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Note

Subalgebras of Incidence Algebras Determined by Equivalence Relations

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The main result of this paper is a necessary and sufficient condition on an equivalence relation R defined on the set of intervals of a locally finite partially ordered set S for the set of R-functions to be a subalgebra of the incidence algebra of S over a field of characteristic zero.

1. INTRODUCTION

The idea of the incidence algebra of a locally finite partially ordered set was proposed by Rota [1] as the basis for a unified study of combinatorial theory. The study of incidence algebras was continued by Smith [2–4]. In Section 5 of [3] Smith considered subalgebras of incidence algebras arising as the set of functions whose values are constant on equivalence classes of intervals for a certain type of equivalence relation. We are concerned with more general equivalence relations. Our main result is a characterization of those equivalence relations which give rise to subalgebras of the incidence algebra in this way. Only incidence algebras over field of characteristic zero are considered.

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2. Preliminaries

Our notation and terminology are similar to those of [2, 3]. For the reader's convenience we present the following relevant details. Throughout, $(S, \leqslant)$ will denote a locally finite partially ordered set. Local finiteness means that every (closed) interval $[x, y] = \{u : x \leqslant u \leqslant y\}$ is finite. The *incidence algebra* $A(S)$ of S over the fixed but arbitrary field K is the set of all functions $f\colon S \times S \to K$ with the property that $f(x, y) = 0$ whenever $x \nleqslant y$. $A(S)$ becomes an associative K-algebra with the pointwise operations of addition and scalar multiplication and with the Dirichlet product:

$$(f * g)(x, y) = \sum_{x \leqslant u \leqslant y} f(x, u)\, g(u, y).$$

The Kronecker delta function δ is the multiplicative identity of $A(S)$. The *zeta function* ζ of S is the element of $A(S)$ defined by $\zeta(x, y) = 1$ if $x \leqslant y$ and $\zeta(x, y) = 0$ otherwise. The multiplicative group of units of $A(S)$ is denoted by $G(S)$. It may be shown that $\zeta \in G(S)$ (Proposition 1, [2]). The multiplicative inverse μ of ζ is called the *Möbius function* of S.

If R is an equivalence relation on the set of (non-empty, closed) intervals $[x, y]$ of S call an element f of $A(S)$ and *R-function* if $[x, y]\, R\, [z, w]$ implies $f(x, y) = f(z, w)$, i.e., if f is constant on R-equivalence classes, and let $A_R(S)$ denote the set of R-functions.

Finally, for any set E, denote its cardinality by $|E|$.

3. Main Theorem

In the following it is to be understood that the field K has characteristic zero.

Theorem. *$A_R(S)$ is a subalgebra of $A(S)$ if and only if whenever $[x, y]\, R\, [z, w]$ there exists a bijection $\phi\colon [x, y] \to [z, w]$ such that $[x, u]\, R\, [z, \phi(u)]$ and $[u, y]\, R\, [\phi(u), w]$ for every $u \in [x, y]$.*

Proof. First suppose that the equivalence relation R has the stated property. $A_R(S)$ is clearly a vector subspace of $A(S)$. Let $f, g \in A_R(S)$ and $[x, y]\, R\, [z, w]$. Let $\phi\colon [x, y] \to [z, w]$ be a bijection as in the statement of the theorem. Then

$$(f * g)(x, y) = \sum_{x \leqslant u \leqslant y} f(x, u)\, g(u, y)$$

$$= \sum_{x \leqslant u \leqslant y} f(z, \phi(u))\, g(\phi(u), w)$$

$$= \sum_{z \leqslant t \leqslant w} f(z, t)\, g(t, w)$$

$$= (f * g)(z, w).$$

Hence $f * g \in A_R(S)$, and $A_R(S)$ is a subalgebra of $A(S)$.

Converseiy, suppose that $A_R(S)$ is a subalgebra of $A(S)$ and that $[x, y]\, R\, [z, w]$. Let R' denote the equivalence relation on $[x, y]$ defined by $uR'v$ if $[x, u]\, R\, [x, v]$ and $[u, y]\, R\, [v, y]$. Denote the corresponding partition of $[x, y]$ by P'. Similarly define an equivalence relation R'' on $[z, w]$ and denote the corresponding partition of $[z, w]$ by P''. For $\xi \in P'$ define the subset $\psi(\xi)$ of $[z, w]$ by

$$\psi(\xi) = \{t \in [z, w]\colon [x, s]\, R\, [z, t] \text{ and } [s, y]\, R\, [t, w] \text{ for some } s \in \xi\}.$$

We show that $|\psi(\xi)| = |\xi|$ for every such ξ. Let f be the R-function defined by $f(a, b) = 1$ if $[a, b]\, R\, [x, s]$ for some $s \in \xi$ and $f(a, b) = 0$ otherwise. Let g be the R-function defined by $g(c, d) = 1$ if $[c, d]\, R\, [s, y]$ for some $s \in \xi$ and $g(c, d) = 0$ otherwise. Then since $f * g \in A_R(S)$, $(f * g)(x, y) = (f * g)(z, w)$. Now

$$(f * g)(x, y) = \sum_{x \leqslant s \leqslant y} f(x, s)\, g(s, y),$$

where each term in the sum is either 0 or 1. The contribution to the sum is 1 if and only if $f(x, s) = g(s, y) = 1$ and this, in turn, is equivalent to s being an element of ξ. Thus $(f * g)(x, y) = |\xi| .1$. Similar reasoning gives $(f * g)(z, w) = |\psi(\xi)| .1$. Thus $|\psi(\xi)| = |\xi|$.

It is easily shown that $\psi(\xi) \in P''$ so we have a mapping $\psi\colon P' \to P''$. It is also easily shown that ψ is a bijection of P' onto P'' satisfying the condition $s \in \xi$, $t \in \psi(\xi)$ implies $[x, s]\, R\, [z, t]$ and $[s, y]\, R\, [t, w]$. Now let $\phi\colon [x, y] \to [z, w]$ be any bijection satisfying $\{\phi(s)\colon s \in \xi\} = \psi(\xi)$ for every $\xi \in P'$. It is readily verified that ϕ has the desired property. This completes the proof.

COROLLARY. *If $A_R(S)$ is a subalgebra of $A(S)$, then it contains δ, ζ and μ and its group of units is $A_R(S) \cap G(S)$.*

Proof. Let $A_R(S)$ be a subalgebra of $A(S)$. It is clear that δ and ζ are R-functions. It is not difficult to show that $[x, y]\, R\, [z, w]$ implies that the partially ordered sets $[x, y]$ and $[z, w]$ are of equal length. The proof is completed, as in Theorem 8 of [3], by showing that the inverse of an invertible R-function is an R-function. The details are omitted.

In particular, if the equivalence relation R has the property that, whenever $[x, y]\, R\, [z, w]$ there exists a bijection $\phi\colon [x, y] \to [z, w]$ such that $[u, v]\, R\, [\phi(u), \phi(v)]$ for all u, v such that $x \leqslant u \leqslant v \leqslant y$, then $A_R(S)$ is a

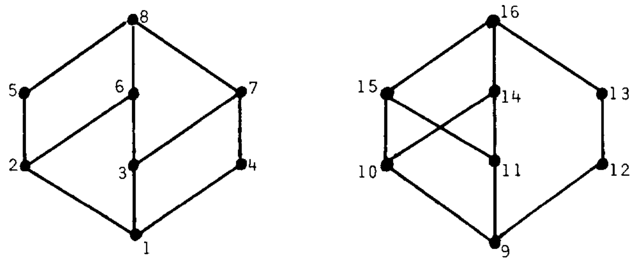

FIGURE 1

subalgebra of $A(S)$ containing δ, ζ and μ and its group of units is $A_R(S) \cap G(S)$. This was proved in [3] where such equivalence relations were called *compatible*. As was pointed out in [3], $[x,y]R[z,w]$ implies that $[x,y]$ and $[z,w]$ are order isomorphic if R is compatible. We show that compatibility of R is not a necessary condition for $A_R(S)$ to be a subalgebra of $A(S)$.

EXAMPLE. Let $S = \{1, 2, 3, ..., 16\}$ and let K be an arbitrary field of characteristic zero. Let the partial order on S be as indicated (in the usual way) by Fig. 1. In particular no element of $\{1, 2, ..., 8\}$ is comparable with any element of $\{9, 10, ..., 16\}$.

The equivalence relation R is defined as follows: say $[x,y]R[z,w]$ if $[x,y]$ and $[z,w]$ are order-isomorphic, and in addition, say $[1,8]R[9,16]$.

To show that $A_R(S)$ is a subalgebra of $A(S)$ we need to demonstrate, for each pair of R-equivalent intervals, the existence of a bijection from one interval to the other having the property specified in the theorem. If $[x,y]$ and $[z,w]$ are order-isomorphic then any order isomorphism from $[x,y]$ to $[z,w]$ has the desired property, and for the R-equivalent intervals $[1,8]$ and $[9,16]$ the mapping $\phi(u) = u + 8$ will suffice.

Thus $A_R(S)$ is a subalgebra, but because $[1,8]$ and $[9,16]$ are not order-isomorphic R is not compatible.

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