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On polynomial mixing bounds for stochastic differential equations

A.Yu. Veretennikov*

Institute of Information Transmission Problems, 19 Bolshoy Karetnii, 101447, Moscow, Russia

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Abstract

Polynomial bounds for the coefficient of β-mixing are established for diffusion processes under weak recurrency assumptions. The method is based on direct evaluations of the moments and certain functionals of hitting-times of the process and on the change of time. © 1997 Elsevier Science B.V.

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1. Introduction

The importance of mixing coefficient bounds for certain classes of stochastic processes is well-known. Such bounds allow to get various limit theorems, there are also applications to parameter estimation, etc. While exponential mixing bounds were obtained by many authors for various classes of processes (see Meyn and Tweedie (1993), Veretennikov (1987), etc.), the polynomial bounds were studied less. It is known, however, that polynomial bounds may be obtained under assumptions like

$$E_x\tau^m \leqslant h(x) \tag{1}$$

and some additional hypotheses, where $\tau = \inf(t \geqslant 0\colon X_t \in D)$ for some "petite" set D, X_t being the process under consideration and h certain function (cf. Gulinsky and Veretennikov (1993), etc.). Tuominen and Tweedie (1994) obtained a criterion for polynomial convergence rate to the invariant measure which is very close to the polynomial mixing rate. Indeed, Ango Nze applied this criterion to get corresponding mixing coefficient bounds (see Ango Nze (1994)). This criterion could provide some good explicit examples for the processes of the type

$$X_{n+1} = f(X_n) + \xi_{n+1} \quad (\xi_n - \text{i.i.d.})$$

* E-mail: ayu@sci.lpi.ac.ru.

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under assumptions like

$$|f(x)| \leqslant |x|(1-|x|^{-\alpha}), \quad 0<\alpha<1, \ |x| \geqslant M_0, \tag{2}$$

and

$$E|\xi_n|^s<\infty, \quad s>0. \tag{3}$$

We consider the solution of the d-dimensional stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}w_t, \quad X_0 = x \in \mathbb{R}^d, \tag{4}$$

either with non-random initial data $X_0 = x \in \mathbb{R}^d$, either with stationary distributed X_0 (however, throughout the paper X_t means a solution with a fixed initial data x, if the other meaning is not noted specially). Here w_t is a d_1-dimensional Wiener process, $d_1 \geqslant d$, $b - d$-dimensional locally bounded Borel function, σ-bounded continuous non-degenerate matrix $d \times d_1$ function. In Section 2 the case $d_1 = d$ and $\sigma \equiv I$ is considered, Section 5 is devoted to a general case.

It is likely that the analogue of condition (2) for the process (4) would be

$$(b(x), x/|x|) \leqslant -|x|^{-\alpha}, \quad |x| \geqslant M_0, \ 0<\alpha<1. \tag{5}$$

On the other hand, there is no analogue for assumption (3) because w_t has all polynomial moments so to say automatically.

We will establish polynomial bounds for "β-mixing" (see below) as well as for the convergence rate to the invariant measure under even less restrictive assumption: there exist constants $M_0 \geqslant 0$ and $r>0$ such that

$$(b(x), x/|x|) \leqslant -r/|x|, \quad |x| \geqslant M_0. \tag{6}$$

The rate of β-mixing and the convergence rate to the invariant measure depend on the value r which plays an important role in the theorems below. The method is based on direct estimation of the left-hand side in (1). Similar bounds may be obtained also for the equation

$$\mathrm{d}x_t = b(t, x_t)\,\mathrm{d}t + \sigma(t, x_t)\,\mathrm{d}w_t.$$

We use the weak existence result and the strong markovian property of solutions of Eq. (4) due to Krylov (1969) and Krylov (1973).

Sections 2 and 5 contain the main results, in Sections 3, 4 and 6 one will find preliminary results and proofs.

Note. After this paper was submitted the article Menshikov and Williams (1996) appeared with estimates for SDEs and martingales very close to the hitting-time estimate of Theorem 3 (below). Moreover, Prof. Menshikov and the referee draw the author's attention to the paper Lamperti (1963), two papers by Aspandiiarov and Iasnogorodski (1994a, b) and the paper Aspandiiarov et al. (1994) with a similar approach and close results for discrete-time case. In three latter papers applications to a random walk in the quadrant on the plane are studied. The approach of this article is different which may also be of interest.

2. Main results for the unit diffusion

Throughout this section $d_1 = d$, $\sigma \equiv I$ (the identity matrix $d \times d$). Let $F^X_{\leqslant s} = \sigma(X_u,\ u \leqslant s)$, $F^X_{\geqslant s} = \sigma(X_u,\ u \geqslant s)$. We recollect the definitions of two mixing coefficients, $\alpha(t)$ and $\beta(t)$:

Strong mixing coefficient or Rosenblatt's coefficient

$$\alpha(t) = \sup_{s \geqslant 0} \sup_{A \in F^X_{\leqslant s}, B \in F^X_{\geqslant t+s}} |P(AB) - P(A)P(B)|;$$

Complete regularity coefficient or Kolmogorov's coefficient

$$\beta(t) = \sup_{s \geqslant 0} E\, \mathrm{var}_{B \in F^X_{\geqslant t+s}} (P(B|F_s) - P(B)).$$

The inequality $\alpha(t) \leqslant \beta(t)$ is well-known.

Denote by $\alpha^x(t)$ and respectively $\beta^x(t)$ these coefficients for fixed non-random initial data $X_0 = x$ and by $\alpha^{inv}(t)$ and respectively $\beta^{inv}(t)$ those for stationary distributed initial data X_0.

Theorem 1. *Under assumption* (6) *with* $r > (d/2) + 1$, *for any* k, $0 < k < r - (d/2) - 1$ *with* $m \in (2k + 2, 2r - d)$,

$$\beta^x(t) \leqslant C(1 + |x|^m)(1 + t)^{-(k+1)}, \tag{7}$$

$$\beta^{inv}(t) \leqslant C(1 + t)^{-(k+1)} \tag{8}$$

(k, m *are not necessarily integers*).

Theorem 2. *Under assumptions of Theorem* 1,

$$\mathrm{var}(\mu^x(t) - \mu^{inv}) \leqslant C(x)(1 + t)^{-(k+1)}, \quad C(x) = C(1 + |x|^m), \tag{9}$$

where $\mu^x(t)$ *is the distribution of* X_t, x *being the initial data, and* μ^{inv} *is the invariant measure for* X_t; *in particular,* μ^{inv} *does exist.*

3. Preliminary results

Theorem 3. *Under assumption* (6) *with* $r > (d/2) + 1$, *for any* $0 < k < r - d/2 - 1$, $m \in (2k + 2, 2r - d)$

$$E_x \tau^{k+1} \leqslant C(1 + |x|^m) \tag{10}$$

(*here constant* C *depends on* m). $\tau = \inf(t \geqslant 0 : |x_t| \leqslant M)$, $\quad M \geqslant M_0$

Lemma 1. *Under assumption* (6) *with* $r > d/2$, *for any* $m < 2r - d$ *there exists such a constant* C *that for any* t *and any* x

$$E_x |X_t|^m \leqslant C(1 + |x|^m).$$

Proof. Follows from Lemmas 2–5. □

Lemma 2. *Let $r' < r$,*

$$\hat{b}(v) = -r'v^{-1} + (d-1)(2v)^{-1}, \quad v \geqslant 0,$$

v_t be a solution of the stochastic differential equation with non-sticky reflecting boundary condition

$$\begin{aligned} & \mathrm{d}v_t = \hat{b}(v_t)\,\mathrm{d}t + \mathrm{d}\hat{w}_t + \mathrm{d}\varphi_t, \quad v_t \geqslant M_0, \quad v_0 = |x|, \\ & \varphi_t = \int_0^t I(v_s = M_0)\,\mathrm{d}\varphi_s, \quad \varphi_0 = 0, \quad E\int_0^\infty I(v_s = M_0)\,\mathrm{d}s = 0, \end{aligned} \tag{11}$$

where φ increases. Here

$$\mathrm{d}\hat{w}_t = \sum_{i=1}^d (X_t^i/|X_t|)\,\mathrm{d}w_t^i, \quad \hat{w}_0 = 0. \tag{12}$$

(Note that $\hat{w}_t$ is a Wiener process). Then

$$P(v_t \geqslant |X_t|,\ t \geqslant 0) = 1. \tag{13}$$

Proof of Lemma 2. We have, for $|X_t| > M_0$ (in fact, for $|X_t| > 0$),

$$\begin{aligned} \mathrm{d}|X_t| &= \mathrm{d}\left(\sum_{i=1}^d (X_t^i)^2\right)^{1/2} \\ &= \left\{\frac{X_t^i b^i(X_t) + d/2}{(\sum_{j=1}^d (X_t^j)^2)^{1/2}} - \frac{X_t^i X_t^i}{2(\sum_{j=1}^d (X_t^j)^2)^{3/2}}\right\}\mathrm{d}t + \frac{X_t^i}{(\sum_{j=1}^d (X_t^j)^2)^{1/2}}\mathrm{d}w_t^i \\ &= [(X_t/|X_t|, b(X_t)) + (d-1)(2|X_t|)^{-1}]\mathrm{d}t + \mathrm{d}\hat{w}_t \end{aligned}$$

($X^i b^i$ means $\sum_i X^i b^i$). Hence, the Itô formula for the process X_t and the function $h(z) = \max(|z|, M_0)$, gives one

$$\begin{aligned} \mathrm{d}h(X_t) &= [(X_t/|X_t|, b(X_t)) + (d-1)(2|X_t|)^{-1}]I(|X_t| > M_0)\,\mathrm{d}t \\ &\quad + I(|X_t| > M_0)\,\mathrm{d}\hat{w}_t + \mathrm{d}\psi_t, \end{aligned}$$

$$\psi \text{ increases}, \quad \psi_t = \int_0^t I(|X_s| = M_0)\,\mathrm{d}\psi_s, \quad \psi_0 = 0.$$

Since $[(X_t/|X_t|, b(X_t)) + (d-1)(2|X_t|)^{-1}] \leqslant -(r-(d-1)/2)/|X_t|$ for $|X_t| > M_0$ and $-(r-(d-1)/2)/|x| < -(r'-(d-1)/2)/|x|$ $\forall |x| > 0$ and, at last, both functions $b_1(v) = -(r-(d-1)/2)/v$ and $b_2(v) = -(r'-(d-1)/2)/v$ are continuous in v ($v > 0$), then the comparison theorem gives one the result.

Indeed, let $b_0(x) = [(x/|x|, b(x) + (d-1)(2|x|)^{-1}]I(|x| > M_0)$. Let us consider the function $(v)_+^2 = v^2 I\ (v > 0)$. Due to the Itô formula, one obtains

$$\begin{aligned} \mathrm{d}(h(X_t) - v_t)_+^2 &= 2(h(X_t) - v_t) + \mathrm{d}(h(X_t) - v_t) \\ &= 2(h(X_t) - v_t)_+(b_0(X_t) - b_2(v_t)) + 2(h(X_t) - v_t)_+(\mathrm{d}\psi_t - \mathrm{d}\psi_t) \\ &\leqslant 2(h(X_t) - v_t)_+(b_1(|X_t|) - b_2(v_t)) + 2(h(X_t) - v_t)_+(\mathrm{d}\psi_t - \mathrm{d}\varphi_t). \end{aligned}$$

We recollect that $h(X_t) - v_t > 0$ implies $I(|X_t| \leqslant M_0) = 0$, so that one has the identity

$$2(h(X_t) - v_t)_+(\mathrm{d}\hat{w}_t - I(|X_t| > M_0)\,\mathrm{d}\hat{w}_t)$$
$$= 2(h(X_t) - v_t)_+ I(|X_t| \leqslant M_0)\,\mathrm{d}\hat{w}_t \equiv 0.$$

Suppose $|X_t| = v_t$ for some *stopping time* t. The equality $|X_t| = v_t$ implies the strict inequality $b_1(|X_s|) < b_2(v_s) - v$ for some right neighbourhood $t < s < s_0$, s_0 being again a stopping time and $v > 0$. Thus, the expression $(h(X_s) - v_s)_+(b_1(|X_s|) - b_2(v_s))$ is strictly negative for $t < s < s_0$ if $|X_t| = v_t$. Further, the second expression $2(h(X_t) - v_t)_+(\mathrm{d}\psi_t - \mathrm{d}\varphi_t)$ may be nonzero only if $h(X_t) > v_t$. But this implies $\mathrm{d}\psi_t = 0$, so $2(h(X_t) - v_t)_+(\mathrm{d}\psi_t - \mathrm{d}\varphi_t) \leqslant 0$. Thus, $P(|X_t| \leqslant v_t,\ t \geqslant 0) = 1$. Lemma 2 is proved.

Lemma 3. *Let $\tilde{v}_t$ be a solution of the stochastic differential equation with nonsticky reflecting boundary conditions*

$$\mathrm{d}\tilde{v}_t = \hat{b}(\tilde{v}_t)\,\mathrm{d}t + \mathrm{d}\hat{w}_t + \mathrm{d}\tilde{\varphi}_t, \quad \tilde{v}_t \geqslant |x|, \ \tilde{v}_0 = |x|,$$
$$\tilde{\varphi}_t = \int_0^t I(\tilde{v}_s = |x|)\,\mathrm{d}\tilde{\varphi}_s, \qquad \tilde{\varphi}_0 = 0, \qquad E\int_0^\infty I(\tilde{v}_s = |x|)\,\mathrm{d}s = 0, \tag{14}$$

$\tilde{\varphi}$ increases. Then

$$P(\tilde{v}_t \geqslant v_t,\ t \geqslant 0) = 1. \tag{15}$$

Proof. Follows from similar comparison arguments and strong uniqueness (see Veretennikov (1981)).

Lemma 4. *Let $\bar{v}_t$ be a solution of the stochastic differential equation with nonsticky reflecting boundary conditions*

$$\mathrm{d}\bar{v}_t = \hat{b}(\bar{v}_t)\,\mathrm{d}t + \mathrm{d}\hat{w}_t + \mathrm{d}\bar{\varphi}_t, \quad \bar{v}_t \geqslant |x|, \ \mathscr{L}(\bar{v}_0) = \bar{\mu}^{inv},$$
$$\bar{\varphi}_t = \int_0^t I(\bar{v}_s = |x|)\,\mathrm{d}\bar{\varphi}_s, \qquad \bar{\varphi}_0 = 0, \qquad E\int_0^\infty I(\bar{v}_s = |x|)\,\mathrm{d}s = 0, \tag{16}$$

$\bar{\varphi}$ increases, where $\mathscr{L}(\bar{v}_0)$ is the distribution of $\bar{v}_0$, r.v. $\bar{v}_0$ is independent of w, $\bar{\mu}^{inv}$ is the invariant measure for this equation (see Lemma 5 below). Then

$$P(\bar{v}_t \geqslant \tilde{v}_t,\ t \geqslant 0) = 1. \tag{17}$$

Proof. Follows from the uniqueness theorem: strong solution of Eq. (14) (or (16)) is unique (see Veretennikov (1981)), hence if there are two solutions of the same equation with $\bar{v}_0 \geqslant \tilde{v}_0$ then after the intersection they should coincide. At any rate, $\bar{v}_t \geqslant \tilde{v}_t$ for all $t \geqslant 0$ a.s.

Lemma 5. *Under condition (6) with $r > d/2$, $r' \in (d/2, r)$, for any $m < 2r' - d$*

$$E|X_t|^m \leqslant E\tilde{v}_t^m \leqslant E\bar{v}_t^m \leqslant C(1 + |x|^m).$$

Proof of Lemma 5. One should only prove the last inequality. Process $\tilde{v}_t$ possesses a density p which satisfies the equation

$$(1/2)p''(v) - (\tilde{r}v^{-1}p)'(v) = 0, \quad v \geqslant |x|, \ \tilde{r} = r' - (d-1)/2.$$

The solution is

$$p(v) = vq(v) = Cv^{-2\tilde{r}}, \quad v \geqslant |x|.$$

The last constant here $C = (2\tilde{r} - 1)|x|^{2\tilde{r}-1}$. Hence,

$$\int_{|x|}^{\infty} v^m p(v)\,\mathrm{d}v < \infty$$

if and only if $-2\tilde{r} + m < -1$, that is, $m < 2r' - d$, and in this case

$$\int_{|x|}^{\infty} v^m p(v)\,\mathrm{d}v = (2\tilde{r} - 1)|x|^{2\tilde{r}-1}(2\tilde{r} - m - 1)|x|^{m-2\tilde{r}+1} = C_r|x|^m.$$

Here $|x| \geqslant M_0$. For small $|x|$ one should use $1 + |x|$ instead. Lemma 5 is proved. □

Lemma 6. *Let assumptions of Theorem* 3 *be satisfied.* $\tau_t = \min(\tau, t)$. *Then* $\forall \varepsilon > 0$ *for any* $\bar{m} > m$ *there exists* $C = C(\bar{m})$ *s.t.*

$$E\int_0^{\tau_t} [(1+s)^{k-1}|X_s|^m + (1+s)^k|X_s|^{m-2}] I(|X_s|^2 > \varepsilon(1+s))\,\mathrm{d}s \leqslant C(1 + |x|^{\bar{m}}).$$

Proof of Lemma 6. One has, with $m < \bar{m} < 2r - d$, $a^{-1} + c^{-1} = 1$, $a, c > 1$, $am = \bar{m}$ and using Hölder's inequality and Lemma 1

$$\begin{aligned} &E\int_0^{\tau_t} (1+s)^{k-1}|X_s|^m I(|X_s|^2 > \varepsilon(1+s))\,\mathrm{d}s \\ &\leqslant \int_0^{\infty} (1+s)^{k-1}(E|X_s|^{ma})^{1/a}[E|X_s|^{\bar{m}}(1+s)^{-\bar{m}/2}]^{1/c}\,\mathrm{d}s \\ &\leqslant C(1 + |x|^{\bar{m}})\int_0^{\infty} (1+s)^{k-1-\bar{m}/(2c)}\,\mathrm{d}s. \end{aligned}$$

The integral here is finite if and only if $k - 1 - \bar{m}/(2c) < -1$, that is, $k < \bar{m}/(2c)$. This can be done, at any rate, if $k < \bar{m}/2$. Then m should satisfy the inequality $m < \bar{m}/a$ which is possible in the case $r > d/2$ (see Lemma 1).

Similarly, with another $a^{-1} + c^{-1} = 1$,

$$E\int_0^{\tau_t} (1+s)^k|X_s|^{m-2} I(|X_s|^2 > \varepsilon(1+s))\,\mathrm{d}s \leqslant C(1 + |x|^{\bar{m}})\int_0^{\infty} (1+s)^{k-\bar{m}/(2c)}\,\mathrm{d}s,$$

and the integral here is finite if $k + 1 < \bar{m}/(2c)$, that is, if $k + 1 < \bar{m}/2$ which is possible if $k + 1 < r - d/2$. Lemma 6 is proved. □

Proof of Theorem 3. Due to the Itô formula one gets,

$$E(1+\tau_t)^k|X_{\tau_t}|^m - |x|^m$$
$$= E\int_0^{\tau_t}(1+s)^{k-1}|X_s|^{m-2}[k|X_s|^2 + (1+s)m(X_s, b(X_s))$$
$$+ m(m+d-2)(1+s)/2] \times \{I(|X_s|^2 \leqslant \varepsilon(1+s)) + I(|X_s|^2 > \varepsilon(1+s))\}\,\mathrm{d}s$$
$$\equiv H_1 + H_2.$$

Due to Lemma 6, for any $\varepsilon > 0$ one has $H_2 \leqslant C(1+|x|^{\bar{m}})$, $\bar{m} > m$. On the other hand, if $\varepsilon > 0$ is small enough then due to the assumptions, $(1+s)m(X_s, b(X_s) + k|X_s|^2 + m(m+d-2)(1+s)/2 \leqslant -c_0(1+s)$ with a certain $c_0 > 0$. Hence, again due to Lemma 6, one gets

$$H_1 \leqslant -c_0E_x\int_0^{\tau_t}(1+s)^k|X_s|^{m-2}I(|X_s|^2 \leqslant \varepsilon(1+s))\,\mathrm{d}s$$
$$= -c_0E\int_0^{\tau_t}(1+s)^k|X_s|^{m-2}\,\mathrm{d}s + c_0E\int_0^{\tau_t}(1+s)^k|X_s|^{m-2}I(|X_s|^2 > \varepsilon(1+s))\,\mathrm{d}s$$
$$\leqslant -c_0M^{m-2}(k+1)^{-1}E_x\tau_t^{k+1} + C(1+|x|^{\bar{m}}).$$

Hence, one has,

$$E_x\tau_t^{k+1} \leqslant C(1+|x|^{\bar{m}}).$$

Fatou's lemma now gives one the same inequality for τ. Theorem 3 is proved. □

Now, let (X_t, Y_t) be a couple of two independent copies of solutions of Eq. (5), only with different initial data, x and y correspondently. Let $\gamma = \inf(t \geqslant 0 : |X_t| \leqslant M$ and $|Y_t| \leqslant M)$ and $\gamma_t = \min(\gamma, t)$.

Lemma 7. *Under Assumption* (6) *with* $r > (d/2) + 1$, *for any* $k < r - 1 - d/2$ *and* $m \in (2k+2, 2r-d)$ *there exists such* $M_1 \geqslant M_0$ *that for any* $M > M_1$

$$E_x\gamma^{k+1} \leqslant C_k(1+|x|^m + |y|^m). \tag{18}$$

Proof. Follows from calculations and bounds similar to those in the proof of Theorem 3 applied to the process $(1+s)^k(|X_s|^m + |Y_s|^m)$. Now one should consider the following possibilities for $|X_t| : |X_t| \leqslant M_0$, $M_0 < |X_t| \leqslant M$, $|X_t| > M$ and also $|X_t|^2 \leqslant \varepsilon(1+t)$ and $|X_t|^2 > \varepsilon(1+t)$, and the same for Y_t. Proceeding in such a way, one obtains,

$$E_{x,y}(1+\gamma_t)^k(|X_{\gamma_t}|^m + |Y_{\gamma_t}|^m) - (|x|^m + |y|^m)$$
$$\leqslant -C_1M^{m-2}E\int_0^{\gamma_t}(1+s)^k[1 - I(|X|^2 > \varepsilon(1+s), |Y_s|^2 > \varepsilon(1+s))]\,\mathrm{d}s$$
$$+ C_2M_0^mE\gamma_t^{k+1} + C_3(M) + E\int_0^{\gamma_t}(1+s)^{k-1}|X_s|^{m-2}$$

$$\times \left[k|X_s|^2 + \frac{m(m+d-2)(1+s)}{2}\right] I(|X_s|^2 > \varepsilon(1+s))\,\mathrm{d}s$$

$$+E\int_0^{\gamma_t}(1+s)^{k-1}|Y_s|^{m-2}\left[k|Y_s|^2 + \frac{m(m+d-2)(1+s)}{2}\right]$$

$$\times I(|Y_s|^2 > \varepsilon(1+s))\,\mathrm{d}s.$$

Here $C_3(M)$ is some polynomial of the variable M. Similar to Lemma 6 one finds that last two integrals do not exceed $C(m')(1+|x|^{m'}+|y|^{m'})$ with any $m'>m$. The same bound holds true for $E\int_0^{\gamma_t}(1+s)^k I(|X|^2>\varepsilon(1+s),\ |Y_s|^2>\varepsilon(1+s))\mathrm{d}s$. So one gets

$$C_1 M^{m-2} E_{x,y}\gamma^{k+1} \leqslant C_2 M_0^m E_{x,y}\gamma^{k+1} + C_3(M) + C(m')(1+|x|^{m'}+|y|^{m'}).$$

Finally, if one chooses M s.t. $C_1 M^{m-2} > C_2 M_0^m + 1$, one gets (18). (Details may be found in Veretennikov (1996); similar exponential bound for a hitting-time of a couple of independent "exponentially recurrent" processes may be found in Veretennikov (1987)). Lemma 7 is proved. □

Lemma 8. *If* $r>(d/2)$ *then the invariant measure* μ^{inv} *for Eq.* (1) *does exist, it is unique and for any* $m<2r-d$

$$E^{inv}|X_t|^m < \infty. \tag{19}$$

Proof. Existence follows from Theorem 3 with $k\geqslant 1$ and Lemma 1 by virtue of Has'minski's criterion (Has'minski, 1980, Theorem 4.4.1). Note that in Has'minski (1980) Lipschitz conditions are assumed for drift and diffusion coefficients (see Remark 3.6.5). Nevertheless, the result holds true without that condition as well due to Harnack inequality for parabolic equation with measurable coefficients (Krylov and Safonov, 1981). Uniqueness follows from Corollary 4.5.2 in Has'minski (1980) due to the same inequalities of Theorem 3 and Lemma 1.

Let v_t^* be a stationary distributed solution of equation with nonsticky boundary condition

$$\mathrm{d}v_t = \hat{b}(v_t)\,\mathrm{d}t + \mathrm{d}\hat{w}_t + \mathrm{d}\varphi_t, \quad v_t \geqslant M_0,$$

$$\varphi_t = \int_0^t I(v_s = M_0)\,\mathrm{d}\varphi_s, \qquad E\int_0^\infty I(v_s = M_0)\,\mathrm{d}s = 0, \ \varphi \text{ increases}$$

(see Lemma 5). Inequality $E^{inv}|X_t|^m<\infty$ follows from comparison arguments if one compares v_t^* and X_t with the same distribution for $|X_0|$: then $\mathscr{L}(\tilde{v}_t)=\mathscr{L}(\tilde{v}_0)$ while $P(|X_t|\leqslant v_t^*,\ t\geqslant 0)=1$. Lemma 8 is proved. □

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. We use the coupling technique, see Nummelin (1984), for SDEs Veretennikov (1987). Consider the couple of independent processes (X_t, Y_t) both being solutions of Eq. (1) with different independent Wiener processes w_t and w'_t and initial data $X_0 = x \in \mathbb{R}^d$, Y_0 distributed with the invariant measure μ^{inv}. Fix $s_0 \geqslant 0$. Define the sequence of stopping-times $\gamma_1 < \gamma_2 < \cdots$ in the following way:

$$\gamma_1 = \inf(t \geqslant s_0\colon |X_t| \leqslant M \text{ and } |Y_t| \leqslant M),$$

for $n \geqslant 1$

$$T_n = \min\{\inf(t \geqslant \gamma_n\colon |X_t| \geqslant M + 1 \text{ or } |Y_t| \geqslant M + 1), \gamma_n + 1\};$$

$$\gamma_{n+1} = \inf(t \geqslant T_n\colon |X_t| \leqslant M \text{ and } |Y_t| \leqslant M).$$

We have due to Lemma 6,

$$E((\gamma_1 - s_0)^{k+1} | F^{X,Y}_{s_0}) \leqslant C(1 + |Y_{s_0}|^m + |X_{s_0}|^m).$$

Similarly,

$$E((\gamma_{n+1} - \gamma_n)^{k+1} | F^{X,Y}_{\gamma_n}) \leqslant C, \quad n \geqslant 1.$$

Let $n(t) := \sup(n \geqslant 0\colon \gamma_n \leqslant t)$. By virtue of the last inequality and a strong markovian property of (X_t, Y_t), one gets

$$P(n(t) \to \infty, t \to \infty) = 1.$$

Using a coupling method for SDEs (Veretennikov, 1987) it is possible to define a new process $\tilde{X}_t$ and a random value $L_{s_0} \geqslant s_0$ on a certain extension of the probability space (Ω, F, P) (we do not change the notation for the probability space, though) s.t.

$$\mathrm{d}\tilde{X}_t = b(\tilde{X}_t)\,\mathrm{d}t + \mathrm{d}\tilde{w}_t, \quad \tilde{X}_0 = x,$$

where $(\tilde{w}_t)$ is some new Wiener process and $(w_t, F^{X,Y,\tilde{X}})$, $(w'_t, F^{X,Y,\tilde{X}})$, $(\tilde{w}_t, F^{X,Y,\tilde{X}})$ are still Wiener processes; moreover,

$$P(\tilde{X}_t = X_t, t \leqslant L_{s_0} - 1) = P(\tilde{X}_t = Y_t, t \geqslant L_{s_0}) = 1,$$

and L_{s_0} is a $\hat{F}_t \equiv F^{X,Y,\tilde{X}}_t$-stopping time. Moreover, there exists $q \in (0,1)$ s.t.

$$\sup_{s_0 \geqslant 0} P(L_{s_0} > \gamma_n | \hat{F}_{s_0}) \leqslant q^n \quad \forall n.$$

Hence, $\forall B \in F^{X,Y,\tilde{X}}_{\geqslant t+s_0}$,

$$|P(B|\hat{F}_{s_0}) - P(B)| \leqslant P(L_{s_0} > t + s_0 | \hat{F}_{s_0}).$$

So,

$$\beta^x(t) \leqslant EP(L_{s_0} > t + s_0 | \hat{F}_{s_0}) = P(L_{s_0} > t + s_0).$$

Now,

$$\begin{aligned} P(L_{s_0} > t + s_0 | \hat{F}_{s_0}) &= \sum_{n=0}^{\infty} E(I(L_{s_0} > t + s_0) I(\gamma_n \leqslant t + s_0 < \gamma_{n+1}) | \hat{F}_{s_0}) \\ &\leqslant \sum_{n=0}^{\infty} P(L_{s_0} > \gamma_n | \hat{F}_{s_0})^{1/a} P(\gamma_{n+1} > t + s_0 | \hat{F}_{s_0})^{1/c} \\ &\leqslant \sum_{n=0}^{\infty} q^{n/a} P(\gamma_{n+1} > t + s_0 | \hat{F}_{s_0})^{1/c} \end{aligned}$$

(here $a^{-1} + c^{-1} = 1$, $a > 1$, $c > 1$; we used Hölder's inequality). Due to Bienaimé–Chebyshev's inequality, one gets

$$\begin{aligned} P(\gamma_{n+1} > t + s_0 | \hat{F}_{s_0}) &\leqslant t^{-(k+1)} E((\gamma_{n+1} - s_0)^{k+1} | \hat{F}_{s_0}) \\ &\leqslant t^{-(k+1)} (n+1)^k \sum_{j=0}^{n} E((\gamma_{j+1} - \gamma_j)^{k+1} | \hat{F}_{s_0}) \\ &\leqslant t^{-(k+1)} (n+1)^k (C(1 + |X_{s_0}|^m + |Y_{s_0}|^m) + Cn). \end{aligned} \tag{20}$$

Therefore, due to Lemma 7

$$P(L_{s_0} > t + s_0) \leqslant \sum_{n \geqslant 0} q^{n/a} [t^{-(k+1)} (n+1)^k (C(1 + |x|^m) + Cn)]^{1/c}. \tag{21}$$

For any $\nu > 0$ there exists such c close to 1 and $C < \infty$ that

$$\beta^x(t) \leqslant P(L_{s_0} > t + s_0) \leqslant C t^{-(k+1-\nu)} (1 + |x|^m).$$

Theorem 1 is proved. □

Proof of Theorem 2. Let $s_0 = 0$. One gets, by virtue of the same arguments,

$$\begin{aligned} \operatorname{var}(\mu^x(t) - \mu^{inv}) &= \sup_A (P(\tilde{X}_t \in A) - P(Y_t \in A)) \\ &\leqslant P(L_0 > t) \leqslant C t^{-(k+1-\nu)} (1 + |x|^m) \end{aligned}$$

for any $\nu > 0$ with some $C = C(\nu)$. Theorem 2 is proved. □

5. Main results for the general case

Now we will study Eq. (5) with $d_1 \geqslant d$ and continuous nondegenerate σ. Denote

$$\lambda_- = \inf_{x \neq 0} \left(\sigma\sigma^*(x) \frac{x}{|x|}, \frac{x}{|x|} \right), \quad \lambda_+ = \sup_{x \neq 0} \left(\sigma\sigma^*(x) \frac{x}{|x|}, \frac{x}{|x|} \right),$$

$$\Lambda = \sup_x \frac{\operatorname{Tr} \sigma\sigma^*(x)}{d}. \tag{22}$$

and

$$r_0 = [r - (d\Lambda - \lambda_-)/2]\lambda_+^{-1}. \tag{23}$$

Theorem 4. *Under Assumptions* (6) *with* $r_0 > 3/2$, *for any* $k \in (0, r_0 - 3/2)$, $m \in (2k + 2, 2r_0 - 1)$

$$E_x \tau^{k+1} \leqslant C(1 + |x|^m). \tag{24}$$

Note that in the case $\sigma = I$ one has $\lambda_- = \lambda_+ = \Lambda$ and $r_0 = r - (d-1)/2$; hence, the assumption $r_0 > 3/2$ in this case corresponds to the assumption $r > (d/2) + 1$. Further, $2r - d$ corresponds to $2r_0 - 1$ and $2k + 2$ remains $2k + 2$.

Theorem 5. *Under Assumptions* (6) *with* $r_0 > 3/2$, *for any* $k \in (0, r_0 - 3/2), m \in (2k + 2, 2r_0 - 1)$

$$\beta^{inv}(t) \leqslant C(1 + t)^{-(k+1)}. \tag{25}$$

Theorem 6. *Under Assumption of Theorem* 5

$$\operatorname{var}(\mu^x(t) - \mu^{inv}) \leqslant C(1 + |x|^m)(1 + t)^{-(k+1)}. \tag{26}$$

In particular, the invariant measure μ^{inv} *does exist.*

6. Proofs of Theorems 4–6

We will show how to reduce Theorems 4–6 to the case of the unit diffusion. Denote $\kappa(x) = |\sigma^*(x)x|/|x|$ and consider the change of time $t' = t'(t) = h^{-1}(t)$ where h^{-1} is the inverse function to $h(t) = \int_0^t \kappa(X_s)^2 \,\mathrm{d}s$. Define $\tilde{X}_t = X_{t'(t)}$. Then one gets,

$$\mathrm{d}\tilde{X}_t = \tilde{b}(\tilde{X}_t)\,\mathrm{d}t + \tilde{\sigma}(\tilde{X}_t)\,\mathrm{d}\tilde{w}_t,$$

where $\tilde{w}$ is a new d_1-dimensional Wiener process,

$$\tilde{b}(x) = b(x)/\kappa(x)^2, \qquad \tilde{\sigma}(x) = \sigma(x)/\kappa(x).$$

Note that $\operatorname{Tr}\tilde{\sigma}\tilde{\sigma}^* \equiv 1$. Then for $\tilde{X}_t \neq 0$

$$\mathrm{d}|\tilde{X}_t| = B(\tilde{X}_t)\,\mathrm{d}t + \mathrm{d}\bar{w}_t,$$

$\bar{w}$ being a (1-dimensional) Wiener process. Here

$$B(x) = \kappa(x)^{-2}\{|x|^{-1}(x, b(x)) - |x|^{-3}x_k x_j(\sigma\sigma^*)_{kj}(x)/2 + |x|^{-1}\operatorname{Tr}(\sigma\sigma^*)(x)/2\}.$$

Due to assumptions on b and σ, one gets

$$\bar{B}(|x|) \equiv \sup_{|\xi|=1}(\xi, B(\xi|x|)) \leqslant -r_0/|x|.$$

Similar to the case $\sigma \equiv 1$ one obtains a comparison type inequality

$$P(|\tilde{X}_t| \leqslant \bar{V}_t,\ t \geqslant 0) = 1, \tag{27}$$

where $\bar{V}$ is a strong solution of the one-dimensional SDE with a nonsticky reflecting boundary condition (see Veretennikov, 1981)

$$\bar{V}_t = |x| + \bar{w}_t + \int_0^t \bar{B}(\bar{V}_s)\,\mathrm{d}s + \bar{\psi}_t, \quad \bar{V}_t \geqslant M_0,$$

$$\bar{\psi}_t = \int_0^t I(\bar{V}_s = M_0)\,\mathrm{d}\bar{\psi}_s, \quad \bar{\psi}_0 = 0, \quad E\int_0^t I(\bar{V}_s = M_0)\,\mathrm{d}s = 0, \quad \bar{\psi} \text{ increases.}$$

Hence, $\tau^{\tilde{X}} \leqslant \tau^{\bar{V}}$. Since we have the bound for $\tau^{\bar{V}}$, we obtain the same bound for $\tau^{\tilde{X}}$. Since $\tau^X \leqslant C\tau^{\tilde{X}}$ with $C^{-1} = \inf_x \kappa(x)^2$, we get immediately the same bound (with a new constant) for τ^X. This implies the existence of (unique) invariant measure for the process X.

Let us show the estimate (19) for X_t. From the bound for $\bar{V}$ (see Lemma 1) and from inequality (27) one gets $E_x|\tilde{X}_t|^m \leqslant C(1+|x|^m)$. As a consequence, applying Lemma 8 to $\tilde{X}$ one obtains the inequality

$$E^{inv}|\tilde{X}_t|^m < \infty.$$

Further, the invariant densities p and $\tilde{p}$ for X and (respectively) $\tilde{X}$ satisfy the equations

$$(a_{ij}p_{x_i})_{x_j} - (b_i p)_{x_i} = 0, \qquad (\tilde{a}_{ij}\tilde{p}_{x_i})_{x_j} - (\tilde{b}_i \tilde{p})_{x_i} = 0,$$

with $(a_{ij}) = a = \sigma\sigma^*/2, (\tilde{a}_{ij}) = a/\kappa^2, \tilde{b} = b/\kappa^2$. One can easily see that the function $\kappa^2 p$ satisfies the second invariant equation. Hence, this function is the invariant density for $\tilde{X}$ upto a normalizing constant $C>0$. So $\tilde{p} = C\kappa^2 p$ because the invariant density of the Markov process $\tilde{X}$ is unique. So one obtains

$$E^{inv}|X_t|^m < \infty. \tag{28}$$

Now, to apply considerations of Section 5 one only needs the analogue of Lemma 7 for the general case. It may appear that the estimate like in Lemma 1 is required for this. However, one can, in fact, use the time change $(s \mapsto r = t'(s))$: for example, $\int_0^\infty (1+s)^{k-1-m/(2c)}(E|X_s|^m)^{1/c}\,\mathrm{d}s \leqslant C\int_0^\infty (1+r)^{k-1-m/(2c)}(E\bar{V}_r^m)^{1/c}\,\mathrm{d}r$ (see the proof of Lemma 6) and then apply the estimate $E_x\bar{V}_r^m \leqslant (1+|x|^m)$ (cf. with Lemma 5). So all technique of Sections 3 and 4 including Lemma 7 can be used. In particular, for the stationary regime one passes from (20) to (21) using (27) instead of Lemma 1. In the proof of Theorem 6 one applies, in fact, the bound (20) with $s_0 = 0$ for the general case which implies (21) for this case directly. This gives assertions of Theorems 4–6.

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