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THE NEXT ADMISSIBLE ORDINAL

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The study of recursive functions on admissible ordinals was initiated by S. Kripke and R. Platek, who, independently, generalized recursion theory from the usual nonnegative integers to certain larger initial segments of the ordinal numbers. It has since been intensively developed—principally from the point of view of determining which theorems of ordinary recursion theory (usually theorems about degrees) are true of α-recursion theory, for various α. Our work here, however, is of an entirely different nature. Roughly speaking, our main concern is the question of when and how α^+ can be defined in terms of α-recursion theoretic concepts, where for a given admissible ordinal α, α^+ is the smallest admissible greater than α.

One very successful answer to this question is given in Barwise et al. [6], where it is shown that α^+ is always the supremum of the closure ordinals of first order positive inductive definitions over L_α.

In this paper we attempt to give another answer. In order to describe it, we let $|A|$, whenever A is a set of relations, be the least ordinal which cannot be represented by a well-ordering in A.

Our starting point is the well-known fact that ω^+ is the least non-recursive ordinal, i.e. $|\omega\text{-recursive}|$. From this it might be thought that α^+ is always the least ordinal not representable as an α-recursive well-ordering of a subset of α, i.e.

$$\alpha^+ = |\alpha\text{-recursive}|. \tag{1}$$

This, however, is not the case. Indeed, as we shall see, (1) sometimes is true and sometimes isn't.

When (1) is true we say that α is *good* and when (1) is false we say that α is *bad*. (These names are simply for identifying purposes, and as such, have no ethical significance.)

In very general terms our results can be summarized as follows:

(a) the good ordinals are at least cofinal with constructible $\aleph_1$,

(b) they do not form an initial segment of the admissibles,

(c) the first bad ordinal is relatively small, and

(d) the bad ordinals are at least cofinal with each constructible cardinal.

After seeing these results, the reader might naturally ask: Is α^+ always

$|\alpha - \Sigma_n|$?, for various $n \in \omega$, or $|\alpha$-arithmetical$|$? or $|\alpha - \Delta_1^1|$? etc. And indeed some of these questions have interesting answers. They will be discussed in a forthcoming paper. Here, however, we shall concern ourselves only with the recursive case.

The organization of the paper is as follows. In Section 1 we review the basic notions of admissibility theory. In Section 2 we give a very powerful sufficient condition for showing admissibles to be good, and then proceed with a long list of examples. In Section 3 we give some sufficient conditions for showing ordinals to be bad. We also give a characterization of the bad ordinals, as well as many examples. We conclude, in Section 4, with a few new proofs, using our methods, of some known results concerning Π_1^1 and Σ_2^1 sets.

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1. Preliminaries

There have been essentially two main approaches to recursion theory on admissible ordinals—the original approach of Kripke via an equation calculus, and the later set theoretic approach of Platek. In the following we shall outline what will be needed from each of these approaches.

To begin, we assume that the reader is familar with the basic notions of ordinary recursion theory (i.e. recursion theory on the natural numbers), as well as the elements of set theory, through the development of ordinals and cardinals.

Ordinals will be understood to be such that each ordinal is equal to the set of all smaller ordinals, and cardinals will be identified with initial ordinals. $\alpha, \beta, \gamma, \delta, \ldots$ will be used to denote ordinals, and κ and λ will be used to denote cardinals.

ω will denote the first infinite ordinal, and will be identified with the set of natural numbers. Subsets of ω will generally be referred to as *reals*. $\aleph_1$ will denote the first uncountable ordinal (as well as the first uncountable cardinal) and ω_1 will denote the first nonrecursive ordinal. When A is a real, ω_1^A will be, as in Spector [27], the first non A-recursive ordinal.

To discuss Kripke's approach to admissibility theory, we recall that in [15] Kripke has devised an equation calculus (modeled after Kleene's equation calculus in [13]) for computing partial functions from any ordinal α, into α. Using this equation calculus Kripke defines the notions of *α-recursive function*, *α-partial recursive function*, *α-recursively ennumerable set*, *α-recursive set* and *α-finite set*, along with the very basic notion of an *admissible ordinal.*

Rather than repeat all of these definitions here, we shall simply assume that the reader is familiar with Kripke [15, 16, 17], and refer to the definitions and theorems of those abstracts as we need them.

In the abstracts, Kripke points out that all of the elementary theorems of ordinary recursion theory have analogues in the α-recursion theory, for each admissible α. These results are all proven in detail in [18], where it is shown, in particular, that for each admissible α, the α-recursion theory analogue to Kleene's T predicate (which we shall write as T^{α}) is α-recursive. This immediately implies that Kleene's normal form theorem lifts to all admissible α. We shall lift Kleene's notation, $\{z\}(x) \simeq y$, to the α-recursion theory by writing $\{z\}^{\alpha}(x) \simeq y$. We will normally surpress the superscript α when the context makes clear which recursion theory we are working in.

Platek's approach to admissibility theory concerns the notion of an admissible set. Before we can define what this is, we must first recall the Lévy classification of set theoretic formulas, defined in [19].

To do so we let $\mathscr{L}$ be the usual first order language of set theory, i.e. the language which contains a single binary relation symbol $\in$. A formula of $\mathscr{L}$ in which all quantifiers are bounded i.e. of the form

$$(\forall x)(x \in y \rightarrow \quad) \quad \text{or} \quad (\exists x)(x \in y \wedge \quad),$$

is called a Σ_0-formula. Bounded quantifiers are abbreviated as

$$(\forall x \in y) \quad \text{or} \quad (\exists x \in y).$$

A Σ_n-formula, with $n > 1$, is a formula of $\mathscr{L}$ of the form $(\exists v_0) \cdots (\exists v_m)\varphi$, where φ is Π_{n-1}. Similarly, a Π_n-formula is a formula of $\mathscr{L}$ of the form $(\forall v_0) \cdots (\forall v_m)\varphi$, where φ is Σ_{n-1}.

This classification places all formulas of $\mathscr{L}$ into a hierarchy beginning with the Σ_0-formulas and continuing with progressively more complex formulas. A useful remark—and one that we shall make tacit reference to many times—is that most of the elementary notions of set theory (e.g. "x is a function", "x is an ordinal", "the domain of the function x is $\cdots$") can be expressed by Σ_0-formulas. An exhaustive list of such notions (along with their definitions) is given in the appendix to Karp [12].

Using Levy hierarchy, we can define various restricted forms of familar axiom schemes from set theory. For example, the axiom schema of Σ_0-*separation* is the set of universal closures of all formulas of the form

$$(\exists x)(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi),$$

where φ is Σ_0 and x does not occur free in φ. In the same way, we could define the schema of Σ_n-*separation* by allowing φ to be Σ_n instead of Σ_0.

The axiom schema of Σ_0-*collection* is the set of universal closures of all formulas of the form

$$(\forall x \in u)(\exists y)\varphi \rightarrow (\exists w)(\forall x \in u)(\exists y \in w)\varphi,$$

where φ is Σ_0 and w does not occur free in φ.

Now let KP consist of the following sentences of $\mathscr{L}$:

extensionality,
regularity,
pairs,
unions,
Σ_0-separation,
Σ_0-collection.

An *admissible set* is simply a transitive set A, such that $\langle A, \in \rangle$ is a model of KP.

Among the nice properties possessed by admissible sets, is the fact that they are always a model of Σ_1-replacement and Δ_1-separation, where Σ_1-*replacement* is the set of universal closures of all formulas of the form

$$(\forall x \in u)(\exists y)\varphi \rightarrow (\exists w)(\forall y)(y \in w \leftrightarrow (\exists x \in u)\varphi),$$

where φ is Σ_1 and w does not occur free in φ, and Δ_1-*separation* is the set of universal closures of all formulas of the form

$$(\forall y)(\varphi(y) \leftrightarrow \sim \psi(y)) \rightarrow (3x)(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi),$$

where φ and ψ are Σ_1 and x does not occur free in φ. Proofs of these facts can be found e.g. in Jensen [11].

Now suppose that A is any set. A subset S of A is $\Sigma_n(\Pi_n)$ *over* A if there is a $\Sigma_n(\Pi_n)$-formula $\varphi(v_0, v_1, \ldots, v_k)$ and elements $b_1, \ldots, b_k \in A$ (called parameters) such that

$$S = \{a \in A \mid \langle A, \in \rangle \models \varphi(a, b_1, \ldots, b_k)\}.$$

The notation $\langle A, \in \rangle \models \varphi(a, b_1, \ldots, b_k)$ means that the sequence $a, b_1, \ldots, b_k$ satisfies the formula φ in $\langle A, \in \rangle$ in the usual sense of satisfaction. In this situation we shall occasionally abuse the reader by saying that $\varphi(a, b_1, \ldots, b_k)$ *is true in* $\langle A, \in \rangle$.

We say that a subset S of A is Δ_n over A if S is both Σ_n and Π_n over A.

We next recall Gödel's constructible hierarchy. This will be the key to the connections between admissible ordinals and admissible sets. The hierarchy is defined by transfinite induction as follows

$$L_0 = \emptyset,$$

$$L_{\beta+1} = \text{the set of all subsets of } L_\beta \text{ which are } \Sigma_n \text{ over } L_\beta \text{ for some } n,$$

$$L_\lambda = \bigcup_{\beta<\lambda} L_\beta \quad \text{for } \lambda \text{ a limit ordinal},$$

$$L = \bigcup_{\beta \in On} L_\beta \quad \text{is the class of all constructible sets.}$$

We remark that, in order for sets of the form L_α to be admissible it is enough to assume that $\langle L_\alpha, \in \rangle$ be a model of Σ_0-collection—since, as is easily verified, all the other axioms will then follow trivially.

The following fundamental theorem summarizes some of the important connections between the Kripke and Platek approaches to admissibility theory. It will be tacitly used again and again.

Theorem 1.1 (Kripke–Platek). (a) *α is an admissible ordinal iff L_α is an admissible set. Now assume α is admissible, then* (b) *A partial function f from α to α is α-partial recursive iff its graph is Σ_1 over L_α,* (c) *A subset S of α is α-r.e. (α-recursive) iff S is $\Sigma_1(\Delta_1)$ over L_α.* (d) *A subset S of α is α-finite iff $S \in L_\alpha$.*

Before continuing with our discussion of admissibles we must recall a few of the elementary notions from model theory. Among these are the notions of *truth*, *satisfaction* and *model* (which we have already used) and the notion of one structure being an *elementary substructure* of another—for which we write $\mathfrak{A} \leqslant \mathfrak{B}$. Moreover if $\mathfrak{A}$ and $\mathfrak{B}$ are such that $\mathfrak{A} \leqslant \mathfrak{B}$ holds for all Σ_n formulas (as opposed to all formulas), then we write $\mathfrak{A} \leqslant_n \mathfrak{B}$ and say that $\mathfrak{A}$ is a *Σ_n-elementary substructure* of $\mathfrak{B}$. Thus $\mathfrak{A} \leqslant \mathfrak{B}$ iff $\mathfrak{A} \leqslant_n \mathfrak{B}$, for all $n \in \omega$.

Also recall that if $\mathfrak{A} = \langle A, R_1, \ldots, R_m\rangle$ is a structure, then the set of all variable free sentences which are true in $\langle A, R_1, \ldots, R_m, a\rangle_{a\in A}$, is called the *diagram* of $\mathfrak{A}$, and is denoted by Diag$(\mathfrak{A})$. The set of all sentences which are true in $\langle A, R_1, \ldots, R_m, a\rangle_{a\in A}$, is called the *complete diagram* of $\mathfrak{A}$.

Another useful notion concerning structures is that of *pointwise definability*. We say that $\mathfrak{A} = \langle A, R_1, \ldots, R_m\rangle$ is *pointwise definable* if for each $a \in A$, there is a formula φ, such that a is the unique element of A, which satisfies φ in $\mathfrak{A}$.

A technique from the model theory of set theory which we shall use on several occasions, is that of collapsing a well-founded model of the axiom of extensionality to a transitive isomorph. We shall assume complete familiarity with the details of this procedure—due essentially to Gödel [10], but explicitly formulated much later by Shepherdson and, independently, Mostowski (See [21]).

For technical reasons we shall find a particular sentence of $\mathscr{L}$, called σ, very useful. σ has the property that any transitive model of σ is equal to $\langle L_\beta, \in\rangle$ for some β, and each $\langle L_\beta, \in\rangle$ is a model of σ. See Boolos [7] for the details of the construction of σ.

When α is admissible, the elements of $L_{\alpha+1}$ are called the *α-arithmetic* sets. They are classified as Σ_n, Π_n or Δ_n according to their defining formulas.

Recall from the introduction that α^+ is always the least admissible ordinal greater than α.

The following results concerning admissible sets and ordinals will be referred to later.

Theorem 1.2 (Kripke–Platek). *ω_1^A is admissible for all $A \subseteq \omega$.*

Theorem 1.3 (Sacks). *Every countable admissible ordinal is of the form ω_1^A., for some $A \subseteq \omega$.*

Theorem 1.4 (Kripke–Platek). *For each admissible α there is a 1–1 α^+-recursive function, f, mapping α^+ into α.*

An ordinal α is said to be *projectible* into β, if there is a 1–1 α-recursive map from α into β. Hence Theorem 1.4 states that α^+ is always projectible into α. α is *non-projectible* if it is not projectible into any $\beta < \alpha$.

Theorem 1.5 (Kripke [18]). *α-is non-projectible iff L_α is the union of its proper Σ_1-elementary sul structures.*

We call an ordinal *recursively inaccessible*, if it is admissible, and a limit of admissibles. This notion is a recursive analogue of the set theoretic notion of weak inaccessibility. Recursive analogues of other large cardinal notions will be discussed later.

An ordinal α is called *locally countable* if every element of L_α can be mapped 1–1 into ω by a function in L_α. It should be noted that admissibles which are projectible into ω are locally countable.

A somewhat technical notion, which will be useful at one point later, is the notion of a *non-standard admissible set*. This is simply a model, $\mathcal{M} = \langle M, E\rangle$, of KP where M is any set.

If M is such a set, and N is the set of all $a \in M$ for which there is no infinite sequence $a_0, \ldots, a_n, \ldots$ of elements of M with $a_0 = a$ and $E(a_{n+1}, a_n)$ for each $n \in \omega$, then N is well-founded. Hence N can be collapsed to a transitive set $\tilde{N}$, which is called the *standard part* of $\mathcal{M}$. The following result is essentially due to F. Ville.

Theorem 1.6. *If $\mathcal{M} \models$ KP, then the standard part of $\mathcal{M}$ is admissible.*

A proof of this can be found in Barwise [5].

We conclude this section with a brief discussion of well-orderings.

For a binary relation R we define, the *field* of R, in symbols Fld_R, to be

$$\{x \mid (\exists y)[R(x, y) \text{ or } R(y, x)]\}.$$

If R satisfies

(i) $(\forall x) R(x, x)$,
(ii) $(\forall x)(\forall y)[R(x, y) \,\&\, R(y, x) \to x = y]$,
(iii) $(\forall x)(\forall y)(\forall z)[R(x, y) \,\&\, R(y, x) \to R(x, z)]$,
(iv) $(\forall x)(\forall y)[x \in \mathrm{Fld}_R \,\&\, y \in \mathrm{Fld}_R \to R(x, y) \text{ or } R(y, x)]$,
(v) $\mathrm{Fld}_R \subseteq A$ (where A is some set),

we say that R is a *linear ordering* of a subset (i.e. Fld_R) of A. If in addition to (i), (ii), (iii), (iv) and (v), R satisfies

(vi) there does not exist an infinite descending chain for R, i.e. a function $F: \omega \to \mathrm{Fld}_R$ such that $(\forall n) R(f(n+1), f(n))$, then we say that R is a *well-ordering* of its field.

If R and S are binary relations we say R is *isomorphic* to S if there is a 1–1 map f, of Fld_R onto Fld_S such that

$$(\forall x)(\forall y)[R(x, y) \leftrightarrow S(f(x), f(y))].$$

The map f is said to be an *isomorphism* of R unto S.

It is a theorem of set theory that well-orderings are isomorphic to ordinals. However, if one examines the proof closely, one sees that all one really needs to prove it is Σ_1-replacement (not full set theory). Moreover if this result is formualted with respect to a model of set theory, and if the well-ordering involved is really a well-ordering (not just a well-ordering in the sense of the model), then all that one need assume is that the model satisfy Σ_1-replacement. This is the content of the following lemma, which will be used repeatedly during the course of this work.

Well-ordering Isomorphism Lemma. *If A is an admissible set and $R \in A$ is a well-ordering, then there exists an ordinal $\gamma \in A$ and a map $f \in A$ such that f is an isomorphism from R onto γ.*

Proof. Let

$$I_R = \{a \in \mathrm{Fld}_R \mid A \models (\exists f)(\exists \gamma)[\mathrm{Ord}(\gamma) \,\&\, \mathrm{Fun}(f) \,\&\, \mathrm{dom}(f) = \{b \mid R(b, a)\}$$
$$\& \ f \text{ is an isomorphism from } \mathrm{dom}(f) \text{ onto } \eta]\}.$$

We first note that $\mathrm{Fld}_R - I_R$ can have no R-minimal element. This is so because if $a \in \mathrm{Fld}_R - I_R$ were an R-minimal element, then I_R would be $\{b \mid R(b, a)\}$. For if $b \in \mathrm{Fld}_R - I_R$, since a is R-minimal, we could not have $R(b, a)$. Hence $R(a, b)$. Conversely, suppose $x \in I_R$ and $\sim R(x, a)$. Then $R(a, x)$. Since $x \in I_R$ there is an isomorphism from $\{b \mid R(b, x)\}$ onto an ordinal. The restriction of this isomorphism to $\{b \mid R(b, a)\}$ is also an isomorphism onto an ordinal. But this would show that $a \in I_R$, contradicting the assumption that $a \in \mathrm{Fld}_R - I_R$. Hence $\{I_R = b \mid R(b, a)\}$.

We now recall the well-known fact, if a relation is isomorphic to an ordinal, then both the isomorphism and the ordinal are unique. Hence

$$A \models (\forall x \in \{b \mid R(b, a)\})(\exists f)$$
$$\times [(\exists \gamma)(\mathrm{Ord}(\gamma) \,\&\, \mathrm{Fun}(f) \,\&\, \mathrm{dom}(f) = \{b \mid R(b, x)\}$$
$$\& \ f \text{ is an isomorphism from } \{b \mid R(b, x)\} \text{ onto } \gamma)].$$

If we note that the last conjunct can be written in Σ_o form as

$$(\forall u \in \{b \mid R(b, x)\})(\forall v \in \{b \mid R(b, x)\})[R(u, v) \leftrightarrow f(u) \in f(v)] \wedge$$
$$[(\forall v \in \gamma)(\exists u \in \{b \mid R(b, x)\})(f(u) = v)],$$

then we see that the statement is an instance of the antecedent of Σ_1-replacement. Since A is admissible, Σ_1-replacement holds, and so we can collect all the isomorphisms to form an element F of A. Since it is easy to see that $\bigcup F$ is an isomorphism from $\{b \mid R(b, a)\}$ onto an ordinal, and $\bigcup F \in A$, we see that $a \in I_R$. Since this is a contradiction, we have shown that $\mathrm{Fld}_R - I_R$ has no R-minimal elements.

Now we observe that if $\mathrm{Fld}_R - I_R = \phi$, then $\mathrm{Fld}_R = I_R$, and so just as in the above paragraph, we can piece together the isomorphisms corresponding to each $a \in I_R$, to get an isomorphism onto an ordinal, which is what we wanted to show.

If $\mathrm{Fld}_R - I_R \neq \phi$, then since $\mathrm{Fld}_R - I_R$ has no R-minimal element, we can use the axiom of choice (in the real world) to extract an infinite descending chain for R. This is a contradiction since R was assumed to be a real world well-ordering. Hence $\mathrm{Fld}_R - I_R = \phi$ and so the remarks of the previous paragraph give the isomorphism.

Note that in the above argument it is important that R really be well-ordered, not just well-ordered "in the sense of A". However, if A is a model of Σ_1-separation as well as a model of KP, then I_R will actually belong to A. Hence in this situation, we could show that linear orderings in A, which are well-ordered in the sense of A, are isomorphic to ordinals in A.

The unique ordinal which R is isomorphic to is called the *order type*, or simply the *type*, of R. It is denoted by $\bar{R}$.

One final definition. For each $n \geq 1$ we let δ_n^1 be the least ordinal which is not the order type of a Δ_n^1 well-ordering of ω. For information about the Δ_n^1 sets see Rogers [25]. Similarly, when $A \subseteq \omega$, we define δ_n^{1A} by relativising everything to A.

2. Good ordinals

If A is any set of relations, $|A|$ will denote the least ordinal not representable by a well-ordering in A. An admissible ordinal α is called *good* if

$$\alpha^+ = |\alpha\text{-recursive}|.$$

In this section we shall show that various countable admissibles are good.

We begin with the observation that the quantifiers in any α-arithmetic relation range over α (or L_α), and so are bounded from the point of view of α^+ (or L_{α^+}). This shows that any α-arithmetic relation is α^+-finite, so that from the well-ordering isomorphism lemma, it follows that any α-arithmetic well-ordering has type $<\alpha^+$. Hence

$$|\alpha\text{-arithmetic}| \leq \alpha^+, \tag{1}$$

and *a fortiori*

$$|\alpha\text{-recursive}| \leq \alpha^+. \tag{2}$$

Hence in order to show that α is good, we need only show that, for each $\gamma < \alpha^+$, there is an α-recursive well-ordering of type γ.

Another general observation which we shall use on several occasions is

$$|\alpha\text{-r.e.}| = |\alpha\text{-recursive}| \tag{3}$$

for all admissible α. This was shown by Spector in [27] for the case of $\alpha = \omega$, but his proof clearly works for all admissible α.

(3) should be contrasted with the fact that

$$|\alpha\text{-arithmetic}| = |\alpha\text{-recursive}|$$

does not hold for all admissible α (as we shall see in Section 3), even though Spector in [27] has proven it in the case of $\alpha = \omega$.

Our next result is an observation which lifts a well-known theorem of R. O. Gandy (see Gandy [9]), from ω to all admissible α.

Proposition 2.1. *Let α be admissible and let R be an α-recursive linear ordering of α, which is not well-ordered, but has no infinite descending chains in L_{α^+}. Then the order type of I_R, the maximal well-ordered initial segment of R, is α^+.*

Proof. First observe that if R is as in the hypothesis, then $\bar{I}_R$ cannot be greater than α^+. For if it were, there would be an $x \in \mathrm{Fld}_R$ such that the R-predecessors of x would form an α-recursive well-ordering of type α^+, which by (1) is impossible.

We next claim that if the type of I_R were less than α^+, I_R would have to belong to L_{α^+}. This would then yield a contradiction, since $\mathrm{Fld}_R - I_R$ would belong to L_{α^+}, so that by the axiom of choice, which holds in L_{α^+}, we could extract an infinite descending chain in L_{α^+}. Hence the type I_R must be α^+.

To prove our claim that $I_R \in L_{\alpha^+}$ if the type of $I_R = \gamma$, is less than α^+, we note that for every $a \in I_R$, R restricted to $\{b \mid R(b, a)\}$ is a well-ordering in L_{α^+} whose type is less than γ. The well-ordering isomorphism lemma then guarantees an isomorphism, $f \in L_{\alpha^+}$, between γ and $\{b \mid R(b, a)\}$. Hence

$$L_{\alpha^+} \models (\forall x \in \gamma)(\exists y)[\exists f)(f \text{ is an isomorphism of } \{\mathrm{b} \mid \mathrm{R}(\mathrm{b}, \mathrm{y})\} \text{ onto } x)].$$

But by Σ_1-replacement, the collection of all these y's is a set in L_{α^+}, and this set is precisely I_R.

As an immediate consequence of Proposition 2.1 and the remarks preceeding it we obtain the following

Corollary. *If α is admissible and there exists an α-recursive linear ordering of α, which is not well-ordering, but has no infinite descending chains in L_{α^+}, then α is good.*

As one might now expect, our method for showing ordinals to be good will be to produce linear orderings of the kind mentioned in Proposition 2.1 and its corollary. Since it turns out to be easier to work with trees rather than linear orderings, we shall next discuss the notion of an α-*tree*, and show how to transform results about trees into results about linear orderings. This will result simply from lifting Kleene's analysis (in [14]) of finite sequences of integers to finite sequences of ordinals less than α.

Let Seq^α be the set of finite sequences of ordinals less than α. We remark, that using Gödel's well-ordering of $On \times On$, restricted to α, it is easy to construct α-recursive pairing and inverse pairing functions J^α, K^α and L^α and then, using standard techniques, code the elements of Seq^α by ordinals less than α in an α-recursive manner. In this way we will sometimes wish to think of the elements of Seq^α as ordinals as well as finite sequences. If $x \in \mathrm{Seq}^\alpha$ we let, as usual, $(x)_i$, $i \in \omega$, be the ith coordinate of the sequence x. $(\ )_i$ is of course α-recursive. Complete details of this are worked out in Kripke [18].

We let $<^T$ be the partial ordering of Seq^α given by reverse inclusion, i.e. for $x, y \in \mathrm{Seq}^\alpha$, $x <^T y \leftrightarrow y \subseteq x$. It is easily seen that $<^T$ is α-recursive.

We now define an α-*tree* as a subset $A \subseteq \mathrm{Seq}^\alpha$ with the property that if $x \in A$ and $x <^T y$, then $y \in A$. A tree A is α-recursive if the set of codes for A is α-recursive. A *path* in the tree A is a maximal linearly ordered subset of A. We can think of a path as a function f, from some n into A, with the property that $f(j) <^T f(i)$ for $i < j$.

Lemma 2.1. *If there exists an α-recursive α-tree which has an infinite path, but no infinite paths in L_{α^+}, then there is an α-recursive linear ordering which is not a well-ordering, but has no infinite descending chains in L_{α^+}.*

Proof. Kleene in [14] has proved this for the case of $\alpha = \omega$. A slight modification of his proof will work in our case. The idea is to linearize the tree by the Kleene–Brouwer ordering (KBO),

$$x <^{k-B} y \leftrightarrow x \in \mathrm{Seq}^\alpha \ \&\ y \in \mathrm{Seq}^\alpha \ \&\ x <^T y$$

or

$$(x)_i < (y)_i \quad \text{where } i \text{ is the least } j \text{ such that } (x)_j \neq (y)_j.$$

The restriction of the KBO to the tree is then easily seen to be an α-recursive linear ordering. It is not a well-ordering since any infinite path f, in the tree is such that

$$\cdots <^T f(n+1) <^T f(n) <^T \cdots <^T f(1) <^T f(0),$$

and so would be an infinite descending chain for the KBO, since $<^T \subseteq <^{K-B}$.

There are, however, no infinite descending chains for the linearized tree in L_{α^+}, for if g were such a chain, then g would belong to L_β for $\beta < \alpha^+$. Now define f, where m, n, t, u range over ω, and γ and δ range over α, as follows:

$$f(t) = \gamma \quad \text{iff } L_\beta \models (\exists u)(\forall m \geq u)[(g(m))_t = \gamma] \ \&$$
$$(\forall \delta < \gamma) \sim (\exists u)(\bigvee m \geq u)[(g(m))_t = \gamma].$$

Then $f \in L_{\beta+1} \subset L_{\alpha}^{+}$, so $f \in L_{\alpha^+}$. f is obviously single valued and must be defined for all integers, for otherwise we could get an infinite descending chain of ordinals. We also note that any initial segment of f is an initial segment of some $g(m)$. Hence f is an infinite path through the tree, which is possible.

The construction of α-recursive α-trees with infinite paths, but none in L_{α^+}, was first carried out by Kleene, for $\alpha = \omega$, using results about hyperarithmetic sets. However, in the general case, such luxuries are not available, and indeed, as we shall see, there do not always exist such trees for each admissible α. Hence some new idea is necessary. Such an idea was provided by H. Friedman to whom I am greatly indebted. He showed that such trees exist, if L_α is the least initial segment of L which is a model of some sentence of $\mathscr{L}$.

The following lemma generalizes Friedman's result and is the strongest result of its kind that we know of. For convenience in stating it, we let $\mathscr{L}[L_\gamma]$ denote the language $\mathscr{L}$ augmented by a constant $\bar{a}$ for each $a \in L_\eta$.

Tree Construction Lemma. *Let α be admissible and let $\delta < \alpha$ be countable in L_α. Also suppose that there is a theory T in the language $\mathscr{L}[L_\delta] \cup \{c\}$ such that $T \in L_\alpha$ (where T is coded as a set of integers), c is some new constant symbol, and every sentence in T is Π_1. Then if α is the least admissible ordinal β such that $\langle \mathscr{L}_{\beta^+}, \in, d, \mathscr{L}_\beta \rangle_{d \in \mathscr{L}_\delta}$ is a model of T, there is an α-recursive α-tree with an infinite path, but no infinite paths in L_{α^+}.*

Proof. Let T be as in the hypothesis. Consider the new theory T' in the same language whose axioms are

T,

KP,

φ^c for $\in$ KP,

σ^c,

extentionally^c,

$\bar{a} \in c$ for each $a \in L_\delta$,

where φ^c represents the sentence resulting from restricting all quantifiers in φ to c.

Note that R' is both consistent (since $\{L_{\alpha^+}, \in, L_\alpha, d\}_{d \in L_\delta}$ is a model of T') and α-finite.

We want to add Henkin constants and axioms to T', but for technical reasons, we do so in the following unorthodox manner.

Let $\mathscr{L}^0$ be the language of T'. Since δ is countable in L_α, $\mathscr{L}^0$ is also countable in L_α. Hence there is an ennumeration, $\varphi_0^0(v)$, $\varphi_1^0(v), \ldots$, in L_α, of all one free variable formulas of $\mathscr{L}^0$. For each sentence of the form $(\exists v)\, \varphi(v)$, where φ is one of the φ^0, we introduce a Henkin constant c_φ, and call the resulting language $\mathscr{L}^1$.

Now suppose that $\mathscr{L}^n$ has been defined. Let $\varphi_0^n(v), \varphi_1^n(v), \ldots,$ be an α-finite enumeration of all one free variable formulas of $\mathscr{L}^n$. Then, just as above, we introduce a Henkin constant for each sentence of the form $(\exists v)\,\varphi(v)$, where φ is one of the φ^n. The resulting language is called $\mathscr{L}^{n+1}$.

The language resulting from addition of all Henkin constants is called $\mathscr{L}^\omega$. Since the construction of $\mathscr{L}^\omega$ preceded from an α-finite enumeration of formulas to an α-finite sequence of constants, it is clear that the set of all Henkin constants can be arranged into an α-finite sequence $c_1, c_2, c_3, \ldots$.

We now wish to add special axioms to the theory T'. For each c_i there was a one free variable formula $\varphi(v)$ of $\mathscr{L}^\omega$ which introduced c_i. For such φ and c_i we take

$$(\exists_1 v)\,\varphi(v) \to \varphi(c_i)\ \&\ \sim(\exists_1 v)\,\varphi(v) \to c_i = \bar{o},$$

as the special axiom for c_i. The theory in $\mathscr{L}^\omega$ resulting from adding all special axioms to T' is denoted by $H(T')$. Since the set of special axioms is α-finite and since $H(T') = T' \cup \{\text{special axioms}\}$, it is clear that $H(T')$ is an α-finite set of axioms.

Now suppose $\varphi(v)$ is a one free variable formula of $\mathscr{L}^\omega$. Let $\varphi'(v)$ be

$$\varphi(v)\ \&\ (\forall w)(w <_L v \to \sim\varphi(w)),$$

where $w <_L v$ is a formula which says "w is constructed before v". The following implications are now provable in $H(T')$:

$$\begin{array}{ll}
(\exists v)\,\varphi(v) \to (\exists_1 v)\,\varphi'(v) & \text{since } V = L \in T', \\
(\exists_1 v)\varphi'(v) \to \varphi'(c_{\varphi'}) & \text{special axiom}, \\
\varphi'(c_{\varphi'}) \to \varphi(c_{\varphi'}) & \text{predicate logic}, \\
(\exists v)\,\varphi(v) \to \varphi(c_{\varphi'}) & \text{hypothetical syllogism}.
\end{array}$$

Hence $H(T')$ is what Shoenfield [26] calls a Henkin theory.

Now consider the structure

$$\mathfrak{A} = \{L_{\alpha^+}, \in, L_\alpha, ⓒ_i, d\}_{d \in L_\alpha}, \qquad i = 1, 2, 3, \ldots,$$

as a structure for $\mathscr{L}^\omega$, where c is interpreted as L_α, $\bar{d}$ is interpreted as d, and c_i is interpreted as ⓒ$_i$. If the ⓒ$_i$ are defined as follows, $\mathfrak{A}$ becomes a model of $H(T')$: let $\varphi(v)$ be the formula of $\mathscr{L}^\omega$ which introduces c_i, then

$$ⓒ_i = \begin{cases} \text{the unique } a \in L_{\alpha^+} \text{ such that } L_{\alpha^+} \models \varphi(a) & \text{if } L_{\alpha^+} \models (\exists_1 v)\varphi(v), \\ \varphi & \text{otherwise}. \end{cases}$$

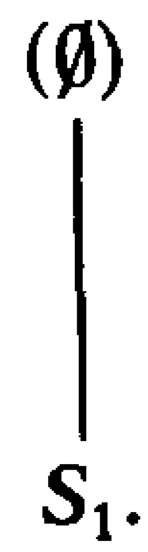
Since $c_i \in L_\alpha$ iff $\mathfrak{A} \models c_i \in c$, and $c_i \in c_j$ iff $\mathfrak{A} \models c_i \in c_j$, we see that $\bigcirc$ is an $\in$-preserving map of $\{c_i \mid \mathfrak{A} \models c_i \in c\}$ into L_α. Furthermore $\bigcirc$ is 1–1 on equivalence classes $[c_i]$, where c_i is equivalent to c_j iff $\mathfrak{A} \models c_i = c_j$. The existence of this map will be important later.

We are now ready to describe the tree. First we let $S_1, S_2, S_3, \ldots$ be an α-finite ennumeration of all sentences of $\mathcal{L}^{\omega}$, with the property that each S_i occurs infinitely often. Again, the L_α-countability of δ insures the existence of such an ennumeration.

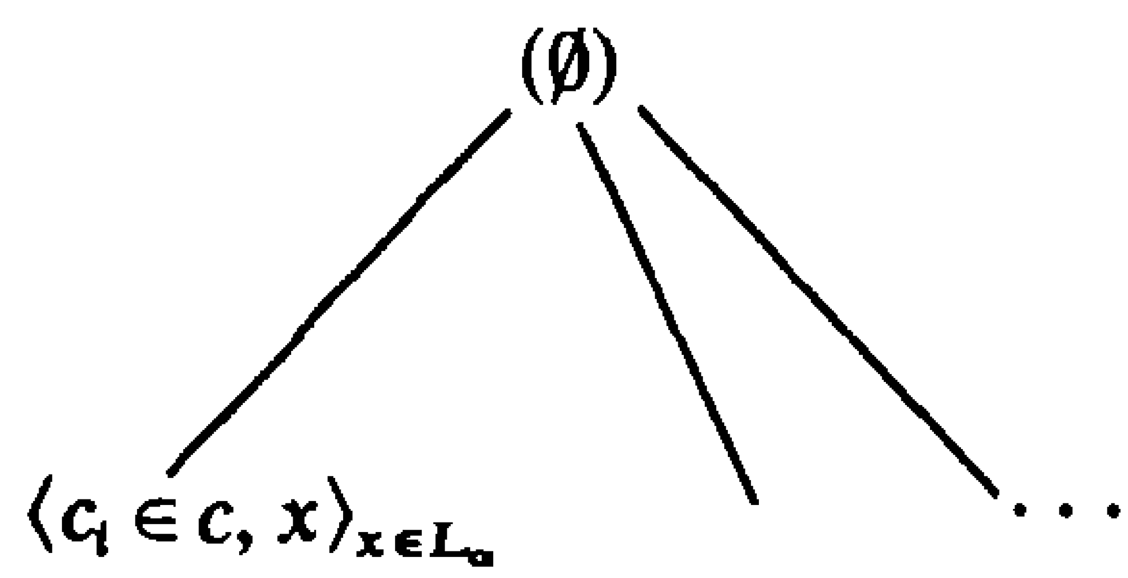
At the top of the tree, at level zero, we place the empty sequence (φ). To define the nodes of level one, we consider S_1.

Case 1. $H(T') \vdash S_1$.

(1a) If there does not exist an i such that S_1 is $c_i \in c$, then S_1 is the only node of level one, i.e.

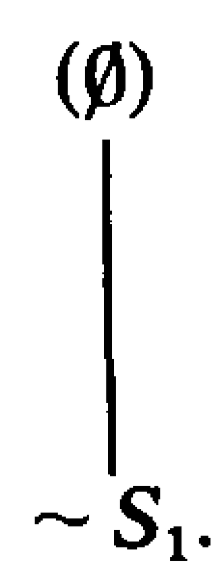


(1b) If there is an i such that S_1 is $c_i \in c$, then there is infinite branching at level zero so that each pair $\langle c_i \in c, x\rangle$, for $x \in L_\alpha$, appears as node of level one, i.e.



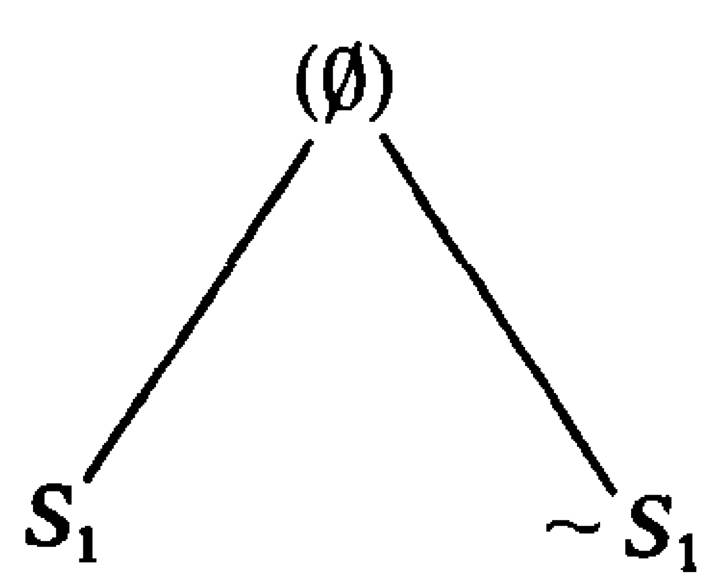
Case 2. $H(T') \vdash \sim S_1$.

Here no matter what the form of S_1, $\sim S_1$ is the only node of level one, i.e.



Case 3. S_1 is undecidable in $H(T')$.

(3a) If there does not exist an i such that S_1 is $c_i \in c$, then there is binary branching at level zero, as follows



(3b) If there is an i such that S_1 is $c_i \in c$, then there is infinite branching at level one as follows

$$\begin{array}{ccc} & (\emptyset) & \\ \swarrow & \downarrow\ \downarrow & \searrow \\ \langle c_i \in c, x\rangle_{x\in L_\alpha} & \cdots & \sim(c_i \in c) \end{array}$$

Now assume that the first n levels have been constructed. Let ψ be any non-terminal (see Case 1_c^n below) node of level n. Denote the set of nodes preceding ψ as Pre_ψ and let $H(T')_\psi = H(T') \cup \{\psi\} \cup \mathrm{Pre}_\psi$. The nodes of level $n+1$ are defined from S_{n+1} by the following cases.

Case 1^{n+1}. $H(T')_\psi \vdash S_{n+1}$.

(1_a^{n+1}) If $S_{n+1} \in \{\psi\} \cup \mathrm{Pre}_\psi$, then S_{n+1} is the only node of level $n+1$ under ψ, i.e.

$$\begin{array}{c} \psi \\ | \\ S_{n+1} \end{array}$$

(1_b^{n+1}) If $S_{n+1} \notin \{\psi\} \cup \mathrm{Pre}_\psi$ and there does not exist an i such that S_{n+1} is $c_i \in c$, then as in 1_a^{n+1}, S_{n+1} is the only node of level $n+1$ under ψ.

(1_c^{n+1}) If $S_{n+1} \notin \{\psi\} \cup \mathrm{Pre}_\psi$ and there is an i such that S_{n+1} is $c_i \in c$, then there is infinite branching from ψ as in 1_b with the proviso that $\langle c_i \in c, x\rangle$ is terminal if there is $\langle c_j \in c, y\rangle \in \{\psi\} \cup \mathrm{Pre}_\psi$ such that one of the following

(i) $c_i = c_j \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $x \neq y$,

(ii) $c_i \neq c_j \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $x = y$,

(iii) $c_i \in c_j \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $x \notin y$,

(iv) $\sim c_i \in c_j \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $x \in y$,

(v) $c_j \in c_i \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $y \notin x$,

(vi) $\sim c_j \in c_j \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $y \in x$,

(vii) there is $\bar{a} \in c_i \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $a \notin x$,

(viii) there is $c_i \in \bar{a} \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $x \notin a$,

(ix) there is $\sim \bar{a} \in c_i \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $a \in x$,

(x) there is $\sim c_i \in \bar{a} \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $x \in a$,

(xi) there is $\bar{a} = c_i \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $a \neq x$,

(xii) there is $\sim c_i = \bar{a} \in \{\psi\} \cup \mathrm{Pre}_\psi$ and $a = x$.

(To say that a node is terminal means that the path on which it lies has been terminated and will never again be considered.)

Case 2^{n+1}. $H(T')_\psi \vdash \sim S_{n+1}$.
This is done exactly as in Case 2 no matter what the form of S_{n+1}, i.e.

$$\begin{array}{c} \psi \\ | \\ | \\ | \\ \sim S_n \end{array}$$

Case 3^{n+1}. S_{n+1} is undecidable in $H(T')_\psi$.

(3^{n+1}_a) Exactly as in (3a) with S_{n+1} instead of S_1.

(3^{n+1}_b) Exactly as in (3b) with S_{n+1} replacing S_1 with the proviso that $\langle c_i \in c, x\rangle$ be terminal if the conditions of 1^{n+1}_c are met.

This completes the inductive definition of the tree. Since an α-tree was defined to be a certain subset of Seq^α and since the tree above was defined in terms of certain sentences of $\mathcal{L}^\omega$, a word of explanation, as to why the above is an α-tree, is in order.

Since any node of the above tree is either a sentence of $\mathcal{L}^\omega$ or a pair $\langle s, x\rangle$, where s is a sentence of $\mathcal{L}^\omega$ and $x \in L_\alpha$, and since sentences of $\mathcal{L}^\omega$ can be coded by integers and elements of L_α can be coded by ordinals less than α, it is clear that each node can be coded by an ordinal less than α. Hence the set of codes for the elements of Pre_ψ, for each node ψ of our tree, is a finite sequence of ordinals less than α. The set of all such sequences constitutes precisely an α-tree. Moreover, since $H(T')$ is an α-finite theory and since the cases of the definition of our tree involved checking only finite sets, our tree is clearly α-recursive. It remains to show that it has infinite paths but no infinite paths in L_{α^+}.

To see it has infinite paths we note that a path is terminated iff there are sentences $c_i \in c$ and $c_j \in c$, in the path, and an assignment of x to c_i and y to c_j by the pairs $\langle c_i \in c, x\rangle$ and $\langle c_j \in c, y\rangle$, which is not "correct", i.e. which either does not preserve $\in$ and $=$, or is not consistent with sentences of the form $(c_i \in \bar{a})$, $(c_i \notin \bar{a})$, $(c_j \in \bar{a})$, $(c_j \notin \bar{a})$, $(\bar{a} \notin c_i)$, and $(\bar{a} \notin c_j)$ which lie in the path. Since we have previously shown that there is a model with a "correct" assignment, we see that there are infinite paths.

Now assume that P is an infinite path in the tree. Then since $H(T')$ is a Henkin theory, and P is a complete extention of $H(T')$, we know that P is a complete Henkin theory. Hence the structure M^P (defined below) is a model of P and, a fortiori, model of T'.

$$M^P = \langle \{[a]^P \mid a \in \mathscr{C}\}, E^P, [a]^P\rangle_{a\in\mathscr{C}}$$

where $\mathscr{C} = L_8 \cup \{c, c_1, c_2, \ldots\}$, $[a]^P$ is the equivalence class of a under the equivalence relation $\sim$, where

$$a \sim b \quad \text{iff } (a = b) \in P,$$

and E^P is defined so that

$$[a]^P E^P [b]^P \quad \text{iff } (a \in b) \in P.$$

Since we shall only be concerned with a fixed path P, we shall normally surpress the superscript P in the above notation. Also for convenience we'll denote $\{[a] \mid [a] E [c]\}$ by c^*.

Now since M is a model of T' and $T' \supseteq \mathrm{KP}$, it follows that the reduct of $\mathcal{M}$ to $\mathcal{L}$ is a non-standard admissible set. Moreover since $\mathcal{M} \models \varphi^c$, for each $\varphi \in \mathrm{KP} \cup \{\text{extentionality}\} \cup \{\sigma\}$, it follows that

$$\langle c^*, E \rangle \models \mathrm{KP} \cup \{\text{extentionality}\} \cup \{\sigma\}. \tag{1}$$

In addition, the assignment map, Ass, is an isomorphism from $\langle c^*, E \rangle$ into $\langle L_\alpha, \in \rangle$. Hence $\langle \mathrm{Ass}''c^*, \in \rangle$ is a well-founded model of extentionality, which can be collapsed to a transitive set, $\overline{\mathrm{Ass}''c^*}$. Then using (1) it follows that $\overline{\mathrm{Ass}''c^*} = L_\gamma$, for some admissible $\gamma \leq \alpha$. Hence $\langle c^*, E \rangle$ is isomorphic to some admissible L_γ.

If f is such an isomorphism, then f', defined by

$$f'(x) = \begin{cases} f(x) & x \in c^*, \\ L_\gamma & x = c^* \end{cases}$$

is clearly an isomorphism of $c^* \cup \{c\}$ onto $L_\gamma \cup \{L_\gamma\}$.

Now since $\mathcal{M}$ is an end extension (see Barwise [4, p. 231 for the definition]) of $\langle c^* \cup \{c\}, E, c, [a] \rangle_{a \in L_\delta}$, it follows that $\mathcal{M}$ is isomorphic to an end extension of $\langle L_\gamma \cup \{L_\gamma\}, \in, L_\gamma, d \rangle_{d \in L_\delta}$. Moreover, γ is admissible, so that Theorem 1.6 implies that $\mathcal{M}$ is actually isomorphic to an end extension of $\langle L_{\gamma^+}, \in, L_\gamma, d \rangle_{d \in L_\delta}$. Then since $\mathcal{M} \models T$, and all $\varphi \in T$ are Π_1, it follows that $\langle L_{\gamma^+}, \in, L_\gamma, d \rangle_{d \in L_\delta} \models T$. Hence γ must equal α (by the minimality condition on α).

Now suppose that $P \in L_{\alpha^+}$. Then clearly, $\mathcal{M}^P \in L_{\alpha^+}$. But this is impossible, since the above paragraph shows that M^P must be isomorphic to an end extension of L_{α^+}, and Nadel (in the corollary to Theorem 1 of [23]) has shown that no β-finite structure can be isomorphic to an end extension of L_β, when β is admissible. Hence our tree can have no infinite paths in L_{α^+}.

By combining the Tree Construction Lemma with Proposition 2.1 and Lemma 2.1 we obtain as an immediate consequence a very general criterion for α to be good. Namely,

Theorem 2.1 *Let α be admissible and let $\delta < \alpha$ be countable in L_α. Also suppose that there is a theory T in the language $\mathcal{L}[L_\delta] \cup \{c\}$ such that $T \in L_\alpha$ and every sentence T is Π_1. Then if α is the least ordinal β such that $\langle L_{\beta^+}, \in, L_\beta, d \rangle_{d \in L_\delta}$ is a model of T, then α is good.*

In most of our applications, the full strength of Theorem 2.1 will not be used. Instead the following two special cases will suffice.

Corollary 2.1.1 *Let α be admissible and let $\delta<\alpha$ be countable in L_α. Also suppose that there is a theory T in the language $\mathscr{L}[L_\delta]$ such that $T\in L_\alpha$ and α is the least ordinal β such that $\langle L_\beta, \in, d\rangle_{d\in L_\delta}$ is a model of T. Then α is good.*

Proof. Let c be a new constant symbol and apply Theorem 2.1 to the theory $T'=\{\varphi^c \mid \varphi\in T\}$.

Corollary 2.1.2. *Let α be admissible greater than ω and let T be a recursive set of sentences in the language of set theory such that α is the least β such that $\langle L_\beta, \in\rangle\models T$. Then α is good.*

Proof. Let $\delta=0$ in Corollary 2.1.1 and recall that every recursive set belongs to $L_{\omega+1}$.

Theorem 2.1 and its corollaries have numerous applications. We will give a few below, by first identifying the ordinal α which we claim to be good, and then displaying the α-recursive set of axioms for which L_α (or L_{α^+}) is the least model. It will normally be clear that T is α-recursive and that L_α(or L_{α^+}) is the least model of T. However, in a few cases some further explanation will be given. We write the axioms in a combination of English and logical symbols, assuming that the reader will have no difficulty in transforming them entirely into the language $\mathscr{L}$, $\mathscr{L}[L_\delta]$, or $\mathscr{L}[L_\delta]\cup\{c\}$, whichever is appropriate,

(1) α admissible, locally countable and not recursively inaccessible:

$$\mathrm{Diag}\,(L_\delta)$$
$$\mathrm{KP}$$

where δ is the least upper bound of admissibles less than α.

Among other things, (i) shows that the first chance to get a bad ordinal is at the first recursively inaccessible. However, the next few results show that, not only is the first recursively inaccessible good, but so is the second, third, . . . , omega-th, etc. recursively inaccessible. Indeed we shall see that no description such as the—th recursively inaccessible will characterize the first bad ordinal.

(ii) The first recursively inaccessible:

$$\mathrm{KP}.$$
$$(\forall x)[\mathrm{Ord}\,(x)\rightarrow(\exists y)(\mathrm{Admiss}\,(y)\wedge y>x)].$$

(iii) The second recursively inaccessible:

$$\mathrm{KP},$$
$$(\exists x)[\mathrm{Admiss}\,(x)\wedge(\forall u\in x)(\exists z)(\mathrm{Admiss}\,(z)\wedge z\geq y],$$
$$(\forall x)[\mathrm{Ord}\,(x)\rightarrow(\exists y)(\mathrm{Admiss}\,(y)\wedge y>x)].$$

(iv) The third, fourth, fifth, ..., omega-d, ... recursively inaccessible:

iterate the procedure going from (ii) to (iii).

In the next few results we give successively more comprehensive processes for generating good ordinals. First, following Kripke, we call an admissible ordinal α, *recursively hyperinaccessible* if α is recursively inaccessible and a limit of recursively inaccessibles. Since it can easily be shown that recursively hyperinaccessibles are fixed points of recursively inaccessibles, the following result includes the information of (ii), (iii) and (iv).

(v) α is recursively inaccessible, locally countable, but not recursively hyperinaccessible:

KP,

$\mathrm{Diag}(L_\delta)$,

$(\forall x)[\mathrm{Ord}(x) \to (\exists y)(\mathrm{Admiss}(y) \wedge y > x)]$,

where δ is the least upper bound of recursively inaccessibles less than α.

At this point we might wish to iterate the procedure going from recursively inaccessible to recursively hyperinaccessible, thereby defining recursively hyper-hyper-inaccessible, recursively hyper-hyper-hyper-inaccessible, etc. We then could prove results analogous to (v). However, let us instead follow Kripke, by calling an ordinal α *recursively Mahlo*, if α is admissible and every α-recursive, closed (in the order topology), α-unbounded subset of α contains an admissible. Since Kripke has shown that recursively Mahlo ordinals are recursively inaccessible of all orders $< \alpha$, it follows that our next result is a far-reaching generalization of any such analogue to (v).

(vi) α admissible, locally countable and not recursively Mahlo:

(1) KP,

(2) $(\forall x)(\varphi(x, \bar{a}) \to \sim \mathrm{Admiss}(x))$,

(3) $\{x \mid \varphi(x, \bar{a})\}$ is closed and unbounded,

where φ is some Σ_1 formula which (with parameter a) defines, over L_α, an α-recursive closed unbounded subset of α containing no admissibles. Let us denote this set by S.

To see that L_α is the least initial segment of L which is a model of (1), (2) and (3), we'll suppose that there is $\beta < \alpha$, such that $a \in L_\beta$ and L_β is a model of (1), (2) and (3). Then β is admissible and, as we show below, $\beta \in S$. This will then contradict the choice of S.

To see that $\beta \in S$, let $S' = \{x \mid L_\beta \models \varphi(x, a)\}$. By (3), S' is closed and unbounded in β. Moreover, since φ is Σ_1, it follows that $S' \subseteq S$. In addition, since β is the least upper bound of the ordinals in S', and S is closed we see that $\beta \in S$.

Since (vi) implies that the first chance to get a bad ordinal is at the first recursively Mahlo, the obvious question to ask is what happens at the first recursively Mahlo. Our next result shows that it also is good.

(vii) The first recursively Mahlo:

$$\mathrm{KP},$$
$$(\forall x)(\forall u)(\forall v)[\varphi(i, x, u) \leftrightarrow \varphi(j, x, v) \wedge$$
$$\{x \mid \varphi(i, x, u)\} \text{ is closed and unbounded} \rightarrow (\exists z)(L_z \models \mathrm{KP} \wedge \varphi(i, z, u)]$$

where $\varphi(i, x, y)$ ennumerates all Σ_1-formulas of $\mathscr{L}$ with two free variables. (In this way all Σ_1-subsets of L_α are obtained by letting i run over ω and y run over α. This is because n-tupling functions are available so that any Σ_1-subset of L_α can be defined from a single parameter.)

Iterations of this idea can be used to show that the second, third, etc, recursively Mahlo ordinal is good. It is also possible to iterate the notion of recursively Mahlo itself. In this way one defines the notions of recursively hyper-Mahlo, recursively hyper-hyper-Mahlo etc. It is then possible to prove analogues to (vi) and (vii). However, rather than proceed in such a tedious fashion, let us consider a much more comprehensive procedure.

Following Aczel and Richter [3], we say that α is Σ_1^1-*reflecting* if every Σ_1^1-sentence φ, with parameters from L_α, is true in some L_β with $\beta < \alpha$, whenever it is true in L_α. Since it is shown in [3] that Σ_1^1 reflecting ordinals are recursively Mahlo of all possible orders, it follows that our next result includes, as well as generalizes, all of the content of (i) to (vii). We remark that this will be the first time we require the full strength of Theorem 2.1.

(viii) α is locally countable and not Σ_1^1-reflecting.

The proof of this splits into two cases—one when α is recursively inaccessible and the other when it is not. Since the latter is just a repetition of (i) we need only prove the former. To handle this we shall require the following slightly strengthened version of a result of Barwise et al. [6].

(BGM) If $\varphi(v_1, \ldots, v_n)$ is a Σ_1^1-formula of $\mathscr{L}$, then there is a Π_1 $\varphi^*(v_0, v_1, \ldots, v_n)$ of $\mathscr{L}$ such that for every nonempty countable transitive set A and every admissible set B such that $A \in B$, if $a_1, \ldots, a_n \in A$, then

$$A \models \varphi(\bar{a}_1, \ldots, \bar{a}_n) \quad \text{iff } B \models \varphi^*(\bar{A}, \bar{a}_1, \ldots, \bar{a}_n).$$

(A proof of this can be found on page 335 of [3].)

Suppose now that α is not Σ_1^1-reflecting, and let φ be some Σ_1^1-sentence (with parameters $a_1, \ldots, a_n \in L_\lambda$ with $\lambda < \alpha$) such that $L_\alpha \models \varphi$ but for no $\gamma < \alpha$ do we have $L_\gamma \models \varphi$. By (BGM) there is some Π_1-sentence φ^* (with parameters $a_1, \ldots, a_n$) such that

$$L_{\alpha^+} \models \varphi^*(\bar{L}_\alpha, \bar{a}_1, \ldots, \bar{a}_n).$$

Moreover α is locally countable, so that λ is countable in L_α. Hence to show that

α is good it suffices to show (by Theorem 1) that α is the least β such that

$$\langle L_{\beta^+}, \in, L_\beta, a_1, \ldots, a_n \rangle \models \varphi^*.$$

To see this, suppose that there is an admissible $\gamma < \alpha$ such that

$$\langle L_{\gamma^+}, \in, L_\gamma, a_1, \ldots, a_n \rangle \models \varphi^*.$$

Then since $\gamma^{++} \leq \alpha^+$, and φ^* is Π_1, we have

$$\langle L_{\gamma^{++}}, \in, L_\gamma, a_1, \ldots, a_n \rangle \models \varphi^*.$$

But since $\gamma^+ \geq \lambda$, it follows from (BGM) that

$$\langle L_{\gamma^+}, \in, L_\gamma, a_1, \ldots, a_n \rangle \models \varphi.$$

However, since α is recursively inaccessible, $\gamma^+ < \alpha$ so that we have contradicted the fact that α is not Σ_1^1-reflecting.

The pattern of what we have done up until now has been to define more and more comprehensive classes of good ordinals. Then each time we considered a new class, we showed that the least ordinal not in that class is also good. Hence the natural next step would be to attempt to show that the first Σ_1^1-reflecting ordinal is good. Unfortunately this is not so, for as we shall see in the next section, all Σ_1^1-reflecting ordinals are bad. Moreover we shall see that the first Σ_1^1-reflecting ordinal is the first bad ordinal.

We now move on to some additional applications of Theorem 2.1, this time concentrating, for the most part, on ordinals larger than the first bad ordinal.

(ix) the first non-projectible:

KP,

Infinity,

Σ_1-separation.

It is easy to show (as Kripke does in [18]) that α is non-projectible iff $L_\alpha \models \Sigma_1$-separation.

(x) the second non-projectible:

KP,

Infinity,

Σ_1-separation,

$(\exists x)(L_x \models \mathrm{KP} + \mathrm{Infinity} + \Sigma_1\text{-separation})$.

The procedure of (x) can be iterated to show that the third, fourth, ..., omega-th, ... non-projectible is good.

(xi) The least α such that $L_\alpha \leq_1 L_{\alpha^+}$:

KP^c,

σ^c,

$(\forall u \in c)[(\exists x)\varphi(x, u) \rightarrow (\exists x \in c)\varphi^c(x, u)]$

for all Σ_1-formulas φ of $\mathscr{L}$ which have at most two free variables. As in (vii) we need only use one parameter since c will be closed under n-tuples. Note that each of the sentences in the third displayed line above is Π_1 so that Theorem 2.1 will apply.

We remark that this is a best possible result, for in the next section we shall show that if $L_\alpha \leq_1 L_{\alpha^+ +1}$, then α is bad.

(xii) β_0 (the closure ordinal of the ramified analytic sets—see Boolos and Putnam [8])

$$\begin{array}{l}\text{KP,}\\ \text{Infinity,}\\ \text{Replacement.}\end{array}$$

Part of the folklore concerning β_0 is that β_0 is the least ordinal satisfying the above set of sentences.

From the main technical lemma of [8] it follows quite easily that all admissibles less than β_0 are locally countable. Hence as a special case of (i), it follows that all admissibles less than β_0 which are not recursively inaccessible are good. Furthermore, suppose there is a real in $L_{\alpha^+} - L_\alpha$. Then again the main technical lemma of [8] implies that α^+ is locally countable. Since α^+ is not recursively inaccessible, (i) implies that all such α^+ are good. We remark that since reals are constructed coefinally with constructible $\aleph_1$, it follows that the good ordinals are at least cofinal with constructible $\aleph_1$.

(xiii) the least α^+ such that there is no real in $L_{\alpha^+} - L_\alpha$:

$$\begin{array}{l}\text{KP,}\\ (\exists x)[L_x \models \text{KP} \wedge (\forall y)(y \subseteq \omega \rightarrow y \in L_x)].\end{array}$$

By the main technical lemma of [8], it is easy to see that the ordinal of (xiii) is the least admissible which is not locally countable.

We remark that the idea of this example can be iterated to show that the least $\alpha^{++}(\alpha^{+++}, \text{etc.})$ such that there is no real in $L_{\alpha^{++}} - L_\alpha (L_{\alpha^{+++}} - L_\alpha, \text{etc.})$ is good.

(xiv) the least α such that there is no real in $L_{\alpha^+} - L_\alpha$:

$$\begin{array}{l}\sigma^c,\\ \text{KP}^c,\\ (\forall x)(x \subseteq \omega \rightarrow x \in c).\end{array}$$

Since these sentences are all Π_1 the result follows from Theorem 2.1.

It is interesting to note that the ordinal of (xiv) is the successor admissible to the ordinal of (xiii). This situation is not unique in that many of the other examples enable us to show that α^+ is good, once we have shown that α is good. These observations make it tempting for us to believe that α^+ is always good whenever α is good. Unfortunately we are not able to prove this. The best we can do is show that if α is good either, on the basis of Corollary 2.1.1, or on the basis of

Theorem 2.1 *with T finite*, then α^+ is also good. To prove the former assertion we shall need the following

Lemma. *Suppose that α is admissible, $\gamma<\alpha$ is countable in L_α, and α is the least β such that $\langle L_\beta, \in, d\rangle_{d\in L_\gamma}$ is a model of T, for some theory $T\in L_\alpha$. Then α is countable in L_{α^+}.*

Proof. Since $\langle L_\alpha, \in, d\rangle_{d\in L_\gamma}\models T$ we can assume that extentionality and σ are elements of T. This will enable us to collapse any well-founded model of T to an initial segment of L.

Now let $\varphi_n(v_0)$ be an enumeration of all formulas of $\mathscr{L}(L_\delta)$, which have exactly v_0 free, and let $\psi_n(v_0)$ be

$$(\varphi_n(v_0)\wedge(\forall v_1)(v_1<_L x\rightarrow \sim\varphi_n(v_1)))\vee\sim(\exists v_1)(\varphi_n(v_1)\wedge v_1)\wedge v_1=\bar{\phi}).$$

Then for each $n\in\omega$ there is a unique $x\in L_\alpha$ such that $L_\alpha\models\psi_n[x]$. Hence if M is the set of all such x, then clearly

$$\langle M, \in, d\rangle_{d\in L_\gamma}\leqslant\langle L_\alpha, \in, d\rangle_{d\in L_\gamma}, \tag{1}$$

so that M, and its collapse $\bar{M}$, are models of T. Moreover, since $M\subseteq L_\alpha$, it follows (from the minimility of α) that $\bar{M}=L_\alpha$.

Now suppose that $x\in M$. Then for some $n\in\omega$, $\langle L_\alpha, \in, d\rangle_{d\in L_\gamma}\models\psi_n[x]$, whence by (1), $\langle M, \in, d\rangle_{d\in L_\gamma}\models\psi_n(\bar{x})$. However, since $\bar{M}=L_\alpha$, it then follows that $x=\bar{x}$. Hence the collapsing map is the identity, so that M is actually equal to L_α.

Note that all we have done so far is show that L_α is pointwise definable, by trivially modifying the usual proof of the fact that the minimal model of ZF is pointwise definable.

To complete the proof of the lemma, we let f be the function from L_α into ω defined by

$$f(x)=\text{least } n \text{ such that } L_\alpha\models\psi_n(x).$$

Since f is a counting of L_α, it suffices to show that $f\in L_{\alpha^+}$. But this is true because f is actually definable over $L_{\alpha+1}$—something which easily follows from the fact that satisfaction for L_α is $L_{\alpha+1}$-definable, together with the fact that (since γ is countable in L_α) the enumeration, $\langle\psi_n\mid n\in\omega\rangle$, belongs to L_α.

Now suppose that α has been shown to be good as an application of Corollary 2.1.1. Then by the lemma above, α (and of course $\alpha+1$) is countable in L_{α^+}. Hence if S is the theory consisting of

KP,

Diag$(L_{\alpha+1})$,

then α^+ is the least β such that $\langle L_\beta, \in, d\rangle_{d\in L_\alpha}\models S$. This shows

(xv) if α is good as a result of Corollary 2.1.1, then α^+ is also good.

Now suppose that α has been shown to be good as a result of *Theorem* 2.1 *with T finite.* Then is S is

$$\text{KP},$$
$$(\exists x)(\varphi^* \bigwedge x \models \{\sigma\} \bigcup \text{KP})$$

where φ^* is the conjunction of the elements of T with each occurrance of c replaced by x (where x is a variable which does not occur in σ or φ), then clearly L_{α^+} is the least initial segment of L which is a model of S. Hence by Corollary 2.1.1 we can conclude

(xvi) if α is good as a result of Theorem 2.1 with T finite, then α^+ is also good.

This list of good ordinals is certainly not exhaustive. But since continuing further would surely exhause the reader, we shall stop here.

It is interesting to note that all of these examples have resulted from applications of Theorem 2.1. Unfortunately we don't know if Theorem 2.1 is strong enough to yield all good ordinals. In particular, two major questions remain open. Namely, (1) are there good ordinals with bad successors? and (2) are there any uncountable good ordinals? (Theorem 1.1, of course, only applies to countable ordinals.)

Although further generalizations of the idea of Theorem 2.1 may lead to positive answers to these questions, we feel that such generalizations are unlikely, in that all three of the hypotheses seem to be needed. We shall close this section by showing this to be the case.

First we note that some condition on δ, such as δ is countable in L_α, is going to be necessary. For if we allowed δ to be any ordinal less than α, then the idea of (i) would show that all non-recursively inaccessible admissibles are good. This, however, as we shall see in the next section, if false.

Next we show that some condition on T, such as $T \in L_\alpha$, is going to be necessary. For if not we could let $T = Th(\langle L_{\delta^1_2}, \in \rangle)$. Then using the fact that there are reals in $L_{\delta^1_2+1} - L_{\delta^1_2}$ (namely all $\Sigma^1_2 - \Delta^1_2$ reals, among other things) we could show that $L_{\delta^1_2}$ is pointwise definable. (The proof is rather similar to part of the proof of the lemma on p. 22.)

This then shows that $L_{\delta^1_2}$ can have no proper elementary submodels, which in turn implies that δ^1_2 is the least β such that $\langle L_\beta, \in \rangle \models T$. The proposed extension of Theorem 2.1 would then show that δ^1_2 is good. However, in the next section we shall show that δ^1_2 is bad.

Finally we remark that the condition that T contain only Π_1 sentences is the strongest possible. For in the next section we shall show that if Π_1 is replaced, in Theorem 2.1, even by Σ_1, the new statement becomes false.

3. Bad ordinals

The previous section was concerned with showing that many admissible ordinals are good. In this section we shall develop techniques for showing that there are many bad admissibles as well. Among other things, we shall answer most questions concerning countable ordinals, give characterizations of both the least bad ordinal and the class of bad ordinals, and answer some questions concerning uncountable admissibles. We shall also make some conjectures on some of the issues we have not been able to settle.

We begin by mentioning that for the past several years many people knew that $\aleph_1$ is bad. The proof, however, which turned on Gödel's result that any countable constructible set of countable ordinals is constructed before $\aleph_1$, was too crude to yield anything more than the badness of regular cardinals. A much more thorough analysis results from exploiting the various beta properties (defined below) in classifying W^α. W^α is the α-recursion theory analogue of Spector's set W (see Spector [27]). It is defined in the following

Definition. Let e, x, y, z range over ordinals less than α and f range over functions from ω into α. LO^α, the set of α-indices of α-recursive linear orderings is defined

$$e \in \mathrm{LO}^\alpha \leftrightarrow \text{(i)} \wedge \text{(ii)} \wedge \text{(iii)} \wedge \text{(iv)}$$

where (i), (ii), (iii) and (iv) are as follows:

(i) $(\forall x)(\forall y)(\exists z) T^\alpha(e, x, y, z)$,

(ii) $(\forall x)(\forall y)(\forall z)[(\{e\}^\alpha(x, z) = 0 \text{ or } \{e\}^\alpha(z, x) = 0) \;\&\; (\{e\}^\alpha(y, z) = 0 \text{ or } \{e\}^\alpha(z, y) = 0) \rightarrow \{e\}^\alpha(x, y) = 0 \text{ or } \{e\}^\alpha(y, x) = 0]$,

(iii) $(\forall x)(\forall y)[\{e\}^\alpha(x, y) = 0 \text{ and } \{e\}^\alpha(y, x) = 0 \rightarrow x = y]$,

(iv) $(\forall x)(\forall y)(\forall z)[\{e\}^\alpha(x, y) = 0 \;\&\; \{e\}^\alpha(y, z) = 0 \rightarrow \{e\}^\alpha(x, z) = 0]$.

W^α, the set of α-indices of the α-recursive well-orderings, is defined as,

$$e \in W^\alpha \leftrightarrow e \in \mathrm{LO}^\alpha \wedge \sim(\exists f)(\forall x \in \omega)[\{e\}^\alpha(f(x+1), f(x)) = 0].$$

The beta properties which we shall require are given in the following

Defiition. An ordinal α will be said to possess the *weak beta property* (w.b.p.) if every linear ordering in L_α, which is not a well-ordering, has an infinite descending chain in L_α. α has the *strong beta property* (s.b.p.) if every constructible linear ordering of α which is not a well-ordering, has an infinite descending chain in L_α. α has the *recursive strong beta property* (r.s.b.p.) if every α-recursive linear ordering of α which is not a well-ordering, has an infinite descending chain in L_α. α has the *recursive strong almost beta property* (r.s.a.b.p.) if every α-recursive linear ordering of α which is not a well-ordering, has an infinite descending chain in L_{α^+}.

All of these beta properties express absoluteness properties for various classes of well-orderings. The have their origin in Mostowski [22], where the weak beta property was studied in the context of models of analysis. Since Mostowski considered no variations, he simply called it the beta property. The variations considered here are new.

Our first result is typical of the way we shall use the β-properties in classifying w^α.

Proposition 3.1. *If α is admissible and has the recursive strong β-property, then W^α is Π_2 (over L_α).*

Proof. For any admissible α, LO^α is clearly Π_2. When α has the r.s.b.p., then to say that an α-recursive linear ordering is a well-ordering, requires saying only that there are no infinite descending chains in L_α. But an infinite descending chain in L_α is just an α-partial recursive function, which has some ordinal index less than α. Hence

$$3\in W^\alpha \leftrightarrow e\in \mathrm{LO}^\alpha \wedge \sim(\exists x)(\forall y\in\omega)[\{e\}^\alpha(\{x\}^\alpha(y+1),\{x\}^\alpha(y))=0]$$

which can easily be seen to be Π_2 (over L_α) using Σ_0-collection.

We next observe that when W^α is α-arithmetical, the well-ordering obtained by taking the sum of the well-orderings represented by the elements of W^α is also α-arithmetical, with an order type greater than the order type of any α-recursive well-ordering. This is essentially the content of

Proposition 3.2. *If α is admissible and has the r.s.b.p., then $|\alpha$-recursive$|<|\alpha$-arithmetical$|$. (Actually $|\alpha$-recursive$|<|\alpha-\Pi_2|$.)*

Proof. The ordering of pairs

$$\begin{aligned} J(x,n)\leq J(y,m)\leftrightarrow n\in W^\alpha\wedge m\in W^\alpha\wedge[\{n\}(x,y)=0 \\ \vee(n<m\wedge(\exists u)(\{n\}(x,u)=0\vee\{n\}(u,x)=0)\wedge \\ (\exists v)(\{m\}(y,v)=0\vee\{m\}(v,y)=0)) \end{aligned}$$

is the sum ordering mentioned above, so its type is greater than or equal to $|\alpha$-recursive$|$. It is easily seen to be Π_2 (over L_α).

By (1) on p. 8, we now obtain the following

Corollary. *If α is admissible and has the recursive strong beta property, then $\alpha^+>|\alpha$-recursive$|$, i.e. α is bad.*

Since the corollary shows that all recursive strong beta ordinals are bad, our next aim is to produce as wide a class of recursive strong beta ordinals as we can. The following result will prove to be quite useful in this direction.

Lemma 3.1. *If α is admissible and R is a linear ordering in L_α with no infinite descending chains in $L_{\alpha+1}$, then R is a well-ordering.*

Proof. Let R be as in the hypothesis and assume that R is not a well-ordering. Consider I_R as in the proof of the Well-ordering Isomorphism Lemma. In that proof we showed that $\mathrm{Fld}_R - I_R$ has no R-minimal element. Hence given any $d \in \mathrm{Fld}_R - I_R$ there is an infinite descending chain f, beginning at d, definable over L_α, i.e.

$$\begin{aligned} f(n) = x \leftrightarrow L_\alpha \models n \in \omega \wedge (\exists g)[\mathrm{dom}\,(g) = n+1 \wedge g(0) = d \\ \& \; g(n) = x \wedge (\forall y)(\forall i \in n+1)(g(i+1) = y \\ \leftrightarrow R(y, g(i)) \wedge (\forall z)(z \in \mathrm{Fld}_R - I_R \\ \wedge z <_L y \rightarrow \sim R(z, g(i))) \wedge y \notin I_R)]. \end{aligned}$$

(As usual $<_L$ is a definable well-ordering of L_α.) Hence $f \in L_{\alpha+1}$.

We remark that the lemma is optimal in that the bound, $\alpha+1$, cannot be improved to α. For if so, then every admissible would have the weak beta property, which is definitely false.

As a first application of Lemma 3.1 we can prove

Theorem 3.1. *If $L_\alpha \leqslant_1 L_{\alpha^+ +1}$, then α has the recursive strong β-property.*

Proof. First note that if $L_\alpha \leqslant_1 L_{\alpha^+ +1}$, then α must be admissible. To see this we need only show that $L_\alpha \models \Sigma_0$-collection. Hence suppose that

$$L_\alpha \models (\forall x \in u)(\exists y)\varphi(x, y)$$

where φ is a Σ_0-formula with parameters from L_α, and $a \in L_\alpha$. Then $(\forall x \in u)(\exists y \in L_\alpha)\varphi^{L_\alpha}(x, y)$. But since φ is Σ_0, it follows that

$$\varphi^{L_\alpha}(x, y) \quad \text{iff } \varphi(x, y),$$

so that

$$(\forall x \in a)(\exists y \in L_\alpha)\varphi(x, y).$$

Then since $L_\alpha \in L_{\alpha^+ +1}$, it follows that

$$L_{\alpha^+ +1} \models (\exists u)(\forall x \in a)(\exists y \in u)\varphi(x, y).$$

However, $L_\alpha \leqslant_1 L_{\alpha^+ +1}$, so that

$$L_\alpha \models (\exists u)(\forall x \in a)(\exists y \in u)\varphi(x, y)$$

Hence $L_\alpha \models \Sigma_0$-collection.

Now let R be an α-recursive linear ordering of α. By the usual argument $R \in L_{\alpha^+}$. Hence if R is not a well-ordering then, by Lemma 3.1, R has an infinite descending chain in L_{α^++1}.

Since R is α-recursive, R is Σ_1 definable over L_α, i.e. there is some Σ_1-formula $\varphi(x, y, v_1, \ldots, v_n)$ such that

$$R(x, y) \leftrightarrow L_\alpha \models \varphi(x, y, a_1, \ldots, a_n).$$

where $a_i \in L_\alpha$. Define R' as

$$R'(x, y) \leftrightarrow L_{\alpha^++1} \models \varphi(x, y, a_1, \ldots, a_n).$$

Since $L_\alpha \leqslant_1 L_{\alpha^++1}$, it is easily seen that $R' \supseteq R$, so that any infinite descending chain for R is also an infinite descending chain for R' (even though R' might not even be a partial ordering). Hence

$$L_{\alpha^++1} \models (\exists f)\mathrm{Fun}\,(f) \wedge \mathrm{dom}\,(f) = \omega \wedge (\forall y \in \omega)\varphi(f(y+1), f(y), a_1, \ldots, a_n)].$$

This is a Σ_1-statement, with parameters from L_α, true in L_{α^++1}, hence true in L_α. It says precisely that R has an infinite descending chain in L_α.

Combining Theorem 3.1 with Proposition 3.2 we get the following immediate corollary.

Corollary 3.1. *If $L_\alpha \leqslant_1 L_{\alpha^++1}$, then α is bad.*

Since the least α such that $L_\alpha \leqslant_1 L_{\alpha^+}$ is good (application xi) of section 2), it follows that Corollary 3.1 is a best possible result.

Before giving further conditions which insure that an ordinal is bad let us pause briefly to consider some of the information given by Corollary 3.1.

Perhaps the first thing one should note is that all stable ordinals are bad. Stable ordinals are the ordinals α, such that $L_\alpha \leqslant_1 L$. They were introduced by Kripke, who showed that the class of stables is rather large. Indeed in [18] it is shown that all uncountable cardinals, as well as all ordinals of the form δ_n^{1A} (with $n \geqslant 2$ and $A \subseteq \omega$) are stable. Moreover it is shown that there are $\aleph_\beta$ stables less than $\aleph_\beta$. Hence just from our knowledge of stables we see that there are lots of bad ordinals.

However, Corollary 3.1 also gives information about ordinals less than the first stable (which incidentaly is δ_2^1). Indeed it follows from Theorem 1.5, that the first ordinal α such that $L_\alpha \leqslant L_{\alpha^++1}$ is much less than the first non-projectible. Hence Corollary 3.1 shows that there are bad ordinals much less than the first non-projectible, so that the first bad ordinal is relatively small.

However, from a recursive point of view, all the ordinals of Corollary 3.1 are large, since as Kripke has shown, whenever α is such that $L_\alpha \leqslant_1 L_\beta$ (with $\beta > \alpha$)

then α is recursively Mahlo, of all orders less than α. This observation is of course consistent with our results in Section 2.

We now move on to a second application of Lemma 3.1, namely

Theorem 3.2. *If α is Σ_1^1-reflecting, then α has the recursive strong β-property.*

Proof. Let R be an α-recursive linear ordering of α. Then there is a Σ_1-formula $\varphi(x, y, v_1, \ldots, v_n)$ and elements $a_1, \ldots, a_n \in L_\alpha$ such that

$$R(x, y) \quad \text{iff } L_\alpha \models \varphi(x, y, a_1, \ldots, a_n).$$

If R is not a well-ordering, then there is some infinite descending chain for R. This can be expressed by a Σ_1^1-statement over L_α, i.e.

$$L_\alpha \models (\exists f \subseteq L_\alpha \times L_\alpha)[f \text{ is a function } \wedge \operatorname{dom}(f) = \omega \wedge (\forall n \in \omega)\varphi(f(n+1), f(n), a_1, \ldots, a_n)].$$

Now since α is Σ_1^1-reflecting, it follows that there is some $\beta < \alpha$ such that

$$L_\beta \models (\exists f \subseteq L_\beta \times L_\beta)[f \text{ is a function } \wedge \operatorname{dom}(f) = \omega \wedge (\forall n \in \omega)\varphi(f(n+1), f(n), a_1, \ldots, a_n)].$$

Moreover since φ is Σ_1 it follows that $R' \subseteq R$, where

$$R'(x, y) \quad \text{iff } L_\beta \models \varphi(x, y, a_1, \ldots, a_n).$$

Hence R' is a linear ordering which is not a well-ordering. Furthermore $R' \in L_{\beta^+}$, since it is definable over L_β.

Now recalling that a Σ_1^1-reflecting ordinals are recursively inaccessible, it follows that $\beta^+ < \alpha$. Hence by Lemma 3.1, R' must have a chain in $L_{\beta^{++}+1}$, and hence a chain in L_α. But any chain for R' is also a chain for R.

Combining Theorem 3.2 with Proposition 3.2 we get, as an immediate corollary a second condition which insures that an ordinal is bad. Namely,

Corollary 3.2. *If α is Σ_1^1-reflecting, then α is bad.*

This corollary now enables us to characterize the first bad ordinal as the first Σ_1^1-reflecting ordinal. For let α_0 be the first bad ordinal. Then, as we have pointed out above, α_0 is projectible into ω and so α_0 must be locally countable. But then α_0 must be Σ_1^1-reflecting. For if not, application (viii) of Section 2 would imply that α_0 is good. Hence we have shown

Theorem 3.3. *The first bad ordinal is the first Σ_1^1-reflecting ordinal.*

Theorem 3.3 enables us to see the truth of the remark made at the end of Section 2, concerning the impossibility of replacing Π_1 by Σ_1 in Theorem 2.1 (or in the tree construction lemma). For suppose this replacement were possible. Then the proof of application (viii) of Section 2 could be repeated mutatus mutandis to obtain

(1) If α is locally countable and not Π^1_1-reflecting, then α is good (where Π^1_1-*reflecting* is defined similarily to Σ^1_1-reflecting except that Π^1_1 is used in place of Σ^1_1).

However, Aczel and Richter in [2] have observed that the proof of Corollary 17 of Lévy [20] will work with Π^1_1-reflecting and Σ^1_1-reflecting instead of Π^2_1-indescribable and Σ^2_1-indescribable, respectively. This has as a consequence

(2) the first Σ^1_1-reflecting ordinals is not Π^1_1-reflecting.

(1) and (2), along with the previously mentioned fact that the first Σ^1_1-reflecting ordinal is locally countable, would then show that the first Σ^1_1-reflecting ordinal is good; contradicting Theorem 3.3.

Another variation on the theme of this section is given in the following theorem, which provides a characterization of the entire class of bad ordinals.

Theorem 3.4. *If α is admissible, then α is bad iff α has the recursive strong almost β-property.*

Proof. The fact that bad admissibles have the recursive strong almost β-property is just the contrapositive of Proposition 2.1. Hence all that remains is to show that admissibles with the recursive strong almost β-property are bad. Again this is done by classifying W^α.

First note that for all admissible α

$$e \in W^\alpha \leftrightarrow e \in \mathrm{LO}^\alpha \;\&\; (\exists f \in L_{\alpha^+})(\exists \gamma \in \alpha^+)[f \text{ is an isomorphism from } \mathrm{Fld}_e \text{ onto } \gamma].$$

This is true by the well-ordering isomorphism lemma. Hence W^α is α^+ r.e. Now if α has the recursive strong almost beta property, then

$$e \in W^\alpha \leftrightarrow e \in \mathrm{LO}^\alpha \;\&\; \sim(\exists f \in L_{\alpha^+})[f \text{ is an infinite descending chain on } \mathrm{Fld}_e].$$

Hence W^α is also Π_1 over L_{α^+}. Since W^α is α^+-bounded, it follows from Theorem 1.1(d) that $W^\alpha \in L_{\alpha^+}$.

Now, just as in the proof of Proposition 3.2 we shall form the sum of all well-orderings with notations in W^α. This is then an α^+-finite well-ordering, R, of type greater than or equal to $|\alpha\text{-recursive}|$. By the well-ordering isomorphism lemma, $R < \alpha^+$.

Unfortunately the characterization given above is not as nice as we would wish—since α^+ appears as part of it (i.e. imbedded in the definition of the recursive strong almost beta property). A much more attractive characterization would result if we could replace the r.s.a.b.p. by the r.s.b.p. in Theorem 3.4. We, however, do not know if this can be done; indeed we conjecture that it cannot.

Theorems 3.1—3.4 illustrate our main techniques for producing bad admissibles. Combined with the results of Section 2, they can be used to settle most questions concerning whether or not a countable admissible is good or bad. For uncountable admissibles, however, the situation seems less tractable.

Indeed for uncountables, we have only been able to extract a few partial results—the most interesting of which are summarized in

Theorem 3.5. *Let α be admissible. If α satisfies either*

(i) *α^+ is projectible into a strong beta ordinal, or*

(ii) *the constructible cofinality of α is greater than ω, then α is bad.*

Since the proof of Theorem 3.5 uses (what should by now be) familiar arguments about constructible well-orderings, we forgo the details.

Instead we simply point out that $(\aleph_\omega^L)^+$ (which has constructible cofinality ω) is probably the most interesting admissible for which Theorem 3.5 gives no information. Using the tree techniques of Section 2, we have been able to show that $(\aleph_\omega^L)^+$ is Δ_2-good—in the sense that the $(\aleph_\omega^L)^+$-Δ_2 well-orderings go out as far as $(\aleph_\omega^L)^{++}$.

Whether or not $(\aleph_\omega^L)^+$ is actually good is an issue which we have not been able to settle. We conjecture, however, that it is bad; moreover it seems very likely that all uncountable admissibles are bad.[1]

Another issue which we have not been able to settle is whether there is a good admissible with a bad successor. (Some remarks were made about this at the end of Section 2.) We conjecture that there isn't—but wish to point out that, if there is, then "recursive strong almost β" cannot be replaced by "recursive strong β", in Theorem 3.4.

4. Some applications

As well as being an interesting subject in its own right, we believe that the theory of good and bad ordinals may be of further interest in terms of applications. In this section we shall take a modest step in this direction by using some of the results and methods of the previous sections to give new proofs of a few results which have been proven elsewhere by other methods.

We begin by establishing that the Π_1^1 well-orderings of subsets of ω go out exactly as far as ω_1^{O}. This is the main new result of Tanaka [28]. In our notation it

[1] Added during revision: This has turned out to be false. Indeed F. Abrahamson and G. E. Sacks have shown that $(\aleph_\omega^L)^+$ is good. Their work should appear elsewhere.

is denoted by $|\Pi_1^1|=\omega_1^O$. To prove it we simply observe the following string of equalities

$$|\Pi_1^1| \overset{(1)}{=} |\omega_1-\text{r.e.}| \overset{(2)}{=} |\omega_1-\text{rec}| \overset{(3)}{=} \omega_1^+ \overset{(4)}{=} \omega_1^O$$

Note that (1) follows because ω_1 is projectible into ω and because the Π_1^1-reals are exactly the ω_1-r.e. reals (see Kripke [17]), (2) is simply (3) on p. 179, (3) is true because ω_1 is good (see (i) of Section 2) and (4) is true because of the following argument. First note that since $O \in L_{\omega_1+1}$ (see Kripke [17]), $\omega_1^O \leq \omega_1^+$. Then we recall that there is a Π_1^1 well-ordering, R, of type ω_1 (see Gandy [9]). Since R is recursive in O it follows that $\omega_1 < \omega_1^O$. But since ω_1^O is admissible it must be the case that $\omega_1^O = \omega_1^+$.

Our second application is the analogue of the above for Σ_2^1 well-orderings. Here the result, also from Tanaka [28], is the opposite of what one might expect after seeing the above—namely $|\Sigma_2^1| < \omega_1^{O^2}$ (where O^2 is any complete Σ_2^1-set). It is proven by establishing the following string of equalities and inequalities

$$|\Sigma_2^1| \overset{(1')}{=} |\delta_2^1\text{-r.e.}| \overset{(2')}{=} |\delta_2^1\text{-rec}| \overset{(3')}{<} \delta_2^{1+} \overset{(4')}{=} \omega_1^{O^2}.$$

Note that (1′) follows because δ_2^1 is projectible into ω and because the Σ_2^1-reals are exactly the δ_2^1-r.e. reals (see Kripke [17]), (2′) is simply (3) on p. 179, (3′) is true because δ_2^1 is bad (Corollary 3.1) and (4′) is true because of the following argument. First note that since $O^2 \in L_{\delta_2^1+1}$ (see Kripke [17]), $\omega_1^{O^2} \leq \delta_2^{1+}$. Then we recall that there is a Σ_2^1 well-ordering, R, of type δ_2^1 (see Rogers [25, p. 417]). Since R is recursive in O^2 it follows that $\delta_2^1 < \omega_1^{O^2}$. But since $\omega_1^{O^2}$ is admissible, it must be the case that $\omega_1^{O^2} = \delta_2^{1+}$.

Our third application concerns the order relation between the least Σ_1^1-reflecting ordinal (which we shall denote by σ_1^1) and the least Π_1^1-reflecting ordinal (which we shall denote by π_1^1). This problem was raised by Aczel and Richter [2] who observed (as mentioned on p. 199) that the proof of Corollary 17 of Lévy [20], suitably modified would show that $\pi_1^1 \neq \sigma_1^1$.

Since Aczel and Richter [2, 3], have conveniently characterized π_1^1 as the least α such that $L_\alpha \leq_1 L_{\alpha^+}$, we see from (xi) of Section 2 that π_1^1 is good. On the other hand σ_1^1 is bad by Theorem 3.4. Hence we have just given an alternative proof of the fact that $\pi_1^1 \neq \sigma_1^1$.

The question of which of these ordinals is actually greater is seemed impervious to Lévy's methods, and indeed was left open by Aczel and Richter. It was subsequently settled by Aanderaa in [1] who showed, using alternate characterizations of these ordinals involving certain inductive definitions, that

$$\pi_1^1 < \sigma_1^1. \tag{1}$$

Since this problem was somewhat tricky (as evidenced by the fact that it was open for a relatively long time) it would be very satisfying to give an alternate proof of (1) using our methods. Although we have not yet succeeded in doing so,[2] we do feel that some variation on the basic tree construction idea would enable us to show that

$$\text{all admissibles less than } \pi_1^1 \text{ are good,} \tag{2}$$

from which (1) would immediately follow.

In addition to eventually succeeding in proving (2), we have hopes that further applications of our methods will be found in the future.

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[2] Added during revision: This has been subsequently shown by the author and K. Hrbacek. Their proof will appear in the Zeitschrift für Mathematische Logik (vol. 25).

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