Available online at www.sciencedirect.com

SciVerse ScienceDirect

ELSEVIER

indagationes mathematicae

Indagationes Mathematicae 23 (2012) 10–18

www.elsevier.com/locate/indag

An epimorphic subgroup arising from Roberts' counterexample

Roman Avdeev

Department of Mechanics and Mathematics, Moscow State University, 1, Leninskie Gory, Moscow, 119992, Russia

Received 17 February 2011; accepted 9 September 2011

Communicated by H.W. Broer

Abstract

In 1994, based on Roberts' counterexample to Hilbert's fourteenth problem, A'Campo-Neuen constructed an example of a linear action of a 12-dimensional commutative unipotent group H_0 on a 19-dimensional vector space V such that the algebra of invariants $\Bbbk[V]^{H_0}$ is not finitely generated. We consider a certain extension H of H_0 by a one-dimensional torus and prove that H is epimorphic in $\mathrm{SL}(V)$. In particular, the homogeneous space $\mathrm{SL}(V)/H$ provides a new example of a homogeneous space with epimorphic stabilizer that admits no projective embeddings with small boundary.

MSC: 14E25; 14M99; 14M17

Keywords: Algebraic group; Epimorphic subgroup; Projective embedding; Hilbert's fourteenth problem

1. Introduction

We work over an algebraically closed field $\Bbbk$ of characteristic zero. Throughout the paper all topological terms relate to the Zarisky topology, all groups are supposed to be algebraic and their subgroups closed.

Let G be a connected affine algebraic group and H a subgroup of it. We recall that a *projective embedding with small boundary* of the homogeneous space G/H is an open G-equivariant embedding $\rho\colon G/H \hookrightarrow X$, where X is an irreducible normal projective G-variety

E-mail address: suselr@yandex.ru.

doi:10.1016/j.indag.2011.09.009

and $\operatorname{codim}_X(X \setminus \rho(G/H)) \geqslant 2$. For a given homogeneous space G/H, the existence of such embedding implies that the algebra $\Bbbk[G/H]$ of regular functions on G/H consists of constants, that is, $\Bbbk[G/H] = \Bbbk$. A subgroup $H \subset G$ with $\Bbbk[G/H] = \Bbbk$ is said to be *epimorphic*. Various characterizations, properties, and examples of epimorphic subgroups, as well as several conjectures and open problems concerning them, can be found in [3–5], and [7, Section 23 B].

It turns out that not every homogeneous space G/H with epimorphic H admits a projective embedding with small boundary. A criterion for this is given by Theorem 1 below. To formulate this theorem, we need to recall some additional notions. A subgroup $H \subset G$ is said to be *observable* if G/H is a quasi-affine variety. An observable subgroup $H \subset G$ is said to be a *Grosshans subgroup* if the algebra $\Bbbk[G/H]$ is finitely generated over $\Bbbk$.

Theorem 1. *Let $H \subset G$ be an epimorphic subgroup. Then the following conditions are equivalent:*
(a) *G/H admits a projective embedding with small boundary;*
(b) *there is a character χ of H such that* Ker χ *is a Grosshans subgroup in G.*

A complete proof of this theorem can be found in [2, Theorem 1.1].

Under a certain assumption on H, a combinatorial classification of all projective embeddings with small boundary for a given homogeneous space G/H is obtained in [2, Section 3]. Another problem arising in connection with Theorem 1 is to construct examples of epimorphic subgroups $H \subset G$ such that the corresponding homogeneous spaces G/H admit no projective embeddings with small boundary. In view of condition (b) of Theorem 1, such examples should be based on examples of observable subgroups $H_0 \subset G$ such that the algebra $\Bbbk[G/H_0]$ is not finitely generated over $\Bbbk$. In their turn, examples of this kind are provided by linear counterexamples to Hilbert's fourteenth problem.

We recall that, in general, Hilbert's fourteenth problem asks whether for a subfield L of the field $K(x_1, \dots, x_n)$ of rational functions in n variables over a field K such that $L \supset K$ the algebra $L \cap K[x_1, \dots, x_n]$ is finitely generated over K. (Here $K[x_1, \dots, x_n]$ is the algebra of polynomials in $x_1, \dots, x_n$.) An important special case of this problem asks whether the algebra $K[x_1, \dots, x_n]^G$ of invariants of a linear action of a group G on an n-dimensional vector space is finitely generated over K. (This special case is obtained from the general one with L being the quotient field of $K[x_1, \dots, x_n]^G$.) Counterexamples to this special case of Hilbert's fourteenth problem are called linear counterexamples.

At this moment, few counterexamples to Hilbert's fourteenth problem are known. The first one, which turns out to be linear, was discovered by Nagata in 1958 [8]. In Nagata's example, a 13-dimensional unipotent group acts on a 32-dimensional vector space. Much later, in 1997, Nagata's counterexample was considerably simplified by Steinberg [11] whose counterexample is now known as Nagata–Steinberg's counterexample. In this example, the dimension of the subgroup is reduced to 6 and that of the vector space to 18. The second counterexample, which is not linear, was constructed in 1990 by Roberts [10] who used an approach completely different from that of Nagata. In 1994, based on Roberts' counterexample, A'Campo-Neuen [1] constructed a linear counterexample involving an action of a 12-dimensional unipotent group on a 19-dimensional vector space. Subsequently, this counterexample was followed by a series of linear counterexamples (see, for instance, [6] and references therein). We note that among linear counterexamples to Hilbert's fourteenth problem the key role is played by counterexamples involving linear actions of unipotent groups.

A natural way of obtaining examples of homogeneous spaces with epimorphic stabilizer that admit no projective embeddings with small boundary is as follows. First, one takes a linear counterexample to Hilbert's fourteenth problem involving an action of a unipotent group H_0 on a vector space V over $\Bbbk$. Second, one fixes a connected reductive group $G \subset \mathrm{GL}(V)$ containing H_0. Third, one chooses an appropriate one-dimensional torus $S \subset G$ normalizing H_0 such that the subgroup $H = SH_0$ is epimorphic in G. In this situation, the homogeneous space G/H admits no projective embeddings with small boundary; see Proposition 1 in Section 2.

The first example of a homogeneous space with epimorphic stabilizer that admits no projective embeddings with small boundary was mentioned in [4, 7(b)]. In this example, $G = \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$ (16 copies) and H is obtained by extending the group in Nagata's counterexample by a one-dimensional torus. An analogous example based on Nagata–Steinberg's counterexample was considered (with complete proofs) in [2, Section 2]. In the present paper, we construct a new example of that kind based on A'Campo-Neuen's counterexample. The precise formulations and construction are given in Section 2.

In connection with these results the following question may be of interest.

Question. *Let a unipotent group H_0 act linearly on a finite-dimensional vector space V and let G be a connected reductive subgroup of* $\mathrm{GL}(V)$ *containing H_0. Suppose that the algebra $\Bbbk[V]^{H_0}$ is not finitely generated over $\Bbbk$. Is there a one-dimensional torus $S \subset G$ normalizing H_0 such that the group $H = SH_0$ is epimorphic in G?*

2. Construction of the subgroup

We put $G = \mathrm{SL}_{19}$ and denote by V the space of the tautological representation of G. We fix a basis $e_1, e_2, \ldots, e_{19}$ in V. Further on, for any element of G, its matrix is considered with respect to this basis.

Let $\mathbb{G}_a$ be the additive group of $\Bbbk$.

We consider a subgroup $H_0 \simeq (\mathbb{G}_a)^{12}$ embedded in G as follows:

$$\overline{\mu} = (\mu_0, \mu_1, \ldots, \mu_{11}) \mapsto h(\overline{\mu}) = \begin{pmatrix} E_4 & 0 \\ M(\overline{\mu}) & E_{15} \end{pmatrix},$$

where E_4, E_{15} are the identity matrices of order 4, 15, respectively, and

$$M(\overline{\mu}) = \begin{pmatrix} \mu_1 & \mu_0 & 0 & 0 \\ \mu_2 & \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ \mu_3 & 0 & \mu_0 & 0 \\ \mu_4 & 0 & \mu_3 & 0 \\ 0 & 0 & \mu_4 & 0 \\ \mu_5 & 0 & 0 & \mu_0 \\ \mu_6 & 0 & 0 & \mu_5 \\ 0 & 0 & 0 & \mu_6 \\ \mu_7 & \mu_0 & 0 & 0 \\ \mu_8 & \mu_7 & 0 & 0 \\ \mu_9 & 0 & \mu_8 & 0 \\ \mu_{10} & 0 & \mu_9 & 0 \\ \mu_{11} & 0 & 0 & \mu_{10} \\ 0 & 0 & 0 & \mu_{11} \end{pmatrix}. \qquad (1)$$

The result of A'Campo-Neuen is as follows.

Theorem 2 ([1]). *The algebra $\Bbbk[V]^{H_0}$ is not finitely generated over $\Bbbk$.*

Remark 1. In [1] this theorem is proved for any ground field of characteristic zero.

To construct our example, we consider the one-dimensional torus

$$S = \{\mathrm{diag}(s^{15}, s^{15}, s^{15}, s^{15}, \underbrace{s^{-4}, \dots, s^{-4}}_{15}) \mid s \in \Bbbk^\times\} \subset G, \tag{2}$$

where $\Bbbk^\times = \Bbbk \setminus \{0\}$ is the multiplicative group of $\Bbbk$. Clearly, S normalizes H_0. We now put $H = SH_0$. The main result of this paper is given by the following theorem.

Theorem 3. *The subgroup H is epimorphic in G.*

This theorem will be proved in Section 3.

Corollary 1. *The homogeneous space G/H admits no projective embeddings with small boundary.*

This corollary follows from Theorem 3 and the following proposition.

Proposition 1. *Suppose we are given a linear action of a unipotent group H_0 on a vector space V over $\Bbbk$ such that the algebra $\Bbbk[V]^{H_0}$ is not finitely generated. Suppose that a connected reductive subgroup $G \subset \mathrm{GL}(V)$ containing H_0 is fixed and there is a one-dimensional torus $S \subset G$ normalizing H_0 such that the group $H = SH_0$ is epimorphic in G. Then the homogeneous space G/H admits no projective embeddings with small boundary.*

Proof. It suffices to show that condition (b) of Theorem 1 does not hold. This is done by the same argument as in [2, Lemma 2.2]. □

3. Proof of Theorem 3

Before we prove Theorem 3, let us fix some notation.

We recall that $G = \mathrm{SL}_{19}$. Let B (resp. U, T) be the Borel subgroup (resp. the maximal unipotent subgroup, the maximal torus) in G consisting of all upper triangular (resp. upper unitriangular, diagonal) matrices contained in G. Let $N_G(T)$ denote the normalizer of T in G, which consists of all monomial matrices contained in G. We denote by $\mathfrak{X}(B)$ the weight lattice of B. The semigroup of dominant weights of B is denoted by $\mathfrak{X}_+(B)$, $\mathfrak{X}_+(B) \subset \mathfrak{X}(B)$. For $i = 1, 2, \dots, 18$ we denote by π_i the i-th fundamental weight of B, which takes every upper triangular matrix to the product of its first i diagonal entries.

The simple G-module with highest weight $\lambda \in \mathfrak{X}_+(B)$ is denoted by $V(\lambda)$, and its highest weight vector with respect to B is denoted by v_λ. Let $P_\lambda \subset G$ be the subgroup that stabilizes the line $\langle v_\lambda \rangle \subset V(\lambda)$. This subgroup is a parabolic subgroup containing the Borel subgroup B. We identify the weight lattice $\mathfrak{X}(P_\lambda)$ of P_λ with a sublattice of $\mathfrak{X}(B)$ by means of the natural embedding $B \hookrightarrow P_\lambda$.

Every dominant weight λ of B has the form $\lambda = a_1\pi_1 + a_2\pi_2 + \cdots + a_{18}\pi_{18}$ for some non-negative integers $a_1, a_2, \dots, a_{18}$. If $a_i > 0$ for some $i \in \{1, 2, \dots, 18\}$ then P_λ stabilizes the line $\langle v_{\pi_i} \rangle \subset V(\pi_i)$. At that, P_λ acts on v_{π_i} by multiplication by the weight π_i. This weight takes every matrix $A \in P_\lambda$ to the minor corresponding to the first i and last i rows and columns of A. (The lower left $(19 - i) \times i$ block of A consists of zero entries.)

In this section, we identify elements $s \in \Bbbk^\times$ and their images in S, see (2).

We now proceed to prove Theorem 3. By [7, Lemma 23.5] it suffices to show that there are no proper observable subgroups of G containing H. In view of [7, Lemma 7.7] the proof will be completed if we check the following two conditions:

(1) for every non-trivial simple G-module $V(\lambda)$ and every Borel subgroup $\widetilde{B} \subset G$ the highest weight vector of $V(\lambda)$ with respect to $\widetilde{B}$ is not invariant under H;
(2) there are no proper reductive subgroups of G containing H.

Condition (1) follows from Lemma 1. Condition (2) will be checked using Lemma 2. We now turn to formulate and prove the lemmas.

Lemma 1. *Let $\widetilde{B} \subset G$ be an arbitrary Borel subgroup and $V(\lambda)$, $\lambda \neq 0$, an arbitrary simple G-module with highest weight vector $\widetilde{v}_\lambda$ with respect to $\widetilde{B}$. Then there is an element $h \in H$ such that $h \cdot \widetilde{v}_\lambda \neq \widetilde{v}_\lambda$.*

Proof. Assume that $h \cdot \widetilde{v}_\lambda = \widetilde{v}_\lambda$ for all $h \in H$. Since $\lambda \neq 0$, we have $\lambda = a_1\pi_{i_1} + a_2\pi_{i_2} + \cdots + a_m\pi_{i_m}$, where $1 \leqslant m \leqslant 18$, $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant 18$, and $a_i > 0$ for all $i = 1, \ldots, m$. The subsequent argument is divided into several steps.

Step 1. Since all Borel subgroups in G are conjugated, there exists an element $g_0 \in G$ such that $\widetilde{B} = g_0 B g_0^{-1}$. Then $\widetilde{v}_\lambda = \alpha g_0 \cdot v_\lambda$ for some $\alpha \neq 0$, whence

$$g_0^{-1} h g_0 \cdot v_\lambda = v_\lambda \tag{3}$$

for all $h \in H$. Consider the Bruhat decomposition of g_0:

$$g_0 = u\sigma b, \tag{4}$$

where $u \in U$, $\sigma \in N_G(T)$, $b \in B$ are some fixed elements. We may assume that $\sigma = \varepsilon\sigma_0$, where σ_0 is a permutation matrix, $\varepsilon = 1$ for $\det \sigma_0 = 1$, and $\varepsilon = e^{\pi\sqrt{-1}/19}$ for $\det \sigma_0 = -1$. We now substitute expression (4) for g_0 in (3). Since b multiplies v_λ by a scalar, we have

$$\sigma^{-1} u^{-1} h u \sigma \cdot v_\lambda = v_\lambda \tag{5}$$

for all $h \in H$. Let $\tau\colon G \to G$ be the map given by the formula $\tau(g) = \sigma^{-1}u^{-1}gu\sigma$. Taking into account (5) we obtain $\tau(H) \subset P_\lambda$.

Step 2. Let ν be the permutation of the set $\{1, 2, \ldots, 19\}$ that corresponds to σ. Then $\sigma(e_j) = \varepsilon e_{\nu(j)}$ for $j = 1, \ldots, 19$. For each pair of matrices $g = (g_{ij}) \in G$, $\overline{g} = \sigma^{-1} g \sigma$ we have $\overline{g}_{ij} = g_{\nu(i),\nu(j)}$ for $i, j = \{1, \ldots, 19\}$. In particular, $\overline{g}_{jj} = g_{\nu(j),\nu(j)}$ for $j = 1, \ldots, 19$. We note that under the map $g \mapsto \overline{g}$ entries of g lying in the same row (resp. column) are transformed into elements of $\overline{g}$ that also lie in the same row (resp. column).

Step 3. Suppose $s \in S$. Then $u^{-1}su$ is an upper triangular matrix whose diagonal entries are s^{15}, s^{15}, s^{15}, s^{15}, $s^{-4}, \ldots, s^{-4}$. Further, the diagonal entries of the matrix $\tau(s) = \sigma^{-1}u^{-1}su\sigma$ are again s^{15}, s^{15}, s^{15}, s^{15}, $s^{-4}, \ldots, s^{-4}$, perhaps in another order. At that, for every $i = 1, \ldots, 18$ the determinant of the upper left $i \times i$ block of $\tau(s)$ is equal to the product of all diagonal entries of this block. Therefore, for every $j = 1, \ldots, m$ we have $\pi_{i_j}(\tau(s)) = s^{b_j}$ for some $b_j \in \mathbb{Z}$. Moreover, $b_j = 15k_j - 4l_j$, where

$$k_j = \#\{k \in \{1, 2, \ldots, i_j\} \mid \nu(k) \in \{1, 2, 3, 4\}\}$$

and

$$l_j = \#\{k \in \{1, 2, \ldots, i_j\} \mid \nu(k) \notin \{1, 2, 3, 4\}\}.$$

Clearly, $0 \leqslant k_j \leqslant 4$, $0 \leqslant l_j \leqslant 15$, and $k_j + l_j = i_j$. The last equality implies that $(k_j, l_j) \notin \{(0,0),(4,15)\}$, whence $b_j \neq 0$ for all $j = 1, \ldots, m$. Further, the condition $\tau(s) \cdot v_\lambda = v_\lambda$ implies that $\lambda(\tau(s)) = 1$ for all $s \in S$ and $a_1 b_1 + a_2 b_2 + \ldots + a_m b_m = 0$. We conclude that there exists $j_0 \in \{1, \ldots, m\}$ with $b_{j_0} > 0$. Put $i^* = i_{j_0}$, $k^* = k_{j_0}$, and $l^* = l_{j_0}$. Since $b_{j_0} > 0$, we have $l^* < 15k^*/4$, in particular, $k^* > 0$. Obviously, for every matrix in $\tau(H)$ its lower left $(19 - i^*) \times i^*$ block consists of zero entries.

Step 4. Suppose that $u = \begin{pmatrix} P & R \\ 0 & Q \end{pmatrix}$ and $u^{-1} = \begin{pmatrix} P^{-1} & R' \\ 0 & Q^{-1} \end{pmatrix}$, where P and Q are upper unitriangular matrices of order 4 and 15, respectively, R and R' are 4×15 matrices, $R' = -P^{-1}RQ^{-1}$. Let $h = h(\overline{\mu}) \in H_0$ be an arbitrary element. Recall that $h = \begin{pmatrix} E_4 & 0 \\ M(\overline{\mu}) & E_{15} \end{pmatrix}$ for some $\overline{\mu} \in \Bbbk^{12}$, where $M(\overline{\mu})$ is the matrix in (1). Then

$$u^{-1}hu = \begin{pmatrix} E_4 + R'M(\overline{\mu})P & P^{-1}R + R'M(\overline{\mu})R + R'Q \\ Q^{-1}M(\overline{\mu})P & E_{15} + Q^{-1}M(\overline{\mu})R \end{pmatrix}.$$

We consider the 15×4 matrix $D = D(h) = Q^{-1}M(\overline{\mu})P$. Note that for every entry d_{pq} of D we have $d_{pq} = m_{pq} + \sum c_{ij} m_{ij}$ where the sum is taken over all pairs (i, j) with $i \geqslant p$, $j \leqslant q$, and $(i, j) \neq (p, q)$, the coefficients c_{ij} being uniquely determined by the matrix u. Now, using the latter observation and the explicit form (1) of the matrix $M(\overline{\mu})$, we can successively choose elements $\mu_{11}, \mu_{10}, \mu_9, \mu_8, \mu_7, \mu_0, \mu_6, \mu_5, \mu_4, \mu_3, \mu_2, \mu_1 \in \Bbbk$ in such a way that, for the corresponding element $h_0 \in H_0$, the submatrix $D(h_0)$ of $u^{-1}h_0u$ has the form

$$D(h_0) = \begin{pmatrix} * & \diamond & \diamond & \diamond \\ * & * & \diamond & \diamond \\ \diamond & * & \diamond & \diamond \\ * & \diamond & \diamond & \diamond \\ * & \diamond & * & \diamond \\ \diamond & \diamond & * & \diamond \\ * & \diamond & \diamond & \diamond \\ * & \diamond & \diamond & * \\ \diamond & \diamond & \diamond & * \\ * & * & \diamond & \diamond \\ * & * & \diamond & \diamond \\ * & \diamond & * & \diamond \\ * & \diamond & * & \diamond \\ * & \diamond & \diamond & * \\ \diamond & \diamond & \diamond & * \end{pmatrix}, \tag{6}$$

where the asterisks stand for non-zero entries and the diamonds stand for the entries that are irrelevant for us.

Step 5. We now turn to the element $h_0 \in H_0$ and the corresponding matrix $D(h_0)$ found at the previous step. For $n = 1, 2, 3, 4$ we define numbers $Z(n)$ as follows. We consider all $15 \times n$ submatrices of $D(h_0)$. For each of them, we count the number of non-zero rows. At last, we put $Z(n)$ to be the minimal among the obtained values. Using the explicit form (6) of $D(h_0)$, we find that $Z(1) \geqslant 4$, $Z(2) \geqslant 8$, $Z(3) \geqslant 12$, $Z(4) = 15$.

Step 6. For $j = 1, \ldots, k^*$ we define (pairwise distinct) numbers $n_1, \ldots, n_{k^*} \in \{1, \ldots, i^*\}$ by the condition $\nu(n_j) = j$. The column j of the matrix $u^{-1}h_0u$ is obtained by applying the permutation ν to the column n_j of the matrix $\tau(h_0) = \sigma^{-1}(u^{-1}h_0u)\sigma$ $(j = 1, \ldots, k^*)$. Therefore, none of the elements of $D(h_0)$ is such that its image under the transformation $u^{-1}h_0u \mapsto \sigma^{-1}(u^{-1}h_0u)\sigma$ is contained in one of the rows $n_1, \dots, n_{k^*}$. Since the lower left $(19 - i^*) \times i^*$ block of $\tau(h_0)$ is zero (*Step* 3), it follows that there is a $15 \times k^*$ submatrix of $D(h_0)$ whose number of non-zero rows is at most $i^* - k^* = l^* < 15k^*/4$.

Step 7. Comparing the results of the previous step with the definition of the numbers $Z(n)$ (*Step* 5) we get the following inequality:

$$Z(k^*) < 15k^*/4. \tag{7}$$

Making use of the estimations of $Z(n)$ obtained at *Step* 5, we find that none of the possible values $k^* = 1, 2, 3, 4$ satisfies (7). This contradiction completes the proof of the lemma. $\square$

Thus condition (1) is checked.

Lemma 2. *Suppose that $F \subset G$ is a reductive subgroup containing H. Then F acts irreducibly on V.*

Proof. Since F is reductive, its action on V is completely reducible. Therefore it suffices to show that V contains no proper subspaces invariant under F.

Let $V_1 \subset V$ be the subspace spanned by the vectors e_1, e_2, e_3, e_4 and $V_2 \subset V$ be the subspace spanned by the vectors $e_5, e_6, \dots, e_{19}$. Clearly, $V = V_1 \oplus V_2$, $\dim V_1 = 4$, and $\dim V_2 = 15$. We note that both of the subspaces V_1, V_2 are invariant under the action of S.

Suppose that $W \subset V$ is a subspace invariant under F and choose an arbitrary vector $w \in W$. Then $w = v_1 + v_2$ for some vectors $v_1 \in V_1$ and $v_2 \in V_2$. Acting on w by the element $(\sqrt{-1}, \sqrt{-1}, \sqrt{-1}, \sqrt{-1}, 1, 1, \dots, 1) \in S \subset H \subset F$, we obtain the vector $w' = \sqrt{-1}v_1 + v_2$ that also lies in W. It follows that both vectors v_1, v_2 lie in W. Therefore W is the direct sum of its projections W_1 and W_2 to the subspaces V_1 and V_2, respectively. Clearly, $W_1 = W \cap V_1$ and $W_2 = W \cap V_2$.

Let $w \in W_1$ be an arbitrary element. Then for every $h \in H_0$ we have $h \cdot w = w + v(w, h)$, where $v(w, h) \in W_2$. Regard the set H_0 as a vector space. We define the map $\varphi_w : H_0 \to W_2$ by $\varphi_w(h) = v(w, h)$. In other words, for $w = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$ and $\overline{\mu} \in \Bbbk^{12}$ we have

$$\varphi_w(h(\overline{\mu})) = M(\overline{\mu}) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}. \tag{8}$$

Evidently, φ_w is a linear map, therefore its image is a subspace in W_2. Besides,

$$\dim \operatorname{Ker} \varphi_w + \dim \operatorname{Im} \varphi_w = \dim H_0 = 12. \tag{9}$$

To find $\dim \operatorname{Ker} \varphi_w$ (and thereby $\dim \operatorname{Im} \varphi_w$), it is sufficient to solve the linear system $\varphi_w(h(\overline{\mu})) = 0$ in variables $\overline{\mu}$. It is not hard to see that the dimension of the solution space of this system depends only on the arrangement of non-zero coordinates of w. The values of $\dim \operatorname{Im} \varphi_w$ for different types of $w \in V_1$ are presented in Table 1. In the first row of Table 1 w's are written as column vectors, the asterisk denotes a non-zero coordinate.

We now assume that $V = W \oplus W'$ for some proper subspaces $W, W' \subset V$ invariant under F. Put $W_1 = W \cap V_1$, $W_2 = W \cap V_2$, $W_1' = W' \cap V_1$, and $W_2' = W' \cap V_2$. It follows from the above that $V_1 = W_1 \oplus W_1'$ and $V_2 = W_2 \oplus W_2'$. Without loss of generality we may assume that $\dim W_1 \geqslant \dim W_1'$. Further we consider three possible cases.

Table 1
The values of $\dim \operatorname{Im} \varphi_w$ for different types of $w \in V_1$.

w	0	*	0	0	0	*	*	*	0	0	0	*	*	*	0	*
	0	0	*	0	0	*	0	0	*	*	0	*	*	0	*	*
	0	0	0	*	0	0	*	0	*	0	*	*	0	*	*	*
	0	0	0	0	*	0	0	*	0	*	*	0	*	*	*	*
$\dim \operatorname{Im} \varphi_w$	0	11	4	5	5	12	12	12	8	8	9	12	12	12	12	12

Case 1. $\dim W_1 = 2$, $\dim W_1' = 2$. Then there are two vectors $w_1 \in W_1$, $w_1' \in W_1'$ such that each of them has at least two non-zero coordinates. From Table 1 we find that $\dim \operatorname{Im} \varphi_{w_1} \geqslant 8$ and $\dim \operatorname{Im} \varphi_{w_1'} \geqslant 8$. On the other hand, $\operatorname{Im} \varphi_{w_1}, \operatorname{Im} \varphi_{w_1'} \subset V_2$ and $\dim V_2 = 15$, whence the space $\operatorname{Im} \varphi_{w_1} \cap \operatorname{Im} \varphi_{w_1'}$ has positive dimension. This is impossible because $\operatorname{Im} \varphi_{w_1} \subset W_2$, $\operatorname{Im} \varphi_{w_1'} \subset W_2'$, and $W_2 \cap W_2' = \{0\}$.

Case 2. $\dim W_1 = 3$, $\dim W_1' = 1$. Then there are a vector $w_1 \in W_1$ with at least three non-zero coordinates and a non-zero vector $w_1' \in W_1'$. From Table 1 we find that $\dim \operatorname{Im} \varphi_{w_1} \geqslant 12$ $\dim \operatorname{Im} \varphi_{w_1'} \geqslant 4$. By the same reason as in *Case* 1, the space $\operatorname{Im} \varphi_{w_1} \cap \operatorname{Im} \varphi_{w_1'}$ has positive dimension, a contradiction.

Case 3. $\dim W_1 = 4$, $\dim W_1' = 0$. It is easy to see that the space $\operatorname{Im} \varphi_{e_1}$ contains the vectors $e_5, e_6, e_8, e_9, e_{11}, e_{12}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}$, the space $\operatorname{Im} \varphi_{e_2}$ contains the vector e_7, the space $\operatorname{Im} \varphi_{e_3}$ contains the vector e_{10}, and the space $\operatorname{Im} \varphi_{e_4}$ contains the vectors e_{13}, e_{19}. Thus all the basis vectors of V lie in W, whence $W = V$ and $W' = 0$, a contradiction.

In all the cases we have come to a contradiction, so the proof of the lemma is completed. $\square$

We now show that a reductive subgroup $F \subset G$ containing H coincides with G. First, we note that there are no non-trivial bilinear forms on V preserved by F because this holds even for S. Next, by Lemma 2 the F-module V is simple. Therefore the center of F is finite and F is semisimple. Moreover, F is simple since otherwise the dimension of V would be a composite number, which is not the case (we have $\dim V = 19$). Obviously, the rank of F is at least two. Further, F contains the unipotent subgroup H_0 of dimension 12, whence $\dim F \geqslant 2+2\cdot 12 = 26$.

Assume that $F \neq G$. Since there are no non-trivial bilinear forms on V preserved by F, it follows that F can only be of type SL_k, Spin_{4l+2}, or E_6 (see [9, Section 4.3]). Further we consider these three cases separately. (In all the cases below, our arguments rely on well-known facts from representation theory of semisimple algebraic groups.)

(1) F is of type SL_k. Clearly, $k \leqslant 18$. Since $\dim F \geqslant 26$, we have $k \geqslant 6$. Every simple SL_k-module W with $\dim W > k+1$ has actually dimension at least $k(k-1)/2$, which is more than 19 for $k \geqslant 7$. It remains to consider the case $k = 6$. Every simple SL_6-module W with $\dim W > 15$ has actually dimension at least 20.
(2) F is of type Spin_{4l+2}. Clearly, $l \leqslant 4$. Since $\dim F \geqslant 26$, we have $l \geqslant 2$. Every simple Spin_{18}-module W with $\dim W > 18$ has actually dimension at least 153. Every simple Spin_{14}-module W with $\dim W > 14$ has actually dimension at least 64. Every simple Spin_{10}-module W with $\dim W > 16$ has actually dimension at least 45.
(3) F is of type E_6. Every non-trivial simple E_6-module has dimension at least 27.

In all the cases we have obtained that V is not a simple F-module. This contradiction implies that $F = G = \mathrm{SL}_{19}$.

Thus, we have checked condition (2). Theorem 3 is proved.

Acknowledgments

The author is grateful to I.V. Arzhantsev for suggesting the problem, his interest, and attention to the work. This work was partially supported by Russian Foundation for Basic Research, grant no. 09-01-00648.

References

[1] A. A'Campo-Neuen, Note on a counterexample to Hilbert's fourteenth problem given by P. Roberts, Indag. Math. 5 (3) (1994) 253–257.

[2] I.V. Arzhantsev, Projective embeddings of homogeneous spaces with small boundary, Izv. RAN. Ser. Mat. 73 (3) (2009) 5–22 (in Russian); Izv. Math. 73 (3) (2009) 437–453 (English transl.), See also arXiv:math.AG/0801.1967.

[3] F. Bien, A. Borel, Sous-groupes épimorphiques de groupes linéaires algébriques I, C. R. Acad. Sci., Paris Sér. I Math. 315 (6) (1992) 649–653.

[4] F. Bien, A. Borel, Sous-groupes épimorphiques de groupes linéaires algébriques II, C. R. Acad. Sci., Paris Sér. I Math. 315 (13) (1992) 1341–1346.

[5] F. Bien, A. Borel, J. Kollár, Rationally connected homogeneous spaces, Invent. Math. 124 (1996) 103–127.

[6] G. Freudenburg, A linear counterexample to the fourteenth problem of Hilbert in dimension eleven, Proc. Amer. Math. Soc. 135 (2007) 51–57.

[7] F.D. Grosshans, Algebraic Homogeneous Spaces and Invariant Theory, in: Lecture Notes in Math., vol. 1673, Springer-Verlag, Berlin, 1997.

[8] M. Nagata, On the 14th problem of Hilbert, Amer. J. Math. 81 (1959) 766–772.

[9] A.L. Onishchik, E.B. Vinberg, Lie Groups and Algebraic Groups, in: Springer Ser. Soviet Math., Springer-Verlag, Berlin, 1990.

[10] P. Roberts, An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem, J. Algebra 132 (1990) 461–473.

[11] R. Steinberg, Nagata's example, in: Algebraic Groups and Lie Groups, in: Austral. Math. Soc. Lect. Ser., vol. 9, Cambridge Univ. Press, Cambridge, 1997, pp. 375–384.