THH(R) \cong R \otimes S^1 \text{ for } E_\infty \text{ ring spectra}

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Abstract

We prove that the topological Hochschild homology spectrum \( THH(R) \) of an \( E_\infty \) spectrum \( R \) is the \( S^1 \)-indexed sum of copies \( R \) in the category of \( E_\infty \) ring spectra. As a consequence, we obtain a natural \( S^1 \)-action and compatible power operations on \( THH(R) \). In addition, \( THH(R) \) admits an \( A_\infty \)-comultiplication making it an \( A_\infty \) Hopf algebra spectrum over \( R \). © 1997 Elsevier Science B.V.

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1. Introduction and main results

Around 1985 Bökstedt introduced the notion of Topological Hochschild Homology \( THH \) of a functor with smash products [2] (for a published version see [4]). The category of such functors is topologically enriched. In a conversation Bökstedt pointed out that if this category were tensored \( THH(R) \) ought to be the tensor \( R \otimes S^1 \) in the commutative case.

Recall that a category is \textit{topologically enriched} if the morphism sets are topologized such that composition is continuous. In a topologically enriched category we have the notions of indexed limits and colimits.

1.1. Definition. Let \( \mathcal{K} \) and \( \mathcal{B} \) be topologically enriched categories and let \( F : \mathcal{K} \to \mathcal{T}_{op} \), \( G : \mathcal{K}^{op} \to \mathcal{T}_{op} \) and \( X : \mathcal{K} \to \mathcal{B} \) be continuous functors. The \textit{limit of } \( X \) \textit{indexed}...
by $F$ is an object $\text{lim}_FX$ in $\mathcal{B}$ together with a natural homeomorphism

$$\mathcal{B}(B, \text{lim}_FX) \cong \text{Funct}(\mathcal{X}, \mathcal{Top})(F, \mathcal{B}(B,X(-))),$$

where $\text{Funct}(\mathcal{X}, \mathcal{Top})$ denotes the category of continuous functors $\mathcal{X} \to \mathcal{Top}$. The colimit of $X$ indexed by $G$ is an object $\text{colim}_G X$ in $\mathcal{B}$ together with a natural homeomorphism

$$\mathcal{B}(\text{colim}_G X, B) \cong \text{Funct}(\mathcal{X}^{\text{op}}, \mathcal{Top})(G, \mathcal{B}(X(-), B)).$$

If $F$ and $G$ are the constant functors to a point we get the usual definitions of limits and colimits with the additional requirement that the natural bijections be homeomorphisms. If $\mathcal{X}$ consists of one object and its identity and $F$ and $G$ take the space $K$ as value while $X$ takes $B \in \text{ob}\mathcal{B}$ as value, we denote $\text{lim}_FX$ by $B^K$ and $\text{colim}_X X$ by $B \otimes K$.

1.2. Definition. If $B \otimes K$ exist for all $B \in \mathcal{B}$ and $K \in \mathcal{Top}$, $B$ is called tensored. If all $B^K$ exist it is called cotensored.

In view of Bökstedt’s remark and its implications tensored and cotensored structures are our central concern. They have the following universal property.

1.3. Let $\mathcal{B}$ be a topologically enriched tensored and cotensored category. Then for $B_1, B_2 \in \mathcal{B}$ and $K \in \mathcal{Top}$ we have natural homeomorphisms

$$\mathcal{B}(B_1 \otimes K, B_2) \cong \mathcal{Top}(K, \mathcal{B}(B_1, B_2)) \cong \mathcal{B}(B_1, B_2^K).$$

This shows that $B \otimes K$ is the $K$-indexed sum of copies $B$ if $K$ is discrete and $B^K$ the $K$-indexed product. Hence, $B \otimes K$ and $B^K$ are topologically parametrized versions of $K$-indexed sums and products.

From (1.3) we immediately deduce

1.4. For $B \in \mathcal{B}$ and $K, L \in \mathcal{Top}$ we have natural isomorphisms

$$(B \otimes K) \otimes L \cong B \otimes (K \times L),$$

$$(B^K)^I \cong B^{K \times I}.$$
Recently Elmendorf et al. [6,7] discovered a category of spectra which admits a strictly (up to natural isomorphisms) associative, commutative and unital smash product making it into a symmetric monoidal category. Moreover, it has the pleasant property that the $A_\infty$ ring spectra and $E_\infty$ ring spectra, which are structured by the linear isometry operad $\mathcal{L}$, are exactly the monoids, respectively, commutative monoids in this category. Hence, it is very simple to define $THH$ in terms of this smash product $\star$ (see Section 4), and therefore we will work in this setting. It has not been established yet that this definition of $THH$ agrees with the one of Bökstedt, but this is very likely to be true [15], at least for $CW$ ring spectra [7, Ch. I].

The restriction to $\mathcal{L}$-structured ring spectra is not substantial: recall from [13] that an $E_\infty$ operad is a $\Sigma$-free operad $\mathcal{C}$ such that each $\mathcal{C}(n)$ is contractible. If $\mathcal{C}$ is an operad without an action of $\Sigma_n$ on $\mathcal{C}(n)$ and each $\mathcal{C}(n)$ is contractible, we call $\mathcal{C}$ an $A_\infty$ operad. Our canonical example is the linear isometries operad $\mathcal{L}$: let $\mathcal{I}$ denote the category of real inner product spaces and linear isometries. Let $\mathcal{U} \cong \mathbb{R}^\infty$ be an object in $\mathcal{I}$, then $\mathcal{L}(n) = \mathcal{I}(\mathcal{U}^\otimes n, \mathcal{U})$ defines an $E_\infty$ operad whose structure maps are given by composition. If we forget the action of $\Sigma_n$ on $\mathcal{U}^\otimes n$, $\mathcal{L}$ reduces to an $A_\infty$ operad.

1.5. Definition. An $A_\infty$ ring spectrum consists of a spectrum $R$, an $A_\infty$ operad $\mathcal{C}$ augmented over $\mathcal{L}$ and structure maps

$$\zeta_n : \mathcal{C}(n) \times R^n \to R,$$

$n \geq 0$, defining an action of $\mathcal{C}$ on $R$. Here $R^n$ is the $n$-fold external smash product (see Section 2 for a recollection of the basic definitions). If $\mathcal{C}$ is an $E_\infty$ operad augmented over $\mathcal{L}$ (as $E_\infty$ operad) and the $\zeta_n$ are $\Sigma_n$-equivariant, $R$ is an $E_\infty$ ring spectrum.

Let $R$ be an $A_\infty$ or $E_\infty$ ring spectrum structured by an operad $\mathcal{C}$. Let $C$ denote its associated monad [10, VII.3]. Since $\mathcal{C}$ augments over $\mathcal{L}$ the monad $C$ acts on the monad $L$ associated with the operad $\mathcal{L}$. Applying the functorial two-sided barconstruction we obtain maps of ring spectra

$$B(L, C, R) \leftarrow B(C, C, R) \to R,$$

which are weak equivalences and homomorphisms with respect to the $C$-structure. Moreover, the left ring spectrum is structured by the linear isometry operad (for the two-sided barconstruction on space level and its properties see [13, Section 9]. The spectrum level construction is similar, details will appear in [7]).

Hence, there is a functorial way to replace each $A_\infty$ or $E_\infty$ ring spectrum by a weakly equivalent one structured by the operad $\mathcal{L}$. This allows the following:

1.6. Convention. $A_\infty$ or $E_\infty$ ring spectrum will always mean a ring spectrum structured by the linear isometry operad.

Let $E_\infty$ denote the category of $E_\infty$ ring spectra and homomorphisms. Our main results are
**Theorem A.** \( \mathcal{E}_\infty \) is topologically enriched in a canonical way and contains all indexed limits and colimits. In particular, \( \mathcal{E}_\infty \) is tensored and cotensored.

A different proof that \( \mathcal{E}_\infty \) is tensored and cotensored will appear in a joint paper by Mike Hopkins and the first author.

**Theorem B.** For \( E_\infty \) ring spectra \( R \) there is a natural isomorphism in \( \mathcal{E}_\infty \)

\[
\text{THH}(R) \cong R \otimes S^1.
\]

In particular, \( \text{THH}(R) \) is again an \( E_\infty \) ring spectrum and its multiplication \( \nu: \text{THH}(R) \star \text{THH}(R) \rightarrow \text{THH}(R) \) is induced by the folding map \( S^1 \sqcup S^1 \rightarrow S^1 \).

This result has a number of interesting consequences. We include just the most straightforward ones; others will be studied in a separate paper.

Since \( R \otimes S^1 \) is a continuous functor in both variables we have a homomorphism of topological monoids

\[
\lambda: \mathcal{F}op(S^1, S^1) \rightarrow \mathcal{E}_\infty(\text{THH}(R), \text{THH}(R))
\]

natural in the \( \mathcal{E}_\infty \) ring spectrum \( R \). The adjoint of the multiplication \( S^1 \times S^1 \rightarrow S^1 \) also defines a homomorphism of topological monoids

\[
\rho: S^1 \rightarrow \mathcal{F}op(S^1, S^1).
\]

The composite \( \lambda \circ \rho = \hat{\alpha} \) defines an \( S^1 \)-action on \( \text{THH}(R) \) and we obtain

**Theorem C.** For an \( E_\infty \) ring spectrum \( R \) there is an \( S^1 \)-action on \( \text{THH}(R) \) through homomorphisms, i.e. a homomorphism of topological monoids

\[
\hat{\alpha}: S^1 \rightarrow \mathcal{E}_\infty(\text{THH}(R), \text{THH}(R)).
\]

There are also obvious power operations

\[
\Phi^r: \text{THH}(R) \rightarrow \text{THH}(R)
\]

of the types considered by Loday [11] and McCarthy [14] defined on \( S^1 \) by

\[
op(r e^{2\pi i r}) = e^{2\pi i r}, \quad r \in \mathbb{Z}.
\]

Since the product of \( \text{THH}(R) \) is given by the folding map \( S^1 \sqcup S^1 \rightarrow S^1 \) the following result can easily be checked by considering the \( S^1 \) factor.

**Theorem D.** For each \( E_\infty \) ring spectrum \( R \) there exist natural power operations

\[
\Phi^r: \text{THH}(R) \rightarrow \text{THH}(R)
\]
one for each \( r \in \mathbb{Z} \) satisfying

(i) \( \Phi^0 \) factors through the natural monomorphism

\[
i_R : R = R \otimes \{1\} \to \text{THH}(R)
\]

induced by the inclusion \( \{1\} \subset S^1 \). It defines a retraction of \( i_R \).

(ii) \( \Phi^1 = \text{id} \).

(iii) \( \Phi^r \circ \Phi^s = \Phi^{r+s} \).

(iv) Each \( \Phi^r \) is multiplicative, i.e. a homomorphism of \( E_\infty \) ring spectra.

(v) Compatibility with the \( S^1 \)-action. The following diagram commutes:

\[
\begin{array}{ccc}
\text{THH}(R) \otimes S^1 & \xrightarrow{\alpha} & \text{THH}(R) \\
\downarrow \Phi^r \otimes \Phi^r & & \downarrow \Phi^r \\
\text{THH}(R) \otimes S^1 & \xrightarrow{\alpha} & \text{THH}(R)
\end{array}
\]

Here \( \alpha \) denotes the adjoint of \( \alpha \).

\( \text{THH}(R) \) with its \( S^1 \)-action has an obvious universal property:

**Theorem E.** The natural monomorphism \( i_R : R = R \otimes \{1\} \to \text{THH}(R) \) has the following universal property: given an \( E_\infty \) ring spectrum \( R' \) with an \( S^1 \)-action through homomorphisms and a homomorphism \( f : R \to R' \) then there exists a unique \( S^1 \)-equivariant homomorphism \( \tilde{f} : \text{THH}(R) \to R' \) such that \( \tilde{f} \circ i_R = f \).

By (1.3) \( - \otimes S^1 : \mathcal{E}_\infty \to \mathcal{E}_\infty \) is left adjoint to \( (-)^{S^1} : \mathcal{E}_\infty \to \mathcal{E}_\infty \). We deduce

**Theorem F.** \( \text{THH}(-) : \mathcal{E}_\infty \to \mathcal{E}_\infty \) preserves colimits. In particular, since \( * \) is the coproduct in \( \mathcal{E}_\infty \),

\[
\text{THH}(R_1 \ast R_2) \cong \text{THH}(R_1) \ast \text{THH}(R_2).
\]

In [2] Bökstedt defined a map

\[
\lambda : R \wedge S^l_+ \to \text{THH}(R),
\]

which plays an important role in the calculations of [3]. The existence of \( \lambda \) is obvious in our set-up. Let \( x : \{\ast\} \to S^1 \) denote the map sending \( \ast \) to \( x \in S^1 \).

**Theorem G.** For any \( E_\infty \) ring spectrum \( R \) there is a natural spectrum level Bökstedt map

\[
\lambda : R \wedge S^l_+ \to \text{THH}(R)
\]
with the following properties:

(i) $\lambda$ is $S^1$-equivariant,

(ii) if $T$ is an $E_\infty$ ring spectrum and $f : R \wedge S^1_+ \to T$ a map of spectra such that

$$f \circ (R \wedge x^+_+) : R = R \wedge \{\ast\}_+ \to R \wedge S^1_+ \to T$$

is an $E_\infty$ homomorphism for each $x \in S^1$, there is a unique $E_\infty$ homomorphism $\tilde{f} : \text{THH}(R) \to T$ such that $f = \tilde{f} \circ \lambda$.

**Theorem H.** For each $x \in S^1$ the $E_\infty$ homomorphism

$$i_x = \text{id} \otimes x : R = R \otimes \{\ast\} \to \text{THH}(R)$$

defines an $E_\infty$ $R$-algebra structure on $\text{THH}(R)$. Hence, if $R$ is an Eilenberg–MacLane spectrum $\text{THH}(R)$ is a product of Eilenberg–MacLane spectra.

Let $*_R$ be the smash product over $R$ as defined in Definition 6.1 below. If $A$ and $B$ are $E_\infty$ $R$-algebras $A *_R B$ is the pushout in $\mathcal{S}_\infty$ of

$$\begin{array}{ccc}
R & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & A *_R B
\end{array}$$

(the proof is the same as the usual one). Give $\text{THH}(R)$ the $R$-algebra structure from $i_R$. Then the projection

$$\text{THH}(R) * \text{THH}(R) \to \text{THH}(R) *_R \text{THH}(R)$$

is induced by the map $S^1 \sqcup S^1 \to S^1 \vee S^1$ with $1 \in S^1$ as base point. The algebra multiplication is given by the based folding map $S^1 \vee S^1 \to S^1$. The pinch map $S^1 \to S^1 \vee S^1$ now defines an $A_\infty$ comultiplication

$$\text{THH}(R) \to \text{THH}(R) *_R \text{THH}(R)$$

with (homotopy) counit induced by $S^1 \to \ast$, and we obtain

**Theorem I.** If $R$ is an $E_\infty$ ring spectrum $\text{THH}(R)$ has an $A_\infty$ Hopf algebra structure over $R$.

1.7. Remark. It might be worth noticing that there is a simplicial $E_\infty$ ring spectrum $E_\ast$ with $E_0 = R$ and $E_n$ the $n$-fold application of $\text{THH}$ to $R$. Its realization is $R \otimes \mathbb{C}P^\infty$.

The paper is organized as follows. In Section 2 we recall the basic definitions and results from [6, 7, 10] which we will use in our constructions and add a few facts left out
in these papers. Section 3 provides the proof of Theorem A. Topological Hochschild homology will be defined and studied in Section 4 which includes the proof of Theorem B. Section 5 deals with the Bökstedt map in a more general context. The paper ends with a section on the extension of the definition of topological Hochschild homology to $E_{\infty}$ algebras over $E_{\infty}$ ring spectra and some remarks on possible variations due to changes of the frame work provided by the work of Elmendorf et al. [7]

2. Spectra and (unital) $S$-modules

Throughout this paper $\mathcal{Top}$ denotes the category of compactly generated weak Hausdorff spaces and $\mathcal{Top}_*$ its based version. Limits, colimits and function spaces are taken in these categories.

We will work with coordinate-free spectra in the sense of [10]. Given a universe, i.e. real inner product space $\mathcal{U} \cong \mathbb{R}^\infty$, a $\mathcal{U}$-indexed prespectrum $D$ assigns to each finite-dimensional subspace $V$ of $\mathcal{U}$ a based space $DV$ and to each orthogonal pair $V, W$ a structure map $\sigma_{V,W}:DV \to WD(V \oplus W)$ satisfying the obvious associativity condition. $D$ is a spectrum if the $\sigma_{V,W}$ are homeomorphisms. A map of $\mathcal{U}$-indexed (pre)-spectra $f:D \to D'$ is a family $fV:DV \to D'V$ of based maps preserving the structure. We denote the resulting categories of prespectra and spectra by $\mathcal{P}_\mathcal{U}$ and $\mathcal{S}_\mathcal{U}$, respectively. Both categories are topologically enriched by giving their morphism sets from $D$ to $D'$ the subspace topology of $\mathcal{V}(DV, D'V)$.

For a prespectrum $D$ and a based space $K$ the assignments $V \mapsto DV \wedge K$ and $V \mapsto \mathcal{S}_\mathcal{U}(K, DV)$ define the small smash product and the small function spectrum functors

$$\mathcal{P}_\mathcal{U} \times \mathcal{Top}_* \to \mathcal{P}_\mathcal{U}, \quad (D,K) \mapsto D \wedge K,$$
$$\mathcal{P}_\mathcal{U} \times \mathcal{Top}_* \to \mathcal{P}_\mathcal{U}, \quad (D,K) \mapsto F(K,D).$$

The small function spectrum functor restricts to a functor

$$\mathcal{S}_\mathcal{U} \times \mathcal{Top}_* \to \mathcal{S}_\mathcal{U}.$$ 

For the small smash product we use the composite

$$\mathcal{S}_\mathcal{U} \times \mathcal{Top}_* \to \mathcal{P}_\mathcal{U} \to \mathcal{P}_\mathcal{U}, \quad (E,K) \mapsto L(E \wedge K),$$

where $L$ is the spectrification [10, p. 13]. In abuse of notation we again write $E \wedge K$ for $L(E \wedge K)$.

2.1. Proposition (Lewis et al. [10, I.3.3; I.3.4]). Let $E, E' \in \mathcal{S}_\mathcal{U}$, $K, L \in \mathcal{Top}_*$.

(1) There are natural homeomorphisms

$$\mathcal{S}_\mathcal{U}(E \wedge K, E') \cong \mathcal{Top}_*(K, \mathcal{S}_\mathcal{U}(E, E')) \cong \mathcal{S}_\mathcal{U}(E, F(K, E')).$$
There are natural isomorphisms
\[ E \wedge S^0 \cong E, \quad (E \wedge K) \wedge L \cong E \wedge (K \wedge L), \]
\[ F(S^0, F) \cong F, \quad F(K \wedge L, F) \cong F(K, F(I, F)). \]

To obtain smash products between spectra we note the existence of an associative, commutative and unital external smash product functor
\[ \mathcal{F}(\mathcal{U} \times \mathcal{U}', \mathcal{U}), \quad (E, E') \mapsto E \wedge E' \]
induced by the spectrification of the functor defined by the formula \((E \wedge E')(V \oplus V') = EV \wedge E'V'). To obtain an internal smash product in \(\mathcal{U}\) we apply the twisted half smash product of [10]: let \(\mathcal{I}\) be the topologically enriched category of universes and \(\mathcal{Top}(\mathcal{U}, \mathcal{U}')\) the category of spaces over \(\mathcal{I}(\mathcal{U}, \mathcal{U}')\), then

2.2. Proposition (Lewis et al. [10, VI.1.1; VI.1.5; VI.3.1]). There are functors
\[ \mathcal{Top}(\mathcal{I}(\mathcal{U}, \mathcal{U}'), \mathcal{U}) \times \mathcal{U} \to \mathcal{U}', \quad (A, E) \mapsto A \wedge E, \]
\[ \mathcal{Top}(\mathcal{I}(\mathcal{U}, \mathcal{U}'), \mathcal{U})^p \times \mathcal{U}' \to \mathcal{U}, \quad (A, E) \mapsto F[A, E], \]
such that, for \(A \in \mathcal{Top}(\mathcal{I}(\mathcal{U}, \mathcal{U}'), \mathcal{U})\), \(B \in \mathcal{Top}(\mathcal{I}(\mathcal{U}', \mathcal{U}''), \mathcal{U})\), \(E \in \mathcal{U}\), \(E' \in \mathcal{U}'\) and \(K \in \mathcal{Top}_K\),

1. there is a natural homeomorphism
\[ \mathcal{U}'(A \wedge E, E') \cong \mathcal{U}(E, F[A, E']), \]
2. \(A \wedge E\) preserves colimits in both variables, \(F[A, E']\) preserves limits in \(E'\) and converts colimits in \(A\) to limits,
3. there are isomorphisms
\[ A \wedge (E \wedge K) \cong (A \wedge E) \wedge K \quad \text{and} \quad F(K, F[A, E']) \cong F[A, F(K, E')], \]
4. for \(B \times A \to \mathcal{I}(\mathcal{U}', \mathcal{U}'') \times \mathcal{I}(\mathcal{U}, \mathcal{U}') \twoheadrightarrow \mathcal{I}(\mathcal{U}, \mathcal{U}'')\) we have a natural isomorphism
\[ B \wedge (A \wedge E) \cong (B \times A) \wedge E. \]

We could now define an internal smash product in \(\mathcal{U}\) by the correspondence
\[ (E, E') \mapsto \mathcal{I}(\mathcal{U} \oplus \mathcal{U}, \mathcal{U}) \wedge (E \wedge E'). \]

By the geometry of the spaces \(\mathcal{I}(\mathcal{U}, \mathcal{U}')\) this internal smash product is coherently homotopy associative, homotopy commutative and homotopy unital. The coherence theory makes any construction involving this smash product cumbersome.
Recently, Elmendorf et al. [6, 7] managed to incorporate the coherence directly into the definition to come up with a smash product on a sufficiently large category of spectra with much better properties:

2.3. Let $\mathcal{L}(n) = \mathcal{I}(\mathbb{N}, \mathbb{N})$ denote the linear isometry operad on $\mathbb{N}$. An $S$-module is a spectrum $M$ with an $\mathcal{L}(1)$-action, i.e. $M$ comes equipped with a map $\xi : \mathcal{L}(1) \times M \to M$ such that

\[
\begin{array}{ccc}
\{\text{id}\} \times M & \xrightarrow{=} & \mathcal{L}(1) \times M \\
\downarrow & & \downarrow \xi \\
M & & M
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{L}(1) \times (\mathcal{L}(1) \times M) & \xrightarrow{=} & (\mathcal{L}(1) \times \mathcal{L}(1)) \times M \\
\downarrow \mathcal{L}(1) \times \xi & & \downarrow \xi \\
\mathcal{L}(1) \times M & & M
\end{array}
\]

commute ($\gamma$ is composition in the operad $\mathcal{L}$). A map of $S$-modules is a map $f : M \to N$ of spectra respecting the $\mathcal{L}(1)$-action.

Recall that the stable categories are obtained by formally inverting weak equivalences, i.e. maps of spectra $f : E \to E'$ for which each $f(V) : E(V) \to E'(V)$ is a weak equivalence.

2.4. Proposition (Elmendorf et al. [6, Theorem 1; 7, Ch. 1]). The category $S\text{-Mod}$ of $S$-modules is complete and cocomplete, with both limits and colimits created in $\mathcal{I}\mathbb{N}$. The forgetful functor $S\text{-Mod} \to \mathcal{I}\mathbb{N}$ induces an equivalence of the associated stable categories.

We need a slightly stronger version of the first part of (2.4) which is implicit in [10]. Let $F : \mathcal{C} \to \mathcal{I}\mathbb{N}$ be a diagram of spectra. The spectrum $\lim F$ is defined by $(\lim F)(V) = \lim F(-)(V)$. The same procedure for colimits produces a prespectrum which we have to spectrify. By [10] spectrification is a continuous functor. Since both functors $\text{Top}_*(-, Y)$ and $\text{Top}_*(X, -)$ are continuous, we have

2.5. Proposition. Let $F : \mathcal{C} \to \mathcal{I}\mathbb{N}$ be a diagram of spectra. Then there are natural homeomorphisms

\[
\begin{align*}
\mathcal{I}\mathbb{N}(\text{colim} F, E) & \cong \lim \mathcal{I}\mathbb{N}(F, E), \\
\mathcal{I}\mathbb{N}(E, \text{lim} F) & \cong \lim \mathcal{I}\mathbb{N}(E, F).
\end{align*}
\]

And, by restricting the morphism spaces to their subspaces of maps of $S$-modules,
2.6. Addendum to (2.4). Let $F : \mathcal{C} \to S\text{-}\mathcal{Mod}$ be a diagram of $S$-modules. Then there are natural homeomorphisms

$$S\text{-}\mathcal{Mod}(\text{colim} F, M) \cong \lim S\text{-}\mathcal{Mod}(F, M),$$

$$S\text{-}\mathcal{Mod}(M, \text{lim} F) \cong \lim S\text{-}\mathcal{Mod}(M, F).$$

As an immediate consequence we have

2.7. Proposition. As a topologically enriched category $\mathcal{P}\mathcal{U}$ contains all indexed limits and colimits.

Proof. By (2.1) $\mathcal{P}\mathcal{U}$ is tensored by $(E, K) \mapsto E \wedge (K_+)$ and cotensored by $(E, K) \mapsto F(K_+, E)$. Together with (2.5) this proves the claim [9, (3.70)]. □

To obtain the same result for $S\text{-}\mathcal{Mod}$ we need to know that it is tensored and cotensored. This is a consequence of the following variant of a result of Linton.

2.8. Lemma. Let $T : \mathcal{C} \to \mathcal{C}$ be a continuous monad on a topologically enriched category $\mathcal{C}$ and let $\mathcal{C}^T$ be its category of algebras. Then

1. the forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$ creates all indexed limits,

2. if $\mathcal{C}^T$ has continuous coequalizers (in the sense of (2.5), (2.6)) and $\mathcal{C}$ is tensored, then $\mathcal{C}^T$ contains all indexed colimits.

Proof. (1) is the topologically enriched version of [12, VI.2, Excercise 2]. If we denote the free functor $\mathcal{C} \to \mathcal{C}^T$ also by $T$, then $T$ is left adjoint to $U$, hence preserves indexed colimits. In particular, $T(C \otimes K) = (TC) \otimes K$ for $K \in \mathcal{Top}$. Let $\mu : T \circ T \to T$ denote the multiplication of $T$. Then for $X \in \mathcal{C}^T$ with structure map $\xi : TX \to X$

$$\begin{array}{ccc}
TTX & \xymatrix{ \mu_X \ar[r] & TX \ar[l]^-{T\xi} } & X \\
\end{array}$$

is a coequalizer in $\mathcal{C}^T$ by Beck's tripleability theorem [12, VI.7]. Hence the coequalizer of

$$\begin{array}{ccc}
TTX \otimes K & \xymatrix{ \mu_X \otimes K \ar[r] & TX \otimes K \ar[l]^-{T\xi \otimes K} } & \\
\end{array}$$

satisfies the universal property of $X \otimes K$. The result (2) now follows [9, (3.70)]. □

By (2.8) we can apply the proof of the corresponding result from [7, Ch. I] to obtain

2.9. Proposition. As topologically enriched category $S\text{-}\mathcal{Mod}$ contains all indexed limits and colimits, and both are created in $\mathcal{P}\mathcal{U}$. 
For $S$-modules $M$ and $N$ we define the smash product over $S$ to be the coequalizer in $\mathcal{F}_U$

\[
\frac{(L(2) \times L(1) \times L(1)) \times (M \wedge N)}{(L(2) \times (M \wedge N)) \rightarrow M \wedge N}.
\]

2.10. Proposition (Elmendorf et al. [6, Theorem 1; 7, Ch. I]). The smash product over $S$ is an associative and commutative bifunctor, and there is a natural map $S \wedge_S M \rightarrow M$ which is a weak equivalence if $M$ is a CW spectrum and an isomorphism if $M = S$. There is also a functorial function spectrum $F_S(M, P)$ of $S$-modules and a natural homeomorphism

\[S \text{-Mod}(M \wedge_S N, P) \cong S \text{-Mod}(N, F_S(M, P)).\]

To obtain a smash product which is also unital Elmendorf et al. consider the category $S \text{-Mod}_*$ of $S$-modules under $S$. Its objects are $S$-modules $M$ with a map of $S$-modules $\eta : S \rightarrow M$. Such an object is called unital $S$-module.

2.11. Proposition. As a topologically enriched category $S \text{-Mod}_*$ contains all indexed limits and colimits. Ordinary colimits of connected diagrams and all indexed limits are created in $\mathcal{F}_U$.

Proof. $M \mapsto M \vee S$ is a monad on $S \text{-Mod}$ whose algebras are precisely the unital $S$-modules: if $M$ is an algebra with structure map $\xi : M \vee S \rightarrow M$ then $M$ is unital via $S \rightarrow M \vee S \rightarrow M$; conversely, if $M$ is unital, $\eta : S \rightarrow M$, an algebra structure map is defined by

\[M \vee S \xrightarrow{id \vee \eta} M \vee M \xrightarrow{fold} M.\]

Hence, the forgetful $S \text{-Mod}_* \rightarrow S \text{-Mod}$ creates all indexed limits. Now let $F : \mathcal{C} \rightarrow S \text{-Mod}_*$ be a diagram in the usual sense, $\text{col} F$ its colimit in $S \text{-Mod}$ and $\text{col} S$ the colimit in $S \text{-Mod}$ of the constant $\mathcal{C}$-diagram on $S$. Then $\text{col} F$ in $S \text{-Mod}_*$ is the pushout in $S \text{-Mod}$ of

\[
\text{col} S \rightarrow S \xrightarrow{\text{col} F}
\]

Hence, $S \text{-Mod}$ has arbitrary indexed colimits by (2.8). If $\mathcal{C}$ is connected $S = \text{col} S$ and hence $\text{col} F = \text{colim} F$. □

2.12. Definition (Elmendorf et al. [6, Section 2; 7, Chap. 1]). Let $M$ and $N$ be unital $S$-modules. The reduced smash product $M \star N$ is the pushout in $S \text{-Mod}$
with the structure map $S \cong S \wedge_S S \rightarrow M \wedge_S N \rightarrow M \star N$.

2.13. Proposition (Elmendorf et al. [6, Section 2]). $S$-$\text{Mod}_*$ is a symmetric monoidal category with tensor product $\star$ and unit $S$. Its monoids and commutative monoids are precisely the $A_\infty$ and $E_\infty$ ring spectra, respectively.

Although $M \star -$ does not preserve sums in the category $S$-$\text{Mod}_*$ we note for later use.


Proof. By (2.9) and (2.11) colimits of connected diagrams are created in $S$-$\text{Mod}$. For an $S$-module $K$ the functor $K \vee -$ : $S$-$\text{Mod} \rightarrow S$-$\text{Mod}$ preserves colimits of connected diagrams while $K \wedge_S -$ : $S$-$\text{Mod} \rightarrow S$-$\text{Mod}$ preserves all colimits because it is a left adjoint (2.10). The result follows since colimits commute with pushouts. \qed

3. Ring spectra and $R$-modules

Before we start with the proof of Theorem A we need to consider $R$-modules.

3.1. Definition. Let $R$ be an $A_\infty$ or $E_\infty$ ring spectrum. A unital (left) $R$-module is a unital $S$-module $M$ with a structure map $\xi : R \star M \rightarrow M$ in $S$-$\text{Mod}_*$ such that

\begin{align*}
(M \wedge_S S) \vee (S \wedge_S N) & \rightarrow M \wedge_S N \\
M \vee N & \rightarrow M \star N
\end{align*}

commute, where $\mu : R \star R \rightarrow R$ is the multiplication and $\eta : S \rightarrow R$ the unit. A map of unital $R$-modules is a map of unital $S$-modules respecting the $R$-structure.

3.2. Proposition. The category $R$-$\text{Mod}_*$ of unital $R$-modules is topologically enriched and contains all indexed limits and colimits. Colimits of connected diagrams and arbitrary indexed limits are created in $S$-$\text{Mod}_*$.

Proof. $M \rightarrow R \star M$ is a continuous monad on $S$-$\text{Mod}_*$ with unit $M \cong S \star M \rightarrow R \star M$ whose algebras are by definition precisely the unital $R$-modules (the module structure
maps are the algebra structure maps of the monad and vice versa). Hence, the forgetful $R-\text{Mod}_* \to S-\text{Mod}_*$ creates all indexed limits (Lemma 2.8(1)). Now let $F: \mathcal{C} \to R-\text{Mod}_*$ be a connected diagram and $Q \in S-\text{Mod}_*$ its colimit in $S-\text{Mod}_*$. Since $R^* \dashv$ preserves colimits of connected diagrams in $S-\text{Mod}_*$ (Lemma 2.14) $Q$ is a unital $R$-module with structure map

$$\xi_Q: R \star Q = \text{colim}(R \star F) \to \text{colim} F = Q.$$  

It is easy to check that $Q$ has the topologized universal property of a colimit in $R-\text{Mod}_*$. Since coequalizers are colimits of connected diagrams the result follows from Propositions 2.8 and 2.11. $\square$

Let $\mathcal{E}_\infty$ be the category of $E_\infty$ ring spectra and their homomorphisms. As always before we topologically enrich $\mathcal{E}_\infty$ by giving $\mathcal{E}_\infty(E, E')$ the subspace topology from $\mathcal{S}\mathcal{W}(E, E')$.

3.3. Proposition (Elmendorf et al. [6, Proposition 2]). $E \star E'$ is the sum of $E$ and $E'$ in $\mathcal{E}_\infty$, and the folding map $E \star E \to E$ is the multiplication of $E$.

3.4. Proof of Theorem A. Let $L: \mathcal{S}\mathcal{W} \to \mathcal{S}\mathcal{W}$ be the monad associated with the linear isometry operad $\mathcal{L}$. Then $L$ is continuous and $\mathcal{E}_\infty$ its category of algebras [10, VIII. 3]. Hence, the forgetful $\mathcal{E}_\infty \to \mathcal{S}\mathcal{W}$ creates all indexed limits (Lemma 2.8(1)). To show the existence of indexed colimits we have to establish the existence of continuous coequalizers (Lemma 2.8(2)). So given

$$Q \xrightarrow{f} R \xrightarrow{g} R$$

in $\mathcal{E}_\infty$. Let $f_1$ denote the composite

$$f_1: R \star Q \xrightarrow{R \star f} R \star R \xrightarrow{\mu_R} R,$$

where $\mu_R$ is the multiplication of $R$. Then $f_1$ is a homomorphism of $R$-modules. Take the coequalizer $T$ in $R-\text{Mod}_*$

$$R \star Q \xrightarrow{f_1} R \xrightarrow{g_1} T.$$  

We show that $T$ is the coequalizer of $f$ and $g$ in $\mathcal{E}_\infty$. Since $T^* \dashv$ preserves coequalizers in $R-\text{Mod}_*$ the spectrum $T \star T$ is the colimit in $R-\text{Mod}_*$ of

$$R \star Q \star R \star Q \xrightarrow{f_1 \star R \star Q} R \star R \star Q \xrightarrow{g_1 \star R \star Q} R \star R \star Q.$$  

$$R \star Q \star R \xrightarrow{f_1 \star R} R \star R \xrightarrow{g_1 \star R} R \star R$$
and the colimits structure maps are induced from

\[ t \ast s : R \ast R \to T \ast T. \]

Since \( R \) is a commutative monoid in \( \mathcal{E}_\infty \) the following diagram commutes (\( \tau \) is the interchange):

\[
\begin{array}{cccccc}
R \ast Q & \to & R \ast R \ast R & \to & R \ast R & \to & R \ast R \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R \ast R \ast Q & \to & R \ast R \ast R & \to & R \ast R & \to & R \\
\mu_R \ast \mu_R & & \mu_R & & \mu_R & & \\
& & & & & & \\
R \ast Q & \to & R \ast R & \to & R & & \\
\mu_R & & \mu_R & & & & \\
\end{array}
\]

Hence, \( f_1 \) and correspondingly \( g_1 \) are \( E_\infty \) homomorphisms, and (3.6) is a diagram in \( \mathcal{E}_\infty \). The folding maps, which define the various multiplications, induce a map of diagram (3.6) to \( T \), giving rise to an \( R \)-module homomorphism \( \mu_T : T \ast T \to T \) satisfying

\[ \mu_T \circ (t \ast s) = t \circ \mu_R. \]

In particular, \( T \) is an \( E_\infty \) ring spectrum and \( t \) an \( E_\infty \) homomorphism. Now given an \( E_\infty \) homomorphism \( h : R \to U \) such that \( h \circ f = h \circ g \) then

\[ h \circ \mu_R \circ (R \ast f) = \mu_U \circ (h \ast h) \circ (R \ast f) = \mu_U \circ (h \ast (h \circ f)) = h \circ \mu_R \circ (R \ast g). \]

Hence, there is a unique \( R \)-module homomorphism

\[ q : T \to U \]

such that \( q \circ t = h \). Since \( h \ast h \) induces a map of (3.6) to \( U \ast U \), it follows that \( q \) is an \( E_\infty \) homomorphism. Conversely, given an \( E_\infty \) homomorphism \( q : T \to U \) such that \( q \circ t = f_1 = q \circ t \circ f_2 \) as \( R \)-module homomorphisms then they agree as \( E_\infty \) homomorphisms, and hence \( q \circ t \circ f = q \circ t \circ g \) because \( R \ast Q \) is the sum in \( \mathcal{E}_\infty \) and \( \mu_R \) is the folding map. Hence, the natural homeomorphism

\[
\text{Equalizer} \left( R\text{-Mod}_*(R, U) \right) \to R\text{-Mod}_*(R \ast Q, U) \cong R\text{-Mod}_*(T, U)
\]

restricts to the subspaces of \( E_\infty \) homomorphisms

\[
\text{Equalizer} \left( \mathcal{E}_\infty(R, U) \right) \to \mathcal{E}_\infty(Q, U) \cong \mathcal{E}_\infty(T, U).
\]

\[ \square \]
4. Topological Hochschild homology

4.1. Definition. Let $R$ be an $A_\infty$ ring spectrum and $M$ be an $R$-bimodule (defined analogously to (3.1)). The topological Hochschild homology $THH(R; M)$ of $R$ with coefficients in $M$ is the topological realization in $\mathcal{SH}$ of the simplicial spectrum $THH(R; M)_*$

$$[n] \mapsto R \star \cdots \star R \star M = R^n \star M$$

($n$ copies of $R$) with the usual Hochschild structure maps

\[
\begin{align*}
  d^0 &: R^n \star M \xrightarrow{\mu} R^{n-1} \star M, \\
  d^i &: R^n \star M \xrightarrow{R^{n-1} \star \mu R^{n-1}} R^{n-1} \star M, \quad 0 < i < n, \\
  d^n &: R^n \star M \xrightarrow{(R^{n-1} \star \eta) \circ \tau} R^{n-1} \star M, \\
  s^i &: R^n \star M \xrightarrow{R^{n-1} \star \eta \star R \star M} R^{n+1} \star M, \quad 0 \leq i \leq n,
\end{align*}
\]

where $\mu : R \star R \to R$ is the multiplication, $\eta : S \to R$ the unit, $\xi_l$ and $\xi_r$ the left and right actions of $R$ on $M$ and $\tau$ the cyclic permutation

$$\tau : R \star \cdots \star R \star M \to R \star \cdots \star R \star M \star R.$$ 

If $M = R$ we write $THH(R)$ for $THH(R; R)$.

The topological realization of a simplicial spectrum $E_*$ in $\mathcal{SH}$ is the coend in $\mathcal{SH}$ of the functor

$$\Delta^p \times \Delta \to \mathcal{SH}, \quad ([m], [n]) \mapsto E_m \wedge \Delta(n)_+,$$

where $\Delta(n)$ is the standard $n$-simplex [5]. Since geometric realization commutes with any monad in $\mathcal{SH}$ (cf. [13, Theorem 12.2] for the space-level version; the spectrum-level argument is similar, cf. [7, Ch. VI]), the geometric realization of a simplicial $E_\infty$ ring is again an $E_\infty$ ring. In particular, if $R$ is an $E_\infty$ ring and hence a commutative monoid in $S \cdot \text{Mod}_*$ the simplicial spectrum $THH(R)_*$ lives in $\mathcal{SH}$ so that $THH(R) \in \mathcal{SH}$. Hence, we have an endofunctor

$$THH : \mathcal{SH} \to \mathcal{SH}.$$ 

Let $S^1_\bullet = \Delta(1)_*/\partial \Delta(1)_*$ denote the simplicial 1-sphere and $f^n_i : [n] \to [1]$ the map determined by $(f^n_i)^{-1}(1) = \{i, \ldots, n\}$. Then

$$\Delta([n], [1]) = \{f^n_0, f^n_1, \ldots, f^n_{n+1}\}.$$
The boundary and degeneracies are

\[
d^j f^n_i = \begin{cases} f_{i-1}^{n+1} & \text{for } j < i, \\ f_i^{n+1} & \text{for } j \geq i, \end{cases}
\]

\[
s^j f^n_i = \begin{cases} f_{i-1}^{n+1} & \text{for } j < i, \\ f_i^{n+1} & \text{for } j \geq i. \end{cases}
\]

Since \( S^1_n = \Delta([n],[1])/(f_0^n \sim f_{n+1}^n) \), and \( \ast \) is the coproduct in \( \mathcal{E}_\infty \), we have the ring spectrum version of a well-known fact of classical Hochschild homology (e.g. see [11, Section 3]).

4.2. If \( R \otimes S^1 \ast \) denotes the simplicial object \([n] \rightarrow R \otimes S^1_n \) in \( \mathcal{E}_\infty \) then \( THH(R) \ast = R \otimes S^1 \ast \).

Theorem B is a consequence of

4.3. Proposition. For any \( \mathcal{E}_\infty \) ring \( R \) the following diagram commutes up to natural isomorphism

\[
\begin{array}{ccc}
\mathcal{Top}^{\Delta^{op}} & \xrightarrow{(R \otimes -)^{\Delta^{op}}} & \mathcal{E}_\infty^{\Delta^{op}} \\
\downarrow T_1 & & \downarrow T_2 \\
\mathcal{Top} & \xrightarrow{(R \otimes -)} & \mathcal{E}_\infty
\end{array}
\]

where we let \( T_i \) denote the respective topological realization functors.

Proof. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{E}_\infty & \xrightarrow{\mathcal{E}_\infty(R,-)} & \mathcal{Top} \\
\downarrow S_2 & & \downarrow S_1 \\
\mathcal{E}_\infty^{\Delta^{op}} & \xrightarrow{\mathcal{E}_\infty(R,-)^{\Delta^{op}}} & \mathcal{Top}^{\Delta^{op}}
\end{array}
\]

where \( S_1 \) is the topologized singular functor right adjoint to \( T_1 \) and

\[
S_2 : \mathcal{E}_\infty \rightarrow \mathcal{E}_\infty^{\Delta^{op}}, \quad Q \mapsto ([n] \mapsto F(\Delta(n)_+, Q))
\]

is the spectrum singular functor right adjoint to \( T_2 \). Since \( \mathcal{E}_\infty(R,-) \) is right adjoint to \( R \otimes - \), the result now follows from
Claim. There is a natural isomorphism

\[ \mathcal{E}_\infty(R, -)^{\Delta^op} \circ S_2 \cong S_1 \circ \mathcal{E}_\infty(R, -). \]

The left side sends an \( E_\infty \) ring \( Q \) to

\[ [n] \mapsto \mathcal{E}_\infty(R, F(\triangle(n)_+, Q)). \]

The right side sends \( Q \) to

\[ [n] \mapsto \mathcal{F}op(\triangle(n), \mathcal{E}_\infty(R, Q)). \]

The required natural isomorphism is the adjointness homeomorphism

\[ \mathcal{E}_\infty(R, F(\triangle(n)_+, Q)) \cong \mathcal{F}op(\triangle(n), \mathcal{E}_\infty(R, Q)). \]

It remains to prove that the multiplication of \( THH(R) \) is induced by the folding map \( S^1 \sqcup S^1 \to S^1 \). Since

\[ R \otimes -: \mathcal{F}op \to \mathcal{E}_\infty \quad \text{and} \quad - \otimes S^1: \mathcal{E}_\infty \to \mathcal{E}_\infty \]

are left adjoints they preserve sums. Hence there are canonical \( E_\infty \) isomorphisms

\[ (R \ast R) \otimes S^1 \cong THH(R) \ast THH(R) \cong R \ast (S^1 \sqcup S^1) \]

and the multiplication on \( THH(R) \) is defined by either folding map

\[ R \ast R \to R \text{ or } S^1 \sqcup S^1 \to S^1 \]

\[ \square \]

4.4. There is also an internal realization

\[ T_3: \mathcal{E}_\infty^{\Delta^op} \to \mathcal{E}_\infty \]

sending a simplicial \( E_\infty \) ring spectrum \( R_* \) to the coend of the functor

\[ \triangle^{op} \times \triangle \to \mathcal{E}_\infty, \quad ([m], [n]) \mapsto R_m \otimes \triangle(n). \]

in \( \mathcal{E}_\infty \). Applying the universal properties of the coend construction and of the tensor in \( \mathcal{E}_\infty \) it is easy to see that the singular functor \( S_2 \) is right adjoint to \( T_2 \). Hence \( T_2 \) and \( T_3 \) are naturally isomorphic.

4.5. Proposition. The topological realization in the sense of [5] of a simplicial \( E_\infty \) ring spectrum \( R_* \) is naturally isomorphic in \( \mathcal{E}_\infty \) to the internal realization in the sense of (4.4). In particular, \( THH(R) \) is the coend in \( \mathcal{E}_\infty \) of

\[ \triangle^{op} \times \triangle \to \mathcal{E}_\infty, \quad ([m], [n]) \mapsto THH(R)_m \otimes \triangle(n). \]
5. Maps of Bökstedt type

Let $X$ be a topological space and $R$ and $T$ be $E_\infty$ ring spectra. Since $R \otimes X$ and $R \wedge X_+$ are the $X$-parametrized coproducts of copies of $R$ in $E_\infty$ and $\mathcal{E}$, respectively, there are $X$-parametrized families of natural $E_\infty$ inclusions $\{i_x : R \to R \otimes X\}$, respectively, spectrum level maps $\{j_x : R \to R \wedge X_+\}$ having the following universal property: given an $X$-parametrized family of $E_\infty$ homomorphisms $f_x : R \to T$, respectively, an $X$-parametrized family of maps of spectra $g_x : R \wedge X_+ \to E$ such that $f \circ i_x = f_x$ for each $x \in X$, respectively, a unique map of spectra $g : R \wedge X_+ \to E$ such that $g \circ j_x = g_x$ for each $x \in X$.

If we take $f_x = \text{id}_R$ for all $x \in X$ we obtain an $E_\infty$ homomorphisms $\rho_x : R \otimes X \to R$ such that $\rho_x \circ i_x = \text{id}_R$.

Considering the $i_x : R \to R \otimes X$ as maps of spectra we obtain an induced map $\lambda(R,X) : R \wedge X_+ \to R \otimes X$ satisfying $\lambda(R,X) \circ j_x = i_x$, $x \in X$. Since $i_x$ admits a retraction, we have

5.1. If $R$ is not contractible and $X$ is not empty $\lambda(R,X)$ is essential.

The universal property also implies

5.2. The following diagram commutes:

\[
\begin{array}{ccc}
(R \wedge X_+) \land Y_+ & \rightarrow & (R \otimes X) \wedge Y_+ \\
\downarrow \cong & & \downarrow \cong \\
R \wedge (X \times Y)_+ & \rightarrow & R \otimes (X \times Y)
\end{array}
\]

Specializing to $X = Y = S^1$ we obtain a commutative diagram

\[
\begin{array}{ccc}
(R \wedge S^1_+) \land S^1_+ & \rightarrow & R \wedge (S^1 \times S^1)_+ \\
\downarrow \cong & & \downarrow \cong \\
(R \otimes S^1) \wedge S^1_+ & \rightarrow & R \otimes (S^1 \times S^1)
\end{array}
\]

where $\gamma$ is the multiplication of $S^1$. Hence,

5.3. $\lambda(R,S^1) : R \wedge S^1_+ \to R \otimes S^1$ is $S^1$-equivariant.
Given a map \( f : R \wedge X_+ \to T \) of spectra such that each \( f_x = f \circ j_x : R \to T \) is an \( E_\infty \) homomorphism, the \( f_x \) induce a unique \( E_\infty \) homomorphism \( \hat{f} : R \otimes X \to T \) such that \( \hat{f} \circ i_x = f_x \) for all \( x \in X \). Since \( f_x = \hat{f} \circ i_x = \hat{f} \circ \lambda(R,X) \circ j_x \) and \( f_x = f \circ j_x \) we conclude

5.4. Let \( X \) be a topological space and \( R \) and \( TE_\infty \) ring spectra. Given a map of spectra \( f : R \wedge X_+ \to T \) such that each \( f \circ j_x \) is an \( E_\infty \) homomorphism there exists a unique \( E_\infty \) homomorphism \( \hat{f} : R \otimes X \to R \) such that \( \hat{f} \circ \lambda(R,X) = f \).

5.5. For each \( x \in X \) we have an \( R \)-module structure on \( R \otimes X \) defined by

\[
R \ast (R \otimes X) \xrightarrow{i_* \ast \text{id}} (R \otimes X) \ast (R \otimes X) \xrightarrow{\text{mult}} R \otimes X.
\]

Clearly, this defines an \( R \)-algebra structure on \( R \otimes X \) turning \( i_x \) into an \( R \)-algebra homomorphism. This proves the first part of Theorem H. The second part is a well-known consequence of the first part [1, II.6.1]. It is straightforward to show that this structure is equivalent to an \( E_\infty \) algebra structure as defined in Definition 6.6 below.

6. Extensions of the result and final remarks

6.1. Definition. Let \( R \) be an \( A_\infty \) or \( E_\infty \) ring spectrum, \( K \) a unital right \( R \)-module and \( L \) a unital left \( R \)-module with structure maps \( \xi_K \) and \( \xi_L \). The reduced smash product over \( R \) of \( K \) and \( L \) is the coequalizer in \( S\text{-Mod}_* \)

\[
K \wedge R \wedge L \xrightarrow{\xi_K} K \wedge L \xrightarrow{\eta} K \wedge_R L.
\]

6.2. If \( K \) is a unital \( Q \)-\( R \)-bimodule, \( K \wedge_R L \) is a unital \( Q \)-module since \( Q \ast \)-preserves coequalizers by Lemma 2.14.

6.3. If \( R \) is an \( E_\infty \) ring spectrum any unital \( R \)-module is an \( R \)-bimodule in the obvious way and \( \ast_R \) defines a bifunctor

\[
R\text{-Mod}_* \times R\text{-Mod}_* \to R\text{-Mod}_*.
\]

6.4. Proposition. Let \( Q \) and \( R \) be \( A_\infty \) or \( E_\infty \) ring spectra, \( K \) a unital right \( Q \)-module, \( L \) a unital \( Q \)-\( R \)-bimodule and \( M \) a unital left \( R \)-module. Then there are natural isomorphisms

1. \( (K \wedge Q L) \ast_R M \cong K \ast_Q (L \ast_R M) \),
2. \( R \ast_R M \cong M \),
3. if \( R \) is an \( E_\infty \) ring spectrum, then \( L \ast_R M \cong M \ast_R L \).
Proof. (1) follows from the fact that $K \star_Q -$ preserves coequalizers and that two coequalizers commute. (3) is trivial, and (2) holds because
\[
R \star R \star M \xrightarrow{\mu \star id} R \star M \xrightarrow{\xi} M
\]
is a coequalizer by [12, VI. 7].

6.5. Corollary. If $R$ is an $E_\infty$ ring spectrum $R-\mathcal{M}od_*$ is a symmetric monoidal category with $\star_R$ as tensor product and $R$ as unit.

6.6. Definition. Let $R$ be an $E_\infty$ ring spectrum. An $(E_\infty,$ respectively) $A_\infty$ $R$-algebra is a (commutative) monoid in the symmetric monoidal category $R-\mathcal{M}od_*$.

We are interested in the commutative case. Let $F_{R-\mathcal{A}lg}$ denote the category of $E_\infty$ $R$-algebras and $R$-algebra homomorphisms. We topologically enrich $E_{R-\mathcal{A}lg}$ in the obvious way so that the forgetful
\[
U : E_{R-\mathcal{A}lg} \to R-\mathcal{M}od_ *
\]
is continuous. $U$ has a continuous left adjoint
\[
F : R-\mathcal{M}od_* \to E_{R-\mathcal{A}lg}
\]
defined as follows: let $\mathcal{S}$ denote the category of finite sets $n = \{1, \ldots , n\} \ n \geq 0$ with $Q = \emptyset$ and injections. For each $M \in R-\mathcal{M}od_*$ define a functor
\[
F_M : \mathcal{S} \to R-\mathcal{M}od_*
\]
by $F_M(n) = M^{\star_R n}$ with $M^{\star_R 0} = R$ and for each ordered injection $f : m \to n$ by
\[
F_M(f) : M^{\star_R m} \cong N_1 \star_R N_2 \star_R \cdots \star_R N_n \xrightarrow{\alpha} M^{\star_R n},
\]
where $N_k = M$ if $k \in \text{im}(f)$ and $N_k = R$ otherwise. $\alpha$ is defined by the identity on $M$ and by the unital structure on $M$
\[
R \cong R \star S \xrightarrow{\xi_M} R \star M \xrightarrow{\xi_M} M
\]
$F_M$ applied to a permutation is the obvious morphism. We define
\[
F(M) = \text{colim} \ F_M
\]
in the category $R-\mathcal{M}od_*$. Since $F_M$ is a connected diagram this colimit is created in $S-\mathcal{M}od$, and since $M \star_R -$ preserves colimits of connected diagrams the concatenation of finite sets induces a multiplication on $F(M)$ making it a commutative monoid in $R-\mathcal{A}lg$. Extending the proofs of Section 3 to $R$-algebras and modules over $R$-algebras we obtain
6.7. Theorem. \( E_{R-\mathcal{A}lg} \) is topologically enriched and contains all indexed limits and colimits. All indexed limits are created in \( \mathcal{F} \).

Next let \( R \) be an \( E_{\infty} \) ring spectrum, \( A \) a commutative \( R \)-algebra and \( M \) an \( A \)-module.

6.8. Definition. The topological Hochschild homology \( \text{THH}_R(A; M) \) of \( A \) with coefficients in \( M \) is the topological realization in \( \mathcal{F} \) of the simplicial \( R \)-module

\[
[n] \mapsto A^{*n} \ast_R M
\]

with the Hochschild structure maps.

As in Section 4 we obtain

6.9. Theorem. Let \( R \) be an \( E_{\infty} \) ring spectrum and \( A \) an \( E_{\infty} \) \( R \)-algebra. Then there is a natural isomorphism in \( E_{R-\mathcal{A}lg} \)

\[
\text{THH}_R(A) \cong A \otimes S^1
\]

and the multiplication of \( \text{THH}_R(A) \) is induced by the folding map \( S^1 \sqcup S^1 \to S^1 \).

Elmendorf et al. [7] are contemplating another variant of a category of \( S \)-modules [8]. To distinguish it from Definition 2.3 we call its objects strong \( S \)-modules.

6.10. Definition (Elmendorf and May [8]). The category \( sS-\mathcal{Mod} \) of strong \( S \)-modules is the subcategory of \( S-\mathcal{Mod} \) of all objects \( M \) for which the map \( S \wedge_S M \to M \) is an isomorphism.

\( sS-\mathcal{Mod} \) is a coreflective subcategory of \( S-\mathcal{Mod} \) in the sense of [12, IV. 3] with the continuous coreflector

\[
S-\mathcal{Mod} \to sS-\mathcal{Mod}, \ R \mapsto S \wedge_S R.
\]

Hence, [12, IV. Exercise 3.7].

6.11. Proposition. As a topologically enriched category \( sS-\mathcal{Mod} \) contains all indexed colimits and they are created in \( S-\mathcal{Mod} \). It also contains all indexed limits which are obtained from the indexed limits in \( S-\mathcal{Mod} \) by applying the coreflector.

Note that for strong unital \( S \)-modules \( \wedge_S \) coincides with \( \ast \) so that the category \( S-\mathcal{Mod}_\ast \) is redundant: we can work in \( sS-\mathcal{Mod} \). \( \mathcal{E}_\infty \) has to be replaced by \( s\mathcal{E}_\infty \), the subcategory of \( E_{\infty} \) ring spectra which are strong \( S \)-modules. If \( R \) is an \( E_{\infty} \) ring spectrum then \( S \wedge_S R \) is an \( E_{\infty} \) ring spectrum in \( s\mathcal{E}_\infty \) which is weakly equivalent to \( R \). Hence, we may restrict our attention to \( s\mathcal{E}_\infty \). Also observe that \( s\mathcal{E}_\infty \) has the same indexed colimits as \( \mathcal{E}_\infty \). If we now replace \( \ast \) by \( \wedge_S \) throughout Sections 3 and 4 and
the singular functor $S_2 : \mathcal{E}_\infty \to \mathcal{E}_\infty^{\Delta op}$ by the functor

$$s\mathcal{E}_\infty \to s\mathcal{E}_\infty^{\Delta op}, \quad Q \mapsto ([n] \mapsto S \wedge S F(\Delta(n)_+, Q))$$

all proofs go through in this setting.

A final word to the notion of unital $R$-module. In [7, Ch. II] Elmendorf et al. showed that $A_\infty$ and $E_\infty$ ring spectra $R$ are $S$-modules with structure maps $\eta : S \to R$ and $\mu : R \wedge S R \to R$ such that $\mu$ is associative (and commutative in the $E_\infty$ case) and that

$$
\begin{array}{c}
S \wedge_s R \\
\downarrow \eta \wedge id \\
R \\
\mu \\
\downarrow id \wedge \eta \\
R \wedge_s S
\end{array}
$$

commutes. A module over an $A_\infty$ or $E_\infty$ ring spectrum $R$ is an $S$-module $M$ with a structure map $\xi : R \wedge S M \to M$ satisfying the obvious associativity and unit conditions. Let $R-\text{Mod}$ denote the category of such $R$-modules and $R$-module homomorphisms. It is not difficult to show that our category $R-\text{Mod}_*$ of unital $R$-modules is isomorphic to the category $R-\text{Mod}$ under $R$.

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References


