,j-1}^{(k)} - \lambda_{n,j}^{(k)} \tag{100}$$

with starting values $\lambda_{n,j}^{(0)} = 1$. This holds for all j if we define $\lambda_{n,j}^{(k)} = 0$ for $j < 0$ or $j > k$. Because $\Phi_n^{(k)} \neq 0$, we have $\lambda_{n,k}^{(k)} \neq 0$ such that $\{\lambda_{n,j}^{(k)}\}$ is a coefficient set for all $k \in \mathbb{N}_0$.

Similarly, the coefficients for algorithm (98) that are defined via $\tilde{D}_n^{(k)} = \sum_{j=0}^{k} \tilde{\lambda}_{n,j}^{(k)}/\omega_{n+j}$, satisfy the recursion

$$\tilde{\lambda}_{n,j}^{(k+1)} = \tilde{\lambda}_{n+1,j-1}^{(k)} - \Psi_n^{(k)}\tilde{\lambda}_{n,j}^{(k)} \tag{101}$$

with starting values $\tilde{\lambda}_{n,j}^{(0)} = 1$. This holds for all j if we define $\tilde{\lambda}_{n,j}^{(k)} = 0$ for $j < 0$ or $j > k$. In this case, we have $\tilde{\lambda}_{n,k}^{(k)} = 1$ such that $\{\tilde{\lambda}_{n,j}^{(k)}\}$ is a coefficient set for all $k \in \mathbb{N}_0$.

Since the $\mathscr{J}$ transformation vanishes for $\{\{s_n\}\} = \{\{c\omega_n\}\}$, $c \in \mathbb{K}$ according to Eq. (95) for all $k \in \mathbb{N}$, it is convex. This may also be shown by using induction in k using $\lambda_{n,1}^{(1)} = -\lambda_{n,0}^{(1)} = 1$ and the equation

$$\sum_{j=0}^{k+1} \lambda_{n,j}^{(k+1)} = \Phi_n^{(k)} \sum_{j=0}^{k} \lambda_{n+1,j}^{(k)} - \sum_{j=0}^{k} \lambda_{n,j}^{(k)} \tag{102}$$

that follows from Eq. (100).

Assuming that the limits $\Phi_k = \lim_{n\to\infty} \Phi_n^{(k)}$ exist for all $k \in \mathbb{N}$ and noting that for $k = 0$ always $\Phi_0 = 1$ holds, it follows that there exists a limiting transformation $\mathring{\mathscr{J}}\,[\mathring{\Lambda}]$ that can be considered as special variant of the $\mathscr{J}$ transformation and with coefficients given explicitly as [46, Eq. (16)]

$$\mathring{\lambda}_j^{(k)} = (-1)^{k-j} \sum_{\substack{j_0+j_1+\dots+j_{k-1}=j,\\ j_0\in\{0,1\},\dots,j_{k-1}\in\{0,1\}}} \prod_{m=0}^{k-1} (\Phi_m)^{j_m} . \tag{103}$$

As characteristic polynomial we obtain

$$\mathring{\Pi}^{(k)}(z) = \sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} z^j = \prod_{j=0}^{k-1} (\Phi_j z - 1). \tag{104}$$

Hence, the $\mathring{\mathscr{J}}$ transformation is convex since $\mathring{\Pi}^{(k)}(1) = 0$ due to $\Phi_0 = 1$.

The ${}_p\mathbf{J}$ *Transformation*: This is the special case of the $\mathscr{J}$ transformation corresponding to

$$\delta_n^{(k)} = \frac{1}{(n+\beta+(p-1)k)_2} \tag{105}$$

or to [46, Eq. (18)][2]

$$\Phi_n^{(k)} = \begin{cases} \left(\dfrac{n+\beta+2}{p-1}\right)_k \Big/ \left(\dfrac{n+\beta}{p-1}\right)_k & \text{for } p \neq 1, \\[2ex] \left(\dfrac{n+\beta+2}{n+\beta}\right)^k & \text{for } p = 1 \end{cases} \tag{106}$$

or to

$$\Psi_n^{(k)} = \begin{cases} \left(\dfrac{n+\beta+k-1}{p-2}\right)_k \Big/ \left(\dfrac{n+\beta+k+1}{p-2}\right)_k & \text{for } p \neq 2, \\[2ex] \left(\dfrac{n+\beta+k-1}{n+\beta+k+1}\right)^k & \text{for } p = 2, \end{cases} \tag{107}$$

that is,

$${}_p\mathbf{J}_n^{(k)}(\beta, \{\{s_n\}\}, \{\{\omega_n\}\}) = \mathscr{J}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{1/(n+\beta+(p-1)k)_2\}). \tag{108}$$

The limiting transformation ${}_p\mathring{\mathbf{J}}$ of the ${}_p\mathbf{J}$ transformation exists for all p and corresponds to the $\mathring{\mathscr{J}}$ transformation with $\Phi_k = 1$ for all k in $\mathbb{N}_0$. This is exactly the Drummond transformation discussed in Section 4.2.2, i.e., we have

$${}_p\mathring{\mathbf{J}}_n^{(k)}(\beta, \{\{s_n\}\}, \{\{\omega_n\}\}) = \mathscr{D}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}). \tag{109}$$

[2] The equation in [46] contains an error.

4.2.2. Drummond transformation

This transformation was given by Drummond [19]. It was also discussed by Weniger [84]. It may be defined as

$$\mathscr{D}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}) = \frac{\triangle^k[s_n/\omega_n]}{\triangle^k[1/\omega_n]}. \tag{110}$$

Using the definition (32) of the forward difference operator, the coefficients may be taken as

$$\lambda_{n,j}^{(k)} = (-1)^j \binom{k}{j}, \tag{111}$$

i.e., independent of n. As moduli, one has $\mu_n^{(k)} = \binom{k}{[\![k/2]\!]} = \tilde{\mu}^{(k)}$. Consequently, the Drummond transformation is given in subnormalized form. As characteristic polynomial we obtain

$$\Pi_n^{(k)}(z) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^j = (1-z)^k. \tag{112}$$

Hence, the Drummond transformation is convex since $\Pi_n^{(k)}(1) = 0$. Interestingly, the Drummond transformation is identical to its limiting transformation:

$$\mathring{\mathscr{D}}^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}) = \mathscr{D}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}). \tag{113}$$

The Drummond transformation may be computed using the recursive scheme

$$\begin{aligned} &N_n^{(0)} = s_n/\omega_n, \quad D_n^{(0)} = 1/\omega_n\,, \\ &N_n^{(k)} = \triangle N_n^{(k-1)}, \quad D_n^{(k)} = \triangle D_n^{(k-1)}, \\ &\mathscr{D}_n^{(k)} = N_n^{(k)}/D_n^{(k)}. \end{aligned} \tag{114}$$

4.2.3. Levin transformation

This transformation was given by Levin [53]. It was also discussed by Weniger [84]. It may be defined as [3]

$$\mathscr{L}_n^{(k)}(\beta,\{\{s_n\}\},\{\{\omega_n\}\}) = \frac{(n+\beta+k)^{1-k}\,\triangle^k\,[(n+\beta)^{k-1}s_n/\omega_n]}{(n+\beta+k)^{1-k}\,\triangle^k\,[(n+\beta)^{k-1}/\omega_n]}. \tag{115}$$

Using the definition (32) of the forward difference operator, the coefficients may be taken as

$$\lambda_{n,j}^{(k)} = (-1)^j \binom{k}{j} (n+\beta+j)^{k-1}/(n+\beta+k)^{k-1}. \tag{116}$$

The moduli satisfy $\mu_n^{(k)} \leqslant \binom{k}{[\![k/2]\!]} = \tilde{\mu}^{(k)}$ for given k. Consequently, the Levin transformation is given in subnormalized form. As characteristic polynomial we obtain

$$\Pi_n^{(k)}(z) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^j (n+\beta+j)^{k-1}/(n+\beta+k)^{k-1}. \tag{117}$$

[3] Note that the order of indices is different from that in the literature.

Since $\Pi_n^{(k)}(1)=0$ because $\triangle^k$ annihilates any polynomial in n with degree less than k, the Levin transformation is convex. The limiting transformation is identical to the Drummond transformation

$$\mathring{\mathscr{L}}^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}) = \mathscr{D}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}). \tag{118}$$

The Levin transformation may be computed using the recursive scheme [21,55,84,14, Section 2.7]

$$\begin{aligned}
&N_n^{(0)} = s_n/\omega_n, \quad D_n^{(0)} = 1/\omega_n,\\
&N_n^{(k)} = N_{n+1}^{(k-1)} - \frac{(\beta+n)(\beta+n+k-1)^{k-2}}{(\beta+n+k)^{k-1}} N_n^{(k-1)},\\
&D_n^{(k)} = D_{n+1}^{(k-1)} - \frac{(\beta+n)(\beta+n+k-1)^{k-2}}{(\beta+n+k)^{k-1}} D_n^{(k-1)},\\
&\mathscr{L}_n^{(k)}(\beta,\{\{s_n\}\},\{\{\omega_n\}\}) = N_n^{(k)}/D_n^{(k)}.
\end{aligned} \tag{119}$$

This is essentially the same as the recursive scheme (98) for the $\mathscr{J}$ transformation with

$$\Psi_n^{(k)} = \frac{(\beta+n)(\beta+n+k)^{k-1}}{(\beta+n+k+1)^k}, \tag{120}$$

since the Levin transformation is a special case of the $\mathscr{J}$ transformation (see Table 1). Thus, the Levin transformation can also be computed recursively using scheme (94)

$$\delta_n^{(k)} = \frac{1}{(n+\beta)(n+\beta+k+1)} \tag{121}$$

or scheme (96) with [46]

$$\Phi_n^{(k)} = (n+\beta+k+1)\frac{(n+\beta+1)^{k-1}}{(n+\beta)^k}. \tag{122}$$

4.2.4. Weniger transformations

Weniger [84,87,88] derived sequence transformations related to factorial series. These may be regarded as special cases of the transformation

$$\mathscr{C}_n^{(k)}(\alpha,\zeta,\{\{s_n\}\},\{\{\omega_n\}\}) = \frac{((\alpha[n+\zeta+k])_{k-1})^{-1}\triangle^k\,[(\alpha[n+\zeta])_{k-1}s_n/\omega_n]}{((\alpha[n+\zeta+k])_{k-1})^{-1}\triangle^k\,[(\alpha[n+\zeta])_{k-1}/\omega_n]}. \tag{123}$$

In particular, the Weniger $\mathscr{S}$ transformation may be defined as

$$\mathscr{S}_n^{(k)}(\beta,\{\{s_n\}\},\{\{\omega_n\}\}) = \mathscr{C}_n^{(k)}(1,\beta,\{\{s_n\}\},\{\{\omega_n\}\}) \tag{124}$$

and the Weniger $\mathscr{M}$ transformation as

$$\mathscr{M}_n^{(k)}(\xi,\{\{s_n\}\},\{\{\omega_n\}\}) = \mathscr{C}_n^{(k)}(-1,\xi,\{\{s_n\}\},\{\{\omega_n\}\}). \tag{125}$$

The parameters β, ξ, and ζ are taken to be positive real numbers. Weniger considered the $\mathscr{C}$ transformation only for $\alpha>0$ [87,88] and thus, he was not considering the $\mathscr{M}$ transformation as a special case of the $\mathscr{C}$ transformation. He also found that one should choose $\xi \geqslant k-1$. In the u variant of the $\mathscr{M}$ transformation he proposed to choose $\omega_n = (-n-\xi)\triangle s_{n-1}$. This variant is denoted as ${}^u\mathscr{M}$ transformation in the present work.

Using the definition (32) of the forward difference operator, the coefficients may be taken as

$$\lambda_{n,j}^{(k)} = (-1)^j \binom{k}{j} (\alpha[n+\zeta+j])_{k-1}/(\alpha[n+\zeta+k])_{k-1} \tag{126}$$

in the case of the $\mathscr{C}$ transformation, as

$$\lambda_{n,j}^{(k)} = (-1)^j \binom{k}{j} (n+\beta+j)_{k-1}/(n+\beta+k)_{k-1} \tag{127}$$

in the case of the $\mathscr{S}$ transformation, and as

$$\lambda_{n,j}^{(k)} = (-1)^j \binom{k}{j} (-n-\xi-j)_{k-1}/(-n-\xi-k)_{k-1} \tag{128}$$

in the case of the $\mathscr{M}$ transformation.

The $\mathscr{S}$ transformation in (124) may be computed using the recursive scheme (98) with [84, Section 8.3]

$$\Psi_n^{(k)} = \frac{(\beta+n+k)(\beta+n+k-1)}{(\beta+n+2k)(\beta+n+2k-1)}. \tag{129}$$

The $\mathscr{M}$ transformation in (125) may be computed using the recursive scheme (98) with [84, Section 9.3]

$$\Psi_n^{(k)} = \frac{\xi+n-k+1}{\xi+n+k+1}. \tag{130}$$

The $\mathscr{C}$ transformation in (123) may be computed using the recursive scheme (98) with [87, Eq. (3.3)]

$$\Psi_n^{(k)} = (\alpha[\zeta+n]+k-2)\frac{(\alpha[n+\zeta+k-1])_{k-2}}{(\alpha[n+\zeta+k])_{k-1}}. \tag{131}$$

Since the operator $\triangle^k$ for $k \in \mathbb{N}$ annihilates all polynomials in n of degree smaller than k, the transformations $\mathscr{S}, \mathscr{M}$, and $\mathscr{C}$ are convex. The moduli satisfy $\mu_n^{(k)} \leqslant \binom{k}{[\![k/2]\!]} = \tilde{\mu}^{(k)}$ for given k. Consequently, the three Weniger transformations are given in subnormalized form.

For $\alpha \to \infty$, the Levin transformation is obtained from the $\mathscr{C}$ transformation [87]. The $\mathscr{S}$ transformation is identical to the ${}_3\mathbf{J}$ transformation. It is also the special case $x_n = n+\beta$ of the $\mathscr{F}$ transformation. Analogously, the $\mathscr{C}$ transformation is obtained for $x_n = \alpha[\zeta+n]$. All these Weniger transformations are special cases of the $\mathscr{J}$ transformation (cf Table 1).

The limiting transformation of all these Weniger transformations is the Drummond transformation.

4.2.5. Levin–Sidi transformations and W algorithms

As noted above in Section 3.2, the $d^{(m)}$ transformations were introduced by Levin and Sidi [54] as a generalization of the u variant of the Levin transformation, and these transformations may implemented recursively using the $\mathbf{W}^{(m)}$ algorithms.

The case $m=1$ corresponding to the $d^{(1)}$ transformation and the $\mathbf{W}^{(1)} = W$ algorithm is relevant for the present survey of Levin-type transformations. In the following, the kth-order transformation $\mathscr{T}^{(k)}$ of Levin-type transformation $\mathscr{T}$ as given by the W algorithm is denoted by $W^{(k)}$ which should not be confused with the $\mathbf{W}^{(m)}$ algorithms of Ford and Sidi [22].

The W algorithm [73] was also studied by other authors [84, Section 7.4], [14, p. 71f, 116f] and may be regarded as a special case of the $\mathscr{J}$ transformation [36]. It may be defined as (cf [78, Theorems 1.1 and 1.2])

$$N_n^{(0)} = \frac{s_n}{\omega_n}, \quad D_n^{(0)} = \frac{1}{\omega_n},$$
$$N_n^{(k)} = \frac{N_{n+1}^{(k-1)} - N_n^{(k-1)}}{t_{n+k} - t_n},$$
$$D_n^{(k)} = \frac{D_{n+1}^{(k-1)} - D_n^{(k-1)}}{t_{n+k} - t_n},$$
$$W_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\{t_n\}\}) = N_n^{(k)}/D_n^{(k)} \tag{132}$$

and computes

$$W_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\{t_n\}\}) = \frac{\square_n^{(k)}(s_n/\omega_n)}{\square_n^{(k)}(1/\omega_n)}, \tag{133}$$

where the divided difference operators $\square_n^{(k)} = \square_n^{(k)}[\{\{t_n\}\}]$ are used. The W algorithm may be used to calculate the Levin transformation on putting $t_n = 1/(n+\beta)$. Some authors call a linear variant of the W algorithm with $\omega_n = (-1)^{n+1} \mathrm{e}^{-n\zeta q} t_n^{\alpha}$ the W transformation, while the $\tilde{t}$ variant of the W algorithm [74,75] is sometimes called mW transformation [31,57,60].

If $t_{n+1}/t_n \to \tau$ for large n, one obtains as limiting transformation the $\mathring{\mathscr{J}}$ transformation with $\Phi_j = \tau^{-j}$ and characteristic polynomial

$$\mathring{\Pi}^{(k)}(z) = \prod_{j=0}^{k-1} (z/\tau^j - 1). \tag{134}$$

For the $d^{(1)}$ transformation, we write

$$(d^{(1)})_n^{(k)}(\alpha, \{\{s_n\}\}, \{\{\xi_n\}\}) = W_n^{(k)}(\{\{s_{\xi_n}\}\}, \{\{(\xi_n+\alpha)(s_{\xi_n} - s_{\xi_n - 1})\}\}, \{\{1/(\xi_n+\alpha)\}\}). \tag{135}$$

Thus, it corresponds to the variant of the W algorithm with remainder estimates chosen as $(\xi_n + \alpha)(s_{\xi_n} - s_{\xi_n - 1})$ operating on the subsequence $\{\{s_{\xi_n}\}\}$ of $\{\{s_n\}\}$ with $t_n = 1/(\xi_n + \alpha)$. It should be noted that this is not(!) identical to the u variant

$$^{u}W_n^{(k)}(\{\{s_{\xi_n}\}\}, \{\{1/(\xi_n+\alpha)\}\}) = W_n^{(k)}(\{\{s_{\xi_n}\}\}, \{\{^{u}\omega_n\}\}, \{\{1/(\xi_n+\alpha)\}\}), \tag{136}$$

neither for $^{u}\omega_n = (n+\alpha)(s_{\xi_n} - s_{\xi_{n-1}})$ nor for $^{u}\omega_n = (\xi_n+\alpha)(s_{\xi_n} - s_{\xi_{n-1}})$, since the remainder estimates are chosen differently in Eq. (135).

The $d^{(1)}$ transformation was thoroughly analyzed by Sidi (see [77,78] and references therein).

4.2.6. Mosig–Michalski transformation

The Mosig–Michalski transformation — also known as "weighted–averages algorithm" — was introduced by Mosig [61] and modified later by Michalski who gave the $\tilde{t}$ variant of the transformation the name $\mathscr{K}$ transformation (that is used for a different transformation in the present article(!)), and applied it to the computation of Sommerfeld integrals [60].

The Mosig–Michalski transformation M may be defined via the recursive scheme

$$s_n^{(0)} = s_n,$$
$$s_n^{(k+1)} = \frac{s_n^{(k)} + \eta_n^{(k)} s_{n+1}^{(k)}}{1+\eta_n^{(k)}},$$
$$M_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\{\{x_n\}\}) = s_n^{(k)} \tag{137}$$

for $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ where $\{\{x_n\}\}$ is an auxiliary sequence with $\lim_{n\to\infty} 1/x_n = 0$ such that $x_{n+\ell} > x_n$ for $\ell \in \mathbb{N}_0$ and $x_0 > 1$, i.e., a diverging sequence of monotonously increasing positive numbers, and

$$\eta_n^{(k)} = -\frac{\omega_n}{\omega_{n+1}}\left(\frac{x_{n+1}}{x_n}\right)^{2k}. \tag{138}$$

Putting $\omega_n^{(k)} = \omega_n/x_n^{2k}$, $N_n^{(k)} = s_n^{(k)}/\omega_n^{(k)}$, and $D_n^{(k)} = 1/\omega_n^{(k)}$, it is easily seen that the recursive scheme (137) is equivalent to the scheme (94) with

$$\delta_n^{(k)} = \frac{1}{x_n^2}\left(1 - \frac{\omega_n x_{n+1}^{2k}}{\omega_{n+1} x_n^{2k}}\right). \tag{139}$$

Thus, the Mosig–Michalski transformation is a special case of the $\mathscr{J}$ transformation. Its character as a Levin-type transformation is somewhat formal since the $\delta_n^{(k)}$ and, hence, the coefficients $\lambda_{n,j}^{(k)}$ depend on the ω_n.

If $x_{n+1}/x_n \sim \xi > 1$ for large n, then a limiting transformation exists, namely $M(\{\{s_n\}\},\{\{\omega_n\}\},\{\{\xi^{n+1}\}\})$. It corresponds to the $\mathring{\mathscr{J}}$ transformation with $\Phi_k = \xi^{2k}$. This may be seen by putting $\widehat{D}_n^{(k)} = 1/\omega_n$, $\widehat{N}_n^{(k)} = s_n^{(k)} D_n^{(k)}$ and $\Phi_n^{(k)} = \xi^{2k}$ in Eq. (96).

4.2.7. $\mathscr{F}$ transformation

This transformation is seemingly new. It will be derived in a later section. It may be defined as

$$\mathscr{F}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\{\{x_n\}\}) = \frac{\Box_n^{(k)}((x_n)_{k-1}s_n/\omega_n)}{\Box_n^{(k)}((x_n)_{k-1}/\omega_n)} = \frac{x_n^k/(x_n)_{k-1}\Box_n^{(k)}((x_n)_{k-1}s_n/\omega_n)}{x_n^k/(x_n)_{k-1}\Box_n^{(k)}((x_n)_{k-1}/\omega_n)}, \tag{140}$$

where $\{\{x_n\}\}$ is an auxiliary sequence with $\lim_{n\to\infty} 1/x_n = 0$ such that $x_{n+\ell} > x_n$ for $\ell \in \mathbb{N}$ and $x_0 > 1$, i.e., a diverging sequence of monotonously increasing positive numbers. Using the definition (39) of the divided difference operator $\Box_n^{(k)} = \Box_n^{(k)}[\{\{x_n\}\}]$, the coefficients may be taken as

$$\lambda_{n,j}^{(k)} = \frac{(x_{n+j})_{k-1}}{(x_n)_{k-1}} \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{x_n}{x_{n+j}-x_{n+i}} = \prod_{m=0}^{k-2} \frac{x_{n+j}+m}{x_n+m}\left(\frac{x_n}{x_{n+j}}\right)^k \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{1}{1-x_{n+i}/x_{n+j}}. \tag{141}$$

Assuming that the following limit exists such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \xi > 1 \tag{142}$$

holds, we see that one can define a limiting transformation $\mathring{\mathscr{F}}^{(k)}$ with coefficients

$$\mathring{\lambda}_j^{(k)} = \lim_{n\to\infty} \lambda_{n,j}^{(k)} = \frac{1}{\xi^j} \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} \frac{1}{1-\xi^{\ell-j}} = (-1)^k \xi^{-k(k+1)/2} \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} \frac{1}{\xi^{-j}-\xi^{-\ell}}, \tag{143}$$

since

$$\prod_{m=0}^{k-2} \frac{x_{n+j}+m}{x_n+m} \left(\frac{x_n}{x_{n+j}}\right)^k \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} \frac{1}{1-x_{n+\ell}/x_{n+j}} \to \xi^{(k-1)j}\xi^{k(-j)} \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} \frac{\xi^{-l}}{\xi^{-\ell}-\xi^{-j}} \tag{144}$$

for $n\to\infty$. Thus, the limiting transformation is given by

$$\mathring{\mathscr{F}}^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\xi) = \frac{\sum_{j=0}^{k} s_{n+j}/\omega_{n+j} \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} 1/(\xi^{-j}-\xi^{-\ell})}{\sum_{j=0}^{k} \frac{1}{\omega_{n+j}} \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} 1/(\xi^{-j}-\xi^{-\ell})}. \tag{145}$$

Comparison with definition (39) of the divided difference operators reveals that the limiting transformation can be rewritten as

$$\mathring{\mathscr{F}}^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\xi) = \frac{\square_n^{(k)}[\{\{\xi^{-n}\}\}](s_n/\omega_n)}{\square_n^{(k)}[\{\{\xi^{-n}\}\}](1/\omega_n)}. \tag{146}$$

Comparison to Eq. (133) shows that the limiting transformation is nothing but the W algorithm for $t_n=\xi^{-n}$. As characteristic polynomial we obtain

$$\mathring{\Pi}^{(k)}(z) = \sum_{j=0}^{k} z^j \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} \frac{1}{\xi^{-j}-\xi^{-\ell}} = \xi^{k(k+1)/2} \prod_{j=0}^{k-1} \frac{1-z\xi^j}{\xi^{j+1}-1}. \tag{147}$$

The last equality is easily proved by induction. Hence, the $\mathring{\mathscr{F}}$ transformation is convex since $\mathring{\Pi}^{(k)}(1)=0$.

As shown in Appendix B, the $\mathscr{F}$ transformation may be computed using the recursive scheme

$$\begin{aligned}
N_n^{(0)} &= \frac{1}{x_n-1}\frac{s_n}{\omega_n}, \quad D_n^{(0)} = \frac{1}{x_n-1}\frac{1}{\omega_n},\\
N_n^{(k)} &= \frac{(x_{n+k}+k-2)N_{n+1}^{(k-1)}-(x_n+k-2)N_n^{(k-1)}}{x_{n+k}-x_n},\\
D_n^{(k)} &= \frac{(x_{n+k}+k-2)D_{n+1}^{(k-1)}-(x_n+k-2)D_n^{(k-1)}}{x_{n+k}-x_n},\\
\mathscr{F}_n^{(k)} &= N_n^{(k)}/D_n^{(k)}.
\end{aligned} \tag{148}$$

It follows directly from Eq. (146) and the recursion relation for divided differences that the limiting transformation can be computed via the recursive scheme

$$\begin{aligned}
\mathring{N}_n^{(0)} &= \frac{s_n}{\omega_n}, \quad \mathring{D}_n^{(0)} = \frac{1}{\omega_n},\\
\mathring{N}_n^{(k)} &= \frac{\mathring{N}_{n+1}^{(k-1)} - \mathring{N}_n^{(k-1)}}{\xi^{-(n+k)}-\xi^{-n}},
\end{aligned}$$

$$\mathring{D}_n^{(k)} = \frac{\mathring{D}_{n+1}^{(k-1)} - \mathring{D}_n^{(k-1)}}{\xi^{-(n+k)} - \xi^{-n}},$$
$$\mathring{\mathscr{F}}_n^{(k)} = \mathring{N}_n^{(k)} / \mathring{D}_n^{(k)}. \tag{149}$$

4.2.8. $\mathscr{J}\mathscr{D}$ transformation

This transformation is newly introduced in this article. In Section 5.2.1, it is derived via (asymptotically) hierarchically consistent iteration of the $\mathscr{D}^{(2)}$ transformation, i.e., of

$$s_n' = \frac{\triangle^2(s_n/\omega_n)}{\triangle^2(1/\omega_n)}. \tag{150}$$

The $\mathscr{J}\mathscr{D}$ transformation may be defined via the recursive scheme

$$N_n^{(0)} = s_n/\omega_n, \quad D_n^{(0)} = 1/\omega_n,$$
$$N_n^{(k)} = \tilde{\nabla}_n^{(k-1)} N_n^{(k-1)}, \quad D_n^{(k)} = \tilde{\nabla}_n^{(k-1)} D_n^{(k-1)},$$
$$\mathscr{J}\mathscr{D}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\zeta_n^{(k)}\}) = N_n^{(k)}/D_n^{(k)}, \tag{151}$$

where the generalized difference operator defined in Eq. (34) involves quantities $\zeta_n^{(k)} \neq 0$ for $k \in \mathbb{N}_0$. Special cases of the $\mathscr{J}\mathscr{D}$ transformation result from corresponding choices of the $\zeta_n^{(k)}$. From Eq. (151) one easily obtains the alternative representation

$$\mathscr{J}\mathscr{D}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\zeta_n^{(k)}\}) = \frac{\tilde{\nabla}_n^{(k-1)} \tilde{\nabla}_n^{(k-2)} \cdots \tilde{\nabla}_n^{(0)} [s_n/\omega_n]}{\tilde{\nabla}_n^{(k-1)} \tilde{\nabla}_n^{(k-2)} \cdots \tilde{\nabla}_n^{(0)} [1/\omega_n]}. \tag{152}$$

Thus, the $\mathscr{J}\mathscr{D}^{(k)}$ is a Levin-type sequence transformation of order $2k$.

4.2.9. $\mathscr{H}$ transformation and generalized $\mathscr{H}$ transformation

The $\mathscr{H}$ transformation was introduced by Homeier [34] and used or studied in a series of articles [35,41–44,63]. Target of the $\mathscr{H}$ transformation are Fourier series

$$s = A_0/2 + \sum_{j=1}^{\infty} (A_j \cos(j\alpha) + B_j \sin(j\alpha)) \tag{153}$$

with partial sums $s_n = A_0/2 + \sum_{j=1}^{n} (A_j \cos(j\alpha) + B_j \sin(j\alpha))$ where the Fourier coefficients A_n and B_n have asymptotic expansions of the form

$$C_n \sim \rho^n n^{\varepsilon} \sum_{j=0}^{\infty} c_j n^{-j} \tag{154}$$

for $n \to \infty$ with $\rho \in \mathbb{K}$, $\varepsilon \in \mathbb{K}$ and $c_0 \neq 0$.

The $\mathscr{H}$ transformation was critized by Sidi [77] as very unstable and useless near singularities of the Fourier series. However, Sidi failed to notice that – as in the case of the $d^{(1)}$ transformation with $\xi_n = \tau n$ – one can apply also the $\mathscr{H}$ transformation (and also most other Levin-type sequence transformations) to the subsequence $\{\{s_{\xi_n}\}\}$ of $\{\{s_n\}\}$. The new sequence elements $s_{\xi_n} = s_{\tau n}$ can be regarded as the partial sums of a Fourier series with τ–fold frequency. Using this τ-fold frequency approach, one can obtain stable and accurate convergence acceleration even in the vicinity of singularities [41–44].

The $\mathscr{H}$ transformation may be defined as

$$N_n^{(0)} = (n+\beta)^{-1} s_n/\omega_n, \quad D_n^{(0)} = (n+\beta)^{-1}/\omega_n,$$

$$N_n^{(k)} = (n+\beta)N_n^{(k-1)} + (n+2k+\beta)N_{n+2}^{(k-1)} - 2\cos(\alpha)(n+k+\beta)N_{n+1}^{(k-1)},$$

$$D_n^{(k)} = (n+\beta)D_n^{(k-1)} + (n+2k+\beta)D_{n+2}^{(k-1)} - 2\cos(\alpha)(n+k+\beta)D_{n+1}^{(k-1)},$$

$$\mathscr{H}_n^{(k)}(\alpha,\beta,\{\{s_n\}\},\{\{\omega_n\}\}) = N_n^{(k)}/D_n^{(k)}, \tag{155}$$

where $\cos\alpha \neq \pm 1$ and $\beta \in \mathbb{R}_+$.

It can also be represented in the explicit form [34]

$$\mathscr{H}_n^{(k)}(\alpha,\beta,\{\{s_n\}\},\{\{\omega_n\}\}) = \frac{\mathscr{P}[P^{(2k)}(\alpha)][(n+\beta)^{k-1}s_n/\omega_n]}{\mathscr{P}[P^{(2k)}(\alpha)][(n+\beta)^{k-1}/\omega_n]}, \tag{156}$$

where the $p_m^{(2k)}(\alpha)$ and the polynomial $P^{(2k)}(\alpha) \in \mathscr{P}^{(2k)}$ are defined via

$$P^{(2k)}(\alpha)(x) = (x^2 - 2x\cos\alpha + 1)^k = \sum_{m=0}^{2k} p_m^{(2k)}(\alpha) x^m \tag{157}$$

and $\mathscr{P}$ is the polynomial operator defined in Eq. (38). This shows that the $\mathscr{H}^{(k)}$ transformation is a Levin-type transformation of order $2k$. It is not convex.

A subnormalized form is

$$\mathscr{H}_n^{(k)}(\alpha,\beta,\{\{s_n\}\},\{\{\omega_n\}\}) = \frac{\sum_{m=0}^{2k} p_m^{(2k)}(\alpha)\frac{(n+\beta+m)^{k-1}}{(n+\beta+2k)^{k-1}}\frac{s_{n+m}}{\omega_{n+m}}}{\sum_{m=0}^{2k} p_m^{(2k)}(\alpha)\frac{(n+\beta+m)^{k-1}}{(n+\beta+2k)^{k-1}}\frac{1}{\omega_{n+m}}}. \tag{158}$$

This relation shows that the limiting transformation

$$\mathring{\mathscr{H}}^{(k)} = \frac{\mathscr{P}[P^{(2k)}(\alpha)][s_n/\omega_n]}{\mathscr{P}[P^{(2k)}(\alpha)][1/\omega_n]} \tag{159}$$

exists, and has characteristic polynomial $P^{(2k)}(\alpha)$.

A generalized $\mathscr{H}$ transformation was defined by Homeier [40,43]. It is given in terms of the polynomial $P^{(k,M)}(\mathbf{e}) \in \mathbb{P}^{(kM)}$ with

$$P^{(k,M)}(\mathbf{e})(x) = \prod_{m=1}^{M}(x-e_m)^k = \sum_{\ell=0}^{kM} p_\ell^{(k,M)}(\mathbf{e})x^\ell, \tag{160}$$

where $\mathbf{e} = (e_1,\ldots,e_M) \in \mathbb{K}^M$ is a vector of constant parameters. Then, the generalized $\mathscr{H}$ transformation is defined as

$$\mathscr{H}_n^{(k,M)}(\beta,\{\{s_n\}\},\{\{\omega_n\}\},\mathbf{e}) = \frac{\mathscr{P}[P^{(k,M)}(\mathbf{e})][(n+\beta)^{k-1}s_n/\omega_n]}{\mathscr{P}[P^{(k,M)}(\mathbf{e})][(n+\beta)^{k-1}/\omega_n]}. \tag{161}$$

This shows that the generalized $\mathscr{H}^{(k,M)}$ is a Levin-type sequence transformation of order kM. The generalized $\mathscr{H}$ transformation can be computed recursively using the scheme [40,43]

$$
\begin{aligned}
&N_n^{(0)} = (n+\beta)^{-1}\, s_n/\omega_n, \quad D_n^{(0)} = (n+\beta)^{-1}/\omega_n,\\
&N_n^{(k)} = \sum_{j=0}^{M} q_j\,(n+\beta+j\,k) N_{n+j}^{(k-1)},\\
&D_n^{(k)} = \sum_{j=0}^{M} q_j (n+\beta+jk) D_{n+j}^{(k-1)},\\
&\mathscr{H}_n(k,M)(\beta,\{\{s_n\}\},\{\{\omega_n\}\},\mathbf{e}) = \frac{N_n^{(k)}}{D_n^{(k)}}.
\end{aligned}
\tag{162}
$$

Here, the q_j are defined by

$$
\prod_{m=1}^{M}(x-e_m) = \sum_{j=0}^{M} q_j x^j. \tag{163}
$$

Algorithm (155) is a special case of algorithm (162). To see this, one observes that $M=2$, $e_1 = \exp(\mathrm{i}\alpha)$ und $e_2 = \exp(-\mathrm{i}\alpha)$ imply $q_0 = q_2 = 1$ and $q_1 = -2\cos(\alpha)$.

For $M=1$ and $e_1 = 1$, the Levin transformation is recovered.

4.2.10. $\mathscr{I}$ transformation

The $\mathscr{I}$ transformation was in a slightly different form introduced by Homeier [35]. It was derived via (asymptotically) hierarchically consistent iteration of the $\mathscr{H}^{(1)}$ transformation, i.e., of

$$
s'_n = \frac{s_{n+2}/\omega_{n+2} - 2\cos(\alpha) s_{n+1}/\omega_{n+1} + s_n/\omega_n}{1/\omega_{n+2} - 2\cos(\alpha)/\omega_{n+1} + 1/\omega_n}. \tag{164}
$$

For the derivation and an analysis of the properties of the $\mathscr{I}$ transformation see [40,44]. The $\mathscr{I}$ transformation may be defined via the recursive scheme

$$
\begin{aligned}
&N_n^{(0)} = s_n/\omega_n, \quad D_n^{(0)} = 1/\omega_n,\\
&N_n^{(k+1)} = \nabla_n^{(k)}[\alpha] N_n^{(k)},\\
&D_n^{(k+1)} = \nabla_n^{(k)}[\alpha] D_n^{(k)},\\
&\mathscr{I}_n^{(k)}(\alpha,\{\{s_n\}\},\{\{\omega_n\}\},\{\Delta_n^{(k)}\}) = \frac{N_n^{(k)}}{D_n^{(k)}},
\end{aligned}
\tag{165}
$$

where the generalized difference operator $\nabla_n^{(k)}[\alpha]$ defined in Eq. (35) involves quantities $\Delta_n^{(k)} \neq 0$ for $k \in \mathbb{N}_0$. Special cases of the $\mathscr{I}$ transformation result from corresponding choices of the $\Delta_n^{(k)}$. From Eq. (165) one easily obtains the alternative representation

$$
\mathscr{I}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\},\{\Delta_n^{(k)}\}) = \frac{\nabla_n^{(k-1)}[\alpha]\,\nabla_n^{(k-2)}[\alpha]\ldots\nabla_n^{(0)}[\alpha][s_n/\omega_n]}{\nabla_n^{(k-1)}[\alpha]\,\nabla_n^{(k-2)}[\alpha]\ldots\nabla_n^{(0)}[\alpha][1/\omega_n]}. \tag{166}
$$

Thus, $\mathscr{I}^{(k)}$ is a Levin-type sequence transformation of order $2k$. It is not convex.

Put $\Theta_n^{(0)} = 1$ and for $k > 0$ define

$$\Theta_n^{(k)} = \frac{\Delta_n^{(0)} \dots \Delta_n^{(k-1)}}{\Delta_{n+1}^{(0)} \dots \Delta_{n+1}^{(k-1)}}. \tag{167}$$

If for all $k \in \mathbb{N}$ the limits

$$\lim_{n\to\infty} \Theta_n^{(k)} = \Theta_k \tag{168}$$

exist (we have always $\Theta_0 = 1$), then one can define a limiting transformation $\mathring{\mathscr{I}}$ for large n. It is a special case of the $\mathscr{I}$ transformation according to [44]

$$\mathring{\mathscr{I}}_n^{(k)}(\alpha, \{\{s_n\}\}, \{\{\omega_n\}\}, \{\{\Theta_k\}\}) = \mathscr{I}_n^{(k)}(\alpha, \{\{s_n\}\}, \{\{\omega_n\}\}, \{\{(\Theta_k/\Theta_{k+1})^n\}\}). \tag{169}$$

This is a transformation of order $2k$. The characteristic polynomials of $\mathring{\mathscr{I}}$ are known [44] to be

$$Q^{(2k)}(\alpha) \in \mathbb{P}^{2k}\colon\ Q^{(2k)}(\alpha)(z) = \prod_{j=0}^{k-1} [(1 - z\Theta_j \exp(\mathrm{i}\alpha))(1 - z\Theta_j \exp(-\mathrm{i}\alpha))]. \tag{170}$$

4.2.11. $\mathscr{K}$ transformation

The $\mathscr{K}$ transformation was introduced by Homeier [37] in a slightly different form. It was obtained via iteration of the simple transformation

$$s_n' = \frac{\zeta_n^{(0)} s_n/\omega_n + \zeta_n^{(1)} s_{n+1}/\omega_{n+1} + \zeta_n^{(2)} s_{n+2}/\omega_{n+2}}{\zeta_n^{(0)} 1/\omega_n + \zeta_n^{(1)} 1/\omega_{n+1} + \zeta_n^{(2)} 1/\omega_{n+2}}, \tag{171}$$

that is exact for sequences of the form

$$s_n = s + \omega_n (cP_n + dQ_n), \tag{172}$$

where c and d are arbitrary constants, while P_n and Q_n are two linearly independent solutions of the three-term recurrence

$$\zeta_n^{(0)} v_n + \zeta_n^{(1)} v_{n+1} + \zeta_n^{(2)} v_{n+2} = 0. \tag{173}$$

The $\mathscr{K}$ transformation may be defined via the recursive scheme

$$\begin{aligned} &N_n^{(0)} = s_n/\omega_n, \quad D_n^{(0)} = 1/\omega_n, \\ &N_n^{(k+1)} = \partial_n^{(k)}[\zeta] N_n^{(k)}, \\ &D_n^{(k+1)} = \partial_n^{(k)}[\zeta] D_n^{(k)}, \\ &\mathscr{K}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\tilde{\Delta}_n^{(k)}\}, \{\zeta_n^{(j)}\}) = \frac{N_n^{(k)}}{D_n^{(k)}}, \end{aligned} \tag{174}$$

where the generalized difference operator $\partial_n^{(k)}[\zeta]$ defined in Eq. (36) involves recursion coefficients $\zeta_{n+k}^{(j)}$ with $j = 0, 1, 2$ and quantities $\tilde{\Delta}_n^{(k)} \neq 0$ for $k \in \mathbb{N}_0$. Special cases of the $\mathscr{K}$ transformation for given recursion, i.e., for given $\zeta_n^{(j)}$, result from corresponding choices of the $\tilde{\Delta}_n^{(k)}$. From Eq. (174) one easily obtains the alternative representation

$$\mathscr{K}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\tilde{\Delta}_n^{(k)}\}, \{\zeta_n^{(j)}\}) = \frac{\partial_n^{(k-1)}[\zeta]\partial_n^{(k-2)}[\zeta] \dots \partial_n^{(0)}[\zeta][s_n/\omega_n]}{\partial_n^{(k-1)}[\zeta]\partial_n^{(k-2)}[\zeta] \dots \partial_n^{(0)}[\zeta][1/\omega_n]}. \tag{175}$$

Thus, $\mathscr{K}^{(k)}$ is a Levin-type sequence transformation of order $2k$. It is not convex. For applications of the $\mathscr{K}$ transformation see [37,40,42,45].

5. Methods for the construction of Levin-type transformations

In this section, we discuss approaches for the construction of Levin-type sequence transformations and point out the relation to their kernel.

5.1. Model sequences and annihilation operators

As discussed in the introduction, the derivation of sequence transformations may be based on model sequences. These may be of the form (10) or of the form (6). Here, we consider model sequences of the latter type that involves remainder estimates ω_n. As described in Section 3.1, determinantal representations for the corresponding sequence transformations can be derived using Cramer's rule, and one of the recursive schemes of the **E** algorithm may be used for the computation. However, for important special choices of the functions $\psi_j(n)$, simpler recursive schemes and more explicit representations in the form (11) can be obtained using the annihilation operator approach of Weniger [84]. This approach was also studied by Brezinski and Matos [13] who showed that it leads to a unified derivation of many extrapolation algorithms and related devices and general results about their kernels. Further, we mention the work of Matos [59] who analysed the approach further and derived a number of convergence acceleration results for Levin-type sequence transformations.

In this approach, an annihilation operator $\mathscr{A}=\mathscr{A}_n^{(k)}$ as defined in Eq. (31) is needed that annihilates the sequences $\{\{\psi_j(n)\}\}$, i.e., such that

$$\mathscr{A}_n^{(k)}(\{\{\psi_j(n)\}\}) = 0 \quad \text{for } j = 0, \ldots, k-1. \tag{176}$$

Rewriting Eq. (6) in the form

$$\frac{\sigma_n - \sigma}{\omega_n} = \sum_{j=0}^{k-1} c_j \psi_j(n) \tag{177}$$

and applying $\mathscr{A}$ to both sides of this equation, one sees that

$$\mathscr{A}_n^{(k)} \left\{ \left\{ \frac{\sigma_n - \sigma}{\omega_n} \right\} \right\} = 0 \tag{178}$$

This equation may be solved for σ due to the linearity of $\mathscr{A}$. The result is

$$\sigma = \frac{\mathscr{A}_n^{(k)}(\{\{\sigma_n/\omega_n\}\})}{\mathscr{A}_n^{(k)}(\{\{1/\omega_n\}\})} \tag{179}$$

leading to a sequence transformation

$$\mathscr{T}_n^{(k)}(\{\{s_n\}\}, \{\{\omega\}\}) = \frac{\mathscr{A}_n^{(k)}(\{\{s_n/\omega_n\}\})}{\mathscr{A}_n^{(k)}(\{\{1/\omega_n\}\})}. \tag{180}$$

Since $\mathscr{A}$ is linear, this transformation can be rewritten in the form (11), i.e., a Levin-type transformation has been obtained.

We note that this process can be reversed, that is, for each Levin-type sequence transformation $\mathscr{T}[\Lambda^{(k)}]$ of order k there is an annihilation operator, namely the polynomial operator $\mathscr{P}[\Pi_n^{(k)}]$ as defined in Eq. (38) where $\Pi_n^{(k)}$ are the characteristic polynomials as defined in Eq. (72). Using this operator, the defining Eq. (66) can be rewritten as

$$\mathscr{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}) = \frac{\mathscr{P}[\Pi_n^{(k)}](s_n/\omega_n)}{\mathscr{P}[\Pi_n^{(k)}](1/\omega_n)}. \tag{181}$$

Let $\phi_{n,m}(k)$ for $m = 0,\ldots,k-1$ be k linearly independent solutions of the linear $(k+1)$–term recurrence

$$\sum_{j=0}^{k} \lambda_{n,j}^{(k)} v_{n+j} = 0. \tag{182}$$

Then $\mathscr{P}[\Pi_n^{(k)}]\phi_{n,m}(k) = 0$ for $m = 0,\ldots,k-1$, i.e., $\mathscr{P}[\Pi_n^{(k)}]$ is an annihilation operator for all solutions of Eq. (182). Thus, all sequences that are annihilated by this operator are linear combinations of the k sequences $\{\{\phi_{n,m}^{(k)}\}\}$.

If $\{\{\sigma_n\}\}$ is a sequence in the kernel of $\mathscr{T}^{(k)}$ with (anti)limit σ, we must have

$$\sigma = \frac{\mathscr{P}[\Pi_n^{(k)}](\sigma_n/\omega_n)}{\mathscr{P}[\Pi_n^{(k)}](1/\omega_n)} \tag{183}$$

or after some rearrangement using the linearity of $\mathscr{P}$

$$\mathscr{P}[\Pi_n^{(k)}]\left(\frac{\sigma_n - \sigma}{\omega_n}\right) = 0. \tag{184}$$

Hence, we must have

$$\frac{\sigma_n - \sigma}{\omega_n} = \sum_{m=0}^{k-1} c_m \phi_{n,m}^{(k)}, \tag{185}$$

or, equivalently

$$\sigma_n = \sigma + \omega_n \sum_{m=0}^{k-1} c_m \phi_{n,m}^{(k)} \tag{186}$$

for some constants c_m. Thus, we have determined the kernel of $\mathscr{T}^{(k)}$ that can also be considered as the set of model sequences for this transformation. Thus, we have proved the following theorem:

Theorem 1. *Let $\phi_{n,m}^{(k)}$ for $m = 0,\ldots,k-1$ be the k linearly independent solutions of the linear $(k+1)$–term recurrence* (182). *The kernel of $\mathscr{T}[\Lambda^{(k)}](\{\{s_n\}\},\{\{\omega_n\}\})$ is given by all sequences $\{\{\sigma_n\}\}$ with* (*anti*)*limit σ and elements σ_n of the form* (186) *for arbitrary constants c_m.*

We note that the $\psi_j(n)$ for $j = 0,\ldots,k-1$ can essentially be identified with the $\phi_{n,j}^{(k)}$. Thus, we have determinantal representations for known $\psi_j(n)$ as noted above in the context of the E algorithm. See also [38] for determinantal representations of the $\mathscr{J}$ transformations and the relation to its kernel.

Examples of annihilation operators and the functions $\psi_j(n)$ that are annihilated are given in Table 2. Examples for the Levin-type sequence transformations that have been derived using the approach of model sequences are discussed in Section 5.1.2.

Table 2
Examples of annihilation operators[a]

Type	Operator	$\psi_j(n),\ j=0,\dots,k-1$
Differences	$\triangle^k$	$(n+\beta)^j$
		$(n+\beta)_j$
		$(\alpha[n+\zeta])^j$
		$(\alpha[n+\zeta])_j$
		$p_j(n),\ p_j \in \mathbb{P}^{(j)}$
Weighted differences	$\triangle^k(n+\beta)^{k-1}$	$1/(n+\beta)^j$
	$\triangle^k(n+\beta)_{k-1}$	$1/(n+\beta)_j$
	$\triangle^k(\alpha[n+\zeta])^{k-1}$	$1/(\alpha[n+\zeta])^j$
	$\triangle^k(\alpha[n+\zeta])_{k-1}$	$1/(\alpha[n+\zeta])_j$
Divided differences	$\square_n^{(k)}[\{\{t_n\}\}]$	t_n^j
	$\square_n^{(k)}[\{\{t_n\}\}]$	$p_j(t_n),\ p_j \in \mathbb{P}^{(j)}$
	$\square_n^{(k)}[\{\{x_n\}\}](x_n)_{k-1}$	$1/(x_n)_j$
Polynomial	$\mathscr{P}[P^{(2k)}(\alpha)]$	$\exp(+\mathrm{i}\alpha n)p_j(n),\ p_j \in \mathbb{P}^{(j)}$
		$\exp(-\mathrm{i}\alpha n)p_j(n),\ p_j \in \mathbb{P}^{(j)}$
	$\mathscr{P}[P^{(2k)}(\alpha)](n+\beta)^{k-1}$	$\exp(+\mathrm{i}\alpha n)/(n+\beta)^j$
		$\exp(-\mathrm{i}\alpha n)/(n+\beta)^j$
	$\mathscr{P}[P^{(2k)}(\alpha)](n+\beta)_{k-1}$	$\exp(+\mathrm{i}\alpha n)/(n+\beta)_j$
		$\exp(-\mathrm{i}\alpha n)/(n+\beta)_j$
	$\mathscr{P}[P^{(k)}]$	$\psi_j(n)$ is solution of $\sum_{m=0}^k p_n^{(k)} v_{n+j} = 0$
	$\mathscr{P}[P^{(k)}](n+\beta)^m$	$(n+\beta)^m\psi_j(n)$ is solution of $\sum_{m=0}^k p_n^{(k)} v_{n+j} = 0$
	L_1 (see (188))	$\frac{\alpha_j^{n+1}}{n!}$
	L_2 (see (189))	$\frac{n^j\alpha^{n+1}}{n!}$
	$\tilde{L}$ (see (191))	$\frac{\Gamma(n+\alpha_j+1)}{n!}$

[a]See also Section 5.1.1.

Note that the annihilation operators used by Weniger [84,87,88] were weighted difference operators $\mathscr{W}_n^{(k)}$ as defined in Eq. (37). Homeier [36,38,39] discussed operator representations for the $\mathscr{J}$ transformation that are equivalent to many of the annihilation operators and related sequence transformations as given by Brezinski and Matos [13]. The latter have been further discussed by Matos [59] who considered among others Levin-type sequence transformations with constant coefficients, $\lambda_{n,j}^{(k)} = \text{const.}$, and with polynomial coefficients $\lambda_{n,j}^{(k)} = \lambda_j(n+1)$, with $\lambda_j \in \mathbb{P}$, and $n \in \mathbb{N}_0$, in particular annihilation operators of the form

$$L(u_n) = (\Omega^l + \lambda_1\Omega^{l-1} + \cdots + \lambda_l)(u_n) \tag{187}$$

with the special cases

$$L_1(u_n) = (\Omega-\alpha_1)(\Omega-\alpha_2)\cdots(\Omega-\alpha_l)(u_n) \quad (\alpha_i \neq \alpha_j \quad \text{for all } i \neq j) \tag{188}$$

and

$$L_2(u_n) = (\Omega-\alpha)^l(u_n), \tag{189}$$

where

$$\Omega^r(u_n) = (n+1)_r u_{n+r}, \quad n \in \mathbb{N}_0 \tag{190}$$

and

$$\tilde{L}(u_n) = (\pi - \alpha_1)(\pi - \alpha_2)\cdots(\pi - \alpha_l)(u_n), \tag{191}$$

where

$$\pi(u_n) = (n+1) \triangle u_n, \quad \pi^r(u_n) = \pi(\pi^{r-1}(u_n)), \quad n \in \mathbb{N}_0 \tag{192}$$

and the λ's and α's are constants. Note that n is shifted in comparison to [59] where the convention $n \in \mathbb{N}$ was used. See also Table 2 for the corresponding annihilated functions $\psi_j(n)$.

Matos [59] also considered difference operators of the form

$$L(u_n) = \triangle^k + p_{k-1}(n) \triangle^{k-1} + \cdots + p_1(n) \triangle + p_0(n), \tag{193}$$

where the functions f_j given by $f_j(t) = p_j(1/t)t^{-k+j}$ for $j = 0, \ldots, k-1$ are analytic in the neighborhood of 0. For such operators, there is no explicit formula for the functions that are annihilated. However, the asymptotic behavior of such functions is known [6,59]. We will later return to such annihilation operators and state some convergence results.

5.1.1. Derivation of the $\mathscr{F}$ transformation

As an example for the application of the annihilation operator approach, we derive the $\mathscr{F}$ transformation. Consider the model sequence

$$\sigma_n = \sigma + \omega_n \sum_{j=0}^{k-1} c_j \frac{1}{(x_n)_j}, \tag{194}$$

that may be rewritten as

$$\frac{\sigma_n - \sigma}{\omega_n} = \sum_{j=0}^{k-1} c_j \frac{1}{(x_n)_j}. \tag{195}$$

We note that Eq. (194) corresponds to modeling $\mu_n = R_n/\omega_n$ as a truncated factorial series in x_n (instead as a truncated power series as in the case of the W algorithm). The x_n are elements of $\{\{x_n\}\}$ an auxiliary sequence $\{\{x_n\}\}$ such that $\lim_{n\to\infty} 1/x_n = 0$ and also $x_{n+\ell} > x_n$ for $\ell \in \mathbb{N}$ and $x_0 > 1$, i.e., a diverging sequence of monotonously increasing positive numbers. To find an annihilation operator for the $\psi_j(n) = 1/(x_n)_j$, we make use of the fact that the divided difference operator $\square_n^{(k)} = \square_n^{(k)}[\{\{x_n\}\}]$ annihilates polynomials in x_n of degree less than k. Also, we observe that the definition of the Pochhammer symbols entails that

$$(x_n)_{k-1}/(x_n)_j = (x_n + j)_{k-1-j} \tag{196}$$

is a polynomial of degree less than k in x_n for $0 \leqslant j \leqslant k-1$. Thus, the sought annihilation operator is $\mathscr{A} = \square_n^{(k)}(x_n)_{k-1}$ because

$$\square_n^{(k)}(x_n)_{k-1}\frac{1}{(x_n)_j} = 0, \quad 0 \leqslant j < k. \tag{197}$$

Hence, for the model sequence (194), one can calculate σ via

$$\sigma = \frac{\square_n^{(k)}((x_n)_{k-1}\sigma_n/\omega_n)}{\square_n^{(k)}((x_n)_{k-1}/\omega_n)} \tag{198}$$

and the $\mathscr{F}$ transformation (140) results by replacing σ_n by s_n in the right-hand side of Eq. (198).

5.1.2. Important special cases

Here, we collect model sequences and annihilation operators for some important Levin-type sequence transformations that were derived using the model sequence approach. For further examples see also [13]. The model sequences are the kernels by construction. In Section 5.2.2, kernels and annihilation operators are stated for important Levin-type transformation that were derived using iterative methods.

Levin transformation: The model sequence for $\mathscr{L}^{(k)}$ is

$$\sigma_n = \sigma + \omega_n \sum_{j=0}^{k-1} c_j/(n+\beta)^j. \tag{199}$$

The annihilation operator is

$$\mathscr{A}_n^{(k)} = \triangle^k (n+\beta)^{k-1}. \tag{200}$$

Weniger transformations: The model sequence for $\mathscr{S}^{(k)}$ is

$$\sigma_n = \sigma + \omega_n \sum_{j=0}^{k-1} c_j/(n+\beta)_j. \tag{201}$$

The annihilation operator is

$$\mathscr{A}_n^{(k)} = \triangle^k (n+\beta)_{k-1}. \tag{202}$$

The model sequence for $\mathscr{M}^{(k)}$ is

$$\sigma_n = \sigma + \omega_n \sum_{j=0}^{k-1} c_j/(-n-\xi)_j. \tag{203}$$

The annihilation operator is

$$\mathscr{A}_n^{(k)} = \triangle^k (-n-\xi)_{k-1}. \tag{204}$$

The model sequence for $\mathscr{C}^{(k)}$ is

$$\sigma_n = \sigma + \omega_n \sum_{j=0}^{k-1} c_j/(\alpha[n+\zeta])_j. \tag{205}$$

The annihilation operator is

$$\mathscr{A}_n^{(k)} = \triangle^k (\alpha[n+\zeta])_{k-1}. \tag{206}$$

W algorithm: The model sequence for $W^{(k)}$ is

$$\sigma_n = \sigma + \omega_n \sum_{j=0}^{k-1} c_j t_n^j. \tag{207}$$

The annihilation operator is

$$\mathscr{A}_n^{(k)} = \square_n^{(k)}[\{\{t_n\}\}]. \tag{208}$$

$\mathscr{H}$ *transformation*: The model sequence for $\mathscr{H}^{(k)}$ is

$$\sigma_n = \sigma + \omega_n \left(\exp(\mathrm{i}\alpha n) \sum_{j=0}^{k-1} c_j^+/(n+\beta)^j + \exp(-\mathrm{i}\alpha n) \sum_{j=0}^{k-1} c_j^-/(n+\beta)^j \right). \tag{209}$$

The annihilation operator is

$$\mathscr{A}_n^{(k)} = \mathscr{P}[P^{(2k)}(\alpha)](n+\beta)^{k-1}. \tag{210}$$

Generalized $\mathscr{H}$ *transformation*: The model sequence for $\mathscr{H}^{(k,m)}$ is

$$\sigma_n = \sigma + \omega_n \sum_{m=1}^{M} e_m^n \sum_{j=0}^{k-1} c_{m,j}(n+\beta)^{-j}. \tag{211}$$

The annihilation operator is

$$\mathscr{A}_n^{(k)} = \mathscr{P}[P^{(k,m)}(\mathbf{e})](n+\beta)^{k-1}. \tag{212}$$

5.2. Hierarchically consistent iteration

As alternative to the derivation of sequence transformations using model sequences and possibly annihilation operators, one may take some simple sequence transformation T and iterate it k times to obtain a transformation $T^{(k)} = T \circ \cdots \circ T$. For the iterated transformation, by construction one has a simple algorithm by construction, but the theoretical analysis is complicated since usually no kernel is known. See for instance the iterated Aitken process where the $\triangle^2$ method plays the role of the simple transformation. However, as is discussed at length in Refs. [36,86], there are usually several possibilities for the iteration. Both problems – unknown kernel and arbitrariness of iteration – are overcome using the concept of hierarchical consistency [36,40,44] that was shown to give rise to powerful algorithms like the $\mathscr{J}$ and the $\mathscr{I}$ transformations [39,40,44]. The basic idea of the concept is to provide a hierarchy of model sequences such that the simple transformation provides a mapping between neighboring levels of the hierarchy. To ensure the latter, normally one has to fix some parameters in the simple transformation to make the iteration consistent with the hierarchy.

A formal description of the concept is given in the following taken mainly from the literature [44]. As an example, the concept is later used to derive the $\mathscr{J}\mathscr{D}$ transformation in Section 5.2.1.

Let $\{\{\sigma_n(\boldsymbol{c},\boldsymbol{p})\}\}_{n=0}^{\infty}$ be a simple "basic" model sequence that depends on a vector $\boldsymbol{c} \in \mathbb{K}^a$ of constants, and further parameters $\boldsymbol{p}$. Assume that its (anti)limit $\sigma(\boldsymbol{p})$ exists and is independent of $\boldsymbol{c}$. Assume that the basic transformation $T = T(\boldsymbol{p})$ allows to compute the (anti)limit exactly according to

$$T(\boldsymbol{p}) : \{\{\sigma_n(\boldsymbol{c},\boldsymbol{p})\}\} \to \{\{\sigma(\boldsymbol{p})\}\}. \tag{213}$$

Let the hierarchy of model sequences be given by

$$\{\{\{\sigma_n^{(\ell)}(\boldsymbol{c}^{(\ell)},\boldsymbol{p}^{(\ell)})|\boldsymbol{c}^{(\ell)} \in \mathbb{K}^{a^{(\ell)}}\}\}\}_{\ell=0}^{L} \tag{214}$$

with $a^{(\ell)} > a^{(\ell')}$ for $\ell > \ell'$. Here, ℓ numbers the levels of the hierarchy. Each of the model sequences $\{\{\sigma_n^{(\ell)}(\boldsymbol{c}^{(\ell)},\boldsymbol{p}^{(\ell)})\}\}$ depends on an $a^{(\ell)}$-dimensional complex vector $\boldsymbol{c}^{(\ell)}$ and further parameters $\boldsymbol{p}^{(\ell)}$. Assume that the model sequences of lower levels are also contained in those of higher levels: For all $\ell < L$ and all $\ell' > \ell$ and $\ell' \leqslant L$, every sequence $\{\{\sigma_n^{(\ell)}(\boldsymbol{c}^{(\ell)},\boldsymbol{p}^{(\ell)})\}\}$ is assumed to be representable as a model sequence $\{\{\sigma_n^{(\ell')}(\boldsymbol{c}^{(\ell')},\boldsymbol{p}^{(\ell')})\}\}$ where $\boldsymbol{c}^{(\ell')}$ is obtained from $\boldsymbol{c}^{(\ell)}$ by the natural injection $\mathbb{K}^{a^{(\ell)}} \to \mathbb{K}^{a^{(\ell')}}$. Assume that for all ℓ with $0 < \ell \leqslant L$

$$T(\boldsymbol{p}^{(\ell)}) : \{\{\sigma_n^{(\ell)}(\boldsymbol{c}^{(\ell)},\boldsymbol{p}^{(\ell)})\}\} \to \{\{\sigma_n^{(\ell-1)}(\boldsymbol{c}^{(\ell-1)},\boldsymbol{p}^{(\ell-1)})\}\} \tag{215}$$

is a mapping between neighboring levels of the hierarchy. Composition yields an iterative transformation

$$T^{(L)} = T(\boldsymbol{p}^{(0)}) \circ T(\boldsymbol{p}^{(1)}) \circ \cdots \circ T(\boldsymbol{p}^{(L)}). \tag{216}$$

This transformation is called "hierarchically consistent" or "consistent with the hierarchy". It maps model sequences $\sigma_n^{(\ell)}(\boldsymbol{c}^{(\ell)},\boldsymbol{p}^{(\ell)})$ to constant sequences if Eq. (213) holds with

$$\{\{\sigma_n^{(0)}(\boldsymbol{c}^{(0)},\boldsymbol{p}^{(0)})\}\} = \{\{\sigma_n(\boldsymbol{c},\boldsymbol{p})\}\}. \tag{217}$$

If instead of Eq. (215) we have

$$T(\boldsymbol{p}^{(\ell)})(\{\{\sigma_n^{(\ell)}(\boldsymbol{c}^{(\ell)},\boldsymbol{p}^{(\ell)})\}\}) \sim \{\{\sigma_n^{(\ell-1)}(\boldsymbol{c}^{(\ell-1)},\boldsymbol{p}^{(\ell-1)})\}\} \tag{218}$$

for $n \to \infty$ for all $\ell > 0$ then the iterative transformation $T^{(L)}$ is called "asymptotically consistent with the hierarchy" or "asymptotically hierarchy-consistent".

5.2.1. Derivation of the $\mathscr{J}\mathscr{D}$ transformation

The simple transformation is the $\mathscr{D}^{(2)}$ transformation

$$s_n' = T(\{\{\omega_n\}\})(\{\{s_n\}\}) = \frac{\triangle^2(s_n/\omega_n)}{\triangle^2(1/\omega_n)} \tag{219}$$

depending on the "parameters" $\{\{\omega_n\}\}$, with basic model sequences

$$\frac{\sigma_n}{\omega_n} = \sigma\frac{1}{\omega_n} + (an + b). \tag{220}$$

The more complicated model sequences of the next level are taken to be

$$\frac{\sigma_n}{\omega_n} = \sigma\frac{1}{\omega_n} + (an + b + (a_1 n + b_1)r_n). \tag{221}$$

Application of $\triangle^2$ eliminates the terms involving a and b. The result is

$$\frac{\triangle^2 \sigma_n/\omega_n}{\triangle^2 r_n} = \sigma \frac{\triangle^2 1/\omega_n}{\triangle^2 r_n} + \left(a_1 n + b_1 + 2a_1 \frac{\triangle r_n}{\triangle^2 r_n} \right) \tag{222}$$

for $\triangle^2 r_n \neq 0$. Assuming that for large n

$$\frac{\triangle r_n}{\triangle^2 r_n} = An + B + \mathrm{o}(1) \tag{223}$$

holds, the result is asymptotically of the same *form* as the model sequence in Eq. (220), namely

$$\frac{\sigma'_n}{\omega'_n} = \sigma \frac{1}{\omega'_n} + (a'n + b' + \mathrm{o}(1)) \tag{224}$$

with renormalized "parameters"

$$1/\omega'_n = \frac{\triangle^2 (1/\omega_n)}{\triangle^2 r_n} \tag{225}$$

and obvious identifications for a' and b'.

We now assume that this mapping between two neighboring levels of the hierarchy can be extended to any two neighboring levels, provided that one introduces ℓ-dependent quantities, especially $r_n \to r_n^{(\ell)}$ with $\zeta_n^{(\ell)} = \triangle^2 r_n^{(\ell)} \neq 0$, $s_n/\omega_n \to N_n^{(\ell)}$, $1/\omega_n \to D_n^{(\ell)}$ and $s'_n/\omega'_n \to N_n^{(\ell+1)}$, $1/\omega'_n \to D_n^{(\ell+1)}$.

Iterating in this way leads to algorithm (151).

Condition (223) or more generally

$$\frac{\triangle r_n^{(\ell)}}{\triangle^2 r_n^{(\ell)}} = A_\ell n + B_\ell + \mathrm{o}(1) \tag{226}$$

for given ℓ and for large n is satisfied in many cases. For instance, it is satisfied if there are constants $\beta_\ell \neq 0$, γ_ℓ and $\delta_\ell \neq 0$ such that

$$\triangle\, r_n^{(\ell)} \sim \beta_\ell \begin{cases} \left(\dfrac{\delta_\ell + 1}{\delta_\ell} \right)^n & \text{for } \gamma_\ell = 0, \\ \left(\dfrac{\delta_\ell + 1}{\gamma_\ell} \right)_n \Big/ \left(\dfrac{\delta_\ell}{\gamma_\ell} \right)_n & \text{otherwise.} \end{cases} \tag{227}$$

This is for instance the case for $r_n^{(\ell)} = n^{\zeta_\ell}$ with $\zeta_\ell(\zeta_\ell - 1) \neq 0$.

The kernel of $\mathscr{J}\mathscr{D}^{(k)}$ may be found inductively in the following way:

$$\begin{aligned} & N_n^{(k)} - \sigma D_n^{(k)} = 0 \\ & \quad \Rightarrow \triangle^2 (N_n^{(k-1)} - \sigma D_n^{(k-1)}) = 0 \\ & \quad \Rightarrow N_n^{(k-1)} - \sigma D_n^{(k-1)} = a_{k-1} n + b_{k-1} \\ & \quad \Rightarrow \triangle^2 (N_n^{(k-2)} - \sigma D_n^{(k-2)}) = (a_{k-1} n + b_{k-1}) \zeta_n^{(k-2)} \\ & \quad \Rightarrow N_n^{(k-2)} - \sigma D_n^{(k-2)}) = a_{k-2} n + b_{k-2} + \sum_{j=0}^{n-2} \sum_{n'=0}^{j} (a_{k-1} n' + b_{k-1}) \zeta_{n'}^{(k-2)} \end{aligned} \tag{228}$$

yielding the result

$$N_n^{(0)} - \sigma D_n^{(0)} = a_0 n + b_0 + \sum_{j=0}^{n-2} \sum_{n_1=0}^{j} \zeta_{n_1}^{(0)} (a_1 n_1 + b_1 + \cdots$$

$$+ (a_{k-2} n + b_{k-2} + \sum_{j_{k-2}=0}^{n_{k-2}-2} \sum_{n_{k-1}=0}^{j_{k-2}} \zeta_{n_{k-1}}^{(k-2)} (a_{k-1} n_{k-1} + b_{k-1}) \Bigg) . \tag{229}$$

Here, the definitions $N_n^{(0)} = \sigma_n/\omega_n$ and $D_n^{(0)} = 1/\omega_n$ may be used to obtain the model sequence $\{\{\sigma_n\}\}$ for $\mathscr{J}\mathscr{D}^{(k)}$, that may be identified as kernel of that transformation, and also may be regarded as model sequence of the kth level according to $\{\{\sigma_n^{(k)}(\boldsymbol{c}^{(k)}, \boldsymbol{p}^{(k)})\}\}$ with $\boldsymbol{c}^{(k)} = (a_0, b_0, \ldots, a_{k-1}, b_{k-1})$ and $\boldsymbol{p}^{(k)}$ corresponds to $\omega_n^{(k)} = 1/D_n^{(k)}$ and the $\{\zeta_n^{(\kappa)} | 0 \leqslant \kappa \leqslant k-2\}$.

We note this as a theorem:

Theorem 2. *The kernel of* $\mathscr{J}\mathscr{D}^{(k)}$ *is given by the set of sequences* $\{\{\sigma_n\}\}$ *such that Eq.* (229) *holds with* $N_n^{(0)} = \sigma_n/\omega_n$ *and* $D_n^{(0)} = 1/\omega_n$.

5.2.2. Important special cases

Here, we give the hierarchies of model sequences for sequence transformations derived via hierarchically consistent iteration.

$\mathscr{J}$ transformation: The most prominent example is the $\mathscr{J}$ transformation (actually a large class of transformations). The corresponding hierarchy of model sequences provided by the kernels that are explicitly known according to the following theorem:

Theorem 3 (Homeier [36]). *The kernel of the* $\mathscr{J}^{(k)}$ *transformation is given by the sequences* $\{\{\sigma_n\}\}$ *with elements of the form*

$$\sigma_n = \sigma + \omega_n \sum_{j=0}^{k-1} c_j \psi_j(n) \tag{230}$$

with

$$\begin{aligned} \psi_0(n) &= 1, \\ \psi_1(n) &= \sum_{n_1=0}^{n-1} \delta_{n_1}^{(0)}, \\ \psi_2(n) &= \sum_{n_1=0}^{n-1} \delta_{n_1}^{(0)} \sum_{n_2=0}^{n_1-1} \delta_{n_2}^{(1)}, \\ &\vdots \\ \psi_{k-1}(n) &= \sum_{n>n_1>n_2>\cdots>n_{k-1}} \delta_{n_1}^{(0)} \delta_{n_2}^{(1)} \cdots \delta_{n_{k-1}}^{(k-2)} \end{aligned} \tag{231}$$

with arbitrary constants $c_0, \ldots, c_{k-1}$.

$\mathcal{I}$ transformation: Since the $\mathcal{I}$ transformation is a special case of the $\mathcal{J}$ transformation (cf. Table 1) and [44], its kernels (corresponding to the hierarchy of model sequences) are explicitly known according to the following theorem:

Theorem 4 (Homeier [44, Theorem 8]). *The kernel of the $\mathcal{I}^{(k)}$ transformation is given by the sequences $\{\{\sigma_n\}\}$ with elements of the form*

$$\sigma_n = \sigma + \exp(-\mathrm{i}\alpha n)\omega_n \Bigg[d_0 + d_1 \exp(2\mathrm{i}\alpha n) + \sum_{n_1=0}^{n-1}\sum_{n_2=0}^{n_1-1} \exp(2\mathrm{i}\alpha(n_1-n_2))(d_2 + d_3\exp(2\mathrm{i}\alpha n_2))\Delta_{n_2}^{(0)} + \cdots + \sum_{n>n_1>n_2>\cdots>n_{2k-2}} \exp(2\mathrm{i}\alpha[n_1-n_2+\cdots+n_{2k-3}-n_{2k-2}]) (d_{2k-2} + d_{2k-1}\exp(2\mathrm{i}\alpha n_{2k-2}))\prod_{j=0}^{k-2}\Delta_{n_{2j+2}}^{(j)}\Bigg] \tag{232}$$

with constants $d_0, \ldots, d_{2k-1}$. Thus, we have $s = \mathcal{I}_n^{(k')}(\alpha, \{\{s_n\}\}, \{\{\omega_n\}\}, \{\Delta_n^{(k)}\})$ for $k' \geqslant k$ for sequences of this form.

5.3. A two-step approach

In favorable cases, one may use a two-step approach for the construction of sequence transformations:

Step 1: Use asymptotic analysis of the remainder $R_n = s_n - s$ of the given problem to find the adequate model sequence (or hierarchy of model sequences) for large n.

Step 2: Use the methods described in Sections 5.1 or 5.2 to construct the sequence transformation adapted to the problem.

This is, of course, a mathematically promising approach. A good example for the two-step approach is the derivation of the $d^{(m)}$ transformations by Levin and Sidi [54] (cf. also Section 3.2).

But there are two difficulties with this approach.

The first difficulty is a practical one. In many cases, the problems to be treated in applications are simply too complicated to allow to perform Step 1 of the two-step approach.

The second difficulty is a more mathematical one. The optimal system of functions $f_j(n)$ used in the asymptotic expansion

$$s_n - s \sim \sum_{j=0}^{\infty} c_j f_j(n) \tag{233}$$

with $f_{j+1}(n) = \mathrm{o}(f_j(n))$, i.e., the optimal *asymptotic scale* [102, p. 2], is not clear a priori. For instance, as the work of Weniger has shown, sequence transformations like the Levin transformation that are based on expansions in powers of $1/n$, i.e., the asymptotic scale $\phi_j(n) = 1/(n+\beta)^j$, are not always superior to, and even often worse than those based upon factorial series, like Weniger's $\mathcal{S}$

transformation that is based on the asymptotic scale $\psi_j(n)=1/(n+\beta)_j$. To find an optimal asymptotic scale in combination with nonlinear sequence transformations seems to be an open mathematical problem.

Certainly, the proper choice of remainder estimates [50] is also crucial in the context of Levin-type sequence transformations. See also Section 9.

6. Properties of Levin-type transformations

6.1. Basic properties

Directly from the definition in Eqs. (65) and (66), we obtain the following theorem. The proof is left to the interested reader.

Theorem 5. *Any Levin-type sequence transformation $\mathcal{T}$ is quasilinear, i.e., we have*

$$\mathcal{T}_n^{(k)}(\{\{As_n+B\}\},\{\{\omega_n\}\}) = A\mathcal{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\})+B \tag{234}$$

for arbitrary constants A and B. It is multiplicatively invariant in ω_n, i.e., we have

$$\mathcal{T}_n^{(k)}(\{\{s_n\}\},\{\{C\omega_n\}\}) = \mathcal{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}) \tag{235}$$

for arbitrary constants $C\neq 0$.

For a coefficient set Λ define the sets $Y_n^{(k)}[\Lambda]$ by

$$Y_n^{(k)}[\Lambda] = \left\{(x_0,\dots,x_k)\in\mathbb{F}^{k+1}\left|\sum_{j=0}^{k}\lambda_{n,j}^{(k)}/x_j\neq 0\right.\right\}. \tag{236}$$

Since $\mathcal{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\})$ for given coefficient set Λ depends only on the $2k+2$ numbers $s_n,\dots,s_{n+k}$ and $\omega_n,\dots,\omega_{n+k}$, it may be regarded as a mapping

$$U_n^{(k)}:\mathbb{C}^{k+1}\times Y_n^{(k)}[\Lambda]\Rightarrow\mathbb{C},\quad (x,y)\mapsto U_n^{(k)}(x\,|\,y) \tag{237}$$

such that

$$\mathcal{T}_n^{(k)} = U_n^{(k)}(s_n,\dots,s_{n+k}\,|\,\omega_n,\dots,\omega_{n+k}). \tag{238}$$

The following theorem is a generalization of theorems for the $\mathcal{J}$ transformation [36, Theorem 5] and the $\mathcal{I}$ transformation [44, Theorem 5].

Theorem 6. (I − 0) *The $\mathcal{T}^{(k)}$ transformation can be regarded as continous mapping $U_n^{(k)}$ on $\mathbb{C}^{k+1}\times Y_n^{(k)}[\Lambda]$ where $Y_n^{(k)}[\Lambda]$ is defined in Eq.* (236):

(I − 1) *According to Theorem* 5, *$U_n^{(k)}$ is a homogeneous function of first degree in the first $(k+1)$ variables and a homogeneous function of degree zero in the last $(k+1)$ variables. Hence, for all vectors $\boldsymbol{x}\in\mathbb{C}^{k+1}$ and $\boldsymbol{y}\in Y_n^{(k)}[\Lambda]$ and for all complex constants s and $t\neq 0$ the equations*

$$U_n^{(k)}(s\boldsymbol{x}\,|\,\boldsymbol{y}) = sU_n^{(k)}(\boldsymbol{x}\,|\,\boldsymbol{y}),$$

$$U_n^{(k)}(\boldsymbol{x}\,|\,t\boldsymbol{y}) = U_n^{(k)}(\boldsymbol{x}\,|\,\boldsymbol{y}) \tag{239}$$

hold.

(I − 2) $U_n^{(k)}$ *is linear in the first* $(k+1)$ *variables. Thus, for all vectors* $\boldsymbol{x} \in \mathbb{C}^{k+1}$, $\boldsymbol{x}' \in \mathbb{C}^{k+1}$, *und* $\boldsymbol{y} \in Y_n^{(k)}[\Lambda]$

$$U_n^{(k)}(\boldsymbol{x} + \boldsymbol{x}' \mid \boldsymbol{y}) = U_n^{(k)}(\boldsymbol{x} \mid \boldsymbol{y}) + U_n^{(k)}(\boldsymbol{x}' \mid \boldsymbol{y}) \tag{240}$$

holds.

(I − 3) *For all constant vectors* $\boldsymbol{c} = (c, c, \ldots, c) \in \mathbb{C}^{k+1}$ *and all vectors* $\boldsymbol{y} \in Y_n^{(k)}[\Lambda]$ *we have*

$$U_n^{(k)}(\boldsymbol{c} \mid \boldsymbol{y}) = c. \tag{241}$$

Proof. These are immediate consequences of the definitions. □

6.2. The limiting transformation

We note that if a limiting transformation $\mathring{\mathscr{T}}\,[\mathring{\Lambda}\,]$ exists, it is also of Levin-type, and thus, the above theorems apply to the limiting transformation as well.

Also, we have the following result for the kernel of the limiting transformation:

Theorem 7. *Suppose that for a Levin-type sequence transformation* $\mathscr{T}^{(k)}$ *of order k there exists a limiting transformation* $\mathring{\mathscr{T}}^{(k)}$ *with characteristic polynomial* $\mathring{\Pi} \in \mathbb{P}^k$ *given by*

$$\mathring{\Pi}^{(k)}(z) = \sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} z^j = \prod_{\ell=1}^{M} (z - \zeta_\ell)^{m_\ell}, \tag{242}$$

where the zeroes $\zeta_\ell \neq 0$ *have multiplicities* m_ℓ. *Then the kernel of the limiting transformation consists of all sequences* $\{\{s_n\}\}$ *with elements of the form*

$$\sigma_n = \sigma + \omega_n \sum_{\ell=1}^{M} \zeta_\ell^n P_\ell(n), \tag{243}$$

where $P_\ell \in \mathbb{P}^{m_\ell - 1}$ *are arbitrary polynomials and* $\{\{\omega_n\}\} \in \mathring{\mathbb{Y}}^{(k)}$.

Proof. This follows directly from the observation that for such sequences $(\sigma_n - \sigma)/\omega_n$ is nothing but a finite linear combination of the solutions $\varphi_{n,\ell,j_\ell}^{(k)} = n^{j_\ell} \zeta_\ell^n$ with $\ell = 1, \ldots, M$ and $j_\ell = 0, \ldots, m_\ell - 1$ of the recursion relation

$$\sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} v_{n+j} = 0 \tag{244}$$

and thus, it is annihilated by $\mathscr{P}[\,\mathring{\Pi}^{(k)}]$. □

6.3. Application to power series

Here, we generalize some results of Weniger [88] that regard the application of Levin-type sequence transformations to power series.

We use the definitions in Eq. (27). Like Padé approximants, Levin-type sequence transformations yield rational approximants when applied to the partial sums $f_n(z)$ of a power series $f(z)$ with terms $a_j = c_j z^j$. These approximations offer a practical way for the analytical continuation of power series to regions outside of their circle of convergence. Furthermore, the poles of the rational approximations model the singularities of $f(z)$. They may also be used to approximate further terms beyond the last one used in constructing the rational approximant.

When applying a Levin-type sequence transformation $\mathscr{T}$ to a power series, remainder estimates $\omega_n = m_n z^{\gamma+n}$ will be used. We note that t variants correspond to $m_n = c_n$, $\gamma = 0$, u variants correspond to $m_n = c_n(n+\beta)$, $\gamma = 0$, $\tilde{t}$ variants to $m_n = c_{n+1}$, $\gamma = 1$. Thus, for these variants, m_n is independent of z (Case A). For v variants, we have $m_n = c_{n+1}c_n/(c_n - c_{n+1}z)$, and $\gamma = 1$. In this case, $1/m_n \in \mathbb{P}^{(1)}$ is a linear function of z (Case B).

Application of $\mathscr{T}$ yields after some simplification

$$\mathscr{T}_n^{(k)}(\{\{f_n(z)\}\}, \{\{m_n z^{\gamma+n}\}\}) = \frac{\sum_{\ell=0}^{n+k} z^\ell \sum_{j=\max(0,k-\ell)}^{k} (\lambda_{n,j}^{(k)}/m_{n+j}) c_{\ell-(k-j)}}{\sum_{j=0}^{k} (\lambda_{n,j}^{(k)}/m_{n+j}) z^{k-j}} = \frac{P_n^{(k)}[T](z)}{Q_n^{(k)}[T](z)}, \tag{245}$$

where in Case A, we have $P_n^{(k)}[T] \in \mathbb{P}^{n+k}$, $Q_n^{(k)}[T] \in \mathbb{P}^k$, and in Case B, we have $P_n^{(k)}[T] \in \mathbb{P}^{n+k+1}$, $Q_n^{(k)}[T] \in \mathbb{P}^{k+1}$. One needs the $k+1+\gamma$ partial sums $f_n(z), \ldots, f_{n+k+\gamma}(z)$ to compute these rational approximants. This should be compared to the fact that for the computation of the Padé approximant $[n+k+\gamma/k+\gamma]$ one needs the $2k+2\gamma+1$ partial sums $f_n(z), \ldots, f_{n+2k+2\gamma}(z)$.

We show that Taylor expansion of these rational approximants reproduces all terms of power series that have been used to calculate the rational approximation.

Theorem 8. *We have*

$$\mathscr{T}_n^{(k)}(\{\{f_n(z)\}\}, \{\{m_n z^{\gamma+n}\}\}) - f(z) = \mathrm{O}(z^{n+k+1+\tau}), \tag{246}$$

where $\tau = 0$ for t and u variants corresponding to $m_n = c_n$, $\gamma = 0$, or $m_n = c_n(n+\beta)$, $\gamma = 0$, respectively, while $\tau = 1$ holds for the v variant corresponding to $m_n = c_{n+1}c_n/(c_n - c_{n+1}z)$, $\gamma = 1$, and for the $\tilde{t}$ variants corresponding to $m_n = c_{n+1}$, $\gamma = 1$, one obtains $\tau = 1$ if $\mathscr{T}$ is convex.

Proof. Using the identity

$$\mathscr{T}_n^{(k)}(\{\{f_n(z)\}\}, \{\{m_n z^{\gamma+n}\}\}) = f(z) + \mathscr{T}_n^{(k)}(\{\{f_n(z) - f(z)\}\}, \{\{m_n z^{\gamma+n}\}\}) \tag{247}$$

that follows from Theorem 5, we obtain after some easy algebra

$$\mathscr{T}_n^{(k)}(\{\{f_n(z)\}\}, \{\{m_n z^{\gamma+n}\}\}) - f(z) = z^{n+k+1} \frac{\sum_{\ell=0}^{\infty} z^\ell \sum_{j=0}^{k} (\lambda_{n,j}^{(k)}/m_{n+j}) c_{\ell+n+j+1}}{\sum_{j=0}^{k} (\lambda_{n,j}^{(k)}/m_{n+j}) z^{k-j}}. \tag{248}$$

This shows that the right-hand side is at least $\mathrm{O}(z^{n+k+1})$ since the denominator is $\mathrm{O}(1)$ due to $\lambda_{n,k}^{(k)} \neq 0$. For the $\tilde{t}$ variant, the term corresponding to $\ell = 0$ in the numerator is $\sum_{j=0}^{k} \lambda_{n,j}^{(k)} = \Pi_n^{(k)}(1)$ that vanishes for convex $\mathscr{T}$. For the v variant, that term is $\sum_{j=0}^{k} \lambda_{n,j}^{(k)}(c_{n+j} - c_{n+j+1}z)/c_{n+j}$ that simplifies to $(-z)\sum_{j=0}^{k} \lambda_{n,j}^{(k)} c_{n+j+1}/c_{n+j}$ for convex $\mathscr{T}$. This finishes the proof. □

7. Convergence acceleration results for Levin-type transformations

7.1. General results

We note that Germain-Bonne [23] developed a theory of the regularity and convergence acceleration properties of sequence transformations that was later extended by Weniger [84, Section 12; 88, Section 6] to sequence transformations that depend explicitly on n and on an auxiliary sequence of remainder estimates. The essential results of this theory apply to convergence acceleration of linearly convergent sequences. Of course, this theory can be applied to Levin-type sequence transformations. However, for the latter transformations, many results can be obtained more easily and also, one may obtain results of a general nature that are also applicable to other convergence types like logarithmic convergence. Thus, we are not going to use the Germain–Bonne–Weniger theory in the present article.

Here, we present some general convergence acceleration results for Levin-type sequence transformations that have a limiting transformation. The results, however, do not completely determine which transformation provides the best extrapolation results for a given problem sequence since the results are asymptotic in nature, but in practice, one is interested in obtaining good extrapolation results from as few members of the problem sequence as possible. Thus, it may well be that transformations with the same asymptotic behavior of the results perform rather differently in practice.

Nevertheless, the results presented below provide a first indication which results one may expect for large classes of Levin-type sequence transformations.

First, we present some results that show that the limiting transformation essentially determines for which sequences Levin-type sequence transformations are accelerative. The speed of convergence will be analyzed later.

Theorem 9. *Assume that the following asymptotic relations hold for large n:*

$$\lambda_{n,j}^{(k)} \sim \mathring{\lambda}_j^{(k)}, \quad \mathring{\lambda}_k^{(k)} \neq 0, \tag{249}$$

$$\frac{s_n - s}{\omega_n} \sim \sum_{\nu=1}^{A} c_\nu \zeta_\nu^n, \quad c_\nu \zeta_\nu \neq 0, \quad \mathring{\Pi}^{(k)}(\zeta_\nu) = 0, \tag{250}$$

$$\frac{\omega_{n+1}}{\omega_n} \sim \rho \neq 0, \quad \mathring{\Pi}^{(k)}(1/\rho) \neq 0. \tag{251}$$

Then, $\{\{\mathcal{T}_n^{(k)}\}\}$ accelerates $\{\{s_n\}\}$ to s, i.e., we have

$$\lim_{n\to\infty} \frac{\mathcal{T}_n^{(k)} - s}{s_n - s} = 0. \tag{252}$$

Proof. Rewriting

$$\frac{\mathcal{T}_n^{(k)} - s}{s_n - s} = \frac{\omega_n}{s_n - s} \frac{\sum_{j=0}^{k} \lambda_{n,j}^{(k)} (s_{n+j} - s)/\omega_{n+j}}{\sum_{j=0}^{k} \lambda_{n,j}^{(k)} \omega_n/\omega_{n+j}}. \tag{253}$$

one may perform the limit for $n \to \infty$ upon using the assumptions according to

$$\frac{\mathscr{T}_n^{(k)} - s}{s_n - s} \to \frac{\sum_{j=0}^{k} \overset{\circ}{\lambda}_j^{(k)} \sum_v c_v \zeta_v^{n+j}}{\sum_v c_v \zeta_v^n \sum_{j=0}^{k} \overset{\circ}{\lambda}_j^{(k)} \rho^{-j}} = \frac{\sum_v c_v \zeta_v^n \, \overset{\circ}{\Pi}^{(k)}(\zeta_v)}{\overset{\circ}{\Pi}^{(k)}(1/\rho) \sum_v c_v \zeta_v^n} = 0 \tag{254}$$

since $\omega_n/\omega_{n+j} \to \rho^{-j}$. $\square$

Thus, the zeroes ζ_v of the characteristic polynomial of the limiting transformation are of particular importance.

It should be noted that the above assumptions correspond to a more complicated convergence type than linear or logarithmic convergence if $|\zeta_1| = |\zeta_2| \geqslant |\zeta_3| \geqslant \cdots$. This is the case, for instance, for the $\mathscr{H}^{(k)}$ transformation where the limiting transformation has the characteristic polynomial $P^{(2k)}(\alpha)$ with k-fold zeroes at $\exp(\imath\alpha)$ and $\exp(-\imath\alpha)$. Another example is the $\mathscr{I}^{(k)}$ transformation where the limiting transformation has characteristic polynomials $Q^{(2k)}(\alpha)$ with zeroes at $\exp(\pm\imath\alpha)/\Theta_j$, $j = 0, \ldots, k-1$.

Specializing to $A = 1$ in Theorem 9, we obtain the following corollary:

Corollary 10. *Assume that the following asymptotic relations hold for large n:*

$$\lambda_{n,j}^{(k)} \sim \overset{\circ}{\lambda}_j^{(k)}, \quad \overset{\circ}{\lambda}_k^{(k)} \neq 0, \tag{255}$$

$$\frac{s_n - s}{\omega_n} \sim c q^n, \quad cq \neq 0, \quad \overset{\circ}{\Pi}^{(k)}(q) = 0, \tag{256}$$

$$\frac{\omega_{n+1}}{\omega_n} \sim \rho \neq 0, \quad \overset{\circ}{\Pi}^{(k)}(1/\rho) \neq 0. \tag{257}$$

Then, $\{\{\mathscr{T}_n^{(k)}\}\}$ *accelerates* $\{\{s_n\}\}$ *to s, i.e., we have*

$$\lim_{n\to\infty} \frac{\mathscr{T}_n^{(k)} - s}{s_n - s} = 0. \tag{258}$$

Note that the assumptions of Corollary 10 imply

$$\frac{s_{n+1} - s}{s_n - s} = \frac{s_{n+1} - s}{\omega_{n+1}} \frac{\omega_n}{s_n - s} \frac{\omega_{n+1}}{\omega_n} \sim \rho \frac{cq^{n+1}}{cq^n} = \rho q \tag{259}$$

and thus, Corollary 10 corresponds to linear convergence for $0 < |\rho q| < 1$ and to logarithmic convergence for $\rho q = 1$.

Many important sequence transformations have convex limiting transformations, i.e., the characteristic polynomials satisfy $\overset{\circ}{\Pi}^{(k)}(1) = 0$. In this case, they accelerate linear convergence. More exactly, we have the following corollary:

Corollary 11. *Assume that the following asymptotic relations hold for large n:*

$$\lambda_{n,j}^{(k)} \sim \overset{\circ}{\lambda}_j^{(k)}, \quad \overset{\circ}{\lambda}_k^{(k)} \neq 0, \tag{260}$$

$$\frac{s_n - s}{\omega_n} \sim c, \quad c \neq 0, \quad \overset{\circ}{\Pi}^{(k)}(1) = 0, \tag{261}$$

$$\frac{\omega_{n+1}}{\omega_n} \sim \rho \neq 0, \quad \mathring{\Pi}^{(k)}(1/\rho) \neq 0. \tag{262}$$

Then, $\{\{\mathscr{T}_n^{(k)}\}\}$ accelerates $\{\{s_n\}\}$ *to* s, *i.e., we have*

$$\lim_{n\to\infty} \frac{\mathscr{T}_n^{(k)} - s}{s_n - s} = 0. \tag{263}$$

Hence, any Levin-type sequence transformation with a convex limiting transformation accelerates linearly convergent sequences with

$$\lim_{n\to\infty} \frac{s_{n+1} - s}{s_n - s} = \rho, \quad 0 < |\rho| < 1 \tag{264}$$

such that $\mathring{\Pi}^{(k)}(1/\rho) \neq 0$ *for suitably chosen remainder estimates* ω_n *satisfying* $(s_n - s)/\omega_n \to c \neq 0$.

Proof. Specializing Corollary 10 to $q = 1$, it suffices to prove the last assertion. Here, the proof follows from the observation that $(s_{n+1} - s)/(s_n - s) \sim \rho$ and $(s_n - s)/\omega_n \sim c$ imply $\omega_{n+1}/\omega_n \sim \rho$ for large n in view of the assumptions. □

Note that Corollary 11 applies for instance to suitable variants of the Levin transformation, the ${}_p\mathbf{J}$ transformation and, more generally, of the $\mathscr{J}$ transformation. In particular, it applies to t, $\tilde{t}$, u and v variants, since in the case of linear convergence, one has $\Delta s_n/\Delta s_{n-1} \sim \rho$ which entails $(s_n - s)/\omega_n \sim c$ for all these variants by simple algebra.

Now, some results for the speed of convergence are given. Matos [59] presented convergence theorems for sequence transformations based on annihilation difference operators with characteristic polynomials with constants coefficients that are close in spirit to the theorems given below. However, it should be noted that the theorems presented here apply to large classes of Levin-type transformations that have a limiting transformation (the latter, of course, has a characteristic polynomial with constants coefficients).

Theorem 12. (C-1) *Suppose that for a Levin-type sequence transformation* $\mathscr{T}^{(k)}$ *of order* k *there is a limiting transformation* $\mathring{\mathscr{T}}^{(k)}$ *with characteristic polynomial* $\mathring{\Pi} \in \mathbb{P}^k$ *given by Eq.* (242) *where the multiplicities* m_ℓ *of the zeroes* $\zeta_\ell \neq 0$ *satisfy* $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_M$. *Let*

$$\lambda_{n,j}^{(k)} \frac{n^{m_1-1}}{(n+j)^{m_1-1}} \sim \mathring{\lambda}_j^{(k)} \left(\sum_{t=0}^{\infty} \frac{e_t^{(k)}}{(n+j)^t} \right), \quad e_0^{(k)} = 1 \tag{265}$$

for $n \to \infty$.

(C-2) *Assume that* $\{\{s_n\}\} \in \mathbb{S}^{\mathbb{K}}$ *and* $\{\{\omega_n\}\} \in \mathbb{O}^{\mathbb{K}}$. *Assume further that for* $n \to \infty$ *the asymptotic expansion*

$$\frac{s_n - s}{\omega_n} \sim \sum_{\ell=1}^{M} \zeta_\ell^n \sum_{r=0}^{\infty} c_{\ell,r} n^{-r} \tag{266}$$

holds, and put

$$r_\ell = \min\{r \in \mathbb{N}_0 \mid f_{\ell,r+m_1} \neq 0\}, \tag{267}$$

where

$$f_{\ell,v} = \sum_{r=0}^{v} e_{v-r}^{(k)} c_{\ell,r} \tag{268}$$

and

$$B_\ell = (-1)^{m_\ell} \frac{\mathrm{d}^{m_\ell}\, \mathring{\Pi}^{(k)}}{\mathrm{d}x^{m_\ell}}(\zeta_\ell) \tag{269}$$

for $\ell = 1,\ldots,M$.

(C-3) *Assume that the following limit exists and satisfies*

$$0 \neq \lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = \rho \notin \{\zeta_\ell^{-1} \,|\, \ell = 1,\ldots,M\}. \tag{270}$$

Then we have

$$\frac{\mathscr{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}) - s}{\omega_n} \sim \frac{\sum_{\ell=1}^{M} f_{\ell,r_\ell+m_1} \zeta_\ell^{n+m_\ell} \binom{r_\ell + m_\ell}{r_\ell} B_\ell / n^{r_\ell+m_\ell-m_1}}{\mathring{\Pi}^{(k)}(1/\rho)} \frac{1}{n^{2m_1}}. \tag{271}$$

Thus, $\{\{\mathscr{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\})\}\}$ *accelerates* $\{\{s_n\}\}$ *to* s *at least with order* $2m_1$, *i.e.*,

$$\frac{\mathscr{T}_n^{(k)} - s}{s_n - s} = \mathrm{O}(n^{-2m_1-\tau}), \quad \tau \geqslant 0 \tag{272}$$

if $c_{\ell,0} \neq 0$ *for all* ℓ.

Proof. We rewrite $\mathscr{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}) = \mathscr{T}_n^{(k)}$ as defined in Eq. (11) in the form

$$\mathscr{T}_n^{(k)} - s = \omega_n \frac{\sum_{j=0}^{k} \lambda_{n,j}^{(k)} (s_{n+j} - s)/\omega_{n+j}}{\sum_{j=0}^{k} \lambda_{n,j}^{(k)} \omega_n/\omega_{n+j}} \sim \omega_n \frac{\sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} \sum_{t=0}^{\infty} \frac{e_t^{(k)}}{(n+j)^t} \frac{(n+j)^{m_1-1}}{n^{m_1-1}} \frac{s_{n+j}-s}{\omega_{n+j}}}{\sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} \frac{1}{\rho^j}} \tag{273}$$

for large n where we used Eq. (265) in the numerator, and in the denominator the relation $\omega_n/\omega_{n+j} \to \rho^{-j}$ that follows by repeated application of Eq. (270). Insertion of (266) now yields

$$\mathscr{T}_n^{(k)} - s \sim \frac{\omega_n}{n^{m_1-1}\, \mathring{\Pi}^{(k)}(1/\rho)} \sum_{\ell=1}^{M} \sum_{r=0}^{\infty} f_{\ell,r+m_1} \sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} \frac{\zeta_\ell^{n+j}}{(n+j)^{r+1}}, \tag{274}$$

where Eq. (268) was used. Also the fact was used that $\mathscr{P}[\,\mathring{\Pi}^{(k)}]$ annihilates any linear combination of the solutions $\varphi_{n,\ell,j_\ell}^{(k)} = n^{j_\ell} \zeta_\ell^n$ with $\ell = 1,\ldots,M$ and $j_\ell = 0,\ldots,m_1 - 1$ of the recursion relation (244) since each ζ_ℓ is a zero with multiplicity exceeding $m_1 - 1$. Invoking Lemma C.1 given in Appendix C one obtains

$$\mathscr{T}_n^{(k)} - s \sim \frac{\omega_n}{n^{m_1-1}\, \mathring{\Pi}^{(k)}(1/\rho)} \sum_{\ell=1}^{M} \sum_{r=0}^{\infty} f_{\ell,r+m_1} \zeta_\ell^{n+m_\ell} \binom{r + m_\ell}{r} \frac{(-1)^{m_\ell}}{n^{r+m_\ell+1}} \frac{\mathrm{d}^{m_\ell}\, \mathring{\Pi}^{(k)}}{\mathrm{d}x^{m_\ell}}(\zeta_\ell). \tag{275}$$

The proof of Eq. (271) is completed taking leading terms in the sums over r. Since $s_n - s \sim \omega_n Z^n \sum_{\ell\in I} (\zeta_\ell/Z)^n c_{\ell,0}$ where $Z = \max\{|\zeta_\ell| \,|\ell = 1,\ldots,M\}$, and $I = \{\ell = 1,\ldots,M \,|\, Z = |\zeta_\ell|\}$, Eq. (272) is obtained where $\tau = \min\{r_\ell + m_\ell - m_1 \,|\, \ell \in I\}$. □

If $\omega_{n+1}/\omega_n \sim \rho$, where $\mathring{\Pi}^{(k)}(1/\rho) = 0$, i.e., if (C-3) of Theorem 12 does not hold, then the denominators vanish asymptotically. In this case, one has to investigate whether the numerators or the denominators vanish faster.

Theorem 13. *Assume that* (C-1) *and* (C-2) *of Theorem* 12 *hold.*
(C-3′) *Assume that for* $n \to \infty$ *the asymptotic relation*

$$\frac{\omega_{n+1}}{\omega_n} \sim \rho \exp(\varepsilon_n), \quad \rho \neq 0 \tag{276}$$

holds where

$$\frac{1}{\lambda!}\frac{\mathrm{d}^\lambda \mathring{\Pi}^{(k)}}{\mathrm{d}x^\lambda}(1/\rho) = \begin{cases} 0 & \text{for } \lambda = 0,\ldots,\mu-1 \\ C \neq 0 & \text{for } \lambda = \mu \end{cases} \tag{277}$$

and

$$\varepsilon_n \to 0, \quad \frac{\varepsilon_{n+1}}{\varepsilon_n} \to 1 \tag{278}$$

for large n. *Define* δ_n *via* $\exp(-\varepsilon_n) = 1 + \delta_n\rho$.
Then we have for large n

$$\frac{\mathscr{T}_n^{(k)}(\{\{s_n\}\},\{\{\omega_n\}\}) - s}{\omega_n} \sim \frac{\sum_{\ell=1}^{M} f_{\ell,r_\ell+m_1}\zeta_\ell^{n+m_\ell}\binom{r_\ell+m_\ell}{r_\ell}B_\ell/n^{r_\ell+m_\ell-m_1}}{C(\delta_n)^\mu}\frac{1}{n^{2m_1}}. \tag{279}$$

Proof. The proof proceeds as the proof of Theorem 12 but in the denominator we use

$$\sum_{j=0}^{k}\lambda_{n,j}^{(k)}\frac{\omega_n}{\omega_{n+j}} \sim C(\delta_n)^\mu \tag{280}$$

that follows from Lemma C.2 given in Appendix C. □

Thus, the effect of the sequence transformation in this case essentially depends on the question whether $(\delta_n)^{-\mu}n^{-2m_1}$ goes to 0 for large n or not. In many important cases like the Levin transformation and the ${}_p\mathbf{J}$ transformations, we have $M = 1$ and $m_1 = k$. We note that Theorem 11 becomes especially important in the case of logarithmic convergence since for instance for $M = 1$ one observes that $(s_{n+1} - s)/(s_n - s) \sim 1$ and $(s_n - s)/\omega_n \sim \zeta_1^n c_{1,0} \neq 0$ imply $\omega_{n+1}/\omega_n \sim 1/\zeta_1$ for large n such that the denominators vanish asymptotically. In this case, we have $\mu = m_1$ whence $(\delta_n)^{-\mu}n^{-2m_1} = \mathrm{O}(n^{-m_1})$ if $\delta_n = \mathrm{O}(1/n)$. This reduction of the speed of convergence of the acceleration process from $\mathrm{O}(n^{-2k})$ to $\mathrm{O}(n^{-k})$ in the case of logarithmic convergence is a generic behavior that is reflected in a number of theorems regarding convergence acceleration properties of Levin-type sequence transformations. Examples are Sidi's theorem for the Levin transformation given below (Theorem 15), and for the ${}_p\mathbf{J}$ transformation the Corollaries 18 and 19 given below, cf. also [84, Theorems 13.5, 13.9, 13.11, 13.12, 14.2].

The following theorem was given by Matos [59] where the proof may be found. To formulate it, we define that a sequence $\{\{u_n\}\}$ has *property M* if it satisfies

$$\frac{u_{n+1}}{u_n} \sim 1 + \frac{\alpha}{n} + r_n \quad \text{with } r_n = \mathrm{o}(1/n), \qquad \triangle^\ell r_n = \mathrm{o}(\triangle^\ell(1/n)) \quad \text{for } n \to \infty. \tag{281}$$

Theorem 14 (Matos [59, Theorem 13]). *Let $\{\{s_n\}\}$ be a sequence such that*

$$s_n - s = \omega_n(a_1 g_1^{(1)}(n) + \cdots + a_k g_1^{(k)}(n) + \rho_n) \tag{282}$$

with $g_1^{(j+1)}(n) = \mathrm{o}(g_1^{(j)}(n))$, $\rho_n = \mathrm{o}(g_1^{(k)}(n))$ for $n \to \infty$. Let us consider an operator L of the form (193) for which we know a basis of solutions $\{\{u_n^{(j)}\}\}$, $j = 1,\ldots,k$, and each one can be written as

$$u_n^{(j)} \sim \sum_{m=1}^{\infty} \alpha_m^{(j)} g_m^{(j)}(n), \qquad g_{m+1}^{(j)}(n) = \mathrm{o}(g_m^{(j)}(n)) \tag{283}$$

as $n \to \infty$ for all $m \in \mathbb{N}$ and $j = 1,\ldots,k$. Suppose that

$$\begin{aligned} &\text{(a) } g_2^{(j+1)}(n) = \mathrm{o}(g_2^{(j)}(n)) \text{ for } n \to \infty,\ j = 1,\ldots,k-1,\\ &\text{(b) } g_2^{(1)}(n) = \mathrm{o}(g_1^{(k)}(n)), \text{ and } \rho_n \sim K g_2^{(1)}(n) \quad \text{for } n \to \infty,\\ &\text{(c) } \{\{g_m^{(j)}(n)\}\} \text{ has property } M \text{ for}\\ &\quad m \in \mathbb{N},\ j = 1,\ldots,k. \end{aligned} \tag{284}$$

Then

1. *If $\{\{\omega_n\}\}$ satisfies $\lim_{n\to\infty} \omega_n/\omega_{n+1} = \lambda \neq 1$, the sequence transformation $\mathcal{T}_n^{(k+1)}$ corresponding to the operator L accelerates the convergence of $\{\{s_n\}\}$. Moreover, the acceleration can be measured by*

$$\frac{\mathcal{T}_n^{(k+1)} - s}{s_n - s} \sim C n^{-k} \frac{g_2^{(1)}(n)}{g_1^{(1)}(n)}, \quad n \to \infty. \tag{285}$$

2. *If $\{\{1/\omega_n\}\}$ has property M, then the speed of convergence of $\mathcal{T}_n^{(k+1)}$ can be measured by*

$$\frac{\mathcal{T}_n^{(k+1)} - s}{s_n - s} \sim C \frac{g_2^{(1)}(n)}{g_1^{(1)}(n)}, \quad n \to \infty. \tag{286}$$

7.2. Results for special cases

In the case that peculiar properties of a Levin-type sequence transformation are used, more stringent theorems can often be proved as regards convergence acceleration using this particular transformation.

In the case of the Levin transformation, Sidi proved the following theorem:

Theorem 15 (Sidi [76] and Brezwski and Redivo Zaglia [14, Theorem 2.32]). *If $s_n = s + \omega_n f_n$ where $f_n \sim \sum_{j=0}^{\infty} \beta_j/n^j$ with $\beta_0 \neq 0$ and $\omega_n \sim \sum_{j=0}^{\infty} \delta_j/n^{j+a}$ with $a > 0$, $\delta_0 \neq 0$ for $n \to \infty$ then, if $\beta_k \neq 0$*

$$\mathcal{L}_n^{(k)} - s \sim \frac{\delta_0 \beta_k}{\binom{-a}{k}} \cdot n^{-a-k} \quad (n \to \infty). \tag{287}$$

For the W algorithm and the $d^{(1)}$ transformation that may be regarded as direct generalizations of the Levin transformation, Sidi has obtained a large number of results. The interested reader is referred to the literature (see [77,78] and references therein).

Convergence results for the Levin transformation, the Drummond transformation and the Weniger transformations may be found in Section 13 of Weniger's report [84].

Results for the $\mathscr{J}$ transformation and in particular, for the ${}_p\mathbf{J}$ transformation are given in [39,40]. Here, we recall the following theorems:

Theorem 16. *Assume that the following holds*:

(A-0) *The sequence* $\{\{s_n\}\}$ *has the* (*anti*)*limit* s.

(A-1a) *For every* n, *the elements of the sequence* $\{\{\omega_n\}\}$ *are strictly alternating in sign and do not vanish.*

(A-1b) *For all* n *and* k, *the elements of the sequence* $\{\{\delta_n^{(k)}\}\} = \{\{\Delta r_n^{(k)}\}\}$ *are of the same sign and do not vanish.*

(A-2) *For all* $n \in \mathbb{N}_0$ *the ratio* $(s_n - s)/\omega_n$ *can be expressed as a series of the form*

$$\frac{s_n - s}{\omega_n} = c_0 + \sum_{j=1}^{\infty} c_j \sum_{n>n_1>n_2>\cdots>n_j} \delta_{n_1}^{(0)} \delta_{n_2}^{(1)} \cdots \delta_{n_j}^{(j-1)} \tag{288}$$

with $c_0 \neq 0$.

Then the following holds for $s_n^{(k)} = \mathscr{J}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\{\delta_n^{(k)}\}\})$:

(a) *The error* $s_n^{(k)} - s$ *satisfies*

$$s_n^{(k)} - s = \frac{b_n^{(k)}}{\nabla_n^{(k-1)} \nabla_n^{(k-2)} \cdots \nabla_n^{(0)} [1/\omega_n]} \tag{289}$$

with

$$b_n^{(k)} = c_k + \sum_{j=k+1}^{\infty} c_j \sum_{n>n_{k+1}>n_{k+2}>\cdots>n_j} \delta_{n_{k+1}}^{(k)} \delta_{n_{k+2}}^{(k+1)} \cdots \delta_{n_j}^{(j-1)}. \tag{290}$$

(b) *The error* $s_n^{(k)} - s$ *is bounded in magnitude according to*

$$|s_n^{(k)} - s| \leqslant |\omega_n b_n^{(k)} \delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}|. \tag{291}$$

(c) *For large* n *the estimate*

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}(\delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}) \tag{292}$$

holds if $b_n^{(k)} = \mathrm{O}(1)$ *and* $(s_n - s)/\omega_n = \mathrm{O}(1)$ *as* $n \to \infty$.

Theorem 17. *Define* $s_n^{(k)} = \mathscr{J}_n^{(k)}(\{\{s_n\}\}, \{\{\omega_n\}\}, \{\{\delta_n^{(k)}\}\})$ *and* $\omega_n^{(k)} = 1/D_n^{(k)}$ *where the* $D_n^{(k)}$ *are defined as in Eq.* (94). *Put* $e_n^{(k)} = 1 - \omega_{n+1}^{(k)}/\omega_n^{(k)}$ and $b_n^{(k)} = (s_n^{(k)} - s)/\omega_n^{(k)}$. *Assume that* (A-0) *of Theorem* 16 *holds and that the following conditions are satisfied*:

(B-1) *Assume that*

$$\lim_{n\to\infty} \frac{b_n^{(k)}}{b_n^{(0)}} = B_k \tag{293}$$

exists and is finite.

(B-2) *Assume that*

$$\Omega_k = \lim_{n\to\infty} \frac{\omega_{n+1}^{(k)}}{\omega_n^{(k)}} \neq 0 \tag{294}$$

and

$$F_k = \lim_{n\to\infty} \frac{\delta_{n+1}^{(k)}}{\delta_n^{(k)}} \neq 0 \tag{295}$$

exist for all $k \in \mathbb{N}_0$. *Hence the limits* $\Phi_k = \lim_{n\to\infty} \Phi_n^{(k)}$ (*cf. Eq.* (97)) *exist for all* $k \in \mathbb{N}_0$. *Then, the following holds*:

(a) *If* $\Omega_0 \notin \{\Phi_0 = 1, \Phi_1, \ldots, \Phi_{k-1}\}$, *then*

$$\lim_{n\to\infty} \frac{s_n^{(k)} - s}{s_n - s} \left\{ \prod_{l=0}^{k-1} \delta_n^{(l)} \right\}^{-1} = B_k \frac{[\Omega_0]^k}{\prod_{l=0}^{k-1} (\Phi_l - \Omega_0)} \tag{296}$$

and, hence,

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}(\delta_n^{(0)} \delta_n^{(1)} \cdots \delta_n^{(k-1)}) \tag{297}$$

holds in the limit $n \to \infty$.

(b) *If* $\Omega_l = 1$ *for* $l \in \{0, 1, 2, \ldots, k\}$ *then*

$$\lim_{n\to\infty} \frac{s_n^{(k)} - s}{s_n - s} \left\{ \prod_{l=0}^{k-1} \frac{\delta_n^{(l)}}{e_n^{(l)}} \right\}^{-1} = B_k \tag{298}$$

and, hence,

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}\left(\prod_{l=0}^{k-1} \frac{\delta_n^{(l)}}{e_n^{(l)}} \right) \tag{299}$$

holds in the limit $n \to \infty$.

This theorem has the following two corollaries for the ${}_p\mathbf{J}$ transformation [39]:

Corollary 18. *Assume that the following holds*:

(C-1) *Let* $\beta > 0$, $p \geqslant 1$ *and* $\delta_n^{(k)} = \triangle[(n+\beta+(p-1)k)^{-1}]$. *Thus, we deal with the* ${}_p\mathbf{J}$ *transformation and, hence, the equations* $F_k = \lim_{n\to\infty} \delta_{n+1}^{(k)}/\delta_n^{(k)} = 1$ *and* $\Phi_k = 1$ *hold for all* k.

(C-2) *Assumptions* (A-2) *of Theorem* 16 *and* (B-1) *of Theorem* 17 *are satisfied for the particular choice* (C-1) *for* $\delta_n^{(k)}$.

(C-3) *The limit* $\Omega_0 = \lim_{n\to\infty} \omega_{n+1}/\omega_n$ *exists, and it satisfies* $\Omega_0 \notin \{0, 1\}$. *Hence, all the limits* $\Omega_k = \lim_{n\to\infty} \omega_{n+1}^{(k)}/\omega_n^{(k)}$ *exist for* $k \in \mathbb{N}$ *exist and satisfy* $\Omega_k = \Omega_0$.

Then the transformation $s_n^{(k)} = {}_p\mathbf{J}_n^{(k)}(\beta, \{\{s_n\}\}, \{\{\omega_n\}\})$ *satisfies*

$$\lim_{n\to\infty} \frac{s_n^{(k)} - s}{s_n - s} \left\{ \prod_{l=0}^{k-1} \delta_n^{(l)} \right\}^{-1} = B_k \left\{ \frac{\Omega_0}{1 - \Omega_0} \right\}^k \tag{300}$$

and, hence,

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}((n+\beta)^{-2k}) \tag{301}$$

holds in the limit $n \to \infty$.

Note that Corollary 18 can be applied in the case of linear convergence because then $0 < |\Omega_0| < 1$ holds.

Corollary 18 allows to conclude that in the case of linear convergence, the ${}_p\mathbf{J}$ transformations should be superior to Wynn's epsilon algorithm [104]. Consider for instance the case that

$$s_n \sim s + \lambda^n n^\theta \sum_{n=0}^{\infty} c_j/n^j, \quad c_0 \neq 0, \quad n \to \infty \tag{302}$$

is an asymptotic expansion of the sequence elements s_n. Assuming $\lambda \neq 1$ and $\theta \notin \{0, 1, \ldots, k-1\}$ it follows that [102, p. 127; 84, p. 333, Eq. (13.4–7)]

$$\frac{\varepsilon_{2k}^{(n)} - s}{s_n - s} = \mathrm{O}(n^{-2k}), \quad n \to \infty. \tag{303}$$

This is the same order of convergence acceleration as in Eq. (301). But it should be noted that for the computation of $\varepsilon_{2k}^{(n)}$ the $2k+1$ sequence elements $\{s_n, \ldots, s_{n+2k}\}$ are required. But for the computation of ${}_p\mathbf{J}_n^{(k)}$ only the $k+1$ sequence elements $\{s_n, \ldots, s_{n+k}\}$ are required in the case of the t and u variants, and additionally s_{n+k+1} in the case of the $\tilde{t}$ variant. Again, this is similar to Levin-type accelerators [84, p. 333].

The following corollary applies to the case of logarithmic convergence:

Corollary 19. *Assume that the following holds*:

(D-1) *Let* $\beta > 0$, $p \geqslant 1$ *and* $\delta_n^{(k)} = \Delta[(n+\beta+(p-1)k)^{-1}]$. *Thus, we deal with the* ${}_p\mathbf{J}$ *transformation and, hence, the equations* $F_k = \lim_{n\to\infty} \delta_{n+1}^{(k)}/\delta_n^{(k)} = 1$ *and* $\Phi_k = 1$ *hold for all* k.

(D-2) *Assumptions* (A-2) *of Theorem* 16 *and* (B-1) *of Theorem* 15 *are satisfied for the particular choice* (C-1) *for* $\delta_n^{(k)}$.

(D-3) *Some constants* $a_l^{(j)}$, $j = 1, 2$, *exist such that*

$$e_n^{(l)} = 1 - \omega_{n+1}^{(l)}/\omega_n^{(l)} = \frac{a_l^{(1)}}{n+\beta} + \frac{a_l^{(2)}}{(n+\beta)^2} + \mathrm{O}((n+\beta)^{-3}) \tag{304}$$

holds for $l = 0$. *This implies that this equation, and hence,* $\Omega_l = 1$ *holds for* $l \in \{0, 1, 2, \ldots, k\}$. *Assume further that* $a_l^{(1)} \neq 0$ *for* $l \in \{0, 1, 2, \ldots, k-1\}$.

Then the transformation $s_n^{(k)} = {}_p\mathbf{J}_n^{(k)}(\beta, \{\{s_n\}\}, \{\{\omega_n\}\})$ *satisfies*

$$\lim_{n\to\infty} \frac{s_n^{(k)} - s}{s_n - s} \left\{ \prod_{l=0}^{k-1} \frac{\delta_n^{(l)}}{e_n^{(l)}} \right\}^{-1} = B_k \tag{305}$$

and, hence,

$$\frac{s_n^{(k)} - s}{s_n - s} = \mathrm{O}((n+\beta)^{-k}) \tag{306}$$

holds in the limit $n \to \infty$.

For convergence acceleration results regarding the $\mathscr{H}$ and $\mathscr{I}$ transformations, see [34,44].

8. Stability results for Levin-type transformations

8.1. General results

We remind the reader of the definition of the *stability indices* $\boldsymbol{\Gamma}_n^{(k)}(\mathscr{T})\geqslant 1$ as given in Eq. (76). We consider the sequence $\{\{\omega_n\}\}\in\mathbb{O}^{\mathbb{K}}$ as given. We call the transformation $\mathscr{T}$ *stable along the path* $\mathscr{P}=\{(n_\ell,k_\ell)\,|\,[n_\ell>n_{\ell-1}$ and $k_\ell\geqslant k_{\ell-1}]$ or $[n_\ell\geqslant n_{\ell-1}$ and $k_\ell>k_{\ell-1}]\}$ in the $\mathscr{T}$ table if the limit of its stability index along the path $\mathscr{P}$ exists and is bounded, i.e., if

$$\lim_{\ell\to\infty}\boldsymbol{\Gamma}_{n_\ell}^{(k_\ell)}(\mathscr{T})=\lim_{\ell\to\infty}\sum_{j=0}^{k_\ell}|\gamma_{n_\ell,j}(k_\ell)(\boldsymbol{\omega}_n)|<\infty, \tag{307}$$

where the $\gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n)$ are defined in Eq. (75). The transformation $\mathscr{T}$ is called *S-stable*, if it is stable along all paths $\mathscr{P}^{(k)}=\{(n,k)\,|\,n=0,1,\ldots\}$ for fixed k, i.e., along all columns in the $\mathscr{T}$ table.

The case of stability along diagonal paths is much more difficult to treat analytically unless Theorem 22 applies. Up to now it seems that such diagonal stability issues have only been analysed by Sidi for the case of the $d^{(1)}$ transformation (see [78] and references therein). We will treat only S-stability in the sequel.

The higher the stability index $\boldsymbol{\Gamma}(\mathscr{T})$ is, the smaller is the numerical stability of the transformation $\mathscr{T}$: If ε_j is the numerical error of s_j,

$$\varepsilon_j=s_j-\mathtt{fl}(s_j), \tag{308}$$

then the difference between the true value $\mathscr{T}_n^{(k)}$ and the numerically computed approximation $\mathtt{fl}(\mathscr{T}_n^{(k)})$ may be bounded according to

$$|\mathscr{T}_n^{(k)}-\mathtt{fl}(\mathscr{T}_n^{(k)})|\leqslant\boldsymbol{\Gamma}_n^{(k)}(\mathscr{T})\left(\max_{j\in\{0,1,\ldots,k\}}|\varepsilon_{n+j}|\right), \tag{309}$$

cf. also [78].

Theorem 20. *If the Levin-type sequence transformation* $\mathscr{T}^{(k)}$ *has a limiting transformation* $\mathring{\mathscr{T}}^{(k)}$ *with characteristic polynomial* $\mathring{\Pi}^{(k)}\in\mathbb{P}^{(k)}$ *for all* $k\in\mathbb{N}$, *and if* $\{\{\omega_n\}\}\in\mathbb{O}^{\mathbb{K}}$ *satisfies* $\omega_{n+1}/\omega_n\sim\rho\neq 0$ *for large* n *with* $\mathring{\Pi}^{(k)}(1/\rho)\neq 0$ *for all* $k\in\mathbb{N}$ *then the transformation* $\mathscr{T}$ *is S-stable. If additionally, the coefficients* $\mathring{\lambda}_j^{(k)}$ *of the characteristic polynomial alternate in sign, i.e., if* $\mathring{\lambda}_j^{(k)}=(-1)^j|\mathring{\lambda}_j^{(k)}|/\tau_k$ *with* $|\tau_k|=1$, *then the limits* $\mathring{\Gamma}^{(k)}(\mathscr{T})=\lim_{n\to\infty}\Gamma_n^{(k)}(\mathscr{T})$ *obey*

$$\mathring{\boldsymbol{\Gamma}}^{(k)}(\mathscr{T})=\tau_k\frac{\mathring{\Pi}^{(k)}(-1/|\rho|)}{|\mathring{\Pi}^{(k)}(1/\rho)|}. \tag{310}$$

Proof. We have for fixed k

$$\gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n)=\lambda_{n,j}^{(k)}\frac{\omega_n}{\omega_{n+j}}\left[\sum_{j'=0}^{k}\lambda_{n,j'}^{(k)}\frac{\omega_n}{\omega_{n+j'}}\right]^{-1}\sim\mathring{\lambda}_j^{(k)}\rho^{-j}\left[\sum_{j'=0}^{k}\mathring{\lambda}_{j'}^{(k)}\rho^{-j'}\right]^{-1}=\frac{\mathring{\lambda}_j^{(k)}\rho^{-j}}{\mathring{\Pi}^{(k)}(1/\rho)}, \tag{311}$$

whence

$$\lim_{n\to\infty} \boldsymbol{\Gamma}_n^{(k)}(\mathscr{T}) = \lim_{n\to\infty} \sum_{j=0}^{k} |\gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n)| = \frac{\sum_{j=0}^{k} |\mathring{\lambda}_j^{(k)}||\rho|^{-j}}{|\mathring{\Pi}^{(k)}(1/\rho)|} < \infty. \tag{312}$$

If the $\mathring{\lambda}_j^{(k)}$ alternate in sign, we obtain for these limits

$$\mathring{\boldsymbol{\Gamma}}^{(k)}(\mathscr{T}) = \tau_k \frac{\sum_{j=0}^{k} \mathring{\lambda}_j^{(k)}(-|\rho|)^{-j}}{|\mathring{\Pi}^{(k)}(1/\rho)|}. \tag{313}$$

This implies Eq. (310). □

Corollary 21. *Assume that the Levin-type sequence transformation $\mathscr{T}^{(k)}$ has a limiting transformation $\mathring{\mathscr{T}}^{(k)}$ with characteristic polynomial $\mathring{\Pi}^{(k)} \in \mathbb{P}^{(k)}$ and the coefficients $\mathring{\lambda}_j^{(k)}$ of the characteristic polynomial alternate in sign, i.e., if $\mathring{\lambda}_j^{(k)} = (-1)^j |\mathring{\lambda}_j^{(k)}|/\tau_k$ with $|\tau_k| = 1$ for all $k \in \mathbb{N}$. The sequence $\{\{\omega_n\}\} \in \mathbb{O}^{\mathbb{K}}$ is assumed to be alternating and to satisfy $\omega_{n+1}/\omega_n \sim \rho < 0$ for large n. Then the transformation $\mathscr{T}$ is S-stable. Additionally the limits are $\mathring{\boldsymbol{\Gamma}}^{(k)}(\mathscr{T}) = 1$.*

Proof. Since

$$\sum_{j'=0}^{k} \frac{\lambda_{n,j'}^{(k)}}{\tau_k^{-1}} \frac{\omega_n}{\omega_{n+j'}} \sim \frac{\mathring{\Pi}^{(k)}(1/\rho)}{\tau_k^{-1}} = \sum_{j'=0}^{k} \frac{\mathring{\lambda}_{j'}^{(k)}}{\tau_k^{-1}}(-1)^{j'}|\rho|^{-j'}$$
$$= \sum_{j'=0}^{k} |\mathring{\lambda}_{j'}^{(k)}||\rho|^{-j'} \geqslant |\mathring{\lambda}_k^{(k)}|/|\rho|^k > 0. \tag{314}$$

$1/\rho$ cannot be a zero of $\mathring{\Pi}^{(k)}$. Then, Theorem 20 entails that $\mathscr{T}$ is S-stable. Furthermore, Eq. (310) is applicable and yields $\mathring{\boldsymbol{\Gamma}}^{(k)}(\mathscr{T}) = 1$. □

This result can be improved if all the coefficients $\lambda_{n,j}^{(k)}$ are alternating:

Theorem 22. *Assume that the Levin-type sequence transformation $\mathscr{T}^{(k)}$ has a characteristic polynomials $\Pi_n^{(k)} \in \mathbb{P}^{(k)}$ with alternating coefficients $\lambda_{n,j}^{(k)}$ i.e., $\lambda_{n,j}^{(k)} = (-1)^j|\lambda_{n,j}^{(k)}|/\tau_k$ with $|\tau_k| = 1$ for all $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. The sequence $\{\{\omega_n\}\} \in \mathbb{O}^{\mathbb{K}}$ is assumed to be alternating and to satisfy $\omega_{n+1}/\omega_n < 0$ for all $n \in \mathbb{N}_0$. Then we have $\boldsymbol{\Gamma}_n^{(k)}(\mathscr{T}) = 1$. Hence, the transformation $\mathscr{T}$ is stable along all paths for such remainder estimates.*

Proof. We have for fixed n and k

$$\gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n) = \frac{\lambda_{n,j}^{(k)}\omega_n/\omega_{n+j}}{\sum_{j'=0}^{k} \lambda_{n,j'}^{(k)}\omega_n/\omega_{n+j'}} = \frac{\lambda_{n,j}^{(k)}\tau_k(-1)^j|\omega_n/\omega_{n+j}|}{\sum_{j'=0}^{k} \lambda_{n,j'}^{(k)}\tau_k(-1)^{j'}\,|\omega_n/\omega_{n+j'}|} = \frac{|\lambda_{n,j}^{(k)}||\omega_n/\omega_{n+j}|}{\sum_{j'=0}^{k} |\lambda_{n,j'}^{(k)}|\,|\omega_n/\omega_{n+j'}|} \geqslant 0. \tag{315}$$

Note that the denominators cannot vanish and are bounded from below by $|\lambda_{n,k}^{(k)}\omega_n/\omega_{n+k}| > 0$. Hence, we have $\gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n) = |\gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n)|$ and consequently, $\boldsymbol{\Gamma}_n^{(k)}(\mathscr{T}) = 1$ since $\sum_{j=0}^{k} \gamma_{n,j}^{(k)}(\boldsymbol{\omega}_n) = 1$ according to Eq. (75). □

8.2. Results for special cases

Here, we collect some special results on the stability of various Levin-type sequence transformations that have been reported in [46] and generalize some results of Sidi on the S-stability of the $d^{(1)}$ transformation.

Theorem 23. *If the sequence ω_{n+1}/ω_n possesses a limit according to*

$$\lim_{n\to\infty} \omega_{n+1}/\omega_n = \rho \neq 0 \tag{316}$$

and if $\rho \notin \{1, \Phi_1, \ldots, \Phi_k, \ldots\}$ such that the limiting transformation exists, the $\mathscr{J}$ transformation is S-stable with the same limiting stability indices as the transformation $\mathring{\mathscr{J}}$, i.e., we have

$$\lim_{n\to\infty} \boldsymbol{\Gamma}_n^{(k)} = \frac{\sum_{j=0}^{k} |\lambda_j^{(k)} \rho^{k-j}|}{\prod_{j'=0}^{k-1} |\Phi_{j'} - \rho|} < \infty. \tag{317}$$

If all Φ_k are positive then

$$\lim_{n\to\infty} \boldsymbol{\Gamma}_n^{(k)} = \prod_{j=0}^{k-1} \frac{\Phi_j + |\rho|}{|\Phi_j - \rho|} < \infty \tag{318}$$

holds.

As corollaries, we get the following results

Corollary 24. *If the sequence ω_{n+1}/ω_n possesses a limit according to*

$$\lim_{n\to\infty} \omega_{n+1}/\omega_n = \rho \notin \{0, 1\}, \tag{319}$$

the ${}_p\mathbf{J}$ transformation for $p > 1$ and $\beta > 0$ is S-stable and we have

$$\lim_{n\to\infty} \boldsymbol{\Gamma}_n^{(k)} = \frac{\sum_{j=0}^{k} \binom{k}{j} |\rho^{k-j}|}{|1-\rho|^k} = \frac{(1+|\rho|)^k}{|1-\rho|^k} < \infty. \tag{320}$$

Corollary 25. *If the sequence ω_{n+1}/ω_n possesses a limit according to*

$$\lim_{n\to\infty} \omega_{n+1}/\omega_n = \rho \notin \{0, 1\}, \tag{321}$$

the Weniger $\mathscr{S}$ transformation [84, Section 8] *for $\beta > 0$ is S-stable and we have*

$$\lim_{n\to\infty} \boldsymbol{\Gamma}_n^{(k)}(\mathscr{S}) = \frac{\sum_{j=0}^{k} \binom{k}{j} |\rho^{k-j}|}{|1-\rho|^k} = \frac{(1+|\rho|)^k}{|1-\rho|^k} < \infty. \tag{322}$$

Corollary 26. *If the sequence* ω_{n+1}/ω_n *possesses a limit according to*

$$\lim_{n\to\infty} \omega_{n+1}/\omega_n = \rho \notin \{0,1\}, \tag{323}$$

the Levin $\mathscr{L}$ *transformation* [53, 84] *is S-stable and we have*

$$\lim_{n\to\infty} \boldsymbol{\Gamma}_n^{(k)}(\mathscr{L}) = \frac{\sum_{j=0}^{k} \binom{k}{j} |\rho^{k-j}|}{|1-\rho|^k} = \frac{(1+|\rho|)^k}{|1-\rho|^k} < \infty. \tag{324}$$

Corollary 27. *Assume that the elements of the sequence* $\{t_n\}_{n\in\mathbb{N}}$ *satisfy* $t_n \neq 0$ *for all n and* $t_n \neq t_{n'}$ *for all* $n \neq n'$. *If the sequence* t_{n+1}/t_n *possesses a limit*

$$\lim_{n\to\infty} t_{n+1}/t_n = \tau \quad \text{with } 0 < \tau < 1 \tag{325}$$

and if the sequence ω_{n+1}/ω_n *possesses a limit according to*

$$\lim_{n\to\infty} \omega_{n+1}/\omega_n = \rho \notin \{0, 1, \tau^{-1}, \ldots, \tau^{-k}, \ldots\}, \tag{326}$$

then the generalized Richardson extrapolation process $\mathscr{R}$ *introduced by Sidi* [73] *that is identical to the* $\mathscr{J}$ *transformation with* $\delta_n^{(k)} = t_n - t_{n+k+1}$ *as shown in* [36], *i.e., the W algorithm is S-stable and we have*

$$\lim_{n\to\infty} \boldsymbol{\Gamma}_n^{(k)}(\mathscr{R}) = \frac{\sum_{j=0}^{k} |\tilde{\lambda}_j^{(k)} \rho^{k-j}|}{\prod_{j'=0}^{k-1} |\tau^{-j'} - \rho|} = \prod_{j'=0}^{k-1} \frac{1 + \tau^{j'}|\rho|}{|1 - \tau^{j'}\rho|} < \infty. \tag{327}$$

Here

$$\tilde{\lambda}_j^{(k)} = (-1)^{k-j} \sum_{\substack{j_0 + j_1 + \ldots + j_{k-1} = j, \\ j_0 \in \{0,1\}, \ldots, j_{k-1} \in \{0,1\}}} \prod_{m=0}^{k-1} (\tau)^{-m\, j_m}, \tag{328}$$

such that

$$\sum_{j=0}^{k} \tilde{\lambda}_j^{(k)} \rho^{k-j} = \prod_{j=0}^{k-1} (\tau^{-j} - \rho) = \tau^{-k(k-1)/2} \prod_{j=0}^{k-1} (1 - \tau^j \rho). \tag{329}$$

Note that the preceding corollary is essentially the same as a result of Sidi [78, Theorem 2.2] that now appears as a special case of the more general Theorem 23 that applies to a much wider class of sequence transformations. As noted above, Sidi has also derived conditions under which the $d^{(1)}$ transformation is stable along the paths $\mathscr{P}_n = \{(n,k) | k = 0, 1, \ldots\}$ for fixed n. For details and more references see [78]. Analogous work for the $\mathscr{J}$ transformation is in progress.

An efficient algorithm for the computation of the stability index of the $\mathscr{J}$ transformation can be given in the case $\delta_n^{(k)} > 0$. Since the $\mathscr{J}$ transformation is invariant under $\delta_n^{(k)} \to \alpha^{(k)}\delta_n^{(k)}$ for any $\alpha^{(k)} \neq 0$ according to Homeier [36, Theorem 4], $\delta_n^{(k)} > 0$ can always be achieved if for given k, all $\delta_n^{(k)}$ have the same sign. This is the case, for instance, for the ${}_p\mathbf{J}$ transformation [36,39].

Theorem 28. *Define*

$$F_n^{(0)} = (-1)^n |D_n^{(0)}|, \qquad F_n^{(k+1)} = (F_{n+1}^{(k)} - F_n^{(k)})/\delta_n^{(k)} \tag{330}$$

and $\hat{F}_n^{(0)} = F_n^{(0)}$, $\hat{F}_n^{(k)} = (\delta_n^{(0)} \cdots \delta_n^{(k-1)}) F_n^{(k)}$. *If all* $\delta_n^{(k)} > 0$ *then*

1. $F_n^{(k)} = (-1)^{n+k} |F_n^{(k)}|$,
2. $\lambda_{n,j}^{(k)} = (-1)^{j+k} |\lambda_{n,j}^{(k)}|$, *and*
3.

$$\boldsymbol{\Gamma}_n^{(k)} = \frac{|\hat{F}_n^{(k)}|}{|\hat{D}_n^{(k)}|} = \frac{|F_n^{(k)}|}{|D_n^{(k)}|}. \tag{331}$$

This generalizes Sidi's method for the computation of stability indices [78] to a larger class of sequence transformations.

9. Application of Levin-type sequence transformations

9.1. Practical guidelines

Here, we address shortly the following questions:

When should one try to use sequence transformations? One can only hope for good convergence acceleration, extrapolation, or summation results if (a) the s_n have some asymptotic structure for large n and are not erratic or random, (b) a sufficiently large number of decimal digits is available. Many problems can be successfully tackled if 13–15 digits are available but some require a much larger number of digits in order to overcome some inevitable rounding errors, especially for the acceleration of logarithmically convergent sequences. The asymptotic information that is required for a successful extrapolation is often hidden in the last digits of the problem data.

How should the transformations be applied? The recommended mode of application is that one computes the highest possible order k of the transformation from the data. In the case of triangular recursive schemes like that of the $\mathscr{J}$ transformation and the Levin transformation, this means that one computes as transformed sequence $\{\mathscr{T}_0^{(n)}\}$. For L-shaped recursive schemes as in the case of the $\mathscr{H}$, $\mathscr{I}$, and $\mathscr{K}$ transformations, one usually computes as transformed sequence $\{\{\mathscr{T}_{n-2[\![n/2]\!]}^{[\![n/2]\!]}\}\}$. The error ε of the current estimate can usually be approximated a posteriori using sums of magnitudes of differences of a few entries of the $\mathscr{T}$ table, e.g.,

$$\varepsilon \approx |\mathscr{T}_1^{(n)} - \mathscr{T}_0^{(n)}| + |\mathscr{T}_0^{(n-1)} - \mathscr{T}_0^{(n)}| \tag{332}$$

for transformations with triangular recursive schemes. Such a simple approach works surprisingly well in practice. The loss of decimal digits can be estimated computing stability indices. An example is given below.

What happens if one of the denominator vanishes? The occurrence of zeroes in the D table for specific combinations of n and k is usually no problem since the recurrences for numerators and denominators still work in this case. Thus, no special devices are required to jump over such singular points in the $\mathscr{T}$ table.

Which transformation and which variant should be chosen? This depends on the type of convergence of the problem sequence. For linearly convergent sequences, t, $\tilde{t}$, u and v variants of the Levin transformation, or the ${}_p\mathbf{J}$ transformation, especially the ${}_2\mathbf{J}$ transformation are usually a good choice [39] as long as one is not too close to a singularity or to a logarithmically convergent problem. Especially well behaved is usually the application to alternating series since then, the stability is very good as discussed above. For the summation of alternating divergent sequences and series, usually the t and the $\tilde{t}$ variants of the Levin transformation, the ${}_2\mathbf{J}$ and the Weniger $\mathscr{S}$ and $\mathscr{M}$ transformations provide often surprisingly accurate results. In the case of logarithmic convergence, t and $\tilde{t}$ variants become useless, and the order of acceleration is dropping from $2k$ to k when the transformation is used columnwise. If a Kummer-related series is available (cf. Section 2.2.1), then K and lu variants leading to linear sequence transformations can be efficient [50]. Similarly, linear variants can be based on some good asymptotic estimates ${}^{\mathrm{asy}}\omega_n$, that have to be obtained via a separate analysis [50]. In the case of logarithmcic convergence, it pays to consider special devices like using subsequences $\{\{s_{\xi_n}\}\}$ where the ξ_n grow exponentially like $\xi_n = [\![\sigma\xi_{n-1}]\!] + 1$ like in the d transformations. This choice can be also used in combination with the $\mathscr{F}$ transformation. Alternatively, one can use some other transformations like the condensation transformation [51,65] or interpolation to generate a linearly convergent sequence [48], before applying an usually nonlinear sequence transformation. A somewhat different approach is possible if one can obtain a few terms a_n with large n easily [47].

What to do near a singularity? When extrapolating power series or, more generally, sequences depending on certain parameters, quite often extrapolation becomes difficult near the singularities of the limit function. In the case of linear convergence, one can often transform to a problem with a larger distance to the singularity: If Eq. (28) holds, then the subsequence $\{\{s_{\tau n}\}\}$ satisfies

$$\lim_{n\to\infty} (s_{\tau(n+1)} - s)/(s_{\tau n} - s) = \rho^{\tau}. \tag{333}$$

This is a method of Sidi that has can, however, be applied to large classes of sequence transformations [46].

What to do for more complicated convergence type? Here, one should try to rewrite the problem sequence as a sum of sequences with more simple convergence behavior. Then, nonlinear sequence transformations are used to extrapolate each of these simpler series, and to sum the extrapolation results to obtain an estimate for the original problem. This is for instance often possible for (generalized) Fourier series where it leads to complex series that may asymptotically be regarded as power series. For details, the reader is referred to the literature [14,35,40–45,77]. If this approach is not possible one is forced to use more complicated sequence transformations like the $d^{(m)}$ transformations or the (generalized) $\mathscr{H}$ transformation. These more complicated sequence transformations, however, do require more numerical effort to achieve a desired accuracy.

9.2. Numerical examples

In Table 3, we present results of the application of certain variants of the $\mathscr{F}$ transformation and the W algorithm to the series

$$S(z,a) = 1 + \sum_{j=1}^{\infty} z^j \prod_{\ell=0}^{j-1} \frac{1}{\ln(a+\ell)} \tag{334}$$

Table 3
Comparison of the $\mathscr{F}$ transformation and the W algorithm for series (334)[a]

n	A_n	B_n	C_n	D_n
14	13.16	13.65	7.65	11.13
16	15.46	15.51	9.43	12.77
18	18.01	17.84	11.25	14.43
20	21.18	20.39	13.10	16.12
22	23.06	23.19	14.98	17.81
24	25.31	26.35	16.89	19.53
26	27.87	28.17	18.83	21.26
28	30.83	30.59	20.78	23.00
30	33.31	33.19	22.76	24.76
n	E_n	F_n	G_n	H_n
14	14.07	13.18	9.75	10.47
16	15.67	15.49	11.59	12.05
18	17.94	18.02	13.46	13.66
20	20.48	20.85	15.37	15.29
22	23.51	23.61	17.30	16.95
24	25.66	25.63	19.25	18.62
26	27.89	28.06	21.23	20.31
28	30.46	30.67	23.22	22.02
29	31.82	32.20	24.23	22.89
30	33.43	33.45	25.24	23.75

[a]Plotted is the negative decadic logarithm of the relative error.
A_n: $\mathscr{F}_0^{(n)}(\{\{S_n(z,a)\}\},\{\{(2+\ln(n+a))\,\triangle\, S_n(z,a)\}\},\{\{1+\ln(n+a)\}\})$,
B_n: $W_0^{(n)}(\{\{S_n(z,a)\}\},\{\{(2+\ln(n+a))\,\triangle\, S_n(z,a)\}\},\{\{1/(1+\ln(n+a))\}\})$,
C_n: $\mathscr{F}_0^{(n)}(\{\{S_n(z,a)\}\},\{\{(n+1)\,\triangle\, S_n(z,a)\}\},\{\{1+n+a\}\})$,
D_n: $W_0^{(n)}(\{\{S_n(z,a)\}\},\{\{(n+1)\,\triangle\, S_n(z,a)\}\},\{\{1/(1+n+a)\}\})$,
E_n: $\mathscr{F}_0^{(n)}(\{\{S_n(z,a)\}\},\{\{\triangle S_n(z,a)\}\},\{\{1+\ln(n+a)\}\})$,
F_n: $W_0^{(n)}(\{\{S_n(z,a)\}\},\{\{\triangle S_n(z,a)\}\},\{\{1/(1+\ln(n+a))\}\})$,
G_n: $\mathscr{F}_0^{(n)}(\{\{S_n(z,a)\}\},\{\{\triangle S_n(z,a)\}\},\{\{1+n+a\}\})$,
H_n: $W_0^{(n)}(\{\{S_n(z,a)\}\},\{\{\triangle S_n(z,a)\}\},\{\{1/(1+n+a)\}\})$.

with partial sums

$$S_n(z,a) = 1 + \sum_{j=1}^{n} z^j \prod_{\ell=0}^{j-1} \frac{1}{\ln(a+\ell)} \tag{335}$$

for $z = 1.2$ and $a = 1.01$. Since the terms a_j satisfy $a_{j+1}/a_j = z/\ln(a+j)$, the ratio test reveals that $S(z,a)$ converges for all z and, hence, represents an analytic function. Nevertheless, only for $j \geqslant -a + \exp(|z|)$, the ratio of the terms becomes less than unity in absolute value. Hence, for larger z the series converges rather slowly.

It should be noted that for cases C_n and G_n, the $\mathscr{F}$ transformation is identical to the Weniger transformation $\mathscr{S}$, i.e., to the ${}_3\mathbf{J}$ transformation, and for cases C_n and H_n the W algorithm is identical to the Levin transformation. In the upper part of the table, we use u-type remainder estimates while

Table 4
Acceleration of $\zeta(-1/10+10\mathrm{i},1,95/100)$ with the $\mathcal{J}$ transformation[a]

n	A_n	B_n	C_n	D_n	E_n	F_n	G_n	H_n
10	2.59e − 05	3.46e + 01	2.11e − 05	4.67e + 01	1.84e − 05	4.14e + 01	2.63e − 05	3.90e + 01
20	1.72e − 05	6.45e + 05	2.53e − 05	5.53e + 07	1.38e − 04	3.40e + 09	1.94e − 05	2.47e + 06
30	2.88e − 05	3.52e + 10	8.70e − 06	2.31e + 14	8.85e − 05	1.22e + 17	2.02e − 05	6.03e + 11
40	4.68e − 06	1.85e + 15	8.43e − 08	1.27e + 20	4.06e − 06	2.78e + 23	1.50e − 06	9.27e + 16
42	2.59e − 06	1.46e + 16	2.61e − 08	1.51e + 21	2.01e − 06	4.70e + 24	6.64e − 07	8.37e + 17
44	1.33e − 06	1.10e + 17	7.62e − 09	1.76e + 22	1.73e − 06	7.85e + 25	2.76e − 07	7.24e + 18
46	6.46e − 07	8.00e + 17	1.80e − 09	2.02e + 23	1.31e − 05	1.30e + 27	1.09e − 07	6.08e + 19
48	2.97e − 07	5.62e + 18	1.07e − 08	2.29e + 24	1.52e − 04	2.12e + 28	4.16e − 08	5.00e + 20
50	1.31e − 07	3.86e + 19	1.51e − 07	2.56e + 25	1.66e − 03	3.43e + 29	1.54e − 08	4.05e + 21

[a] A_n: relative error of ${}_1\mathbf{J}_0^{(n)}(1,\{\{s_n\}\},\{\{(n+1)(s_n-s_{n-1})\}\})$, B_n: stability index of ${}_1\mathbf{J}_0^{(n)}(1,\{\{s_n\}\},\{\{(n+1)(s_n-s_{n-1})\}\})$, C_n: relative error of ${}_2\mathbf{J}_0^{(n)}(1,\{\{s_n\}\},\{\{(n+1)(s_n-s_{n-1})\}\})$, D_n: stability index of ${}_2\mathbf{J}_0^{(n)}(1,\{\{s_n\}\},\{\{(n+1)(s_n-s_{n-1})\}\})$, E_n: relative error of ${}_3\mathbf{J}_0^{(n)}(1,\{\{s_n\}\},\{\{(n+1)(s_n-s_{n-1})\}\})$, F_n: stability index of ${}_3\mathbf{J}_0^{(n)}(1,\{\{s_n\}\},\{\{(n+1)(s_n-s_{n-1})\}\})$, G_n: relative error of $\mathcal{J}_0^{(n)}(\{\{s_n\}\},\{\{(n+1)(s_n-s_{n-1})\}\},\{1/(n+1)-1/(n+k+2)\})$, H_n: Stability index of $\mathcal{J}_0^{(n)}(\{\{s_n\}\},\{\{(n+1)(s_n-s_{n-1})\}\},\{1/(n+1)-1/(n+k+2)\})$.

in the lower part, we use $\tilde{t}$ variants. It is seen that the choices $x_n = 1+\ln(a+n)$ for the $\mathcal{F}$ transformation and $t_n = 1/(1+\ln(a+n))$ for the W algorithm perform for both variants nearly identical (columns A_n, B_n, E_n and F_n) and are superior to the choices $x_n = 1+n+a$ and $t_n = 1/(1+n+a)$, respectively, that correspond to the Weniger and the Levin transformation as noted above. For the latter two transformations, the Weniger ${}^{\tilde{t}}\mathcal{S}$ transformation is slightly superior the ${}^{\tilde{t}}\mathcal{L}$ transformation for this particular example (columns G_n vs. H_n) while the situation is reversed for the u-type variants displayed in colums C_n and D_n.

The next example is taken from [46], namely the "inflated Riemann ζ function", i.e., the series

$$\zeta(\varepsilon,1,q) = \sum_{j=0}^{\infty} \frac{q^j}{(j+1)^{\varepsilon}}, \tag{336}$$

that is a special case of the Lerch zeta function $\zeta(s,b,z)$ (cf. [30, p. 142, Eq. (6.9.7); 20, Section 1.11]). The partial sums are defined as

$$s_n = \sum_{j=0}^{n} \frac{q^j}{(j+1)^{\varepsilon}}. \tag{337}$$

The series converges linearly for $0<|q|<1$ for any complex ε. In fact, we have in this case $\rho = \lim_{n\to\infty}(s_{n+1}-s)/(s_n-s) = q$. We choose $q=0.95$ and $\varepsilon = -0.1+10\mathrm{i}$. Note that for this value of ε, there is a singularity of $\zeta(\varepsilon,1,q)$ at $q=1$ where the defining series diverges since $\Re(\varepsilon)<1$.

The results of applying u variants of the ${}_p\mathbf{J}$ transformation with $p=1,2,3$ and of the Levin transformation to the sequence of partial sums is displayed in Table 4. For each of these four variants of the $\mathcal{J}$ transformation, we give the relative error and the stability index. The true value of the series (that is used to compute the errors) was computed using a more accurate method described below. It is seen that the ${}_2\mathbf{J}$ transformation achieves the best results. The attainable accuracy for this transformation is limited to about 9 decimal digits by the fact that the stability index displayed in the column D_n of Table 4 grows relatively fast. Note that for $n=46$, the number of digits (as given by

Table 5
Acceleration of $\zeta(-1/10+10\mathrm{i},1,95/100)$ with the $\mathcal{J}$ transformation ($\tau=10$)[a]

n	A_n	B_n	C_n	D_n	E_n	F_n	G_n	H_n
10	2.10e − 05	2.08e + 01	8.17e − 06	3.89e + 01	1.85e − 05	5.10e + 01	1.39e − 05	2.52e + 01
12	2.49e − 06	8.69e + 01	1.43e − 07	3.03e + 02	9.47e − 06	8.98e + 02	1.29e − 06	1.26e + 02
14	1.93e − 07	3.11e + 02	5.98e − 09	1.46e + 03	8.24e − 07	4.24e + 03	6.86e − 08	5.08e + 02
16	1.11e − 08	9.82e + 02	2.02e − 11	6.02e + 03	6.34e − 08	2.09e + 04	2.57e − 09	1.77e + 03
18	5.33e − 10	2.87e + 03	1.57e − 12	2.29e + 04	4.08e − 09	9.52e + 04	7.81e − 11	5.66e + 03
20	2.24e − 11	7.96e + 03	4.15e − 14	8.26e + 04	2.31e − 10	4.12e + 05	2.07e − 12	1.73e + 04
22	8.60e − 13	2.14e + 04	8.13e − 16	2.89e + 05	1.16e − 11	1.73e + 06	4.95e − 14	5.08e + 04
24	3.07e − 14	5.61e + 04	1.67e − 17	9.87e + 05	5.17e − 13	7.07e + 06	1.10e − 15	1.46e + 05
26	1.04e − 15	1.45e + 05	3.38e − 19	3.31e + 06	1.87e − 14	2.84e + 07	2.33e − 17	4.14e + 05
28	3.36e − 17	3.69e + 05	6.40e − 21	1.10e + 07	3.81e − 16	1.13e + 08	4.71e − 19	1.16e + 06
30	1.05e − 18	9.30e + 05	1.15e − 22	3.59e + 07	1.91e − 17	4.43e + 08	9.19e − 21	3.19e + 06

[a] A_n: relative error of ${}_1\mathbf{J}_0^{(n)}(1,\{\{s_{10\,n}\}\},\{\{(10\,n+1)(s_{10\,n}-s_{10\,n-1})\}\})$, B_n: stability index of ${}_1\mathbf{J}_0^{(n)}(1,\{\{s_{10\,n}\}\},\{\{(10\,n+1)(s_{10\,n}-s_{10\,n-1})\}\})$, C_n: relative error of ${}_2\mathbf{J}_0^{(n)}(1,\{\{s_{10\,n}\}\},\{\{(10\,n+1)(s_{10\,n}-s_{10\,n-1})\}\})$, D_n: stability index of ${}_2\mathbf{J}_0^{(n)}(1,\{\{s_{10\,n}\}\},\{\{(10\,n+1)(s_{10\,n}-s_{10\,n-1})\}\})$, E_n: relative error of ${}_3\mathbf{J}_0^{(n)}(1,\{\{s_{10\,n}\}\},\{\{(10\,n+1)(s_{10\,n}-s_{10\,n-1})\}\})$, F_n: stability index of ${}_3\mathbf{J}_0^{(n)}(1,\{\{s_{10\,n}\}\},\{\{(10\,n+1)(s_{10\,n}-s_{10\,n-1})\}\})$, G_n: relative error of $\mathcal{J}_0^{(n)}(\{\{s_{10\,n}\}\},\{\{(10\,n+1)(s_{10\,n}-s_{10\,n-1})\}\},\{1/(10\,n+10)-1/(10\,n+10\,k+10)\})$ H_n: Stability index of H_n: $\mathcal{J}_0^{(n)}(\{\{s_{10\,n}\}\},\{\{(10\,n+1)(s_{10\,n}-s_{10\,n-1})\}\},\{1/(n+1)-1/(n+k+2)\})$.

the negative decadic logarithm of the relative error) and the decadic logarithm of the stability index sum up to approximately 32 which corresponds to the maximal number of decimal digits that could be achieved in the run. Since the stability index increases with n, indicating decreasing stability, it is clear that for higher values of n the accuracy will be lower.

The magnitude of the stability index is largely controlled by the value of ρ, compare Corollary 24. If one can treat a related sequence with a smaller value of ρ, the stability index will be smaller and thus, the stability of the extrapolation will be greater.

Such a related sequence is given by putting $\check{s}_\ell = s_{\xi_\ell}$ for $\ell \in \mathbb{N}_0$, where the sequence ξ_ℓ is a monotonously increasing sequence of nonnegative integers. In the case of linear convergent sequences, the choice $\xi_\ell = \tau\ell$ with $\tau \in \mathbb{N}$ can be used as in the case of the $d^{(1)}$ transformation. It is easily seen that the new sequence also converges linearly with $\rho = \lim_{n\to\infty}(\check{s}_{n+1}-s)/(\check{s}_n-s) = q^\tau$. For $\tau > 1$, both the effectiveness and the stability of the various transformations are increased as shown in Table 5 for the case $\tau = 10$. Note that this value was chosen to display basic features relevant to the stability analysis, and is not necessarily the optimal value. As in Table 4, the relative errors and the stability indices of some variants of the $\mathcal{J}$ transformation are displayed. These are nothing but the ${}_p\mathbf{J}$ transformation for $p = 1,2,3$ and the Levin transformation as applied to the sequence $\{\{\check{s}_n\}\}$ with remainder estimates $\omega_n = (n\tau+\beta)(s_{n\tau}-s_{n\tau-1})$ for $\beta = 1$. Since constant factors in the remainder estimates are irrelevant since the $\mathcal{J}$ transformation is invariant under any scaling $\omega_n \to \alpha\omega_n$ for $\alpha \neq 0$, the same results would have been obtained for $\omega_n = (n+\beta/\tau)(s_{n\tau}-s_{n\tau-1})$.

If the Levin transformation is applied to the series with partial sums $\check{s}_n = s_{\tau n}$, and if the remainder estimates $\omega_n = (n+\beta/\tau)(s_{\tau n}-s_{(\tau n)-1})$ are used, then one obtains nothing but the $d^{(1)}$ transformation with $\xi_\ell = \tau\ell$ for $\tau \in \mathbb{N}$ [46,77].

Table 6
Stability indices for the ${}_2\mathbf{J}$ transformation ($\tau = 10$)

n	$\Gamma_n^{(1)}$	$\Gamma_n^{(2)}$	$\Gamma_n^{(3)}$	$\Gamma_n^{(4)}$	$\Gamma_n^{(5)}$	$\Gamma_n^{(6)}$	$\Gamma_n^{(7)}$
20	3.07	9.26	$2.70\,10^1$	$7.55\,10^1$	$2.02\,10^2$	$5.20\,10^2$	$1.29\,10^3$
30	3.54	$1.19\,10^1$	$3.81\,10^1$	$1.16\,10^2$	$3.36\,10^2$	$9.36\,10^2$	$2.51\,10^3$
40	3.75	$1.33\,10^1$	$4.49\,10^1$	$1.44\,10^2$	$4.42\,10^2$	$1.30\,10^3$	$3.71\,10^3$
41	3.77	$1.34\,10^1$	$4.54\,10^1$	$1.46\,10^2$	$4.51\,10^2$	$1.34\,10^3$	$3.82\,10^3$
42	3.78	$1.35\,10^1$	$4.59\,10^1$	$1.49\,10^2$	$4.60\,10^2$	$1.37\,10^3$	$3.93\,10^3$
43	3.79	$1.36\,10^1$	$4.64\,10^1$	$1.51\,10^2$	$4.69\,10^2$	$1.40\,10^3$	$4.05\,10^3$
44	3.80	$1.37\,10^1$	$4.68\,10^1$	$1.53\,10^2$	$4.77\,10^2$	$1.43\,10^3$	
45	3.81	$1.38\,10^1$	$4.73\,10^1$	$1.55\,10^2$	$4.85\,10^2$		
46	3.82	$1.39\,10^1$	$4.77\,10^1$	$1.57\,10^2$			
47	3.83	$1.39\,10^1$	$4.81\,10^1$				
48	3.84	$1.40\,10^1$					
49	3.85						
Extr.	4.01	$1.59\,10^1$	$6.32\,10^1$	$2.52\,10^2$	$1.00\,10^3$	$4.00\,10^3$	$1.59\,10^4$
Corollary 24	3.98	$1.59\,10^1$	$6.32\,10^1$	$2.52\,10^2$	$1.00\,10^3$	$4.00\,10^3$	$1.59\,10^4$

It is seen from Table 5 that again the best accuracy is obtained for the ${}_2\mathbf{J}$ transformation. The $d^{(1)}$ transformation is worse, but better than the ${}_p\mathbf{J}$ transformations for $p = 1$ and 3. Note that the stability indices are now much smaller and do not limit the achievable accuracy for any of the transformations up to $n = 30$. The true value of the series was computed numerically by applying the ${}_2\mathbf{J}$ transformation to the further sequence $\{\{s_{40n}\}\}$ and using 64 decimal digits in the calculation. In this way, a sufficiently accurate approximation was obtained that was used to compute the relative errors in Tables 4 and 5. A comparison value was computed using the representation [20, p. 29, Eq. (8)]

$$\zeta(s,1,q) = \frac{\Gamma(1-s)}{z}(\log 1/q)^{s-1} + z^{-1}\sum_{j=0}^{\infty}\zeta(s-j)\frac{(\log q)^j}{j!} \tag{338}$$

that holds for $|\log q| < 2\pi$ and $s \notin \mathbb{N}$. Here, $\zeta(z)$ denotes the Riemann zeta function. Both values agreed to all relevant decimal digits.

In Table 6, we display stability indices corresponding to the acceleration of $\check{s}_n$ with the ${}_2\mathbf{J}$ transformation columnwise, as obtainable by using the sequence elements up to $\check{s}_{50} = s_{500}$. In the row labelled Corollary 24, we display the limits of the $\Gamma_n^{(k)}$ for large n, i.e., the quantities

$$\lim_{n\to\infty} \Gamma_n^{(k)} = \left(\frac{1+q^\tau}{1-q^\tau}\right)^k, \tag{339}$$

that are the limits according to Corollary 24. It is seen that the values for finite n are still relatively far off the limits. In order to check numerically the validity of the corollary, we extrapolated the values of all $\Gamma_n^{(k)}$ for fixed k with n up to the maximal n for which there is an entry in the corresponding column of Table 6 using the u variant of the ${}_1\mathbf{J}$ transformation. The results of the extrapolation are displayed in the row labelled *Extr* in Table 6 and coincide nearly perfectly with the values expected according to Corollary 24.

Table 7
Extrapolation of series representation (340) of the $F_m(z)$ function using the ${}_2\mathbf{J}$ transformation ($z = 8,\ m = 0$)

n	s_n	${}^u\omega_n$	${}^t\omega_n$	${}^K\omega_n$
5	−13.3	0.3120747	0.3143352	0.3132981
6	−14.7	0.3132882	0.3131147	0.3133070
7	−13.1	0.3132779	0.3133356	0.3133087
8	11.4	0.3133089	0.3133054	0.3133087
9	−8.0	0.3133083	0.3133090	0.3133087

As a final example, we consider the evaluation of the $F_m(z)$ functions that are used in quantum chemistry calculations via the series representation

$$F_m(z) = \sum_{j=0}^{\infty} (-z)^j / j!(2m+2j+1) \tag{340}$$

with partial sums

$$s_n = \sum_{j=0}^{n} (-z)^j / j!(2m+2j+1). \tag{341}$$

In this case, for larger z, the convergence is rather slow although the convergence finally is hyperlinear. As a K variant, one may use

$${}^k\omega_n = \left(\sum_{j=0}^{n} (-z)^j/(j+1)! - (1-\mathrm{e}^{-z})/z \right). \tag{342}$$

since $(1-\mathrm{e}^{-z})/z$ is a Kummer related series. The results for several variants in Table 7 show that the K variant is superior to u and t variants in this case.

Many further numerical examples are given in the literature [39,41–44,50,84].

Appendix A. Stieltjes series and functions

A Stieltjes series is a formal expansion

$$f(z) = \sum_{j=0}^{\infty} (-1)^j \mu_j z^j \tag{A.1}$$

with partial sums

$$f_n(z) = \sum_{j=0}^{n} (-1)^j \mu_j z^j. \tag{A.2}$$

The coefficients μ_n are the moments of an uniquely given positive measure $\psi(t)$ that has infinitely many different values on $0 \leqslant t < \infty$ [4, p. 159]:

$$\mu_n = \int_0^{\infty} t^n \, \mathrm{d}\psi(t), \quad n \in \mathbb{N}_0. \tag{A.3}$$

Formally, the Stieltjes series can be identified with a Stieltjes integral

$$f(z) = \int_0^\infty \frac{\mathrm{d}\psi(t)}{1+zt}, \quad |\arg(z)| < \pi. \tag{A.4}$$

If such an integral exists for a function f then the function is called a Stieltjes function. For every Stieltjes function there exist a unique asymptotical Stieltjes series (A.1), uniformly in every sector $|\arg(z)| < \theta$ for all $\theta < \pi$. For any Stieltjes series, however, several different corresponding Stieltjes functions may exist. To ensure uniqueness, additional criteria are necessary [88, Section 4.3].

In the context of convergence acceleration and summation of divergent series, it is important that for given z the tails $f(z) - f_n(z)$ of a Stieltjes series are bounded in absolute value by the next term of the series,

$$|f(z) - f_n(z)| \leqslant \mu_{n+1} z^{n+1} \quad z \geqslant 0. \tag{A.5}$$

Hence, for Stieltjes series the remainder estimates may be chosen as

$$\omega_n = (-1)^{n+1} \mu_{n+1} z^{n+1}. \tag{A.6}$$

This corresponds to $\omega_n = \Delta f_n(z)$, i.e., to a $\tilde{t}$ variant.

Appendix B. Derivation of the recursive scheme (148)

We show that for the divided difference operator $\square_n^{(k)} = \square_n^{(k)}[\{\{x_n\}\}]$ the identity

$$\square_n^{(k+1)}((x)_{\ell+1} g(x)) = \frac{(x_{n+k+1} + \ell)\square_{n+1}^{(k)}((x)_\ell g(x)) - (x_n + \ell)\square_n^{(k)}((x)_\ell g(x))}{x_{n+k+1} - x_n} \tag{B.1}$$

holds. The proof is based on the Leibniz formula for divided differences (see, e.g., [69, p. 50]) that yields upon use of $(x)_{\ell+1} = (x+\ell)(x)_\ell$ and $\square_n^{(k)}(x) = x_n \delta_{k,0} + \delta_{k,1}$

$$\begin{aligned} \square_n^{(k+1)}((x)_{\ell+1} g(x)) &= \ell \square_n^{(k+1)}((x)_\ell g(x)) + \sum_{j=0}^{k+1} \square_n^{(j)}(x) \square_{n+j}^{(k+1-j)}((x)_\ell g(x)) \\ &= (x_n + \ell)\square_n^{(k+1)}((x)_\ell g(x)) + \square_{n+1}^{(k)}((x)_\ell g(x)). \end{aligned} \tag{B.2}$$

Using the recursion relation of the divided differences, one obtains

$$\square_n^{(k+1)}((x)_{\ell+1} g(x)) = (x_n + \ell)\frac{\square_{n+1}^{(k)}((x)_\ell g(x)) - \square_n^{(k)}((x)_\ell g(x))}{x_{n+k+1} - x_n} + \square_{n+1}^{(k)}((x)_\ell g(x)). \tag{B.3}$$

Simple algebra then yields Eq. (B.1).

Comparison with Eq. (140) shows that using the interpolation conditions $g_n = g(x_n) = s_n/\omega_n$ and $\ell = k-1$ in Eq. (B.1) yields the recursion for the numerators in Eq. (148), while the recursion for the denominators in Eq. (148) follows for $\ell = k-1$ and using the interpolation conditions $g_n = g(x_n) = 1/\omega_n$. In each case, the initial conditions follow directly from Eq. (140) in combination with the definition of the divided difference operator: For $k=0$, we use $(a)_{-1} = 1/(a-1)$ and obtain $\square_n^{(k)}(x_n)_{k-1} g_n = (x_n)_{-1} g_n = g_n/(x_n - 1)$.

Appendix C. Two lemmata

Lemma C.1. *Define*

$$A = \sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} \frac{\zeta^{n+j}}{(n+j)^{r+1}}, \tag{C.1}$$

where ζ *is a zero of multiplicity* m *of* $\mathring{\Pi}^{(k)}(z) = \sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} z^j$. *Then*

$$A \sim \zeta^{n+m} \binom{r+m}{r} \frac{(-1)^m}{n^{r+m+1}} \frac{\mathrm{d}^m \mathring{\Pi}^{(k)}}{\mathrm{d}x^m}(\zeta) \quad (n \to \infty). \tag{C.2}$$

Proof. Use

$$\frac{1}{a^{r+1}} = \frac{1}{r!} \int_0^\infty \exp(-at) t^r \, \mathrm{d}t, \quad a > 0 \tag{C.3}$$

to obtain

$$A = \frac{1}{r!} \int_0^\infty \sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} \zeta^{n+j} \exp(-(n+j)t) t^r \, \mathrm{d}t = \frac{\zeta^n}{r!} \int_0^\infty \exp(-nt) \, \mathring{\Pi}^{(k)}(\zeta \exp(-t)) t^r \, \mathrm{d}t. \tag{C.4}$$

Taylor expansion of the polynomial yields due to the zero at ζ

$$\mathring{\Pi}^{(k)}(\zeta \exp(-t)) = \frac{(-\zeta)^m}{m!} \left. \frac{\mathrm{d}^m \mathring{\Pi}^{(k)}(x)}{\mathrm{d}x^m} \right|_{x=\zeta} t^m (1 + \mathrm{O}(t)). \tag{C.5}$$

Invoking Watson's lemma [6, p. 263ff] completes the proof. □

Lemma C.2. *Assume that assumption* (C-3′) *of Theorem* 13 *holds. Further assume* $\lambda_{n,j}^{(k)} \to \mathring{\lambda}_j^{(k)}$ *for* $n \to \infty$. *Then, Eq.* (280) *holds.*

Proof. We have

$$\frac{\omega_{n+j}}{\omega_n} \sim \rho^j \exp\left(\varepsilon_n \sum_{t=0}^{j-1} \frac{\varepsilon_{n+t}}{\varepsilon_n} \right) \sim \rho^j \exp(j\varepsilon_n) \tag{C.6}$$

for large n. Hence,

$$\sum_{j=0}^{k} \lambda_{n,j}^{(k)} \frac{\omega_n}{\omega_{n+j}} \sim \sum_{j=0}^{k} \mathring{\lambda}_j^{(k)} (\rho \exp(\varepsilon_n))^{-j} = \mathring{\Pi}^{(k)}(1/\rho + \delta_n) \tag{C.7}$$

Since the characteristic polynomial $\mathring{\Pi}^{(k)}(z)$ has a zero of order μ at $z = 1/\rho$ according to the assumptions, Eq. (280) follows using Taylor expansion. □

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