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Eigenvalue and stability of singular differential delay systems ☆

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Abstract

The eigenvalue and the stability of singular differential systems with delay are considered. Firstly we investigate some properties of the eigenvalue, then give the exact exponential estimation for the fundamental solution, and finally discuss the necessary and sufficient condition of uniform asymptotic stability.

Keywords: Singular differential systems with delay; Eigenvalue; Exponential exact estimation; Stability

1. Introduction

Now day, the phenomena of time delay in many practical systems have had many scholars' much attention. And many excellent results for the systems with time delay have been obtained [1–16]. Specially for the relationship between eigenvalue and stability of differential systems with delay, much achievement have been gotten. But we notice that a lot of practical systems, such as economic systems, power systems and so on, are singular differential systems with delay. In [3–11,17–23], authors have discussed the singular differential systems, even the singular differential systems with delay, and have gotten some consequences. But up to now, for the relationship between eigenvalue and stability of singular differential systems with delay, there is hardly effective verdict.

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In this paper, we consider singular differential system with delay

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geqslant t_0, \\ x(t) = \varphi(t), & t_0 - \tau \leqslant t \leqslant t_0, \end{cases} \tag{1}$$

where $x(t) \in R^n$ is a state vector; $E \in R^{n\times n}$ is a singular matrix, $A, B \in R^{n\times n}$ are matrices; $\tau > 0$ is time delay; and $\varphi(t)$ the initial state function.

Definition 1. If $\det(\lambda E - A) \not\equiv 0$, we call matrix couple (A, E) regular. If (A, E) is regular, we call system (1) regular.

From [3], we known that if (E, A) is regular, the system (1) is solvable.

Definition 2. Let E be a square matrix. If there exists a matrix E^d satisfying

(1) $EE^d = E^d E$,
(2) $E^d E E^d = E^d$,
(3) $(I - E^d E)E^l = 0$,

then we call E^d the Drazin inverse matrix of matrix E, simply D-inverse matrix. Here l is the index of matrix E, it is the smallest nonnegative integer which make

$$\operatorname{rank}(E^{l+1}) = \operatorname{rank}(E^l)$$

be true.

Definition 3. Let $X(t) \in R^{n\times n}$, and satisfy equation

$$\begin{cases} E\dot{X}(t) = AX(t) + BX(t-\tau), & t \geqslant t_0, \\ X(t) = \begin{cases} EE^d, & t = t_0, \\ 0, & t_0 - \tau \leqslant t \leqslant t_0. \end{cases} \end{cases} \tag{2}$$

Then we call $X(t)$ the fundamental solution of (1).

Definition 4. Let

$$H(\lambda) = (\lambda E - A - Be^{-\lambda\tau}). \tag{3}$$

We call equation

$$\det\big(H(\lambda)\big) = |\lambda E - A - Be^{-\lambda\tau}| = 0 \tag{4}$$

the proper equation of (1). If λ satisfies Eq. (4), then we call it the eigenvalue of system (1).

In this paper, we consider the eigenvalue and the stability of singular differential systems with delay. Firstly we give some properties of the eigenvalue, then give the exact exponential estimation for the fundamental solution, and finally give the necessary and sufficient condition of uniform asymptotic stability.

For the problems of singular neutral differential systems, by study we found that it can be changed to be the problems of singular differential systems with delay.

2. The properties of the eigenvalue

Now we discuss the proper equation (4) and give some properties of the eigenvalue. For system (1), by well-chosen algebraic substitution, we can make it as the form as

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geqslant t_0, \\ x(t) = \varphi(t), & t_0 - \tau \leqslant t \leqslant t_0. \end{cases}$$

Here

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

If (A, E) is regular, then

$$|E\lambda - A| = \begin{vmatrix} I\lambda - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{vmatrix} \not\equiv 0.$$

According to the decreasing power method to spread out it, we have that there is a positive integer $l > 0$ such that proper equation

$$|E\lambda - A| = \begin{vmatrix} I\lambda - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{vmatrix} = 0$$

becomes that

$$P_l\lambda^l + P_{l-1}\lambda^{l-1} + \cdots + P_1\lambda + P_0 = 0, \tag{5}$$

and $P_l \neq 0$.

Obviously, if (A, E) is regular, and the real part of λ is large enough, we have

$$|E\lambda - A - Be^{-\lambda\tau}| = \begin{vmatrix} I\lambda - A_{11} - B_{11}e^{-\lambda\tau} & -A_{12} - B_{12}e^{-\lambda\tau} \\ -A_{21} - B_{21}e^{-\lambda\tau} & -A_{22} - B_{22}e^{-\lambda\tau} \end{vmatrix} \not\equiv 0.$$

Again, according to the decreasing power method to spread out it, we have that there is a positive integer k $(k \geqslant l > 0)$, such that proper equation

$$|E\lambda - A - Be^{-\lambda\tau}| = \begin{vmatrix} I\lambda - A_{11} - B_{11}e^{-\lambda\tau} & -A_{12} - B_{12}e^{-\lambda\tau} \\ -A_{21} - B_{21}e^{-\lambda\tau} & -A_{22} - B_{22}e^{-\lambda\tau} \end{vmatrix} = 0$$

becomes that

$$R_k(e^{-\lambda\tau})\lambda^k + R_{k-1}(e^{-\lambda\tau})\lambda^{k-1} + \cdots + R_1(e^{-\lambda\tau})\lambda + R_0(e^{-\lambda\tau}) = 0. \tag{6}$$

Here $R_h(t)$, $h = 0, 1, \ldots, k$, are polynomials.

Since when $e^{-\lambda\tau} = 0$, (5) and (6) are the same equations, it is not difficult to know that if $h > l$ then $R_h(0) = 0$, and $R_l(0) = P_l \neq 0$.

We can write (6) as

$$\begin{aligned} &R_k(e^{-\lambda\tau})\lambda^k + R_{k-1}(e^{-\lambda\tau})\lambda^{k-1} + \cdots + R_{l+1}(e^{-\lambda\tau})\lambda^{l+1} \\ &\quad + R_l(e^{-\lambda\tau})\lambda^l + \cdots + R_1(e^{-\lambda\tau})\lambda + R_0(e^{-\lambda\tau}) = 0. \end{aligned} \tag{7}$$

Since $R_k(0) = R_{k-1}(0) = \cdots = R_{l+1}(0) = 0$, and $R_k(t), R_{k-1}(t), \ldots, R_{l+1}(t)$ are polynomials, there are polynomials $\bar{R}_k(t), \bar{R}_{k-1}(t), \ldots, \bar{R}_{l+1}(t)$ such that

$$R_k(t) = t\bar{R}_k(t), \quad R_{k-1}(t) = t\bar{R}_{k-1}(t), \quad \ldots, \quad R_{l+1}(t) = t\bar{R}_{l+1}(t).$$

Now (7) can be rewritten as

$$\begin{aligned}&\bar{R}_k(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^k + \bar{R}_{k-1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-1} + \cdots + \bar{R}_{l+1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{l+1}\\&\quad + R_l(e^{-\lambda\tau})\lambda^l + \cdots + R_1(e^{-\lambda\tau})\lambda + R_0(e^{-\lambda\tau}) = 0.\end{aligned}$$

We have that

$$\begin{aligned}&\bar{R}_k(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-l} + \bar{R}_{k-1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-l-1} + \cdots + \bar{R}_{l+1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda\\&\quad + R_l(e^{-\lambda\tau}) + R_{l-1}(e^{-\lambda\tau})\frac{1}{\lambda} + \cdots + R_1(e^{-\lambda\tau})\frac{1}{\lambda^{l-1}} + R_0(e^{-\lambda\tau})\frac{1}{\lambda^l} = 0.\end{aligned} \tag{8}$$

Theorem 1. *For system* (1), *there exists a large enough positive number* p, *such that for all the eigenvalues of system* (1), $\{\lambda_j\}$, *we have*

$$\mathrm{Re}(\lambda_j) \leqslant p. \tag{9}$$

Proof. Since $R_l(0) \neq 0$, and $R_h(t)$, $h = 0, 1, \ldots, k$, are polynomials, there exist $a > 0$ and $p > 0$, such that when $\mathrm{Re}(\lambda) > p$,

$$\left|R_l(e^{-\lambda\tau})\right| > a \tag{10}$$

and

$$\begin{aligned}&\Bigg|\bar{R}_k(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-l} + \bar{R}_{k-1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-l-1} + \cdots + \bar{R}_{l+1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda\\&\quad + R_{l-1}(e^{-\lambda\tau})\frac{1}{\lambda} + \cdots + R_1(e^{-\lambda\tau})\frac{1}{\lambda^{l-1}} + R_0(e^{-\lambda\tau})\frac{1}{\lambda^l}\Bigg| < a.\end{aligned} \tag{11}$$

From (10) and (11), we have

$$\begin{aligned}&\Bigg|\bar{R}_k(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-l} + \bar{R}_{k-1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-l-1} + \cdots + \bar{R}_{l+1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda\\&\qquad + R_l(e^{-\lambda\tau}) + R_{l-1}(e^{-\lambda\tau})\frac{1}{\lambda} + \cdots + R_1(e^{-\lambda\tau})\frac{1}{\lambda^{l-1}} + R_0(e^{-\lambda\tau})\frac{1}{\lambda^l}\Bigg|\\&\quad \geqslant \left|R_l(e^{-\lambda\tau})\right| - \Bigg|\bar{R}_k(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-l} + \bar{R}_{k-1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda^{k-l-1} + \cdots\\&\qquad + \bar{R}_{l+1}(e^{-\lambda\tau})e^{-\lambda\tau}\lambda + R_{l-1}(e^{-\lambda\tau})\frac{1}{\lambda} + \cdots + R_1(e^{-\lambda\tau})\frac{1}{\lambda^{l-1}} + R_0(e^{-\lambda\tau})\frac{1}{\lambda^l}\Bigg|\\&\quad > 0.\end{aligned}$$

The left of (8) is always not equal to zero, and so does Eq. (6). We have that Theorem 1 is true. □

From [1] and [2], we know that for common system

$$\begin{cases}\dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geqslant t_0,\\ x(t) = \varphi(t), & t_0 - \tau \leqslant t \leqslant t_0,\end{cases}$$

and for any real numbers α, β, there exist only finite eigenvalues in $\alpha \leqslant \mathrm{Re}(\lambda) \leqslant \beta$. But for singular system (1), this result is not true.

Example 1. Consider the singular system

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} x(t-\tau).$$

We have

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

then

$$|E\lambda - A - Be^{-\lambda\tau}| = \begin{vmatrix} \lambda + 1 & 0 \\ 1 - e^{-\lambda\tau} & 1 - e^{-\lambda\tau} \end{vmatrix} = (\lambda+1)(1-e^{-\lambda\tau}) = 0. \tag{12}$$

For $\alpha = -1$, $\beta = 1$, in $\alpha \leqslant \mathrm{Re}(\lambda) \leqslant \beta$, proper equation (12) there exists infinite eigenvalues: -1, $-\frac{2k\pi}{\tau} i$ $(k = 0, \pm 1, \pm 2, \ldots)$.

3. The exact exponential estimation for the fundamental solution

We consider the exact exponential estimation for the fundamental solution.

From [3], we have two lemmas.

Lemma 1. *For any square matrix E, the Drazin inverse matrix E^d exists and is unique, and if the Jordan normalized form is*

$$E = T \begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix} T^{-1},$$

where J_0 is a nilpotent matrix, J_1 and T are invertible matrices, then

$$E^d = T \begin{pmatrix} J_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}.$$

Lemma 2. *If (E, A) is regular, there are two invertible matrices P and Q, such that*

$$PEQ = \begin{pmatrix} I_1 & 0 \\ 0 & N \end{pmatrix}, \qquad PAQ = \begin{pmatrix} A_1 & 0 \\ 0 & I_2 \end{pmatrix}.$$

Here $I_1 \in R^{n_1 \times n_1}$ and $I_2 \in R^{n_2 \times n_2}$ are identity matrices, $N \in R^{n_2 \times n_2}$ is a nilpotent matrix, $A_1 \in R^{n_1 \times n_1}$ is a matrix.

For $\det(\lambda E - A) \not\equiv 0$, from Lemma 2, we can easily have that there exist nonsingular matrices $P, Q \in R^{n \times n}$, such that (1) is equivalent to canonical system

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_{11} x_1(t-\tau) + B_{12} x_2(t-\tau), & t \geqslant t_0, \\ N\dot{x}_2(t) = x_2(t) + B_{21} x_1(t-\tau) + B_{22} x_2(t-\tau), & t \geqslant t_0, \\ x_1(t) = \varphi_1(t), & t_0 - \tau \leqslant t \leqslant t_0, \\ x_2(t) = \varphi_2(t), & t_0 - \tau \leqslant t \leqslant t_0. \end{cases} \tag{13}$$

Here $x = Q \cdot \mathrm{col}(x_1', x_2') = x$, $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $n_1 + n_2 = n$, $PEQ = \mathrm{diag}(I_1, N)$, $PAQ = \mathrm{diag}(A_1, I_2)$,

$$PBQ = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

$\varphi = Q \cdot \mathrm{col}(\varphi_1', \varphi_2')$.

The stability of (1) and (13) is equivalent, so we might as well let

$$E = \begin{pmatrix} I_1 & 0 \\ 0 & N \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & I_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

in (1).

Then we have

$$H(\lambda) = (E\lambda - A - Be^{-\lambda\tau}) = \begin{pmatrix} \lambda I - A_1 - B_{11}e^{-\lambda\tau} & -B_{12}e^{-\lambda\tau} \\ -B_{21}e^{-\lambda\tau} & \lambda N - I - B_{22}e^{-\lambda\tau} \end{pmatrix}. \tag{14}$$

Theorem 2. *Let $X(t)$ be the fundamental solution of* (1), *the set of the eigenvalues to be* $\{\lambda_j\}$, *and let $\alpha_0 = \max\{\mathrm{Re}\,\lambda_j\}$, then for any $\alpha > \alpha_0$, there exists a constant $K(\alpha)$ such that*

$$\left|X(t)\right| \leqslant Ke^{\alpha t}, \quad t \geqslant t_0. \tag{15}$$

Proof. From (2) by taking Laplace transformation we have

$$(E\lambda - A - Be^{-\lambda\tau})L\bigl(X(t)\bigr) = EE^d,$$

that is

$$\begin{aligned} L\bigl(X(t)\bigr) &= (E\lambda - A - Be^{-\lambda\tau})^{-1}EE^d \\ &= \begin{pmatrix} \lambda I - A_1 - B_{11}e^{-\lambda\tau} & -B_{12}e^{-\lambda\tau} \\ -B_{21}e^{-\lambda\tau} & \lambda N - I - B_{22}e^{-\lambda\tau} \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda I - A_1 - B_{11}e^{-\lambda\tau} & -B_{12}e^{-\lambda\tau} \\ -B_{21}e^{-\lambda\tau} & \lambda N - I - B_{22}e^{-\lambda\tau} \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{16}$$

Let

$$\begin{pmatrix} X_{11}(\lambda) & X_{12}(\lambda) \\ X_{21}(\lambda) & X_{22}(\lambda) \end{pmatrix}$$

be the adjoint matrix of

$$\begin{pmatrix} \lambda I - A_1 - B_{11}e^{-\lambda\tau} & -B_{12}e^{-\lambda\tau} \\ -B_{21}e^{-\lambda\tau} & \lambda N - I - B_{22}e^{-\lambda\tau} \end{pmatrix}^{-1}.$$

Because $|\lambda N - I| \neq 0$, when $\mathrm{Re}(\lambda)$ is large enough,

$$d(\lambda) \triangleq \begin{vmatrix} \lambda I - A_1 - B_{11}e^{-\lambda\tau} & -B_{12}e^{-\lambda\tau} \\ -B_{21}e^{-\lambda\tau} & \lambda N - I - B_{22}e^{-\lambda\tau} \end{vmatrix} \neq 0.$$

From (16) we have

$$
\begin{aligned}
L\big(X(t)\big) &= (E\lambda - A - Be^{-\lambda\tau})^{-1}EE^d \\
&= \begin{pmatrix} \frac{1}{d(\lambda)}X_{11}(\lambda) & \frac{1}{d(\lambda)}X_{12}(\lambda) \\ \frac{1}{d(\lambda)}X_{21}(\lambda) & \frac{1}{d(\lambda)}X_{22}(\lambda) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{d(\lambda)}X_{11}(\lambda) & 0 \\ \frac{1}{d(\lambda)}X_{21}(\lambda) & 0 \end{pmatrix}.
\end{aligned}
$$

For

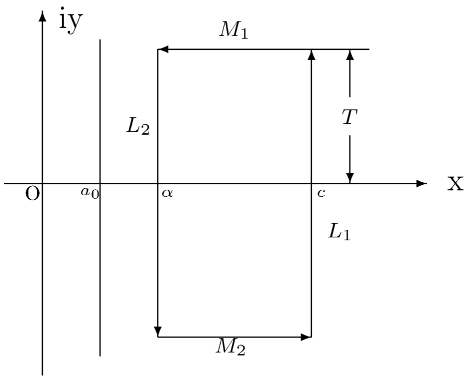
$$X(t) = \int\limits_{(c)} H(\lambda)^{-1}EE^d e^{\lambda t}\,d\lambda,$$

where c is a large enough real number. Now we prove

$$X(t) = \int\limits_{(a)} H(\lambda)^{-1}EE^d e^{\lambda t}\,d\lambda. \tag{17}$$

From following figure, we have that in the rectangle $L_1M_1L_2M_2$ and its boundary, all the elements of matrix $H(\lambda)$ there exist no zero point, that is all the elements of $H(\lambda)^{-1}EE^d e^{\lambda t}$ are analytic. Then we have

$$\int\limits_{L_1} + \int\limits_{M_1} + \int\limits_{L_2} + \int\limits_{M_2} = 0.$$



We know that all the elements of matrices X_{11}, X_{12} can be regarded as the polynomial of λ. On M_1 and M_2, $e^{\pm\lambda t}$ is bounded. It is obvious that the largest power about λ of the elements of matrices $X_{11}(\lambda)$, $X_{12}(\lambda)$ are smaller than that of $d(\lambda)$. Then when $T \to \infty$,

$$\int\limits_{M_1} H(\lambda)^{-1}EE^d e^{\lambda t}\,d\lambda \to 0, \qquad \int\limits_{M_2} H(\lambda)^{-1}EE^d e^{\lambda t}\,d\lambda \to 0,$$

(17) is true, and there exists a constant $K(\alpha)$ such that

$$\big|X(t)\big| \leqslant Ke^{\alpha t}, \quad t \geqslant t_0.$$

The proof of Theorem 2 is completed. □

4. The stability of singular differential systems with delay

Now we use the results in above sections to discuss the stability of singular differential systems with delay.

Firstly we give the concept of stability [2].

Definition 5. The singular differential system with delay (1) is called uniform asymptotic stable, simply called stable, if there exist scalars $\alpha < 0$, $\beta > 0$ such that for $t > t_0$ its state $x(t)$ satisfies

$$\left|x(t)\right| \leqslant \beta e^{\alpha t}\left|x(0)\right|, \quad t \geqslant t_0. \tag{18}$$

Lemma 3. *In singular differential system with delay* (1), *if* (A, E) *is regular and for all* $t \leqslant t_0 - \tau$, $\varphi(t) \equiv 0$, *we have that*

$$\varphi(t) = EE^d\varphi(t), \quad t_0 - \tau \leqslant t \leqslant t_0. \tag{19}$$

Proof. Let $y(t) = Px(t)$, and when $t_0 - \tau \leqslant t \leqslant t_0$ let $y(t) = \psi(t) = P\varphi(t)$. Left multiply the equation of (1) by the matrix Q^{-1}, from Lemma 2 we have that the singular differential system with delay (1) will become

$$\begin{cases} \dot y_1(t) = A_1y_1(t) + B_{11}y_1(t-\tau) + B_{12}y_2(t-\tau), & t \geqslant t_0, \\ N\dot y_2(t) = y_2(t) + B_{21}y_1(t-\tau) + B_{22}y_2(t-\tau), & t \geqslant t_0, \\ y(t) = \psi(t), & t_0 - \tau \leqslant t \leqslant t_0. \end{cases}$$

Then we have

$$N\dot\psi_2(t) = \psi_2(t) + B_{21}\psi_1(t-\tau) + B_{22}\psi_2(t-\tau), \quad t_0 - \tau \leqslant t \leqslant t_0.$$

That is when $t_0 - \tau \leqslant t \leqslant t_0$,

$$\begin{aligned} \psi_2(t) &= -\bigl(B_{21}\psi_1(t-\tau) + NB_{21}\dot\psi_1(t-\tau) + \cdots + N^{l-1}B_{21}\overset{(l-1)}{\psi_1}(t-\tau)\bigr) \\ &\quad - \bigl(B_{22}\psi_2(t-\tau) + NB_{22}\dot\psi_2(t-\tau) + \cdots + N^{l-1}B_{22}\overset{(l-1)}{\psi_2}(t-\tau)\bigr) \\ &\equiv 0, \\ \psi(t) &= \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} \psi_1(t) \\ 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\psi(t). \end{aligned}$$

That is

$$\begin{aligned} \varphi(t) &= P^{-1}\psi(t)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = P^{-1}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\psi(t) \\ &= P^{-1}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}P\varphi(t) = P^{-1}\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}P\varphi(t) \\ &= P^{-1}\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}Q^{-1}Q\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}P\varphi(t) = EE^d\varphi(t). \qquad \square \end{aligned}$$

Theorem 3. *In singular differential system with delay* (1)*, if* (A, E) *is regular,* $BEE^d = EE^dB$ *and for all* $t \leqslant t_0 - \tau$*,* $\varphi(t) \equiv 0$*, we get that the solution of* (1) *can be written as*

$$x(t) = X(t)\varphi(t_0) + \int_{t_0-\tau}^{t_0} X(t-\theta-\tau)B\varphi(\theta)\,d\theta,$$

where $X(t)$ *is the fundamental solution of* (1).

Proof. From (1) by taking Laplace transformation we have

$$(E\lambda - A - Be^{-\lambda\tau})L\big(x(t)\big) = \varphi(t_0) + \int_{t_0-\tau}^{t_0} B\varphi(t_1)e^{-\lambda(t_1+\tau)}\,dt_1,$$

that is

$$\begin{aligned} L\big(x(t)\big) &= (E\lambda - A - Be^{-\lambda\tau})^{-1}\varphi(t_0) \\ &\quad + (E\lambda - A - Be^{-\lambda\tau})^{-1}\int_{t_0-\tau}^{t_0} B\varphi(t_1)e^{-\lambda(t_1+\tau)}\,dt_1 \\ &= (E\lambda - A - Be^{-\lambda\tau})^{-1}EE^d\varphi(t_0) \\ &\quad + (E\lambda - A - Be^{-\lambda\tau})^{-1}\int_{t_0-\tau}^{t_0} BEE^d\varphi(t_1)e^{-\lambda(t_1+\tau)}\,dt_1 \\ &= (E\lambda - A - Be^{-\lambda\tau})^{-1}EE^d\varphi(t_0) \\ &\quad + (E\lambda - A - Be^{-\lambda\tau})^{-1}EE^d\int_{t_0-\tau}^{t_0} B\varphi(t_1)e^{-\lambda(t_1+\tau)}\,dt_1 \\ &= L\big(X(t)\big)\varphi(t_0) + L\big(X(t)\big)\int_{t_0-\tau}^{t_0} B\varphi(t_1)e^{-\lambda(t_1+\tau)}\,dt_1. \end{aligned} \tag{20}$$

Let

$$\omega(t_1) = \begin{cases} 0, & t_1 \geqslant t_0, \\ 1, & t_1 < t_0, \end{cases}$$

and

$$\hat{\varphi}(t_1) = \begin{cases} \varphi(t_1), & t_1 \geqslant t_0, \\ \varphi(t_0), & t_1 < t_0, \end{cases}$$

we get

$$L\big(X(t)\big)\int_{t_0-\tau}^{t_0} B\varphi(t_1)e^{-\lambda(t_1+\tau)}\,dt_1 = L\big(X(t)\big)\int_{t_0-\tau}^{+\imath} B\hat{\varphi}(t_1)\omega(t_1)e^{-\lambda(t_1+\tau)}\,dt_1$$

$$
\begin{aligned}
&= L\big(X(t)\big) \int_{t_0}^{+\infty} B\hat{\varphi}(s-\tau)\omega(s-\tau)e^{-\lambda s}\,ds \\
&= L\big(X(t)\big) \int_{t_0}^{+\infty} B\hat{\varphi}(s-\tau)\omega(s-\tau)e^{-\lambda s}\,ds \\
&= L\big(X(t)\big)(\lambda) L\big(B\hat{\varphi}(t-\tau)\omega(t-\tau)\big) \\
&= L\left(\int_{t_0}^{t} X(t-s)B\hat{\varphi}(s-\tau)\omega(s-\tau)\,ds\right) \\
&= L\left(\int_{t_0}^{t_0-\tau} X(t-s)B\varphi(s-\tau)\,ds\right) \\
&= L\left(\int_{t_0-\tau}^{t_0} X(t-\theta-\tau)B\varphi(\theta)\,d\theta\right).
\end{aligned}
\tag{21}
$$

From (20) and (21), we get

$$
x(t) = X(t)\varphi(t_0) + \int_{t_0-\tau}^{t_0} X(t-\theta-\tau)B\varphi(\theta)\,d\theta.
$$

The proof of Theorem 3 is completed. □

From Theorems 2 and 3 we can easily prove

Theorem 4. *For singular differential system with delay* (1), *if* (A, E) *is regular,* $BEE^d = EE^dB$ *and for all* $t \leqslant t_0 - \tau$, $\varphi(t) \equiv 0$, *the necessary and sufficient condition of* (1) *is uniformly asymptotic stable is that for all its eigenvalues* $\{\lambda_j\}$ *we have* $\alpha_0 = \max\{\operatorname{Re}\lambda_j\} < 0$.

5. About the singular neutral differential systems

For singular neutral differential systems, paper [22] had given some results, but by study we found that it can be changed to be the problems of singular differential systems with delay.

Usually, the singular neutral differential systems can be written as

$$
\begin{cases}
E\dot{x}(t) = C\dot{x}(t-\tau) + Ax(t) + Bx(t-\tau), & t \geqslant t_0, \\
x(t) = \varphi(t), & t_0 - \tau \leqslant t \leqslant t_0, \\
\dot{x}(t) = \theta(t), & t_0 - \tau \leqslant t \leqslant t_0,
\end{cases}
\tag{22}
$$

where $x(t) \in R^n$ is a state vector; $E \in R^{n\times n}$ is a singular matrix, $A, B, C \in R^{n\times n}$ are matrices; $\tau > 0$ is time delay; and $\varphi(t), \theta(t)$ the initial functions.

Let $y(t) = \dot{x}(t)$. System (22) will become

$$\begin{cases} \dot{x}(t) = y(t), & t \geqslant t_0, \\ 0 = Ax(t) - Ey(t) + Bx(t-\tau) + Cy(t-\tau), & t \geqslant t_0, \\ x(t) = \varphi(t), & t_0 - \tau \leqslant t \leqslant t_0, \\ y(t) = \theta(t), & t_0 - \tau \leqslant t \leqslant t_0. \end{cases} \tag{23}$$

Let

$$\bar{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \bar{\varphi}(t) = \begin{pmatrix} \varphi(t) \\ \theta(t) \end{pmatrix},$$

and

$$\bar{E} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 0 & I \\ A & -E \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 & 0 \\ B & C \end{pmatrix}.$$

Then (23) could be written as

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{x}(t-\tau), & t \geqslant t_0, \\ \bar{x}(t) = \bar{\varphi}(t), & t_0 - \tau \leqslant t \leqslant t_0. \end{cases} \tag{24}$$

Obviously, this is one form of singular differential systems with delay (1).

From that we can see that the results of this paper could be used to the neutral differential systems (22).

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