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A Nielsen theory for coincidences of iterates

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ABSTRACT

As the title suggests, this paper gives a Nielsen theory of coincidences of iterates of two self maps $f, g: X \to X$ of a closed manifold X. The idea is, as much as possible, to generalize Nielsen type periodic point theory, but there are many obstacles. Familiar results as in periodic point theory are obtained, but often require stronger hypotheses.

1. Introduction

This paper seeks to generalize Nielsen periodic point theory to iterates of coincidences. In many settings, fixed point theory can be thought of as a special case of coincidence theory of a pair of maps f and g, with g taken to be the identity map. At times however, this viewpoint is overly simplistic, and as this paper will show, this is certainly the case with respect to the relationship between periodic point theory, and the theory of iterates of coincidences.

There is much written in the literature concerning both Nielsen coincidence theory, and Nielsen periodic point theory. Let $f, g: X \to Y$ be maps (continuous functions) of closed manifolds X and Y of the same dimension. We use the symbol $\Phi(f, g)$ to denote the set of coincidences of f and g, that is $\Phi(f, g) = \{x \in X \mid f(x) = g(x)\}$. The aim of Nielsen coincidence theory is to define a lower bound (as sharp as possible), for the set $M\Phi(f, g)$, which is defined to be $\min(\#\{\Phi(f_1, g_1) \mid f_1 \simeq f,\ g_1 \simeq g\})$, where $\simeq$ denotes homotopy, and # cardinality. In a similar setting for a fixed positive integer n, Jiang [19] introduced two Nielsen type numbers $NP_n(f)$ and $N\Phi_n(f)$ where f is a self map of X. These two numbers are homotopy invariant lower bounds respectively for the number $MP_n(f)$, which is the cardinality of the smallest among the sets $P_n(f_1) = \Phi(f_1^n) - \bigcup_{m|n\, m\neq n} \Phi(f_1^m)$, as f_1 ranges over all maps homotopic to f, and of $M\Phi_n(f) = \min\{\#\Phi(f_1^n) \mid f_1 \simeq f\}$ where $\Phi(f_1^n) = \{x \in X \mid f_1^n(x) = x\}$ is the fixed point set of f_1^n. It is important in the definitions of $MP_n(f)$ and $M\Phi_n(f)$ to note that we only allow homotopies of f, not of f^n.

In this paper then, we define two Nielsen type numbers $NP_n(f, g)$ and $N\Phi_n(f, g)$ (in the context that $X = Y$), both of which are homotopy invariant lower bounds for the cardinalities of appropriate sets. The first of these, $MP_n(f, g)$, of which $NP_n(f, g)$ is a lower bound, is a straightforward generalization of $MP_n(f)$. In fact

$$MP_n(f, g) = \min \#\{P_n(f_1, g_1) \mid f_1 \simeq f \text{ and } g_1 \simeq g\},$$

where $P_n(f, g)$ denotes the set of points x with $f^n(x) = g^n(x)$ but $f^m(x) \neq g^m(x)$ for any $m \mid n$. We will often write that x is a coincidence *at level n* when $f^n(x) = g^n(x)$.

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Before we discuss the number of which $N\Phi_n(f,g)$ is a lower bound, we ask the reader to observe the following equalities which hold automatically in periodic point theory

$$\Phi(f^n) = \bigcup_{m|n} P_m(f) = \bigcup_{m|n} \Phi(f^m).$$

Note also that the $P_m(f)$ are disjoint. When it comes to coincidences of iterates, as we will see, the above equalities that we take for granted in periodic point theory do not universally hold true. We will show for coincidences of iterates that there are examples that variously illustrate the following possibilities:

$$\Phi(f^n, g^n) \neq \bigcup_{m|n} \Phi(f^m, g^m) = \bigcup_{m|n} P_m(f,g) \neq \bigsqcup_{m|n} P_m(f,g), \tag{1}$$

where $\bigsqcup$ denotes disjoint union. We will see immediately an example where $\Phi(f^n,g^n)$ and $\bigcup_{m|n}\Phi(f^m,g^m)$ can be very different. However it is not until Section 5 that we will see an example where $\bigcup_{m|n} P_m(f,g) \neq \bigsqcup_{m|n} P_m(f,g)$. To put this last statement another way, we are saying for coincidences of iterates, that the P_m may not be disjoint. The middle equality of the display (1) is of course, always true.

The differences shown in the numbered display above give us three possibilities for defining the minimum number $M\Phi_n(f,g)$ of which $N\Phi_n(f,g)$ is to be a homotopy invariant a lower bound. Perhaps counter intuitively we define it to be:

$$M\Phi_n(f,g) := \min \# \left\{ \bigsqcup_{m|n} P_m(f_1,g_1) \,\middle|\, f_1 \simeq f \text{ and } g_1 \simeq g \right\}.$$

We will not discuss the technical reason why we choose to minimize the disjoint union rather than the ordinary union until Section 5. However we feel it is instructive to give an example now, to show why we need to reject the number

$$MC_n(f,g) := \min\{\#\Phi(f_1^n, g_1^n) \mid f_1 \simeq f \text{ and } g_1 \simeq g\}$$

as a possible candidate for $M\Phi_n(f,g)$.

In fact, the example below also allows us to introduce the reader to some of the other hurdles one has to overcome in order to generalize Nielsen periodic point theory to our context. It illustrates a key difference in the behaviour of coincidences of iterates over that of periodic points. In particular, if x is a periodic point of f, then so is $f^j(x)$ for any positive integer j. On the other hand, if x is a coincidence of f and g, there is no guarantee that it is a coincidence of f^n and g^n for any $n > 1$.

Example 1.1. Let f be the self map of S^1 of degree -1, and let g be the rotation: $g(e^{i\theta}) = e^{i(\theta+\epsilon)}$, where $\epsilon \neq 0$ is small. Since $N(f,g) = 2$, it is easy to see that the pair f, g has two nonremovable coincidence points at level 1. On the other hand $\Phi(f^2, g^2) = \emptyset$, that is, there are no coincidence points of f^2 and g^2 at all.

To see this, note that if $f^2(e^{i\theta}) = g^2(e^{i\theta})$, then since $f^2(e^{i\theta}) = e^{i\theta}$, and $g^2(e^{i\theta}) = e^{i(\theta+2\epsilon)}$, then $\theta = \theta + 2\epsilon + 2k\pi$, which is impossible for small epsilon. Thus for this example $\Phi(f^2,g^2) = \emptyset$, while $\bigcup_{m|2}\Phi(f^m,g^m) = \Phi(f,g)$ has two elements. Furthermore $M\Phi_2(f,g) = \min\#\{P_1(f_1,g_1) \sqcup P_2(f_1,g_1) \mid f_1 \simeq f,\ g_1 \simeq g\} = 2$ and $MC_2(f,g) = \min\{\#\Phi(f_1^2, g_1^2) \mid f_1 \simeq f,\ g_1 \simeq g\} = 0$. As we will see when we have defined $N\Phi_n(f,g)$, this example shows that we can have $MC_n(f,g) < N\Phi_n(f,g)$. The point then, is that $MC_n(f,g)$ can fail to account for coincidences at levels other than n.

This example also shows, if we take ϵ to be irrational, that the trajectories $\{x, f(x), f^2(x), \ldots\}$ and $\{x, g(x), g^2(x), \ldots\}$ of a coincidence point x can be infinite.

One implication of Example 1.1 is that the orbit definitions that make sense in Nielsen periodic point theory, make no sense here (see also Example 3.4). Our fall back position is to define our Nielsen theory of coincidences of iterates in terms of classes rather than orbits. This will not then strictly speaking be a generalization of periodic point theory. In particular, as we show in Proposition 4.3, $NP_n(f) \geqslant NP_n(f, \mathrm{id})$, and the inequality can be strict (take f to be the map $g^{-1}f$ of Example 6.9). However as in periodic point theory when we work with tori, nilmanifolds and solvmanifolds orbits would have no real advantage. In fact for tori, nilmanifolds and model solvmanifolds (see [12]), which are the vast majority of our examples, the Nielsen type numbers we define are Wecken (the lower bounds are sharp see Section 8).

One further comment about Example 1.1 is that the fact that $MC_2(f,1) \neq M\Phi(f^2)$ may at first sight, appear to contradict the well known result of Brooks [2] that, under mild conditions, any variance in the cardinality of a coincidence set can be obtained by varying only one of the maps. The apparent contradiction is resolved when we point out that Brooks' result does not apply to iterates of maps when the only homotopies of f^n and g^n that we allow come from level 1 homotopies of f and g.

As it turns out, if f and g commute with each other (i.e. $fg = gf$) then the distinctions in the display (1) both disappear. In particular in such cases we have that $\Phi(f^m, g^m) \subset \Phi(f^{qm}, g^{qm})$ for all positive q (see Lemma 3.2). However restricting to commuting maps and their homotopies would be far too restrictive. What we do need however (in order even for our boosting functions to be well defined on Reidemeister sets), is that the two induced homomorphisms f_* and g_* commute at the level of the fundamental group $\pi_1(X)$ (see Definition 3.5). In fact in all our examples the homotopy class of the maps discussed contain representatives that commute with each other, and this is more than enough for our purposes.

The lack of geometric boostings is just one of several roadblocks one needs to navigate in the process of generalizing periodic point theory to coincidences of iterates. Another hurdle we want to mention here, is the fact that orbits don't work. In periodic point theory, orbits play an important role in certain examples, but not in fact, in the vast majority of the spaces we use in our examples here (but see Section 6.2). We will give additional details in Section 3.1, but for now let $x \in \Phi(f^n)$ for some map f. Then the trajectory (or orbit) $\{x, f(x), f^2(x), \ldots\}$ of x is finite and of length less than n. However as Example 1.1 shows, for coincidences, the trajectories $\{x, f(x), f^2(x), \ldots\}$ and $\{x, g(x), g^2(x), \ldots\}$ of a coincidence point $x \in \Phi(f^n, g^n)$ need not be finite, let alone less than or equal to its level n (see also Example 3.4). Since orbits are not available, we define the new numbers in terms of classes, rather than, as in periodic point theory, in terms of depth of orbit (see Definition 4.1). For most of the examples we use, this is in fact no disadvantage since the spaces are the equivalent of being essentially toral (the definition makes no sense for coincidences). We refer the reader to a related discussion in Section 6.2.

We come next to the question of imitating some of the familiar results of periodic point theory that hold true for all maps on tori, and more generally on nil and certain maps on solvmanifolds (see [13,14,9]). These results compare $N\Phi_n(f)$ to combinations of the $N(f^m)$ for $m \mid n$, and when $N(f^n) \neq 0$, give Möbius inversion type results for $NP_n(f)$. A key property that allows these periodic point results to go through is called essentially reducible to the gcd. In fact all maps on nil and solvmanifolds are essentially reducible to the CGD [9, Corollary 4.12]. On the other hand, the analogous property (essentially coincidence reducible to the gcd) is not even universally true on S^1 (where of course the commuting property of f_* and g_* is automatic). We refer the reader to Example 4.12.

We do however have a new but weaker result, that for all maps that are "coincidence essentially reducible" (see Theorem 4.6), and we investigate conditions for which the full generalizations hold. We give a complete characterization of "coincidence essentially reducible to the gcd" for maps on S^1 (Theorem A.1), and show that this property also holds when (in addition to the commutativity of f_* and g_*), we have that g_* is invertible (Theorem 4.16). In particular this is true if the linearization of a self map g of a torus is a matrix which is invertible over $\mathbb{Z}$. In addition, by giving examples on the Klein Bottle, we hint at the generalizations of our findings to solvmanifolds. We hint rather than prove, since giving full details would be beyond the scope of the paper.

Our theory is, of course, applicable to the case where g is the constant map, in other words to roots of iterates. This subject has already been studied by Brown, Jiang and Schirmer in [1]. One might suspect that many of the results of that paper could be obtained simply by putting g equal to a constant map in this one. This however is not the case. In fact the two theories are deeply incompatible. We will explore this in greater detail in Section 7, but for now we wish to mention two things that emphasize this incompatibility. The first is that the root theory in [1] is heavily dependent on the various choices of the image of the constant map "g". To put this another way, the theory in [1] is not homotopy invariant with respect to homotopies of the constant map g. And of course ours is. To describe the second incompatibility, we want to say first, that it should be clear that the concept of reducibility is foundational to both theories. However, even with respect to this very basic concept, the two theories are not the same. To say it in just a few words, in our work we consider reductions only for $m \mid n$. In [1] reductions are considered for certain m with $m < n$ but $m \nmid n$.

We outline the paper as follows. Following this introduction we give a preliminary section in which among other things, we establish our notation using the modified fundamental group approach. We briefly recall standard coincidence theory as applied to iterates of self maps. We include a short subsection on linearization of maps on tori, illustrating it with an example which will be useful later. Next in Section 3 we discuss the various relationships among the iterates, separating the geometry and the algebra. In the process we recall some of the basic concepts of Nielsen periodic point theory, and show the necessary detours and conditions we need to make in order to proceed with our "generalization" to coincidences. In Section 4, we define our number $NP_n(f, g)$, give some of its properties, show where direct generalizations fail, and introduce conditions under which many of the familiar results of periodic point theory can be generalized to coincidences of iterates. In Section 5, we do the same thing with respect to the number $N\Phi_n(f)$. In Section 6 we consider coincidence theory of iterates under the additional assumption that the map g is invertible. We indicate that under these conditions the notion of orbit is well defined, and that for essentially toral spaces our theory and Nielsen periodic point theory for the map $g^{-1}f$ coincide. In Section 7 we discuss the relationship of our work to the theory of roots of iterates given in [1]. We close the main part of the paper with a short section where we discuss two open questions both related to Wecken type considerations. The final section of the paper is an appendix where we prove the result on S^1, that for maps f and g represented by integers a and b respectively, that f and g are coincidence reducible to the gcd if and only if a and b are relatively prime. This property is foundational to the main computational theorems in Sections 4 and 5.

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2. Preliminaries, standard Nielsen coincidence theory of iterates of self maps

In this section we review standard Nielsen coincidence theory as it applies to iterates of self maps. We use the modified fundamental group approach as in [5] (see also [13,14]). In this approach we separate the geometry from the algebra, and by assigning an index (or semi-index) to the Reidemeister classes, we are able to deal with the possibility of having different empty classes, the supposed advantage of the covering space approach. This section then will establish our notation. At the end of the section we remind the reader of the concept of the linearization of a map on a torus. We will later make an oblique reference to linearization on solvmanifolds, but since we consider only the Klein Bottle, we will not go into the details.

Throughout the paper X will denote a closed manifold, and $f, g : X \to X$ will be self maps of X. As mentioned in the introduction, there are many settings in which fixed point theory can be thought of as a special case of coincidence theory where g is taken to be the identity map. With the proviso that we consider only manifolds, we indicate in this section some of the places where generalizations are entirely straightforward.

2.1. Geometric classes of iterates of self maps

In this subsection we remind the reader of standard coincidence theory, and make some straightforward applications of it to the iterates f^n and g^n of f and g respectively.

We say that $x, y \in \Phi(f^n, g^n)$ are *Nielsen equivalent at level n* provided that there is a path c from x to y so that (relative end points) $f^n(c) \simeq g^n(c)$. For $n = 1$ this is the ordinary Nielsen coincidence relation. The set of equivalence classes thus generated will be denoted by $\Phi(f^n, g^n)/{\sim}$. We call this the set of [*geometric*] *Nielsen classes* for f^n and g^n.

Using either the standard coincidence index (see for example [21]) or the semi-index in case X is not orientable (see [4,8]), we may for each geometric class $\mathbf{A}^n \in \Phi(f^n, g^n)/{\sim}$ associate an integer denoted $\mathrm{ind}(\mathbf{A}^n)$. The classes for which this integer is nonzero are called *essential* Nielsen classes. For any positive integer n, the *Nielsen number* $N(f^n, g^n)$ of f^n and g^n is then the number of essential classes of f^n and g^n. This number is a lower bound for $M\Phi(f^n, g^n)$ which should be carefully distinguished from $M\Phi_n(f, g)$. For $M\Phi(f^n, g^n)$ we consider homotopies that range over all maps h and k that are homotopic respectively to f^n and g^n. Of course this will include homotopies induced by homotopies $f \simeq f_1$ and $g \simeq g_1$ of f and g respectively. This last kind of homotopy, and this kind only, is the kind we consider in the main body of the paper. When g is the identity and we disallow homotopies of g, then the number of essential classes of f, denoted $N(f)$, is the ordinary Nielsen number of f.

2.2. Algebraic and geometric classes and their relationship

We come now to the algebraic side of the story. In what follows we shall not distinguish between a path and its path class in the fundamental groupoid $\pi_1(X)$. Thus c can denote both a path and a path class in $\pi_1(X)$. In addition if $h : X \to X$ is a map, $h(c)$ will denote either a path or class. If c is a path from a to b, then c^{-1} is the path from b to a defined by $c^{-1}(t) = c(1 - t)$.

Choose a base point $x_0 \in X$. For simplicity we work with base point preserving maps $f, g : X \to X$ with respect to this chosen base point. We can do this without loss of generality, since in manifolds base points are always closed and non-degenerate. In particular each homotopy class has a representative that is base point preserving with respect to the chosen base point.

In this way, for each positive integer n we have induced homomorphisms $f_*^n, g_*^n : \pi_1(X, x_0) \to \pi_1(X, x_0)$, and these in turn determine an equivalence relation on $\pi_1(X, x_0)$ (doubly-twisted conjugacy) defined by the rule that $\alpha \sim \beta$ in $\pi_1(X, x_0)$ if and only if there exists $\gamma \in \pi_1(X, x_0)$ with $\alpha = g_*^n(\gamma)\beta f_*^n(\gamma^{-1})$. The resulting classes are called Reidemeister classes. The Reidemeister class containing α is denoted by $[\alpha]^n$. The set of all Reidemeister classes is denoted by $\mathcal{R}(f_*^n, g_*^n)$, and its cardinality is the *Reidemeister number* $R(f^n, g^n)$. There is an exact sequence of based sets,

$$\pi_1(X, x_0) \to \pi_1(X, x_0) \xrightarrow{j_n} \mathcal{R}\big(f_*^n, g_*^n\big) \to 1$$

where the first function takes an element α to $g_*^n(\alpha) f_*^n(\alpha^{-1})$, and j_n places an element β in its Reidemeister class $[\beta]^n$. If $\pi_1(X, x_0)$ is Abelian there is a canonical group structure on $\mathcal{R}(f_*^n, g_*^n)$, moreover in this case the sequence consists of groups and homomorphisms. All the above constructions are independent of the choice of base point and path classes in the sense that there exists bijections between the various Reidemeister sets (see [5]). When $\pi_1(X)$ is Abelian we write composition of functions additively, and we have:

Theorem 2.1. *(Guo and Heath [5,8]) Let $f, g : X \to X$ be maps with $\pi_1(X, x_0)$ Abelian, then the sequence*

$$0 \to \mathrm{Coin}\big(f_*^n, g_*^n\big) \to \pi_1(X, x_0) \xrightarrow{g_*^n - f_*^n} \pi_1(X, x_0) \xrightarrow{j_n} \mathcal{R}\big(f_*^n, g_*^n\big) \to 0$$

is an exact sequence of Abelian groups and homomorphisms, where

$$\mathrm{Coin}\big(f_*^n, g_*^n\big) = \big\{\alpha \in \pi_1(X, x_0) \bigm| f_*^n(\alpha) = g_*^n(\alpha)\big\}. \qquad \square$$

The algebraic and geometric components of the theory are related by an injective function

$$\rho_n = \rho : \Phi\big(f^n,\ g^n\big)/{\sim} \to \mathcal{R}\big(f_*^n, g_*^n\big)$$

defined as follows: Given $x \in \mathbf{A}^n$ we choose a path c from the base point x_0 to x. We can then define $\rho_n(\mathbf{A}^n) = [g^n(c) f^n(c^{-1})]^n$. This will be independent of c and of the choice of x within $\mathbf{A}^n$. An algebraic class $[\alpha]^n$ is said to be *nonempty* if it lies in the image of ρ_n. Following the modified fundamental group approach as in [5,8,13,14], we next assign an index (or semi-index for our Klein Bottle examples) to the Reidemeister classes. The index (semi-index) $\mathrm{Ind}([\alpha]^n)$ of a class $[\alpha]^n \in \mathcal{R}(f_*^n, g_*^n)$ is defined as follows

$$\mathrm{Ind}\big([\alpha]^n\big) = \begin{cases} \mathrm{ind}(\mathbf{A}^n) & \text{if } [\alpha]^n = \rho_n(\mathbf{A}^n), \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{ind}(\mathbf{A}^n)$ is the integer defined in the geometric section, the usual coincidence index, or the coincidence semi-index of [4]. As with the geometric classes, an algebraic class is *essential* provided it has nonzero index (semi-index). We denote the set of essential algebraic classes by $\mathcal{R}_{\mathcal{E}}(f_*^n, g_*^n) \subseteq \mathcal{R}(f_*^n, g_*^n)$. Clearly $N(f^n, g^n) = \#(\mathcal{R}_{\mathcal{E}}(f_*^n, g_*^n))$.

2.3. Linearization and weakly Jiang maps

Although most of our examples in this paper will be on tori, we will be using the Klein Bottle as a kind of representative solvmanifold. We will not go into details of linearizations of maps on these spaces, but refer the reader to [9–11] for the fixed point case, and to [8] for some aspects of the coincidence case. Linearization of maps on tori are very simple. If $f : T^r \to T^r$ is a map of an r torus, then we can identify the linearization of f with the induced homomorphism on $\pi_1(T^r) \cong \mathbb{Z}^r$. Using the standard basis for $\mathbb{Z}^r$ we can then identify this homomorphism with a matrix F. This same matrix can then be used to define a map in the homotopy class of f, namely that map which is induced from $F : \mathbb{R}^r \to \mathbb{R}^r$ defined by matrix multiplication on vectors.

Theorem 2.2. *(Jezierski [15, Lemma 7.3]) Let $f, g : T^r \to T^r$ be maps of the r torus with linearizations F and G respectively. Then $N(f, g) = |\det(G - F)|$. If $\det(G - F) \neq 0$ then the linear maps F and G have $\#(\Phi(F, G)) = N(f, g)$.* □

For maps f, g of tori, we will as above often identify f with F and g with G, and write $N(F, G)$ for $N(f, g)$.

Definition 2.3. We say that a pair f, g is *coincidence weakly Jiang* provided that either $N(f, g) = 0$ or else $N(f, g) = R(f, g)$. If all pairs of maps on a space are coincidence weakly Jiang, we say that the space itself is coincidence weakly Jiang.

Recall that a Jiang space is one for which the induced map $x_0^* : \pi_1(X^X, 1_X) \to \pi_1(X, x_0)$ is surjective. These spaces have the property that if the Lefschetz number $L(f, g) = 0$, then $N(f, g) = 0$, and otherwise $N(f, g) = R(f, g)$. A Jiang space will be coincidence weakly Jiang, and in particular tori are coincidence weakly Jiang. On the other hand, there are many pairs of maps that will be coincidence weakly Jiang, where the spaces are not actual Jiang spaces. Our primary example of this phenomenon occurs on the Klein Bottle (see Example 4.14).

Example 2.4. Let $f, g : S^1 \to S^1$ be maps of degree 6 and 2 respectively. For maps on circles our fundamental groups will be $\mathbb{Z}$, and our maps f_*, g_* are multiplication by 6 and 2, respectively. By exactness in Theorem 2.1 we have $\mathcal{R}(f_*^n, g_*^n) \cong \mathbb{Z}_{|6^n - 2^n|}$, and since S^1 is a Jiang space and $L(6^n, 2^n) \neq 0$ we have

$$\mathcal{R}(f_*, g_*) \cong \mathbb{Z}_4, \qquad \mathcal{R}\big(f_*^2, g_*^2\big) \cong \mathbb{Z}_{32}, \qquad \mathcal{R}\big(f_*^3, g_*^3\big) \cong \mathbb{Z}_{208}, \quad \text{and} \quad \mathcal{R}\big(f_*^6, g_*^6\big) \cong \mathbb{Z}_{46\,592},$$

with respective Nielsen numbers 4, 32, 208 and 46 592.

3. Relations among iterates. Geometric and algebraic reductions

In order to avoid giving too much detail, we are assuming that the reader has a basic familiarity with Nielsen periodic point theory. In this regard, we would point the reader to the survey article [8], which we also use as our main reference. At times we will also refer to Jiang's original ground breaking work [19], as well as early expositions and expansions of it [13,14].

We do of course need to give the coincidence analogues of the foundational definitions in periodic point theory, and we do so in this section. Also when we want specifically to compare periodic point concepts with the theory we develop here, we will recall the appropriate definitions. Our point is to indicate where and why straightforward generalizations of periodic point theory to coincidences fail.

3.1. Reducible and irreducible Nielsen classes

In the preliminary section we outlined existing coincidence Nielsen theory as it applies to iterates of self maps $f, g : X \to X$. However, just as Nielsen periodic point theory is much more than the study of the Nielsen numbers of iterates, so our work is more involved than the study of the Nielsen coincidence numbers of iterates. As mentioned in the introduction, our work is also complicated by a number of obstacles we encounter in our attempt to generalize periodic point theory. In Example 1.1 we saw a case where

$$\Phi\big(f^n, g^n\big) \neq \bigcup_{m|n} \Phi\big(f^m, g^m\big).$$

On the other hand, the corresponding equality in periodic point theory always holds. In fact, the equality holds for coincidences when the two maps commute, but we need to state this in the following slightly different form:

Lemma 3.1. *If the self maps f, g of X commute, then for all positive integers q we have that*

$$\Phi\big(f^m, g^m\big) \subset \Phi\big(f^{qm}, g^{qm}\big).$$

In particular if $f(x) = g(x)$ for some x then $f^n(x) = g^n(x)$ for all positive n.

Proof. The inductive step starts with the proof that $f^m(x) = g^m(x)$ implies that $f^{2m}(x) = g^{2m}(x)$. The rest is straightforward. So let $f^m(x) = g^m(x)$, then $f^{2m}(x) = f^m(f^m(x)) = f^m(g^m(x)) = g^m(f^m(x)) = g^m(g^m(x)) = g^{2m}(x)$. □

This respects Nielsen equivalence to give:

Lemma 3.2. *If the self maps f, g of X commute, then for all $m \mid n$, the inclusion map induces a function $\gamma_{m,n} : \Phi(f^m, g^m)/{\sim} \to \Phi(f^n, g^n)/{\sim}$ which takes the class of x at the mth level to the class of x at the nth level.*

So unlike the fixed point case the $\gamma_{m,n}$ need not exist, and as Example 1.1 shows it is not enough that the induced homomorphisms f_* and g_* commute (in that example we have $\pi_1(X) \cong \mathbb{Z}$ and so all maps commute in the algebra). This will complicate our discussion of the relationship between the algebra, which is homotopy invariant, and the geometry, which is not. As in the fixed point case the $\gamma_{m,n}$ need not be injective even when they exist (replace g in Example 1.1 with the identity, and consider $n = 2$).

There is another immediate obstacle to our attempt to generalize Nielsen periodic point theory. In particular Nielsen periodic point theory of a self map f works with f orbits of classes, rather than simply with classes. Recall that the period of $x \in \Phi(f^n)$, is the smallest positive integer $m \mid n$ such that $f^m(x) = x$. The list $\{x, f(x), \ldots, f^{m-1}(x)\}$ is called *the orbit* of x. The algebraic orbit is the list $\{\rho_n([x]^n), \rho_n([f(x)]^n), \ldots\}$, and its length will divide m. Nielsen periodic point theory counts depth of algebraic orbits, rather than classes. This is because there can be Nielsen equivalences (at level m) between different elements $f^i(x)$ and $f^j(x)$ in the above list. When this happens the algebraic length of the orbit (the number of classes in an orbit counted algebraically) is shorter than the geometric length of *points*, and so counting classes gives an inadequate count of the actual minimum number of points present. This idea is encapsulated in the following fundamental lemma from [13, Proposition 1.3], and is the primary reason we consider orbits in Nielsen periodic point theory rather than classes. The notation is taken from [8], where the angle brackets denote f_* orbits of classes. We will be discussing algebraic reductions of coincidence classes later in this section.

Lemma 3.3. *([13, Proposition 1.3]) In Nielsen periodic point theory, the (algebraic) length of an orbit $\langle[\alpha]^n\rangle$ divides its depth (the minimum integer to which $\langle[\alpha]^n\rangle$ reduces algebraically). If $\langle[\alpha]^n\rangle$ is essential and has depth d, then $\langle[\alpha]^n\rangle$ contains at least d periodic points.*

So then in Nielsen periodic point theory, the use of orbits is seen to come into its own when the length of orbit is strictly less than its depth (d say). At the risk of being repetitious what this means is that such orbits contain at least d points, but the number of classes is strictly less than d. If in the definition of the first periodic point number (denoted $NP_n(f)$) we simply counted the number of irreducible essential classes, we would be defining a Nielsen number that, in general, had no chance of being a sharp lower bound. This was the fundamental mistake that Halpern made in his famous innovative and useful, but unpublished preprint [6]. The standard example, due to Jiang, comes from a self map of $\mathbb{R}\mathcal{P}^3$ (see [19,13]), where for certain n a single irreducible class (which contains entire geometric orbits) contains at least n periodic points.

One implication of all of this in periodic point theory is that the length of an orbit at level 1 must necessarily be equal to 1. This can also be seen by the equation $\alpha = \alpha f_*(\alpha) f_*(\alpha^{-1})$, which shows that α and $f_*(\alpha)$ are Reidemeister equivalent at level 1. This is not the case for iterates of coincidence classes, and the difficulty in producing a cohesive theory of orbits in a Nielsen theory of coincidences of iterates is revealed at this very first level (level 1). To say more, let $x \in \Phi(f, g)$. Consider the list $\{x, f(x), \ldots, f^{m-1}(x), \ldots\}$, or we could look at the list $\{x, g(x), \ldots, g^{m-1}(x), \ldots\}$. We call these lists the *trajectories* of x under f and g respectively. By Lemma 3.2 these trajectories are the same when f and g commute. The trajectories are perhaps the obvious candidates for orbits in any generalization of periodic point theory, but they do not have the desired properties. In particular, even at the first level the algebraic trajectory length need not be 1.

Example 3.4. Let $X = S^1$, and define self maps f and g of X by $f(e^{i\theta}) = e^{4i\theta}$, and $g(e^{i\theta}) = e^{-3i\theta}$. Then f and g commute. Now $\Phi(f, g) = \{e^{2k\pi i/7} \mid k = 0, 1, \ldots, 6\}$, and the trajectory of $2\pi i/7$ is the set $\{e^{2\pi i/7}, e^{8\pi i/7}, e^{4\pi i/7}\}$.

Since we cannot use the notion of orbit to count coincidence points, we must fall back on counting appropriate classes. In light of the fundamental lemma for periodic points (Lemma 3.3) this may appear as a severe disadvantage. In fact for the vast majority of the spaces we use in our examples this is not the case. This is because in our examples when the appropriate ordinary coincidence Nielsen numbers are non-zero the classes can be made into singletons. Actually the only exception is Example 6.9 which we use to illustrate that we can define orbits when g is invertible. Without going into too much detail, the point is that the orbit definition has no advantage over definitions that count classes when the spaces are tori or nil- or solvmanifolds. The technical name for this in periodic point theory is essential torality (see Section 4). This definition does not make sense in our theory, but what we want to say is that for these spaces even if we could define orbits, it would not increase the number of coincidence points we could detect (see Remark 6.4). In other words for these spaces there would be no advantage in using orbits anyway.

3.2. Reducible and irreducible Reidemeister classes

We come now to the algebraic counterpart $\iota_{m,n}$ of the geometric "boosting functions" $\gamma_{m,n}$. As with the $\gamma_{m,n}$, we need conditions on f and g in order for the $\iota_{m,n}$ to be well defined on (Reidemeister) classes. It is of course an algebraic condition, and what we require here is only that the induced homomorphisms $f_*, g_* : \pi_1(X) \to \pi_1(X)$ commute.

Definition 3.5. Suppose for given self maps $f,\ g : X \to X$ that the induced homomorphisms $f_*, g_* : \pi_1(X) \to \pi_1(X)$ commute. Let $m \mid n$ be integers, then the *m to n level coincidence boosting functions* (or simply boosting functions) $\iota_{m,n}$ are defined by the equation

$$\iota_{m,n}(\alpha) = \prod_{\ell=0}^{\frac{n}{m}} g_*^{n-(\ell+1)m} f_*^{\ell}(\alpha) \quad \left(= \sum_{\ell=0}^{\frac{n}{m}} g_*^{n-(\ell+1)m} f_*^{\ell}(\alpha) \text{ when } \pi_1 \text{ is Abelian} \right)$$
$$= g_*^{n-m}(\alpha) g_*^{n-2m} f_*^{m}(\alpha) \cdots g_*^{m} f_*^{n-2m}(\alpha) f_*^{n-m}(\alpha).$$

The following lemma is straightforward and its proof is left to the reader.

Lemma 3.6. *When f_* and g_* commute, then the $\iota_{m,n}$ are well defined on Reidemeister classes.* □

We abuse notation and use $\iota_{m,n}$ to denote boosting function on both $\pi_1(X)$ and on Reidemeister classes.

Example 3.7. Continuing Example 2.4 we considered maps of S^1 of degrees 6 and 2 which clearly commute at the level of $\pi_1(S^1)$. From now on we will identify each of these maps with its respective integer. In addition in this example, the boosting functions ι can be represented by multiplication by an integer (mod $(6^n - 2^n)$), and we will further abuse notation by identifying them with the said integer. Thus

$$\iota_{1,6} = 6^5 + 2 \cdot 6^4 + 2^2 \cdot 6^3 + 2^3 \cdot 6^2 + 2^4 \cdot 6 + 2^5 = 11\,648,$$
$$\iota_{2,6} = 6^4 + 2^2 \cdot 6^2 + 2^4 = 1456, \quad \text{and}$$
$$\iota_{3,6} = 6^3 + 2^3 = 224.$$

The proof of the following lemma is an easy generalization of the periodic point case in [7, Lemma 3.1].

Lemma 3.8 *(On the nose boosting). Let $[\alpha]^n \in \mathcal{R}(f^n, g^n)$ be a Reidemeister class that reduces to a class $[\beta]^m$ at level m. Then for any $\sigma \in [\alpha]^n$, there is a $\tau \in [\beta]^m$ for which $\iota_{m,n}(\tau) = \sigma$ in $\pi_1(X)$.* □

The following can be verified by an easy calculation from the definitions of the ρ_k, the geometric and algebraic boosts and Lemma 3.2 (see [13, Proposition 1.14] for the corresponding proof in periodic point theory).

Lemma 3.9. *Suppose that $f, g : X \to X$ induce commuting homomorphisms on $\pi_1(X)$. Let $k \mid m \mid n$ be integers, then we have that $\iota_{k,n} = \iota_{m,n}\iota_{k,m}$. Furthermore if f and g commute as functions, then the following diagram exists*

$$\begin{array}{ccc} \Phi(f^m, g^m)/\sim & \xrightarrow{\rho_m} & \mathcal{R}(f_*^m, g_*^m) \\ \gamma_{m,n}\downarrow & & \downarrow\iota_{m,n} \\ \Phi(f^n, g^n)/\sim & \xrightarrow{\rho_n} & \mathcal{R}(f_*^n, g_*^n), \end{array}$$

and is commutative. □

It would of course, be possible to define a theory of coincidences of iterates by restricting to maps f and g that commute geometrically. In light of Lemma 3.9, this might make the theory look closer to periodic point theory. But this is far too restrictive an assumption, and it is also poorly behaved under homotopy. In fact commutativity of the maps on $\pi_1(X)$ is sufficient to allow boosting (Lemma 3.6) and, as we shall see, is still general enough to do interesting examples.

Definition 3.10. Let $f, g : X \to X$ be maps. We say $[\alpha]^n \in \mathcal{R}(f_*^n, g_*^n)$ is *reducible* to $[\beta]^m \in \mathcal{R}(f_*^m, g_*^m)$ if $\iota_{m,n}([\beta]^m) = [\alpha]^n$. If $[\alpha]^n$ is not reducible to any level $m < n$ then it is *irreducible*. We say that $[\alpha]^n$ has *depth d* if d is the smallest integer for which there is a class $[\delta]^d$ to which $[\alpha]^n$ reduces.

As in periodic point theory, even without the assumption that f and g commute geometrically, if $[\alpha]^n$ is in the image of no $\iota_{m,n}$ for any $m \mid n$, then there can be no geometric coincidence points of f^m and g^m whose Reidemeister class in $\mathcal{R}(f_*^m, g_*^m)$ boosts to $[\alpha]^n$. In fact there can be no coincidence points of f_1^m and g_1^m whose class boosts to $[\alpha]^n$ for any $f_1 \simeq f$ and $g_1 \simeq g$. So then our algebraic constructions are still useful in detecting geometric behaviour even when the maps are not geometrically commutative.

4. The analogue $NP_n(f, g)$ of the periodic point number $NP_n(f)$

As in the introduction we use the symbols $P_n(f, g)$ to denote the set of points x with $f^n(x) = g^n(x)$ but $f^m(x) \neq g^m(x)$ for any $m \mid n$, and $MP_n(f, g)$ to denote the minimum $\min \#\{P_n(f_1, g_1) \mid f_1 \simeq f \text{ and } g_1 \simeq g\}$. The aim in this section is to define a suitable lower bound for $MP_n(f, g)$, give some of its properties together with a number of examples.

Definition 4.1. Let $f, g : X \to X$ be maps with $f_* g_* = g_* f_* : \pi_1(X) \to \pi_1(X)$. We define $NP_n(f, g)$ as the number of irreducible essential Reidemeister classes of $\mathcal{R}(f_*^n, g_*^n)$.

Theorem 4.2. *$NP_n(f, g)$ is homotopy invariant in f and g, and*

$$NP_n(f, g) \leqslant MP_n(f, g) \leqslant \#\bigl(P_n(f, g)\bigr).$$

Proof. The homotopy invariance holds because NP_n is defined only in terms of the induced maps f_* and g_*, and the inequality on the right is obvious.

For the other inequality, let f and g be arbitrary. We need only show that $NP_n(f, g) \leqslant \#(P_n(f, g))$. Accordingly, let $[\alpha]^n \in \mathcal{R}(f_*^n, g_*^n)$ be an essential irreducible class. Because $[\alpha]^n$ is essential, there is a coincidence point x with coincidence class $\mathbf{A}^n$ with $f^n(x) = g^n(x)$ and $\rho_n(\mathbf{A}^n) = [g^n(c) f^n(c^{-1})]^n = [\alpha]^n$, where c is any path from x_0 (the base point) to x. It suffices to show that $x \in P_n(f, g)$. If this is so, then this process will define an injection from the irreducible essential Reidemeister classes into $P_n(f, g)$ establishing the inequality.

For the sake of deducing a contradiction, assume that $x \notin P_n(f, g)$, that is, there is some m such that $f^m(x) = g^m(x)$ with $m \mid n$, and $m \neq n$. Then we will have $x \in \mathbf{B}^m$ where $\mathbf{B}^m$ is the coincidence class of x at level m. Let $[\beta]^m = \rho_m(\mathbf{B}^m) = [g^m(c) f^m(c^{-1})]^m$. Then by the definition of $\iota_{m,n}$ we have that $\iota_{m,n}(\rho_m(\mathbf{B}^m)) = \iota_{m,n}([g^m(c) f^m(c^{-1})]^m) = [g^n(c) f^n(c^{-1})]^n = [\alpha]^n$, contradicting the irreducibility of $[\alpha]$. $\square$

Our first task is to compare $NP_n(f, \mathrm{id})$ with $NP_n(f)$ (id is the identity). Recall, in the context of periodic point theory, that a map $f : X \to X$ is said to be *essentially toral* [9] if, for all $m \mid n$ and every $[\alpha]^m \in \mathcal{R}_{\mathcal{E}}(f_*^m)$, the depth of $[\alpha]^m$ and the orbit length of $[\alpha]^m$ coincide, and the boosting functions are injective on essential boosts. As already discussed, the first part of this simply means we may as well define $NP_n(f)$ to be the number of irreducible essential *classes*. For all but one of our examples (Example 6.9), the maps involved are (individually) essentially toral. See [9, Corollary 4.6].

Proposition 4.3. *Let $f : X \to X$ be a self map, then $NP_n(f) \geqslant NP_n(f, \mathrm{id})$. If f is essentially toral, then $NP_n(f, \mathrm{id}) = NP_n(f)$.* $\square$

At this point in the exposition of periodic point theory in [13] we would be working towards a Möbius inversion type result on a certain class of maps on Jiang spaces, in particular on tori. The generalization of this result applied to Example 2.4, would say that $NP_6(f, g) = N(f^6, g^6) - N(f^3, g^3) - N(f^2, g^2) + N(f, g)$. As we shall see below, the coincidence version of this is false in general (even when f and g commute). We do have a weaker version of this result, but before we can state it, we need the following coincidence analogue of a definition from [9].

Definition 4.4. (Cf. [9]) We say that the pair f, g is *coincidence essentially reducible* provided that for any essential class $[\alpha]^n$ of f^n and g^n, if $[\alpha]^n$ reduces to some class $[\beta]^m$, then $[\beta]^m$ is also essential. If for a given space X any pair of self maps is coincidence essentially reducible then we say that X is coincidence essentially reducible.

For $g = \mathrm{id}$, the identity on a torus, the pair f, g is always coincidence essentially reducible. In periodic point theory there are simple examples of maps that are not essentially reducible, but these tend to be maps on non-manifolds, and so do not provide examples in our setting (the coincidence index is in general not defined for non-manifolds). For coincidences of iterates (consistent with Definition 3.5), we need the assumption that the induced maps commute at the level of the fundamental group.

Theorem 4.5. *If f and g are maps of tori for which the induced maps on the fundamental groups commute, then the pair f, g is coincidence essentially reducible.*

The proof in [9], that tori are essentially reducible for periodic point theory, uses the linearization of the self map f under consideration. In particular it uses the linearization F of f, and the fact that $N(f^n) = |\det(F - I)|$, where I is the identity matrix. We need a slightly different proof than the one given in [9], but we will use Theorem 2.2. We also use the fact that with respect to both fixed and coincidence point theory tori are Jiang spaces and hence also weakly Jiang (Definition 2.3).

Proof. Suppose that some class at level m say, is essential. Then every class at level m is essential. We show that every class at level $k \mid m$ is also essential, or equivalently that $N(f^k, g^k) \neq 0$. Accordingly let F and G be the linearizations of f and g respectively. Now we have that $N(f^m, g^m) = |\det(F^m - G^m)| \neq 0$. Let $r = \frac{m}{k}$. By hypothesis F and G commute, so $F^m - G^m = (F^k - G^k)(F^r + F^{r-1}G + \cdots + G^r)$. So $N(f^k, g^k) = |\det(F^k - G^k)| \neq 0$ or else we would have that $|\det(F^m - G^m)| = 0$, a contradiction. □

The next result is our weaker version of Möbius inversion, which requires only essential reducibility. The periodic point analogy of the second inequality holds true, but this is not the case for the first inequality (see also Example 6.9, and the discussion in Section 6).

Theorem 4.6. *Let $P(n) = \{p(1), p(2), \ldots, p(k)\}$ be the set of primes dividing n, and suppose that the pair f, g is essentially reducible. Then*

$$N\big(f^n, g^n\big) \geqslant NP_n(f, g) \geqslant N\big(f^n, g^n\big) - \sum_{i=1}^{k} N\big(f^{n:i}, g^{n:i}\big),$$

where $n : i = n \cdot (p(i))^{-1}$.

Proof. If $N(f^n, g^n) = 0$ there is nothing to prove. When $N(f^n, g^n) \neq 0$ we write $\mathcal{R}_{\mathcal{E}}(f_*^n, g_*^n) = \mathcal{R}_{\mathcal{IE}}(f_*^n, g_*^n) \cup \mathcal{RE}(f_*^n, g_*^n)$, where $\mathcal{R}_{\mathcal{IE}}(f^n, g^n)$ is the set (with cardinality $NP_n(f, g)$) of essential irreducible algebraic classes in $\mathcal{R}(f_*^n, g_*^n)$. The set $\mathcal{RE}(f^n, g^n)$ consists of the reducible essential classes, and of course the union is disjoint.

We show that $\#\mathcal{RE}(f_*^n, g_*^n) \leqslant \# \bigsqcup_{i=1}^k \mathcal{R}_{\mathcal{E}}(f_*^{n:i}, g_*^{n:i})$ (where $\bigsqcup$ denotes disjoint union). In fact we construct an injection $\psi : \mathcal{RE}(f_*^n, g_*^n) \to \bigsqcup_{i=1}^k \mathcal{R}_{\mathcal{E}}({}_*f^{n:i}, g_*^{n:i})$. So let $[\alpha]^n \in \mathcal{RE}(f_*^n, g_*^n)$, and let $m \mid n$ with $m \neq n$ be the maximal integer for which there exists a $[\beta]^m \in \mathcal{R}_{\mathcal{E}}(f_*^m, g_*^m)$ with $\iota_{m,n}([\beta]^m) = [\alpha]^n$. Necessarily $m = n : i$ for some i. Define $\psi([\alpha]^n) = [\beta]^m$. Clearly $\iota_{m,n}\psi$ is the identity, thus ψ is injective, and $\#\mathcal{RE}(f_*^n, g_*^n) \leqslant \sum_{i=1}^k \#\mathcal{R}_{\mathcal{E}}(f_*^{n:i}, g_*^{n:i})$. Thus

$$\begin{aligned}
N\big(f^n, g^n\big) &= NP_n(f, g) + \#\mathcal{RE}\big(f^n, g^n\big) \\
&\leqslant NP_n(f, g) + \sum_{i=1}^{k} \#\mathcal{R}_{\mathcal{E}}\big(f_*^{n:i}, g_*^{n:i}\big) \\
&= NP_n(f, g) + \sum_{i=1}^{k} N\big(f^{n:i}, g^{n:i}\big),
\end{aligned}$$

which implies the result. □

Actually Theorem 4.6 gives a new result in periodic point theory namely:

Corollary 4.7. *If $f : X \to X$ is a map of a solvmanifold X, then with the notation of Theorem 4.6 we have that*

$$N\big(f^n\big) \geqslant NP_n(f) \geqslant N\big(f^n\big) - \sum_{i=1}^{k} N\big(f^{n:i}\big).$$

The following example further illustrates Theorem 4.6, and will provide a counterexample to Möbius inversion of the type found in [13].

Example 4.8. Continuing Examples 2.4 and 3.7 where we considered maps of S^1 of degrees 6 and 2 with $n=6$ then from Theorem 4.6 we have that

$$46\,592 \geqslant NP_6(f,g) \geqslant 46\,592 - 32 - 208 = 46\,352.$$

We now show that $NP_6(f,g) \neq N(f^6,g^6) - N(f^3,g^3) - N(f^2,g^2) + N(f,g) = 46\,352 + 4 = 46\,356$. From the proof of Theorem 4.6, we have that $NP_6(f,\ g) = R(f^n,g^n) - \#\bigcup_{m|n} \mathrm{Im}(\iota_{m,n}) = 46\,592 - \#(\mathrm{Im}(\iota_{2,6}) \cup \mathrm{Im}(\iota_{3,6}))$. (Im denotes the image of a homomorphism.) So in particular, we need to compute the cardinality of the intersection $\mathrm{Im}(\iota_{2,6}) \cap \mathrm{Im}(\iota_{3,6})$. In Example 3.7 we computed the boosts from 2 to 6 and from 3 to 6 to be multiplication by 1456 and 224 respectively. Thus $\mathrm{Im}(\iota_{3,6})$ is the subgroup of $\mathbb{Z}_{46\,592}$ of order 208 generated by 224, and similarly $\mathrm{Im}(\iota_{2,6})$ is the subgroup of $\mathbb{Z}_{46\,592}$ of order 32 generated by 1456. So $\mathrm{Im}(\iota_{2,6}) \cap \mathrm{Im}(\iota_{3,6})$ is the subgroup of $\mathbb{Z}_{46\,592}$ generated by 2912, the least common multiple of 224 and 1456. This subgroup is of order $46\,592/2912 = 16$. So then by the principle of inclusion and exclusion $NP_6(f,g) = 46\,592 - 32 - 208 + 16 = 46\,368 \neq 46\,356$.

We now start to work our way towards a full Möbius inversion formula. In many ways we are generalizing directly from [9], but of course looking at classes not orbits. This changes the definitions slightly.

Definition 4.9. Let $f,g : X \to X$ be coincidence essentially reducible. We say that the pair f, g is *injective on essential boosts to level n* if for all $m \mid n$ and for any classes $[\beta_1]^m$, $[\beta_2]^m$ at level m with $\iota_{m,n}([\beta_1]^m) = \iota_{m,n}([\beta_2]^m) \in \mathcal{R}_{\mathcal{E}}(f_*^n, g_*^n)$ then we have that $[\beta_1]^m = [\beta_2]^m$. If f, g are injective on essential boosts for all n, we say that is f, g *are injective on essential boosts*.

Lemma 4.10. *If $\pi_1(X)$ is Abelian, f_* and g_* commute, and* $\mathrm{Coin}(f_*^n, g_*^n) = 0$*, then the pair f, g is injective on essential boosts to level n. In particular if X is a torus and* $\det(F^n - G^n) \neq 0$*, then the pair F, G is injective on essential boosts to level n.*

Proof. Since $\pi_1(X)$ is Abelian, for all $m \mid n$ the functions $\iota_{m,n}$ are homomorphisms, and from Theorem 2.1 since $\mathrm{Coin}(f_*^n, g_*^n) = 0$, we have that $g_*^n - f_*^n : \pi_1(X) \to \pi_1(X)$ is injective. Now let $m \mid n$ and $[\beta]^m \in \mathcal{R}_{\mathcal{E}}(f_*^m, g_*^m)$ be such that $\iota_{m,n}([\beta]^m) = [0]^n$. Since the $\iota_{m,n}$ are homomorphisms we need only show that $[\beta]^m = [0]^m$.

So let $\beta \in [\beta]^m$. Since $\iota_{m,n}([\beta]^m) = [0]^n$, then $\iota_{m,n}([\beta]^m) \in \mathrm{Ker}(j_n) = \mathrm{Im}(g_*^n - f_*^n)$ from exactness in Theorem 2.1. So then there is a $\gamma \in \pi_1(X)$ such that $(g_*^n - f_*^n)(\gamma) = \iota_{m,n}(\beta)$. Composing with $g_*^n - f_*^n$ we have:

$$\big(g_*^n - f_*^n\big)\big(g_*^m - f_*^m\big)(\gamma) = \big(g_*^m - f_*^m\big)\big(g_*^n - f_*^n\big)(\gamma) = \big(g_*^m - f_*^m\big)\iota_{m,n}(\beta) = \big(g_*^n - f_*^n\big)(\beta).$$

But $g_*^n - f_*^n$ is injective, so actually $\beta = (g^m - f^m)(\gamma)$. But this means (again from exactness in Theorem 2.1) that $[\beta]^m = [0]^m$ as required.

For tori we have that $\pi_1(X) \cong \mathbb{Z}^r$ for some r. So when $\det(F^n - G^n) \neq 0$ then $\mathrm{Ker}(g_*^n - f_*^n) \cong \mathrm{Coin}(f_*^n, g_*^n) = 0$, and the result follows from the first part. □

Definition 4.11. Let $f,g : X \to X$ be maps. We say that the pair f, g is *coincidence essentially reducible to the gcd*, if they are coincidence essentially reducible, and whenever $[\alpha]^n \in \mathcal{R}_{\mathcal{E}}(f_*^n, g_*^n)$ reduces to both $[\beta]^m \in \mathcal{R}_{\mathcal{E}}(f^m, g^m)$ and $[\gamma]^k \in \mathcal{R}_{\mathcal{E}}(f_*^k, g_*^k)$, then there is a $[\delta]^d \in \mathcal{R}_{\mathcal{E}}(f^d, g^d)$ with $d = \gcd(m,k)$ to which both $[\beta]^m$ and $[\gamma]^k$ reduce. If every pair f,g is coincidence essentially reducible to the gcd, we say that X is *coincidence essentially reducible to the gcd*.

Example 4.12. The continuing Examples 2.4, 3.7 and 4.8, where $f=6$ and $g=2$ show that not all maps of tori satisfy the above definition. It is here that the Möbius formula breaks down. In particular if this example were essentially reducible to the gcd then we would have that the intersection $\mathrm{Im}(\iota_{2,6}) \cap \mathrm{Im}(\iota_{3,6})$ would coincide with $\mathrm{Im}(\iota_{1,6})$ which as Example 4.8 shows it does not.

We will prove the following theorem in Appendix A.

Theorem 4.13. *Let $f,g : S^1 \to S^1$ be maps of degrees $a,b \in \mathbb{Z}$ respectively. Then f, g is coincidence essentially reducible to the gcd if and only if either* $\gcd(a,b) = 1$*, or both a and b are zero.*

Example 4.14 *(The Klein Bottle example, Part I).* Let K^2 denote the Klein Bottle. We regard K^2 as the quotient space of $\mathbb{R}^2$ under the equivalence relation defined by $(s,t) \sim ((-1)^k s, t+k)$ and $(s,t) \sim (s+k,t)$ for any $k \in \mathbb{Z}$. The Klein Bottle fibres as $S^1 \to K^2 \xrightarrow{p} S^1$, where p is induced by projection on the second factor. Given any pair of integers q, r for which r is odd, or r is even and $q=0$, the correspondence $(s,t) \to (qs, rt)$ modulo the equivalence relation defined above, induces a well defined, fibre preserving map on K^2. We abuse notation and denote this map by (q,r). Since p is the projection on the second factor, the map on the base is the standard map of degree r, and the restriction to the principle fibre over the base point has degree q. We point the reader to [9,10] for details of all this. Let (a,c) and (b,d) be two such well defined maps on K^2. If $\gcd(a,b) = 1$ and $\gcd(c,d) = 1$, then the pair of maps (a,c) and (b,d) is injective on essential boosts and essentially reducible to the gcd.

Remark 4.15. A rigorous verification of this example is beyond the scope of this paper. The proofs however do not contain anything that is really new. They rely on coincidence versions of the fibre techniques used in [9], where a number of different properties of the fibre preserving maps are deduced from the very same properties on each of the fibres and the base (see for example [9, 4.4, 4.12]).

In Section 6 we will indicate the general theme in the coincidence theory of a pair f, g that the analogues of the strong results of periodic point theory hold when g is invertible. The following theorem shows that simply requiring g_* (but not g) to be invertible can also often yield strong results.

Theorem 4.16. *Suppose that $\pi_1(X)$ is Abelian, the pair f, g is coincidence essentially reducible, that f_*, g_* commute at the level of $\pi_1(X)$, are injective on essential boosts, and that $g_* : \pi_1(X) \to \pi_1(X)$ is invertible. Then the pair f, g is coincidence essentially reducible to the gcd.*

The proof below is a modification of Boju Jiang's proof of a similar result in periodic point theory (see [19, Proposition 4.4]). Our analogue of reducible to the gcd is more specific than Jiang's concept, in that he does not require δ in the proof below to boost to β and γ. It is here where we need injectivity on essential boosts.

Proof. For this proof (and actually for Appendix A as well), we want to refer both to coincidence boostings and periodic point boostings. So for these two places, we use the notation $\iota_{p,q}^{f_*,g_*}$ to refer to the sum (product) in Definition 3.5 (π_1 is Abelian here), and we use the notation $\iota_{p,q}^{t,1}$ to refer to the sum (to the corresponding product in Section 6)

$$\iota_{p,q}^{x,1} = 1 + x^p + x^{2p} + \cdots + x^{q-p},$$

where x is an arbitrary function on $\pi_1(X)$ and of course 1 is the identity function. Now let $[\alpha]^n \in \mathcal{R}(f_*^n,\ g_*^n)$ be essential and reducible to both $[\beta]^k \in \mathcal{R}(f_*^k, g_*^k)$, and $[\gamma]^m \in \mathcal{R}(f_*^m, g_*^m)$. As in [19] we may without loss assume that $d = \gcd(k,m) = 1$ (or we can work with the maps f^d and g^d). By Lemma 3.8, we may assume without loss, that we have representatives α, β and γ of $[\alpha]^n$, $[\beta]^k$ and $[\gamma]^m$ respectively, in $\pi_1(X)$ such that $\iota_{k,n}^{f_*,g_*}(\beta) = \alpha$ and $\iota_{m,n}^{f_*,g_*}(\gamma) = \alpha$. Now since g_* is invertible, we have that

$$\begin{aligned}
\iota_{k,n}^{f_*,g_*}(\beta) &= \left(g_*^{n-k} + g_*^{n-2k} f_*^k + \cdots + g_*^k f_*^{n-2k} + f_*^{n-k}\right)(\beta) \\
&= g_*^{n-k}\left(1 + \left(g_*^{-1} f_*\right)^k + \cdots + \left(g_*^{-1} f_*\right)^{n-2k} + \left(g_*^{-1} f_*\right)^{n-k}\right)(\beta) \\
&= g_*^{n-k} \iota_{k,n}^{g_*^{-1} f_*,1}(\beta) = \alpha.
\end{aligned}$$

Similarly $\iota_{m,n}^{f_*,g_*}(\gamma) = g_*^{n-m} \iota_{m,n}^{g_*^{-1} f_*,1}(\gamma) = \alpha$, and $\iota_{1,n}^{f_*,g_*}(\delta) = g_*^{n-1} \iota_{1,n}^{g_*^{-1} f_*,1}(\delta)$ for any δ.

Since $\gcd(k,m) = 1$ we may again without loss, assume we are given positive integers a and b with $ak - bm = 1$. Let $P(x) = 1 + x^k + \cdots + x^{(a-1)k}$ and $Q(t) = -x(1 + x^m + \cdots + x^{(b-1)m})$, then

$$P(x)\left(1 + x + \cdots + x^{k-1}\right) + Q(x)\left(1 + x + \cdots + x^{m-1}\right) = 1,$$

or $P(x)\iota_{1,k}^{x,1}(\gamma) + Q(x)\iota_{1,m}^{x,1}(\gamma) = \gamma$ for all x.

Set $\delta = P(g_*^{-1} f_*) g^{-k+1}(\beta) + Q(g_*^{-1} f_*) g^{-m+1}(\gamma)$. We will use $x = g_*^{-1} f_*$ below. Note that this commutes with everything in sight, and we have

$$\begin{aligned}
\iota_{1,n}^{f_*,g_*}(\delta) &= g_*^{n-1} \iota_{1,n}^{g_*^{-1} f_*,1}(\delta) \\
&= g_*^{n-1} \iota_{1,n}^{g_*^{-1} f_*,1} P\left(g_*^{-1} f_*\right) g^{-k+1}(\beta) + g_*^{n-1} \iota_{1,n}^{g_*^{-1} f_*,1} Q\left(g_*^{-1} f_*\right) g^{-m+1}(\gamma) \\
&= P\left(g_*^{-1} f_*\right) g_*^{n-k} \iota_{1,k}^{g_*^{-1} f_*,1} \iota_{k,n}^{g_*^{-1} f_*,1}(\beta) + Q\left(g_*^{-1} f_*\right) g_*^{n-m} \iota_{1,m}^{g_*^{-1} f_*,1} \iota_{m,n}^{g_*^{-1} f_*,1}(\gamma) \\
&= P\left(g_*^{-1} f_*\right) \iota_{1,k}^{g_*^{-1} f_*,1} g_*^{n-k} \iota_{k,n}^{g_*^{-1} f_*,1}(\beta) + Q\left(g_*^{-1} f_*\right) \iota_{1,m}^{g_*^{-1} f_*,1} g^{n-m} \iota_{m,n}^{g_*^{-1} f_*,1}(\gamma) \\
&= P\left(g_*^{-1} f_*\right) \iota_{1,k}^{g_*^{-1} f_*,1}(\alpha) + Q\left(g_*^{-1} f_*\right) \iota_{1,m}^{g_*^{-1} f_*,1}(\alpha) = \alpha.
\end{aligned}$$

Since the pair f, g is coincidence essentially reducible, then δ is essential. That $\iota_{1,k}^{f_*,g_*}(\delta) = \beta$ and $\iota_{1,m}^{f_*,g_*}(\delta) = \gamma$ follows from injectivity on essential boosts. □

A correct proof of the periodic point analogue of the following theorem can be found in [9] (there are errors in the proof in [14]). Since the proof contains no new ideas, it is omitted.

Theorem 4.17. *Suppose that the pair f, g is coincidence essentially reducible to the gcd, that f_*, g_* commute, and are injective on essential boosts. If f^n, g^n are coincidence weakly Jiang and $N(f^n, g^n) \neq 0$, then for all $m \mid n$ we have*

$$NP_m(f) = \sum_{\tau \subset \mathbf{p}(m)} (-1)^{\#\tau} N\big(f^{m:\tau}, g^{m:\tau}\big)$$

where $\mathbf{p}(m)$ denotes the set of prime divisors of m and $m : \tau = m \prod_{p \in \tau} p^{-1}$. □

Example 4.18 *(The Klein Bottle example, Part II).* Continuing Example 4.14 assume that we have maps (a, c) and (b, d) with $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$, then from Example 4.14 the pair is injective on essential boosts, and essentially reducible to the gcd. Moreover $a \neq \pm b$ and $c \neq \pm d$, and a coincidence version of [9, Corollary 4.26] will give that the pair (a, c), (b, d) is weakly Jiang with $N((a, c)^n, (b, d)^n) \neq 0$ for any n. Thus the pair (a, c), (b, d) satisfies the hypothesis of 4.17 for all n. Using the naïve addition formula for coincidences ([15,8] and coincidence distribution considerations analogous to [11, Theorem 4.8]) we have

$$N\big((a, c)^n, (b, d)^n\big) = \frac{|c^n - d^n|}{2} \sum_{j=0}^{1} \big|a^n + (-1)^j b^n\big|.$$

Let $(a, b) = (2, 3)$ and $(c, d) = (3, 5)$, then we have

$$N(f, g) = \frac{2}{2}(1 + 5) = 6, \qquad N\big(f^2, g^2\big) = \frac{16}{2}(5 + 13) = 144,$$
$$N\big(f^3, g^3\big) = \frac{98}{2}(19 + 35) = 266, \qquad N\big(f^6, g^6\big) = \frac{14\,896}{2}(665 + 793) = 10\,859\,184,$$

so $NP_6(f, g) = 10\,856\,400$.

Example 4.19. Consider the following commuting matrices

$$F = \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix},$$

which we regard as maps of T^2. Note that G is invertible over $\mathbb{Z}$, and that F and G commute. So F and G are essentially reducible to the gcd at level n by Theorem 4.16. Recall that $N(F^n, G^n) = |\det(F^n - G^n)|$, so by Theorem 4.17 we have

$$\begin{aligned} NP_{30}(F, G) &= \big|\det\big(F^{30} - G^{30}\big)\big| - \big|\det\big(F^{15} - G^{15}\big)\big| - \big|\det\big(F^{10} - G^{10}\big)\big| - \big|\det\big(F^{6} - G^{6}\big)\big| \\ &\quad + \big|\det\big(F^{5} - G^{5}\big)\big| + \big|\det\big(F^{3} - G^{3}\big)\big| + \big|\det\big(F^{2} - G^{2}\big)\big| - \big|\det(F - G)\big| \\ &= 221\,073\,919\,719\,792\,987\,930\,625 - 470\,183\,304\,961 - 60\,450\,625 - 46\,225 \\ &\quad + 7561 + 181 + 25 - 1 = 221\,073\,919\,719\,322\,744\,136\,580. \end{aligned}$$

5. The analogue $N\Phi_n(f, g)$ of the periodic point number $N\Phi_n(f)$

In Nielsen periodic point theory the second number $N\Phi_n(f)$ satisfies:

$$\sum_{m \mid n} NP_m(f) \leqslant N\Phi_n(f) \leqslant M\Phi_n(f) = \min\big\{\#\Phi\big(f_1^n\big) \bigm| f_1 \simeq f\big\}.$$

In the introduction we saw in Example 1.1, that the number $MC_n(f, g) = \min\{\#\Phi(f_1^n, g_1^n) \mid f_1 \simeq f,\ g_1 \simeq g\}$ may not take into account coincidences of iterates at levels other than n. So what then is our second number $N\Phi_n(f, g)$ to measure, or to put it another way, of what is $N\Phi_n(f, g)$ to be a lower bound? In answering this question, we will need to take into account the phenomenon encountered in the next example, which shows that we can have

$$P_m(f, g) \cap P_k(f, g) \neq \emptyset \quad \text{for } m \neq k.$$

Example 5.1. For this example we consider the circle S^1 as the interval $[0, 1]$ with endpoints identified. Let $\bar{0} = \{0, 1\}$ be the base point, and f the standard map of degree 2 on the usual presentation of S^1 as a quotient of $\mathbb{R}$, conjugated with a homeomorphism between the two presentations. Let $\epsilon \in (0, 1)$ be the real number defined below, and let g be the degree one map $g(x) = x^\epsilon$. Note that $g([0, 1]) = [0, 1]$ with $g(0) = 0$ and $g(1) = 1$ and so g is well defined on S^1.

We will show that there is a point $q \in P_2(f, g) \cap P_7(f, g)$. It can be verified numerically that there are $q, \epsilon \in [0, 1]$ with $q \approx 0.00989471$ and $\epsilon \approx 0.836457$ satisfying:

$$4q = q^{\epsilon^2},$$
$$128q = q^{\epsilon^7} + 1.$$

Then we have that $f^2(q) = 8q = q^{\epsilon^2} = g^2(q)$ and

$$f^7(q) = 128q = q^{\epsilon^7} + 1 = q^{\epsilon^7} = g^7(q) \quad \text{in } S^1,$$

but $f(q) \neq g(q)$. We have shown for this example that $P_2(f, g) \cap P_7(f, g) \neq \emptyset$.

The phenomenon of the example cannot of course happen in periodic point theory. If $f^2(x) = x = f^7(x)$, then $f(x) = f(f^7(x)) = f^2(f^2(f^2(f^2(x)))) = x$! We will say more later. For now it should be clear that in defining $N\Phi_{14}(f, g)$ in this example, we would need to take account of the classes of q at both level 2 and at level 7. It is this that lies behind our definition of $M\Phi_n(f, g)$ which we recall from the introduction is defined to be:

$$M\Phi_n(f, g) = \min\left\{ \# \bigsqcup_{m|n} P_m(f_1, g_1) \;\middle|\; f_1 \simeq f, g_1 \simeq g \right\}.$$

The main goals of this section are to define a Nielsen type number $N\Phi_n(f, g)$, show it is a lower bound for $M\Phi_n(f, g)$, and give some of its properties together with a number of examples. We adapt the definition of the periodic point number $N\Phi_n(f)$ taking account of the following: In periodic point theory $N\Phi_n(f)$ is defined in terms of sets of n-representatives of orbits and of the heights and depths thereof. Since (as in the definition of $NP_n(f, g)$) we cannot use orbits (and so neither height nor depth), we simply work with classes. Apart from this the definitions are entirely analogous.

For a fixed positive integer n, a set

$$\mathcal{G} \subset \bigsqcup_{m|n} \mathcal{R}\big(f_*^m, g_*^m\big)$$

is called a *coincidence set of n-representatives for f and g* if each class of $\bigsqcup_{m|n} \mathcal{R}_{\mathcal{E}}(f_*^m, g_*^m)$ reduces to, or is equal to, some element of $\mathcal{G}$, where as usual $\mathcal{R}_{\mathcal{E}}(f_*^m, g_*^m)$ denotes the set of essential Reidemeister classes at level m.

Definition 5.2. Let $f, g : X \to X$ be maps with $f_* g_* = g_* f_* : \pi_1(X) \to \pi_1(X)$. The *full Nielsen type number $N\Phi_n(f, g)$* is defined to be the minimal size among all sets of coincidence n representatives.

We are grateful to the referee for the following example.

Example 5.3. Let $f : S^2 \to S^2$ be the antipodal map defined on $x \in S^2$ to be $f(x) = -x$, and let g be the identity. Since $\pi_1(S^2)$ is trivial, there is just one Reidemeister class at each level. At level 2, the class is essential and reducible. At level 1, the class is inessential, and in fact empty. There are two minimum sets of 2-representatives each consisting of one of the classes just defined. So then $N\Phi_2(f, 1) = N\Phi_2(f) = 1$.

It seems worth noting that the pair f, 1 is not essentially reducible.

Theorem 5.4. *The number $N\Phi_n(f, g)$ is homotopy invariant, and satisfies the inequalities*:

$$\sum_{m|n} NP_m(f, g) \leqslant N\Phi_n(f, g) \leqslant M\Phi_n(f, g).$$

If the pair f, g is coincidence essentially reducible then

$$N\Phi_n(f, g) = \sum_{m|n} NP_m(f, g).$$

Example 5.3 shows that the left inequality above may be strict.

Proof. The proof of homotopy invariance is analogous to the proof in [14] for periodic points, and involves considering what turn out to be isomorphic systems of coincidence n-representatives. In fact, the only real difference is that we deal here with classes, rather than with orbits.

For the second inequality, recall that $M\Phi_n(f, g)$ is defined in terms of the disjoint union of the $P_m(f, g)$. By homotopy invariance we need only show, for an arbitrary pair f, g that $\#(\bigsqcup_{m|n} P_m(f, g)) \geqslant N\Phi_n(f, g)$. We use $\bigsqcup_{m|n} P_m(f, g)$ to define a set of n representatives as follows. Let $x \in \bigsqcup_{m|n} P_m(f, g)$, then $x \in P_j(f, g)$ for some $j \mid n$. So $\rho_j([x]) \in \mathcal{R}(f_*^j, g_*^j)$. Let $\mathcal{S}$ be the set of all such $\rho_j([x])$ for all x and for all j, and let $[\alpha]^k \in \mathcal{R}_{\mathcal{E}}(f_*^k, g_*^k)$ be arbitrary, where $k \mid n$. To show that $\mathcal{S}$ is a set of n-representatives, we must show that $[\alpha]^k$, reduces to some element of $\mathcal{S}$. Since $[\alpha]^k$ is essential, then

$[\alpha]^k = \rho_k([x])$ for some $x \in \Phi(f^k, g^k)$. Let $m \mid k$ be the least positive integer such that $x \in \Phi(f^m, g^m)$, then $x \in P_m(f, g)$. It is not hard to see from the definitions that $\rho_m([x])$ boosts to $[\alpha]^k$, even if f and g do not commute (so we cannot use Lemma 3.9). We can of course use the same path to define the ρ at levels m and k. Thus $\mathcal{S}$ is indeed a set of n-representatives and

$$\#\left(\bigsqcup_{m|n} P_m(f, g)\right) \geqslant \#(\mathcal{S}) \geqslant N\Phi_n(f, g),$$

as required.

The first inequality follows since any set of n-representatives will contain the set of all irreducible essential classes. In particular we will always have that $\#\mathcal{G} \geqslant \sum_{k|n} NP_n(f, g)$ for any set of n-representatives $\mathcal{G}$. Equality occurs exactly when $\mathcal{G}$ is the set of all irreducible essential classes, and this happens when the pair f, g is coincidence essentially reducible. □

Remark 5.5. We want to make a comment about what could happen in this proof if we defined $M\Phi_n(f, g)$ in terms of the ordinary rather than the disjoint union. The main point is that because, as in Example 5.1, the P_m need not be disjoint, nor can we (automatically) assume that coincidence classes are not singletons, so then neither can we deduce that $\#(\bigcup_{m|n} P_m(f, g)) \geqslant \#(\mathcal{S})$. We will come back to this point in Section 8 where we discuss related open questions.

By analogy with Proposition 4.3, we compare $N\Phi_n(f, \mathrm{id})$ and $N\Phi_n(f)$.

Proposition 5.6. *If $f : X \to X$ is essentially toral, then $N\Phi_n(f, \mathrm{id}) = N\Phi_n(f)$.* □

Theorem 5.7. *We have $N\Phi_n(f, g) \geqslant N(f^m, g^m)$ for all $m \mid n$. Moreover if f, g are coincidence essentially reducible to the gcd at level n with f^n and g^n coincidence weakly Jiang, $N(f^n, g^n) \neq 0$ and the $\iota_{m,n}$ injective on essential boosts for all $m \mid n$, then*

$$N\Phi_n(f, g) = N\big(f^n, g^n\big).$$

Proof. The first part is easy, since any set of n-representatives must contain at least one class to which each of the essential class of $\mathcal{R}(f^m, g^m)$ reduce. Apart from the fact that we are working with classes rather than orbits, the second part of the proof works without modification from periodic point theory (see [9,10]). □

Examples 5.8. Continuing Example 4.18 on the Klein Bottle, we have that $N\Phi_6(f, g) = N(f^6, g^6) = 10\,859\,184$. Continuing Example 4.19 on the 2 torus, we have that $N\Phi_{30}(F, G) = N(F^{30}, G^{30}) = 221\,073\,919\,719\,792\,987\,930\,625$.

The next result generalizes its analogue in [14], and shows what happens, in Theorem 5.7, when we allow $N(f^n, g^n)$ to be equal to zero. Its proof contains nothing new and is omitted. For given $f, g : X \to X$, and a fixed natural number n, we define the set $M(f, g, n)$ as the set of $m \mid n$ with $N(f^m, g^m) \neq 0$.

Theorem 5.9. *Suppose for a fixed positive integer n we have that for each $m \in M(f, g, n)$ the pair f, g is essentially reducible to the gcd at level m, that the f^m, g^m are weakly Jiang, and furthermore that the $\iota_{q,m}$ are injective on essential boosts for each $q \mid m$. Then*

$$N\Phi_n(f, g) = \sum_{\emptyset \neq \mu \subseteq M(f,g,n)} (-1)^{\#\mu - 1} N\big(f^{\xi(\mu)}, g^{\xi(\mu)}\big),$$

where $\xi(\mu)$ is the gcd of the elements of μ. Furthermore, Möbius inversion can be used to obtain the $NP_q(f, g)$ for each $q \mid m$ for $m \in M(f, g, n)$.

6. Connections with periodic point theory when g is invertible

This section should be thought of as an extended remark, rather than a section where rigorous proofs are given. Our intention is simply to inform intuition. We start with the fact that when g is invertible and f and g commute, the set of coincidences of iterates for the pair f, g is exactly the same as the set of periodic points of the map $g^{-1}f$. If in addition X is essentially toral (where in the context of periodic point theory, counting points by depth of orbit has no advantage), then there is an "isomorphism" of the coincidence Nielsen theory of iterates given here, and the Nielsen periodic point theory of $g^{-1}f$. With the exception of Example 6.9 which is given in this section for the purpose of illustration, all examples given in this paper are on spaces that in the periodic point sense are essentially toral.

6.1. *Relationships for invertible g, with the periodic point numbers $NP_n(g^{-1}f)$ and $N\Phi_n(g^{-1}f)$*

If we specifically assume that g is invertible, and that f and g commute, then the coincidence points of f^n and g^n are exactly the same as the fixed points of $(g^{-1}f)^n$. To see this, suppose that $g^n(x) = f^n(x)$, then (composing both sides with g^{-n}) we have $x = g^{-n}(f^n(x)) = (g^{-1}f)^n(x)$. This of course is reversible. Note that this requires the commutativity of f and g as maps. On the other hand, we need only the invertibility of g_*, together with the commutativity of f_* and g_*, to effect a well defined correspondence between $\mathcal{R}(f_*^n, g_*^n)$ and $\mathcal{R}((g_*^{-1}f_*)^n)$. In fact this one to one correspondence goes much deeper, as we will now state, but do not prove. We use the notation of [8].

Theorem 6.1. *If g_* is invertible, and f_* and g_* commute at the π_1 level, then for all $m \mid n$ the homomorphisms $g_*^{-m} : \pi_1(X) \to \pi_1(X)$ induce well-defined bijections $g_*^{-m} : \mathcal{R}(f_*^m, g_*^m) \to \mathcal{R}((g_*^{-1}f)^m)$, and the left-hand diagram below commutes. If g itself is invertible, and f and g commute as maps, then the equality in the right-hand diagram exists, and the diagram commutes.*

$$\begin{array}{ccc} \mathcal{R}(f_*^m, g_*^m) & \xrightarrow{\iota_{m,n}^{f_*,g_*}} & \mathcal{R}(f_*^n, g_*^n) \\ {\scriptstyle g_*^{-m}}\downarrow & & \downarrow{\scriptstyle g_*^{-n}} \\ \mathcal{R}((g_*^{-1}f_*)^m) & \xrightarrow{\iota_{m,n}^{g_*^{-1}f_*,1}} & \mathcal{R}((g_*^{-1}f_*)^n) \end{array} \qquad \begin{array}{ccc} \Phi(f^n, g^n)/\sim & \xrightarrow{\rho_n} & \mathcal{R}(f_*^n, g_*^n) \\ \| & & \downarrow{\scriptstyle g_*^{-n}} \\ \Phi((g^{-1}f)^n)/\sim & \xrightarrow{\rho_n} & \mathcal{R}((g_*^{-1}f_*)^n) \end{array}$$

where notations for the boosts are the product versions of those in the proof of Theorem 4.16. □

In order to avoid getting into too many technical considerations, we make the following definition.

Definition 6.2. Let f_* and g_* commute and g_* be invertible. We say that the function $g_*^{-m} : \mathcal{R}(f_*^m, g_*^m) \to \mathcal{R}((g_*^{-1}f)^m)$ is *essentiality preserving at level m* if a class $[\alpha]^m \in \mathcal{R}(f_*^m, g_*^m)$ is essential if and only if the class $g_*^{-m}([\alpha]^m)$ is essential in $\mathcal{R}((g_*^{-1}f)^m)$.

In the context of tori, when $N(f,g)$ or $N(g^{-1}f) \neq 0$ (see below), then all fixed or coincidence classes of linearized maps are singletons and essential. So then all the examples in this paper on tori are easily seen to be essentiality preserving at every level. It is a bit more work to see that the definition is satisfied for pairs of commuting maps on nil- and solvmanifolds. The product theorem for semi-index [16], together with the usual product formula for the fixed point index can be used in this regard. The definition holds in Example 6.9 for a different reason.

The main result which we state, but do not prove, is the following:

Theorem 6.3. *Let f and g be such that f_* and g_* commute, with g invertible, and suppose that the homotopy class of f and g contains a pair of commuting maps. If g_*^{-m} is essentiality preserving for all $m \mid n$ and $g^{-1}f$ is essentially toral, then*

$$NP_n(f,g) = NP_n\big(g^{-1}f\big) \quad \textit{and} \quad N\Phi_n(f,g) = N\Phi_n\big(g^{-1}f\big). \qquad \square$$

We cannot rescind the hypothesis of essential torality (though it can be weakened). To see this, we can use Jiang's classical example on $\mathbb{R}P^3$ [19, Example 4, p. 67], where of course g is the identity. Jiang's example illustrates why we need to use orbits in periodic point theory. The equalities in the theorem can be observed in practice with Example 4.19, by using, in the case of the NP_n, the periodic point Möbius inversion formula for tori [9, Theorem 1.2]. In the case of the $N\Phi_n$ we need the analogue of the last part of Theorem 5.7 (also [9, Theorem 1.2]). For both numbers we also need the following observation for invertible G:

$$\begin{aligned} N\big(F^n, G^n\big) &= \big|\det\big(G^n - F^n\big)\big| \\ &= \big|\det\big(G^n\big)\big|\big|\det\big(I - G^{-n}F^n\big)\big| \\ &= \big|\det\big(I - G^{-n}F^n\big)\big| = N\big(G^{-n}F^n\big). \end{aligned}$$

The penultimate step follows since $|\det(G^n)| = 1$. This is because when G is invertible over $\mathbb{Z}$, we must have that $\det(G) = \pm 1$.

Remark 6.4. The result of Theorem 6.3 that for invertible g we have that $NP_n(f,g) = NP_n(g^{-1}f)$ and $N\Phi_n(f,g) = N\Phi_n(g^{-1}f)$ when the spaces are essentially toral, gives a strong indication that we have not really lost anything by not being able to work with orbits on these spaces. The earlier comments about classes being singletons also confirm this, for the advantage of orbits only works when there is more than one non-removable point in each Nielsen class. We will not explore this here, but these considerations are related to the question of these spaces being Wecken, that is when the homotopy classes of our maps contain representatives that attain the given lower bound. As the work of You in periodic point theory demonstrates [22], in the presence of Nielsen numbers that are zero, the proofs get complicated even on tori.

6.2. Invertible g, and orbits

When g is invertible, and the pair f and g contains a pair of commuting maps within their homotopy classes, then there is a way to define orbits. We simplify this subsection by making the blanket assumption that the commuting pair has been chosen, leaving the subtler details of the theory as given in section three, to the reader. In fact we will leave a great deal to the reader, dealing only with the NP number. Consistent with the stated goal of this section we only include enough to inform intuition. The definition and lemma below are intended to indicate that the invertibility of g allows us to define orbits. In particular the lemma gives that part of [13, Proposition 1.14] that was missing from Lemma 3.9 in Section 3.

Definition 6.5. Let f, g be a pair of commuting self maps, with g invertible as a map, and let $x \in \Phi(f^n, g^n)$. Then the *geometric orbit* of x is the set

$$\left\{x, g^{-1}f(x), g^{-2}f^2(x), \ldots, g^{-n+1}f^{n-1}(x)\right\}.$$

If $x \in P_n(f, g)$ then the elements of the above list are all distinct, and clearly since $g^n(x) = f^n(x)$, then $g^{-n}f^n(x) = x$. The other analogous definitions of periodic point theory are forthcoming.

Lemma 6.6. *(Cf. [13, Proposition 1.14].) Under the conditions of the definition the function $g^{-1}f$ induces well defined functions $g^{-1}f : \Phi(f^n, g^n) \to \Phi(f^n, g^n)$. This function respects both the Nielsen and Reidemeister relationships, and induces essentiality preserving functions (denoted $g^{-1}f$ and $(g^{-1}f)_*$ respectively) on the respective sets of classes. Moreover, the following diagram is commutative*

$$\begin{array}{ccc} \Phi(f^n, g^n)\sim & \xrightarrow{\rho_n} & \mathcal{R}(f^n_*, g^n_*) \\ {\scriptstyle g^{-1}f}\downarrow & & \downarrow{\scriptstyle (g^{-1}f)_*} \\ \Phi((g_{-1}f)^n)\sim & \xrightarrow{\rho_n} & \mathcal{R}((g_{-1}f)^n_*), \end{array}$$

so that both geometric and algebraic orbits are well defined. Thus the notion of irreducible and essential orbit is well defined. □

Definition 6.7. Let f, g be a pair of self maps whose homotopy classes contain a commuting pair, and suppose that g is invertible. We define $NP_n^{\mathcal{INV}}(f, g)$ to be n times the number of irreducible essential periodic point orbits of the map $g^{-1}f$. That is $NP_n^{\mathcal{INV}}(f, g) := NP_n(g^{-1}f)$, where $NP_n(g^{-1}f)$ is the usual periodic point number of the map $g^{-1}f$ defined in [19].

Theorem 6.8. *Under the conditions of Definition 6.7 we have that $NP_n^{\mathcal{INV}}(f, g)$ is homotopy invariant in f and g, and*

$$NP_n(f, g) \leqslant NP_n^{\mathcal{INV}}(f, g) \leqslant MP_n(f, g) \leqslant \#\big(P_n(f, g)\big).$$

If X is a torus, a nilmanifold or a solvmanifold, then

$$NP_n^{\mathcal{INV}}(f, g) = NP_n(f, g). \qquad \square$$

The last part of Theorem 6.8 follows, since in the indicated spaces, when the Nielsen numbers are non-zero all classes can be homotoped to singletons. Thus even when orbits can be defined there will be no advantage gained by using them. We say more in Section 8.

In Theorem 4.6 we showed that $N(f^n, g^n) \geqslant NP_n(f, g)$. But this does not generalize to the $NP_n^{\mathcal{INV}}(f, g)$ numbers, as the following example shows:

Example 6.9. Let $\tilde{f}, \tilde{g} : S^2 \to S^2$ be maps of degree 3 and -1 respectively. We can think of them as the respective suspensions of the same degree maps on S^1. In this way $\tilde{f}$ and $\tilde{g}$ are seen to be $\mathbb{Z}_2$ equivariant maps that induce self maps f and g respectively on Real Projective space $\mathbb{R}P^2$. It follows easily that f and g induce identity homomorphisms f^n_* and g^n_* on $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, for all positive integers n. Jezierski in [17, Corollary 5.1] has worked out $N(f, g)$ in detail for all pairs of self maps f, g of $\mathbb{R}P^2$. In particular for our f and g we have that $N(f^n, g^n) = 2$ for all positive integers n. Furthermore since g is invertible, f and g commute, and $(g^{-1}f)_*$ is the identity, then each periodic point orbit of $g^{-1}f$ has length 1 at every level. Now let $n = 2^r$ for some positive integer r. Since for any $m \mid n$ the number n/m must be even, it is not hard to see that $\iota_{m,n}$ is multiplication by an even integer, that is, it is the zero function on $\mathbb{Z}_2$. In particular the orbit $\langle [1]^n \rangle$ is irreducible and essential. From above

$$NP_n^{\mathcal{INV}}(f, g) = NP_n\big(g^{-1}f\big) = n = 2^r > N\big(f^n, g^n\big) = 2.$$

7. Roots of iterates and the work of Brown, Jiang and Schirmer

As mentioned in the introduction, Brown, Jiang and Schirmer [1] have already given a Nielsen theory of roots of iterates. One could be forgiven for assuming that putting g equal to a constant map in our theory would give many of the results of [1]. But this is false! Actually there is very little intersection between the two theories. In fact they are profoundly incompatible in at least two important ways. The first is related to the incompatibility of the work in [1] with the periodic point theory we are seeking to generalize. The second incompatibility is reflected in the fact that the root theory one obtains by putting g equal to a constant map in our theory is homotopy invariant with respect to such g, and the work in [1] is not. We illustrate these differences with the following example, in which our f is exactly the same as the f used in [1, Example 6.1].

Example 7.1. Let $f, a : S^1 \to S^1$ be maps with $f(z) = z^2$ and a the constant map $a(z) = 1$. Then $N(f^n, a^n) = 2^n$. When p is a prime and $n = p^k$ where $k \neq 0$ is a positive integer, then any reducible class at level p^k in our theory will be in the image of ι_{p^{k-1},p^k}. Thus there are $2^{p^{k-1}}$ reducible classes at level n. So then

$$NP_{p^k}(f, a) = 2^{p^k} - 2^{p^{k-1}},$$

and by Theorem 5.4 we have

$$N\Phi_{p^k}(f, a) = \sum_{i=0}^{k} NP_{p^i}(f, a) = 2 + \sum_{i=1}^{k} NP_{p^i}(f, a) = 2 + \sum_{i=1}^{k} 2^{p^i} - 2^{p^{i-1}} = 2^{p^k}.$$

In [1], the authors define a Nielsen type number denoted $NI_n(f, a)$, which they call the *Nielsen number of irreducible roots at level n of f at a*. As we will see below, while this sounds like our $NP_n(f, a)$, it is not. We say more below, but we give neither the definition, nor details of their computations.

With the very same f and a, the number $NI_n(f, a)$ is computed in [1, Example 6.1] to be

$$NI_n(f, a) = \begin{cases} 2 & \text{for } n = 1, \\ 2^{n-1} & \text{for } n > 1. \end{cases}$$

If n is prime, then $NP_n(f, a) = 2^n - 2$, and $N\Phi_n(f, a) = 2^n$, while $NI_n(f, a) = 2^{n-1}$, so $NI_n(f, a)$ is different from both $NP_n(f, a)$ and $N\Phi_n(f, a)$.

Remark 7.2 *(Homotopy invariance incompatibility).* As we mentioned above, the number $NI_n(f, a)$ depends on the variable "a". In this regard, we were careful to state in the last part of the example above, that the computation of $NI_n(f, a)$ from [1] was with respect to the very same f and a as in the first part of the example. Since S^1 is path connected, any two constant maps are homotopic. In our theory this means that the root theory obtained by putting g equal to constant map at a is independent of the choice of $a \in S^1$. This is not the case with the root theory in [1]. In [1], the number $NI_n(f, a)$ is dependent on the period of the chosen a under f (which can be either finite or infinite). In particular in the very same example in [1], when $a(z) = -1$ (so a has period 2), then with the same f, we have that $NI_n(f, a) = 2^n$.

Remark 7.3 *(Incompatible definitions of "boosting").* A second way the two theories are incompatible is to be found in the very definition of reducible classes. Though we use the same words to describe $NI_n(f, a)$ and $NP_n(f, a)$ (each is defined to be the number of irreducible essential classes), the two theories are very different because these words mean different things in the two papers. The main point is that in [1] there are "boosts" from levels m to n for certain m which do not divide n.

Consider a hypothetical situation where we have an f with $f(a) = a$ for some $a \in X$. As above, by abuse of notation we use a to denote the function with $a(x) = a$ for all x. Note that whenever $f^m(x) = a$ then $f^{m+1}(x) = f(f^m(x)) = f(a) = a$ too. Suppose, for example, that x is a root of f at a of least period 4. Then $f^4(x) = a$ but $f^j(x) \neq a$ for any $j < 4$. But $f^5(x) = f(a) = a$, and of course $4 \nmid 5$. Theorem 2.1 of [1] (about which we will say more below) implies that this kind of boosting extends to Nielsen classes in a natural way. So then in [1] the class at level 5 could be geometrically reducible for Brown, Jiang and Schirmer, but geometrically irreducible in our theory.

The algebraic side of this type of reducibility for constant maps a can also work. To see this, note that a_* is the trivial homomorphism. In this case, Definition 3.5 simply becomes $\iota_{m,n}(\alpha) = f_*^{n-m}(\alpha)$, and this can be well defined on Reidemeister classes even when m does not divide n.

We wish to say one more thing about incompatibility of the two theories. It pertains to the difficulty of generalizing the theory of roots of iterates of the type given in [1], to coincidences. The problems start with the fundamental result of [1] (Theorem 2.1) upon which their whole paper is based. That theorem states that if R^m and R^n are root classes with $m < n$, and if $R^m \cap R^n \neq \emptyset$, then $R^m \subseteq R^n$. Example 5.1 provides a ready made counter example to a coincidence generalization of this phenomenon. Recall that we constructed a $q \in P_2(f, g) \cap P_7(f, g)$. We contend that 0 and q are Nielsen equivalent at level 2, but not at level 7. To see this let c be the linear path from 0 to q. Since neither of the paths $g^2(c)$ nor $f^2(c)$

traverse $\bar{0}$, we have that $g^2(c) \sim f^2(c)$ at level 2. On the other hand while $g^7(c)$ does not traverse $\bar{0}$, the path $f^7(c)$ does (in fact $g^7(c^{-1})f^7(c)$ is a generator of $\pi_1(S^1)$), and so 0 and q cannot be Nielsen equivalent at level 7.

8. Wecken type questions

We conclude the paper with two open questions. The first asks when $M\Phi_n(f,g)$ can be written in terms of the ordinary (rather than disjoint) union, the second is the obvious Wecken question, however it should be clear that the two questions are related.

Open Question 8.1. On manifolds, do we have

$$M\Phi_n(f,g) = \min \#\left\{ \bigcup_{m|n} P_m(f_1,g_1) \,\middle|\, f_1 \sim f \text{ and } g_1 \sim g \right\}?$$

The question we are asking is if maps that actually attain the minimum number at all levels simultaneously would ever contain "stray" coincidences of iterates (like q in Example 5.1) that lie in the intersection of different $P_m(f,g)$. Our feeling (guess) is that such points are rare, and that they do not form essential singleton classes. In other words it is our feeling that (again like q in Example 5.1) such stay points lie in Nielsen classes that are either not essential, or not singletons. In either case we should be able by some homotopy, either to remove them, or move them even slightly, so that they no longer belong to more than one of the $P_m(f,g)$. However as we will indicate below, the two numbers coincide on tori, nilmanifolds and model solvmanifolds [12] when the Nielsen numbers are not zero.

Open Question 8.2. Under what conditions are the numbers $NP_n(f,g)$ and $N\Phi_n(f,g)$ Wecken, so that there exist maps $f_1 \simeq f$ and $g_1 \simeq g$ such that $NP_n(f,g) = MP_n(f,g)$ and $N\Phi_n(f,g) = M\Phi_n(f,g)$?

Question 8.2 should probably be two questions, one for each number. The following example shows that such conditions do exist.

Example 8.3. Let $f, g : T^2 \to T^2$ be maps of the 2 torus with linearizations F and G respectively, where F and G are as given in Example 4.19. Then f and g satisfy the hypotheses of the last part of Theorem 5.7, and so

$$N\Phi_n(f^n, g^n) = \left|\det(F^n - G^n)\right| = \#\left(\Phi\left(F^n, G^n\right)\right) = M\Phi_n(f,g).$$

The second equality holds, since when in situations like this $\det(F^n - G^n) \neq 0$, the linear representations of f and g have singleton coincidence classes. The last part follows, since $N\Phi_n(f^n, g^n) \leqslant M\Phi_n(f,g) \leqslant \#(\Phi(F^n, G^n))$. This works because firstly when $fg = gf$, then

$$\Phi\left(f^n, g^n\right) = \bigcup_{m|n} \Phi\left(f^m, g^m\right) = \bigcup_{m|n} P_m(f,g) = \bigsqcup_{m|n} P_m(f,g),$$

and secondly that the minimum numbers occur exactly when $fg = gf$.

In fact whenever the Nielsen numbers are non-zero, the same phenomenon occurs on tori, nilmanifolds and on the model solvmanifolds introduced in [12] (models have the advantage that there are necessary and sufficient conditions for maps to exist, and as is shown in [12] the matrices produced as the linearizations of actual maps will indeed produce genuine maps on these solvmanifolds). But this type of phenomenon is hard to generalize. In particular in dealing with the case that the Nielsen numbers are zero, the analogue of the technique used in periodic point Wecken theorems (see [18]), which produces periodic points in empty classes that boost to essential classes at a higher level, will not work unless the modified maps commute.

Appendix A. Reduction to the gcd and the circle

In this section we determine exactly which pairs of maps on the circle are essentially reducible to the gcd. As in the previous sections, we identify a map f on the circle with its integer degree. Our goal is the following theorem:

Theorem A.1. *Given integers a and b, the maps $f = a$ and $g = b$ are essentially reducible to the gcd if and only if either* $\gcd(a,b) = 1$, *or both a and b are zero.*

Though Theorem 4.16 is, from one point of view, more general than the above theorem in that it includes all tori, it is, from another point of view, less general in that it requires the map g to be invertible. The proof of Theorem 4.16, which involved factoring certain polynomials, does not generalize to the setting where g_* is not invertible. So then we need to give a separate proof of this theorem.

Our proof of Theorem A.1 relies heavily on a lemma concerning the factors of the polynomials $\iota^{x,1}_{p,q}$ given in the proof of Theorem 4.16. Since we want to emphasize in this proof that the $\iota^{x,1}_{p,q}$ are polynomials in x, we change notation and denote $\iota^{x,1}_{p,q}$ by $\sigma_{p,q}(x)$. So then

$$\sigma_{p,q}(x) = 1 + x^p + x^{2p} + \cdots + x^{(q-1)p} \quad \text{and} \quad \sigma_{1,q}(x) = 1 + x + \cdots + x^{q-1}$$

and both are polynomials in $\mathbb{Z}[x]$. As in Theorem 4.16 we can, without loss of generality consider only the case when the gcd of the levels to which we consider reductions, is 1. With this in mind, our factoring result is as follows:

Lemma A.2. *When* $\gcd(k,m) = 1$ *with* $n = km$*, for the above polynomials* $\sigma_{p,q}(x) \in \mathbb{Z}[x]$ *there is a polynomial* $p(x) \in \mathbb{Z}[x]$ *satisfying*:

$$\sigma_{k,n}(x) = p(x)\sigma_{1,m}(x), \tag{A.1}$$

$$\sigma_{m,n}(x) = p(x)\sigma_{1,k}(x), \tag{A.2}$$

$$\sigma_{1,n}(x) = p(x)\sigma_{1,k}(x)\sigma_{1,m}(x). \tag{A.3}$$

Proof. Our approach is to fully factor $\sigma_{1,k}$, $\sigma_{m,n}$ and $\sigma_{k,n}$ in $\mathbb{Z}[x]$, and compare the factors. Consider the two factorizations of $x^k - 1$:

$$x^k - 1 = \prod_{d|k} \Phi_d(x) = (1-x)\sigma_{1,k}(x),$$

where the $\Phi_d(x)$ are the cyclotomic polynomials (see [20]). Cancelling $x - 1$ we have that $\sigma_{1,k}(x)$ is the unique product

$$\sigma_{1,k}(x) = \prod_{d|k,\, d\neq 1} \Phi_d(x) \tag{A.4}$$

of distinct (non-repeating) irreducible polynomials over the UFD $\mathbb{Z}[x]$.

We next factorize $\sigma_{k,n}$ as follows:

$$\sigma_{k,n}(x) = 1 + x^k + x^{2k} + \cdots + x^{(m-1)k} = \sigma_{1,m}\big(x^k\big) = \prod_{c|m,\, c\neq 1} \Phi_c\big(x^k\big).$$

Lemma 1 of [3] states that when $\gcd(c,k) = 1$, we have

$$\Phi_c\big(x^k\big) = \prod_{\mathrm{lcm}(r,k)=ck} \Phi_r(x),$$

so then our complete factorization of $\sigma_{k,n}(x)$ is:

$$\sigma_{k,n}(x) = \prod_{c|m,\, c\neq 1} \prod_{\mathrm{lcm}(r,k)=ck} \Phi_r(x).$$

We need an alternative characterization of this product: So let

$$R = \big\{r \mid \text{there exists } c \neq 1 \text{ with } c \mid m \text{ and } \mathrm{lcm}(r,k) = ck\big\}$$
$$\text{and let } S = \{ab \mid a \mid m \text{ and } b \mid k \text{ and } a \neq 1\}.$$

We claim that $R = S$. To show that $R \subseteq S$, let $r \in R$ together with a chosen $c \mid m$ with $ck = \mathrm{lcm}(r,k)$ Since $\mathrm{lcm}(r,k) = rk/\gcd(r,k)$, then $c = r/\gcd(r,k)$ and so $r = c\gcd(r,k)$. So if we let $a = c$, then $a \mid m$ and if $b = \gcd(r,k) \mid k$, then since c is assumed to be nontrivial, we have $a \neq 1$, and thus $r = ab \in S$.

Now we show $S \subseteq R$: Let $r \in S$ with $r = ab$ with a and b factors of m and k respectively such that $a \neq 1$. Let $c = a$, and then we automatically have $c \mid m$ and $c \neq 1$. It remains to show $\mathrm{lcm}(r,k) = ck$. Since k and m have no common divisors, we have $\gcd(r,k) = b$. Then

$$\mathrm{lcm}(r,k) = \frac{rk}{\gcd(r,k)} = \frac{rk}{b} = \frac{r}{b}k = ak = ck$$

as desired.

This now allows us to write $\sigma_{k,n}(x)$ as:

$$\sigma_{k,n}(x) = \prod_{\substack{r=ab \\ a|m,\, b|k,\, a\neq 1}} \Phi_r(x).$$

Similarly

$$\sigma_{m,n}(x) = \prod_{\substack{r=ab \\ a|m,\, b|k,\, b\neq 1}} \Phi_r(x).$$

By uniqueness of the factorizations above it is clear that any factor of $\sigma_{1,k}(x)$ appearing in (A.4), is also a factor of $\sigma_{m,n}(x)$ (one in which $a = 1$). Thus we have $\sigma_{1,k} \mid \sigma_{m,n}$, with quotient

$$p(x) = \sigma_{m,n}(x)/\sigma_{1,k}(x) = \prod_{\substack{r=ab \\ a|k,\, b|m \\ a\neq 1, b\neq 1}} \Phi_r(x).$$

We have established (A.1), and by symmetry that

$$\sigma_{k,n}(x)/\sigma_{1,m}(x) = p(x).$$

This establishes (A.2).

For statement (A.3), observe that

$$\sigma_{1,n}(x) = \prod_{r|n,\, r\neq 1} \Phi_r(x) = \prod_{\substack{r=ab \\ a|k,\, b|m}} \Phi_r(x).$$

As above, we see that any factor of $\sigma_{1,k}(x)$ appearing in (A.4), is a factor in this last product (one in which $b = 1$). Similarly any factor of $\sigma_{1,m}(x)$ is a factor in this last product in which $a = 1$. Thus

$$\frac{\sigma_{1,n}(x)}{\sigma_{1,k}(x)\sigma_{1,m}(x)} = p(x),$$

and this establishes (A.3). □

Lemma A.2 is interesting in its own right because it provides a strategy for factoring the algebraic boosts in periodic points theory (in that setting, the algebraic boost $\mathcal{R}(f^p) \to \mathcal{R}(f^q)$ is exactly $\sigma_{p,q}(f)$). As with maps f and g of S^1, we can identify the $\iota_{p,q}$ with multiplication by some integer, and for the rest of the section we identify each $\iota_{p,q}$ with its corresponding integer. We do as follows: Let $f = a$ and $g = b$ with neither a nor b is zero, then

$$\iota_{p,q} = a^{q-p}\sigma_{p,q}(b/a).$$

Note that, even though we evaluate $\sigma_{p,q}$ at a non-integer, multiplication by a^{q-p} causes the result to be integral. Applying all this evaluation to the above lemma gives the following factorization result for the $\iota_{r,s}$:

Lemma A.3. *Suppose that neither a nor b is zero. When* $\gcd(k, m) = 1$ *with* $n = km$*, there is* $p \in \mathbb{Z}$ *satisfying*:

$$\iota_{k,n} = p\iota_{1,m},$$

$$\iota_{m,n} = p\iota_{1,k},$$

$$\iota_{1,n} = p\iota_{1,k}\iota_{1,m}.$$

We use these factorizations to prove the following lemma.

Lemma A.4. *Let* $n = km$ *with* $\gcd(k, m) = 1$*. Given integers a and b and maps* $f = a$ *and* $g = b$ *with neither a nor b zero, the following are equivalent*:

1. *Two arbitrary classes* $[w]^m$ *and* $[z]^k$ *at levels m and k respectively, which boost to the same class* $[\iota_{m,n}w]^n = [\iota_{k,n}z]^{mn}$ *reduce to level* 1.
2. $\operatorname{lcm}(\iota_{m,n}, \iota_{k,n}) = \iota_{1,n}$.
3. $\gcd(\iota_{1,k}, \iota_{1,m}) = 1$.

Proof. Note that the first condition of the lemma is equivalent to saying that $\text{Im}(\iota_{m,n}) \cap \text{Im}(\iota_{k,n}) = \text{Im}(\iota_{1,n})$. Now the ι are injective, and $\text{Im}(\iota_{m,n})$ is a subgroup of $\mathcal{R}(a^n, b^n)$ of order $|b^m - a^m|$, generated by the integer $\iota_{m,n}$. Similarly $\text{Im}(\iota_{k,n})$ is the subgroup of $\mathcal{R}(a^n, b^n)$ of order $|b^k - a^k|$, generated by the integer $\iota_{k,n}$. So $\text{Im}(\iota_{m,n}) \cap \text{Im}(\iota_{k,n})$ is the subgroup of $\mathcal{R}(a^n, b^n)$ generated by the integer $\text{lcm}(\iota_{m,n}, \iota_{k,n})$. On the other hand $\text{Im}(\iota_{1,n})$ is the subgroup of $\mathcal{R}(a^n, b^n)$ generated by $\iota_{1,n}$. The first equivalence follows.

To prove the equivalence of the second two statements, consider the second statement. By Lemma A.3, the integers $\iota_{m,n}$ and $\iota_{k,n}$ have a common factor p, with respective quotients $\iota_{1,k}$ and $\iota_{1,m}$. Thus

$$\text{lcm}(\iota_{m,n}, \iota_{k,n}) = p\,\text{lcm}(\iota_{1,k}, \iota_{1,m}),$$

and the second statement is equivalent to $p\,\text{lcm}(\iota_{1,k}, \iota_{1,m}) = \iota_{1,n}$. But Lemma A.3 also gives $\iota_{1,n} = p\iota_{1,k}\iota_{1,m}$, and so the second statement is equivalent to $\text{lcm}(\iota_{1,k}, \iota_{1,m}) = \iota_{1,k}\iota_{1,m}$, which is to say that $\gcd(\iota_{1,k}, \iota_{1,m}) = 1$. □

We are now ready to prove the main result of this section.

Proof of Theorem A.1. We deal first with the case that neither a nor b is zero, and assume that $\gcd(a, b) \neq 1$, and show that f and g are not essentially reducible to the gcd. If a and b have a nontrivial common divisor, then $\iota_{1,m}$ and $\iota_{1,k}$ will share this divisor as well for any m and k. This is because each ι is a sum of terms, each of which is divisible by a or b. Thus we have $\gcd(\iota_{1,k}, \iota_{1,m}) \neq 1$ for any k and m. In particular we can choose k and m to be relatively prime, and then Lemma A.4 will apply to show that f and g do not reduce to the gcd at level $n = km$.

For the converse, assume that f and g are not essentially reducible to the gcd, and that $\gcd(a, b) = 1$. We deduce a contradiction. Since $\gcd(a, b) = 1$ if and only if $\gcd(a^d, b^d) = 1$ for all positive integers d, then by the same argument used in the proof of Theorem 4.16, we may assume, without loss of generality, that the failure to reduce comes at levels k and m with $\gcd(k, m) = 1$.

By Lemma A.4, we have that $\iota_{1,k}$ and $\iota_{1,m}$ have a common prime factor p. It is easy to see that

$$(a - b)\iota_{1,k} = a^k - b^k, \quad \text{and} \quad (a - b)\iota_{1,m} = a^m - b^m.$$

Thus p divides $a^k - b^k$ and $a^m - b^m$, and so

$$a^k = b^k \mod p, \quad \text{and} \quad a^m = b^m \mod p.$$

Now since $\gcd(a, b) = 1$, the prime p cannot divide both a and b. Thus one of a and b is invertible mod p, without loss of generality we assume that b is invertible mod p. Then the above can be written

$$(a/b)^k = 1 \mod p, \quad \text{and} \quad (a/b)^m = 1 \mod p.$$

Thus both k and m are divisible by the order of the element a/b in the multiplicative group $\mathbb{Z}_p^*$. If this order is not 1, then this will contradict the assumption that $\gcd(k, m) = 1$.

For the case where the order of a/b is 1, we must have that $a = b \mod p$, and the definition of $\iota_{1,k}$ simplifies modulo p as follows:

$$\iota_{1,k} = ka^{k-1} = kb^{k-1} \mod p,$$

and since $p \mid \iota_{1,k}$, we have $p \mid ka^{k-1}$ and $p \mid kb^{k-1}$. Since p is prime, either p divides k, or p divides both of a and b. But $\gcd(a, b) = 1$, and so we conclude that p divides k. The same argument of the previous paragraph applied to $\iota_{1,m}$ shows that p also divides m, which contradicts the assumption that $\gcd(k, m) = 1$.

Next we address the special case where one of a or b is zero. Without loss we assume that $b = 0$ and a is nonzero. In this case we have $\gcd(a, 0) = a$ (all numbers are divisors of 0), and $\iota_{p,q} = a^{q-p}$. So we need to show that $f = a$ and $g = 0$ are essentially reducible to the gcd if and only if $a = 1$. It is easy to check that when $a = 1$, all boosts are the identity, and that in this case f and g are essentially reducible to the gcd. Suppose next that $a \neq 1$, and that $[\alpha]^n \in \mathcal{R}_{\mathcal{E}}(f_*^n, g_*^n)$ reduces to both $[\beta]^m \in \mathcal{R}_{\mathcal{E}}(f_*^m, g_*^m)$ and $[\gamma^k] \in \mathcal{R}_{\mathcal{E}}(f_*^k, g_*^k)$. As in Theorem 4.16 we may assume without loss that $\gcd(m, k) = 1$, and again without loss that $m > k > 1$. In this case there will always be multiples of a^{n-m} which are not also multiples of a^{n-k}. Such elements will be in the images of $\iota_{m,n} = a^{n-m}$ and $\iota_{k,n} = a^{n-k}$, but not in the image of $\iota_{1,n} = a^{n-1}$. Thus f and g are not essentially reducible to the gcd.

Finally suppose that $f = g = a = b = 0$. Note that $\gcd(0, 0)$ is not defined so it cannot be equal to 1. However all boosts are 0, so that only the trivial element is reducible at each level, and it always reduces to any other level. Thus f and g are essentially reducible to the gcd. □

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