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Semigroups of I-Type

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Assume that S is a semigroup generated by $\{x_1, \dots, x_n\}$, and let $\mathscr{U}$ be the multiplicative free commutative semigroup generated by $\{u_1, \dots, u_n\}$. We say that S is of *I-type* if there is a bijective $v : \mathscr{U} \to S$ such that for all $a \in \mathscr{U}$, $\{v(u_1 a), \dots, v(u_n a)\} = \{x_1 v(a), \dots, x_n v(a)\}$. This condition appeared naturally in the work on Sklyanin algebras by John Tate and the second author. In this paper we show that the condition for a semigroup to be of I-type is related to various other mathematical notions found in the literature. In particular we show that semigroups of I-type appear in the study of the set-theoretic solutions of the Yang–Baxter equation, in the theory of Bieberbach groups, and in the study of certain skew binomial polynomial rings which were introduced by the first author.

1. INTRODUCTION

In the sequel k will be a field. Our starting point for this paper are certain semigroups which were introduced in [3]. Let $X = \{x_1, \dots, x_n\}$ be a

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set of generators. In [3] the first author considers semigroups S of the form $\langle X; R \rangle$ where R is a set of quadratic relations

$$R = \{x_j x_i = u_{ij} \mid i = 1, \dots, n; j = i+1, \dots, n\}$$

satisfying

Condition (*). (1) $u_{ij} = x_{i'} x_{j'}$, $i' < j'$, $i' < j$.

(2) As we vary (i, j), every pair (i', j') occurs exactly once.

(3) The overlaps $x_k x_j x_i$ for $k > j > i$ do not give rise to new relations in S.

The motivation for (*) is developed in [3]. Condition (*)(1) says that the semigroup algebra kS is a *binomial skew polynomial ring*, so the theory of (non-commutative) Gröbner bases applies to it. Condition (*)(3) says that as sets

$$S = \{x_1^{a_1} \cdots x_n^{a_n} \mid (a_1, \dots, a_n) \in \mathbb{N}^n\}.$$

Furthermore it is shown in [3, Theorem II] that (*)(2) is equivalent with kS being noetherian (assuming (*)(1), (3)).

However, conditions (*)(1), (2), (3) are also natural for intrinsic reasons. There are exactly as many monomials $x_j x_i$ with $j > i$ as there are monomials $x_{i'} x_{j'}$ with $i' < j'$. This provides the motivation for imposing (*)(2). Furthermore, it follows from [3, Theorem 3.16] that (*)(1), (2), (3) imply $j, j' > i, i'$ for the relations in R. Thus conditions (*)(1), (2), (3) are actually symmetric, in the sense that if they are satisfied by $S = \langle X; R \rangle$ then they are also satisfied by S°.

The purpose of this paper is to show that the semigroups defined in the previous paragraphs are intimately connected with various other mathematical notions which are currently of some interest. In particular we show that they are related to

(1) Set theoretic solutions of the Yang–Baxter equation [2].

(2) Bieberbach groups [1].

(3) Rings of I-type [6].

We will now sketch these connections. We start by proving the following proposition.

THEOREM 1.1. *Assume that R satisfies* (*)(1), (2), (3). *Define $r : X^2 \to X^2$ as follows: r is the identity on quadratic monomials and if $(x_j x_i = x_{i'} x_{j'}) \in R$ then $r(x_j x_i) = x_{i'} x_{j'}$, $r(x_{i'} x_{j'}) = x_j x_i$. Then r satisfies*

(1) $r^2 = \mathrm{id}_{X^2}$.

(2) *r satisfies the set-theoretic Yang–Baxter equation. That is, one has*

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

where as usual $r_i : X^m \to X^m$ is defined as $\mathrm{id}_{X^{i-1}} \times r \times \mathrm{id}_{X^{m-i-1}}$.

(3) *Given* $a, b \in \{1, \ldots, n\}$ *there exist unique* c, d *such that*

$$r(x_c x_a) = x_d x_b.$$

Furthermore if $a = b$ *then* $c = d$.

In view of this theorem it is natural to consider semigroups of the form $\langle X; x_i x_j = r(x_i x_j)\rangle$ where r is a set-theoretic solution of the Yang–Baxter equation. We will show that some of these are of "I-type" [6]. Being of I-type is a technical condition which is very useful for computations. Let us recall the definition here. We start with a set of variables $u_1, \ldots, u_n$ and we let $\mathscr{U}$ be the free *commutative* multiplicative semigroup generated by $u_1, \ldots, u_n$. Let S be a semigroup generated by $X = \{x_1, \ldots, x_n\}$. S is said to be of (left) I-type if there exists a bijection $v : \mathscr{U} \to S$ (an I-structure) such that $v(1) = 1$ and such that for all $a \in \mathscr{U}$

$$\{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\}. \tag{1.1}$$

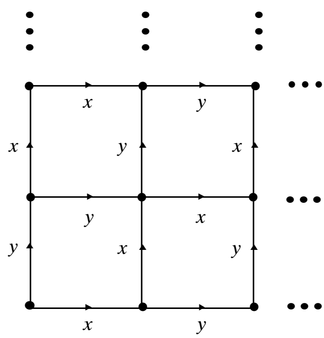
It is clear that if S is of I-type then kS is of I-type in the sense of [6].

Assume that S is of I-type with I-structure v. Equation (1.1) implies that for every $a \in \mathscr{U}$, $i \in \{1, \ldots, n\}$ there exists a unique $x_{a,i} \in X$ such that

$$x_{a,i} v(a) = v(au_i)$$

and $\{x_{a,i} \mid i = 1, \ldots, n\} = X$.

EXAMPLE 1.2. Let S be the semigroup $\langle x, y; x^2 = y^2\rangle$ and consider the following double infinity graph.



Define $v(u_1^{a_1} u_2^{a_2})$ as one (or all) of the paths from (0, 0) to (a_1, a_2), written in reverse order (for example, $v(u_1^2 u_2) = xy^2 = x^3 = y^2 x$). Then it is clear that this v defines a I-structure on S.

We have the following result

THEOREM 1.3. *Assume that S is of I-type. Define* $r : X^2 \to X^2$ *by*

$$r(x_{u_i, j} x_{1, i}) = x_{u_j, i} x_{1, j}.$$

Then r satisfies the conclusions of Theorem 1.2. *Conversely if* $r : X^2 \to X^2$ *satisfies Theorem* 1.1(1)–(3), *then the semigroup* $S = \langle X; x_i x_j = r(x_i x_j) \rangle$ *is of I-type.*

From Theorems 1.1 and 1.3 it follows that semigroups defined by relations satisfying (*)(1), (2), (3) are of I-type. The proof of the following result is similar to the proof of [6, Theorems 1.1, 1.2].

For a cocycle $c : S^2 \to k^*$ we use the notation $k_c S$ for the twisted semi-group algebra associated to (S, c). Thus $k_c S$ is the k-algebra with basis S and with multiplication $x \cdot y = c(x, y) xy$ for $x, y \in S$.

THEOREM 1.4. *Assume that S is of I-type and let* $A = k_c S$ *for some cocycle* $c : S^2 \to k^*$. *Then*

(1) *A has finite global dimension.*

(2) *A is Koszul.*

(3) *A is noetherian.*

(4) *A satisfies the Auslander condition.*

(5) *A is Cohen–Macaulay.*

(6) *If c is trivial then* $k_c S$ *is finite over its center.*

For the definition of "Cohen–Macaulay" and the "Auslander condition" see [4].

COROLLARY 1.5. *Assume that S is a semigroup of I-type. Then* $k_c S$ *is a domain, and in particular S is a cancellative.*

This corollary follows from [4].

Let S be a semi-group of I-type with I-structure $v : \mathcal{U} \to S$. Since S is a cancellative semigroup of subexponential growth, it is Öre. Denote its quotient group by $\overline{S}$. We identify $\mathcal{U}$ in the natural way with $\mathbb{N}^n$, and in this way we embed it in $\mathbb{R}^n$. We will prove the following

THEOREM 1.6. *Assume that S is of I-type with I-structure* $v : \mathcal{U} \to S$. *Let S act on the right of* $\mathcal{U}$ *by pulling back under v the action of S on itself by right translation. Then this action extends to a free right action of* $\overline{S}$ *on* $\mathbb{R}^n$ *by Euclidean transformations and for this action* $[0, 1]^n$ *is a fundamental domain. In particular* $\overline{S}$ *is a Bieberbach group.*

EXAMPLE 1.7. If we take for S the semigroup of Example 1.2 then using (5.3) one checks that x and y act on $\mathbb{R}^2$ by glide reflections along parallex axes. Hence $\mathbb{R}^2/\overline{S}$ is the Klein bottle!

2. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. The notations will be as in the Introduction. So S is a semigroup of the form $\langle X; R\rangle$ where R is a set of relations satisfying (*). It is clear that Theorem 1.1(1) is true by definition. So we concentrate on Theorem 1.1(2), (3).

Below we denote the diagonal of X^m by Δ_m. Clearly

$$r_1(\Delta_3) = \Delta_3, \qquad r_2(\Delta_3) = \Delta_3.$$

Furthermore it follows from the "cyclic condition" [3, Theorem 3.16] that

$$r_1 r_2(\Delta_2 \times X) = X \times \Delta_2. \tag{2.1}$$

LEMMA 2.1. *The relation*

$$r(zt) = xy$$

defines bijections between X^2 and itself given by

$$(t, y) \leftrightarrow (z, t) \leftrightarrow (x, y) \leftrightarrow (z, x).$$

Proof. That $(z, t) \leftrightarrow (x, y)$ defines a bijection is clear. Now consider the map which assigns (t, y) to (z, t). We claim that it is an injection. If this is so then by looking at the cardinality of the source and the target (which are both X^2) we see that it must be a bijection.

To prove the claim we compute $r_2 r_1(xy^2) = r_2(zty) = z^2 *$ where the last equality follows from (2.1). Thus $r(ty) = z*$ and hence z is uniquely determined by t, y. This proves the claim.

That $(z, t) \leftrightarrow (z, x)$ is a bijection is proved similarly. ∎

Note that Lemma 2.1 contains Theorem 1.1(3) as a special case. Hence we are left with proving Theorem 1.1(2).

Let us call $w, w' \in \langle X\rangle$ equivalent if they have the same image in S. Notation: $w \sim w'$. Clearly $w \sim w'$ iff

$$w' = r_{i_1} r_{i_2} \cdots r_{i_p} w$$

for some $p, i_1, \ldots, i_p$.

Concerning the structure of the equivalence classes there is the following easy lemma.

LEMMA 2.2. *Every equivalence class for* $\sim$ *in* X^m *contains exactly one monomial of the form* $x_{a_1} \cdots x_{a_m}$, $a_1 \leq \cdots \leq a_m$.

Proof. This is a consequence of the Bergman diamond lemma. ∎

After these preliminaries we prove the Yang–Baxter relation for r. The proof is based upon a careful examination of the equivalence classes in X^3, together with a counting argument.

Let D be the infinite dihedral group $\langle r_1, r_2; r_1^2 = r_2^2 = e\rangle$. D acts on X^3 and it is clear that the equivalence classes correspond to D-orbits. Let O be such an orbit. There are three possibilities.

(A) $O \cap \Delta_3 = \emptyset$. In this case clearly $|O| = 1$.

(B) $O \cap ((\Delta_2 \times X \cup X \times \Delta_2) \setminus \Delta_3) \neq \emptyset$. In this case it follows from (2.1) that $|O| = 3$.

(C) $O \cap (\Delta_2 \times X \cup X \times \Delta_2) = \emptyset$. Now $O = \{w, r_1 w, r_2 r_1 w, \ldots\}$. Thus a general member of O is of the form $(r_2 r_1)^a w$ or $r_1(r_2 r_1)^a w$.

We claim that $(r_2 r_1)^a w \neq r_1(r_2 r_1)^b w$ for $a, b \in \mathbb{Z}$. To prove this, assume the contrary and define

$$w_1 = \begin{cases} r_1(r_2 r_1)^{\left\lfloor \frac{a+b}{2} \right\rfloor} w & \text{if } a+b \text{ is odd} \\ (r_2 r_1)^{\left\lfloor \frac{a+b}{2} \right\rfloor} w & \text{if } a+b \text{ is even.} \end{cases}$$

Thus $r_1 w_1 = w_1$ or $r_2 w_1 = w_1$ (depending on whether $a + b$ is even or odd), whence $w_1 \in \Delta_2 \times X \cup X \times \Delta_2$, contradicting the hypotheses.

Let p be the smallest positive integer such that $(r_2 r_1)^p w = w$. Then

$$O = \left\{w, (r_2 r_1) w, \ldots, (r_2 r_1)^{p-1} w, r_1 w, r_1(r_2 r_1) w, \ldots, r_1(r_2 r_1)^{p-1} w\right\}.$$

In particular $|O| = 2p$ is even. We claim $|O| \geq 6$. To prove this we have to exclude $|O| = 2, 4$. The case $|O| = 2$ is easily excluded using Theorem 1.2(3). Hence we are left with $|O| = 4$. This means that O looks like

$$\begin{array}{ccc} x_a x_b x_c & \xrightarrow{r_2} & x_a x_d x_e \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ x_f x_g x_c & \xrightarrow{r_2} & x_f x_h x_e \end{array}$$

which implies that R contains relations

$$x_b x_c = x_d x_e \tag{2.3}$$

$$x_a x_b = x_f x_g \tag{2.4}$$

$$x_a x_d = x_f x_h \tag{2.5}$$

$$x_g x_c = x_h x_e. \tag{2.6}$$

Now in a relation $x_u x_v = x_w x_t$ the couples (u, v) and (v, t) determine each other (Lemma 2.1). So looking at (2.4), (2.5) we find $b = d$, $g = h$. This implies that (2.3) is actually of the form $x_d x_c = x_d x_c$, which is a contradiction. Hence $|O| \geq 6$.

An alternative classification of these orbits goes through the elements they contain of the form $x_a x_b x_c$, $a \leq b \leq c$. A unique such element exists in every orbit by Lemma 2.2.

If O contains an element of the form $x_a x_b x_c$, $a < b < c$ then it is of type (C) because if not, it contains an element of the form $x_d x_d x_e$ or $x_d x_e x_e$ with $d \geq e$. Using (2.1) and (*)(1) such elements are equivalent to elements of the form $x_f x_g x_g$, $x_f x_f x_g$ with $f \leq g$, a contradiction.

If O contains an element of the form $x_a x_a x_b$ or of the form $x_a x_b x_b$ with $a < b$ then O is clearly of type (B). Finally O is of type (A) iff it contains an element of the form $x_a x_a x_a$.

Thus we find that there are n orbits of type (A), $n(n-1)$ orbits of type (B), and $n(n-1)(n-2)/6$ orbits of type (C). From the equality

$$|X^3| = n^3 = 1 \cdot n + 2 \cdot n(n-1) + 6 \cdot \frac{n(n-1)(n-2)}{6}$$

we deduce that the orbits of type (C) contain exactly 6 elements.

Now Yang–Baxter easily follows. If w has orbit of type (C) then from (2.2) we deduce that $(r_2 r_1)^3 w = w$. If the orbit is of type (B) then $(r_2 r_1)^3 w = w$ follows directly from (2.1). Finally if the orbit is of type (A) then $r_1 w = r_2 w = w$ and there is nothing to prove.

This concludes the proof of Theorem 1.1.

3. PROOF OF THEOREM 1.3

In this section we prove Theorem 1.4. One direction is trivial, so we concentrate on the other direction. That is, given r satisfying Theorem 1.1(1)–(3), we will construct $v : \mathscr{U} \to S$ and $x_{b,i} \in X$ for $b \in \mathscr{U}$, $i =$

$\{1, \dots, n\}$ in such a way that

(a) v is a bijection.

(b) $v(u_i b) = x_{b,i} v(b)$

(c) $\{x_{b,i} \mid i = 1, \dots, n\} = \{x_1, \dots, x_n\}$

(d) $r(x_{bu_j, i} x_{b,j}) = x_{bu_i, j} x_{b,i}$.

The construction is inductive. To start we put $v(1) = 1$ and $v(u_i) = x_{\sigma(i)}$ for an arbitrary element σ of Sym_n. From here on everything will be uniquely defined. Assume that we have constructed $v(b)$ for $\deg b \leq m - 1$, $x_{b,i}$ for $\deg b \leq m - 2$ satisfying (a)–(d). We will define $x_{a,i}$ for $\deg a = m - 1$ such that (c), (d) hold.

Case 1. $a \neq u_i^{m-1}$. So $a = bu_j$, $j \neq i$. Computing $v(bu_i u_j)$ in two ways (as a heuristic device, since $v(bu_i u_j)$ is still undefined) we find that $x_{a,i}$ must be defined by

$$r(x_{a,i} x_{b,j}) = * x_{b,i}. \tag{3.1}$$

This indeed defines $x_{a,i}$ uniquely thanks to Theorem 1.1(3). However, one still must deal with the possibility that the $x_{a,i}$ might depend on j. To analyze this assume $k \neq i$, $a = du_j u_k$. Put $b = du_k$, $c = du_j$, $e = du_i$. We now define p, q, p', q' by

$$r(px_{b,j}) = qx_{b,i} \tag{3.2}$$

$$r(p' x_{c,k}) = q' x_{c,i}. \tag{3.3}$$

We have to show $p = p'$. By induction we have the identities

$$r(x_{b,j} x_{d,k}) = x_{c,k} x_{d,j} \tag{3.4}$$

$$r(x_{b,i} x_{d,k}) = x_{e,k} x_{d,i} \tag{3.5}$$

$$r(x_{c,i} x_{d,j}) = x_{e,j} x_{d,i}. \tag{3.6}$$

We can now construct a “Yang–Baxter diagram”

$$\begin{array}{ccc} px_{b,j} x_{d,k} & \xrightarrow{r_1} & qx_{b,i} x_{d,k} \\ {\scriptstyle r_2}\downarrow & & \downarrow{\scriptstyle r_2} \\ px_{c,k} x_{d,j} & & qx_{e,k} x_{d,i} \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ XYx_{d,j} & \xrightarrow{r_2} & XZx_{d,i} \end{array}$$

with X, Y, Z unknown so far.

Comparing $r(Yx_{d,j}) = Zx_{d,i}$ with (3.6) yields $Y = x_{c,i}$, $Z = x_{e,j}$.
So we find that

$$r(px_{c,k}) = Xx_{c,i}$$

and comparing this with (3.3) yields $p = p'$.

Hence we can now legally define $x_{a,i} = p$. Furthermore (3.2) can also be read as

$$r(qx_{b,i}) = px_{b,j}.$$

Since obviously $bu_i \neq u_j^{m-1}$ we obtain $q = x_{bu_i,j}$. We conclude that with our present definitions we have for $j \neq i$, $\deg b \leq m-2$,

$$r(x_{bu_j,i}x_{b,j}) = x_{bu_i,j}x_{b,i}. \tag{3.7}$$

We claim that this relation holds more generally under the hypotheses that $\deg b \leq m-2$ and $bu_j \neq u_i^{m-1}$ (or equivalently $bu_i \neq u_j^{m-1}$).

The only case that still has to be checked is $i = j$, $\deg b = m-2$, $b \neq u_i^{m-2}$. In this case we may put $b = cu_k$, $k \neq i$. We construct again a Yang–Baxter diagram

$$\begin{array}{ccc}
x_{cu_iu_k,i}x_{cu_k,i}x_{c,k} & \xrightarrow{r_1} & x_{cu_iu_k,i}Yx_{c,k} \\
r_2\downarrow & & \uparrow r_2 \\
x_{cu_iu_k,i}x_{cu_i,k}x_{c,i} & & x_{cu_iu_k,i}x_{cu_i,k}x_{c,i} \\
r_1\downarrow & & \uparrow r_1 \\
x_{cu_i^2,k}x_{cu_i,i}x_{c,i} & \xrightarrow{r_2} & x_{cu_i^2,k}x_{cu_i,i}x_{c,i}.
\end{array}$$

From the relation

$$r(x_{cu_i,k}x_{c,i}) = Yx_{c,k}$$

we deduce $Y = x_{cu_k,i}$. Looking at the top row of (3.8) finishes the proof of (3.7) under the hypotheses that $bu_j \neq u_i^{m-1}$.

Now we claim that if $\deg a = m-1$, $i \neq j$ and $a \neq u_i^{m-1}$, u_j^{m-1} then $x_{a,i} \neq x_{a,j}$. Assume the contrary and write $a = bu_l$. Then by (3.7) we have

$$\begin{aligned}
r(x_{bu_l,i}x_{b,l}) &= x_{bu_i,l}x_{b,i} \\
r(x_{bu_l,j}x_{b,l}) &= x_{bu_j,l}x_{b,j}.
\end{aligned} \tag{3.9}$$

Since the left-hand sides of (3.9) are the same and this is not the case with the right-hand sides we obtain a contradiction.

Case 2. $a = u_i^{m-1}$. In this case we take $x_{a,i}$ different from $x_{a,j}$, $j \neq i$. This defines $x_{a,i}$ uniquely, and obviously (c) is satisfied if $\deg b \leq m - 1$.

Now we prove (3.7) in th remaining case $b = u_i^{m-2}$, $i = j$.

Since we already know (c) we can write

$$r(x_{bu_k,l} x_{b,k}) = x_{bu_i,i} x_{b,i}$$

for some k, l and we have to show $k = l = i$. Assume on the contrary that $k \neq i$ or $l \neq i$. By what we know so far we have

$$r(x_{bu_k,l} x_{b,k}) = x_{bu_l,k} x_{b,l}.$$

But then $k = l = i$, a contradiction.

So up to this point we have defined $x_{b,i}$ and we have proved (c), (d) for $\deg b \leq m - 1$. Now if $a = bu_i$ has length m then we define

$$v(a) = x_{b,i} v(b) \tag{3.10}$$

so that (b) certainly holds. That (3.10) is well defined follows easily from (d).

Hence to complete the induction step it suffices to show that (a) holds. That is, v should define a bijection on words of length m. Let $U = \{u_1, \ldots, u_n\}$ and let U^m be the words of length m in U. Furthermore let $r_i : U^m \to U^m$ be given by exchanging the $(i, i+1)$st letter. Define a map $\tilde{v}: U^m \to X^m$ by

$$\tilde{v}(u_{i_1} \cdots u_{i_m}) = x_{u_{i_2} \cdots u_{i_m}, i_1} \cdots x_{u_{i_{m-1}} u_{i_m}, i_{m-2}} x_{u_{i_m}, i_{m-1}} x_{1, i_m}.$$

By (c), $\tilde{v}$ is clearly a bijection.

From (d) we obtain the commutative diagram

$$\begin{array}{ccc} U^m & \xrightarrow{\tilde{v}} & X^m \\ {\scriptstyle r_i}\downarrow & & \downarrow{\scriptstyle r_i} \\ U^m & \xrightarrow{\tilde{v}} & X^m \end{array}$$

So $\tilde{v}$ defines a bijection between the orbits U^m/Sym_m and X^m/Sym_m. We have

$$\mathscr{U}_m = U^m/\mathrm{Sym}_m, \qquad S_m = X^m/\mathrm{Sym}_m,$$

where $\mathscr{U}_m$, S_m are the elements of degree m in $\mathscr{U}$ and S, respectively. Furthermore the map $\mathscr{U}_m \to S_m$ induced by $\tilde{v}$ is precisely v. This finishes the proof of Theorem 1.3.

4. SEMIGROUPS OF I-TYPE

Below S will be a semigroup of I-type, with I-structure $v : \mathcal{U} \to S$ (as defined in the Introduction). In this section we will give some properties of S, and in particular we will prove Theorem 1.4.

First observe that every element of $\langle X \rangle$ can be written uniquely in the form

$$x_{u_1 \cdots u_{i_{m-1}}, i_m} \cdots x_{u_{i_1}, i_2} x_{1, i_1}.$$

Two different elements w, w' in X^2 have the same image in S iff there exist $i \neq j$ such that

$$w = x_{u_i, j} x_{1, i}, \qquad w' = x_{u_j, i} x_{1, j}.$$

The following lemma summarizes some observations in [6], translated into the language of semigroups.

LEMMA 4.1. (1) *The natural grading by degree on $\mathcal{U}$ induces via v a grading on S such that* $\deg(x_i) = 1$.

(2) *The map $s \mapsto sv(\mu)$ for a given $\mu \in \mathcal{U}$ induces a bijection between S and $\{v(a\mu) \mid a \in \mathcal{U}\}$.*

(3) *S is right cancellative.*

(4) *S is a quotient of $\langle X \rangle$ by $n/(n-1)/2$ different relations in degree* 2 *given by*

$$x_{u_i, j} x_{1, i} = x_{u_j, i} x_{1, j}, \qquad j > i.$$

If $\sigma \in \mathrm{Sym}_n$ then we extend σ to $\mathcal{U}$ via

$$\sigma(u_{i_1} \cdots u_{i_p}) = u_{\sigma i_1} \cdots u_{\sigma i_p}.$$

LEMMA 4.2. *Every bijection $w : \mathcal{U} \to S$, satisfying* (1.1), *is of the form* $v \circ \sigma$, $\sigma \in \mathrm{Sym}_n$.

Proof. Clearly there exist $\sigma \in \mathrm{Sym}_n$ such that w and $v \circ \sigma$ take the same values on $\{u_1, \dots, u_n\}$. Hence to prove the lemma we have to show that a map v satisfying (1.1) is uniquely determined by the values it takes on $\{u_1, \dots, u_n\}$. This was part of the proof of Theorem 1.3. ∎

Now we want to develop some kind of calculus for semigroups of I-type. Consider the arrows

$$\begin{array}{ccc} S & \xrightarrow{s \mapsto sv(b)} & \{v(ab) \mid b \in \mathscr{U}\} \\ & & \Big\uparrow \begin{smallmatrix} v(ab) \\ \uparrow \\ b \end{smallmatrix} \\ & & \mathscr{U} \end{array}$$

It is clear that the vertical map is a bijection and so is the horizontal map by Lemma 4.1. Thus we may define a bijection $w : \mathscr{U} \to S$ which makes (4.1) commutative. Furthermore w obviously satisfies (1.1), so according to Lemma 4.2, $w = v \circ \phi(b)$ where $\phi(b) \in \mathrm{Sym}_n$. We view ϕ as a map from $\mathscr{U}$ to Sym_n. Expressing the fact that w completes (4.1) to a commutative diagram yields

$$v(ab) = v(\phi(b)(a))v(b). \tag{4.2}$$

If we now compute $v(abc)$ in two ways we find

$$v(abc) = v(\phi(\phi(c)(b))(\phi(c)(a)))v(\phi(c)(b))v(c)$$

and

$$v(abc) = v(\phi(bc)(a))v(\phi(c)(b))v(c).$$

Using the fact that S is right cancellative we obtain

$$\phi(\phi(c)(b))(\phi(c)(a)) = \phi(bc)(a)$$

or put differently

$$(\phi(\phi(c)(b)) \circ \phi(c))(a) = \phi(bc)(a).$$

Since this is true for all a we obtain

$$\phi(bc) = \phi(\phi(c)(b)) \circ \phi(c). \tag{4.3}$$

Let us define $\ker \phi$, $\operatorname{im} \phi$ in the usual way (even though ϕ is clearly not a semigroup homomorphism),

$$\ker \phi = \{a \in \mathscr{U} \mid \phi(a) = \mathrm{id}\}$$
$$\operatorname{im} \phi = \{\phi(a) \mid a \in \mathscr{U}\}.$$

To simplify the notation we put $P = \ker \phi$, $G = \operatorname{im} \phi$.

Then (4.2), (4.3) yield the following lemma.

LEMMA 4.3. (1) *If* $b \in P$ *then*

$$\phi(ab) = \phi(a) \tag{4.4}$$

$$v(ab) = v(a)v(b). \tag{4.5}$$

(2) *P is a saturated subsemigroup of $\mathscr{U}$ $(a \in P \Rightarrow (ab \in P \Leftrightarrow b \in P))$.*

(3) *G is a subgroup of* Sym_n (*note that a finite subsemigroup of a group is itself a group*).

(4) *If $b \in G$ and $a \in P$ then $b(a) \in P$.*

LEMMA 4.4. *There exist $t_1, \dots, t_n > 0$ such that $u_i^{t_i} \in P$.*

Proof. Since Sym_n is finite there exist $r_i < s_i$ such that

$$\phi(u_i^{r_i}) = \phi(u_i^{s_i}). \tag{4.6}$$

Put $a = \prod_i u_i^{r_i}$, $t_i' = s_i - r_i$.

Now if $\phi(p) = \phi(q)$ then (4.3) implies that $\phi(rp) = \phi(rq)$. Applying this with $p = u^{r_i}$, $q = u^{s_i}$, and $r = \prod_{j \neq i} u_j^{r_i}$ yields $\phi(a) = \phi(au^{t_i'}) = \phi(\phi(a)(u_i^{t_i}))\phi(a)$ and thus

$$\phi(a)(u_i)^{t_i'} \in \ker \phi.$$

Now $\phi(a)(u_i) = u_{\phi(a)(i)}$ so if we put $t_i = t'_{\phi(a)(i)}$ then $\phi(u_i^{t_i}) = \mathrm{id}$. ∎

COROLLARY 4.5. *Let P_0 be the subsemigroup of $\mathscr{U}$ generated by $u_i^{t_i}$. Then*

(1) *$v(P_0)$ is a free abelian subsemigroup of S, generated by $v(u_i^{t_i})$.*

(2) $S = \bigcup_a v(a)v(P_0)$.

where the union runs over those $a = u_1^{p_1} \cdots u_n^{p_n}$ with $0 \leq p_i \leq t_i - 1$.

Proof. The corresponding statements for $\mathscr{U}$ are obvious. To obtain them for S one applies v and uses (4.5). ∎

Proof of Theorem 1.4. This is entirely similar to the proof of [6, Theorems 1.1, 1.2] so we content ourselves with a quick sketch. Note that by [6, Corollary 3.6] an algebra of I-type is automatically Koszul and has finite global dimension, so we only have to prove Theorem 1.4(3)–(6).

Note that the equations of $k_c S$ are given by $x_{u_i, j} x_{1,i} = d_{ij} x_{u_j, i} x_{1,j}$ for some $d_{ij} \in k^*$. We first assume that the $(d_{ij})_{ij}$ are roots of unity. Then (using (4.5)) we can take P_0 so small that $v(P_0)$ is commutative in $k_c S$. Thus by Corollary 4.5, $k_c S$ is finite on the left over a commutative ring, and hence is PI. This proves in particular (6) and using the same results of

Stafford and Zhang [5] as in the proof of [6, Theorem 1.1] also yields (2)–(5) in this case.

The general case is now proved using reduction to a finite field as in [6]. ∎

5. PROOF OF THEOREM 1.6

In this section we use the same notations and assumptions as in the previous sections.

Since S is cancellative (Corollary 1.5) and has subexponential growth it is (left and right) Öre. For an Öre semigroup T denote by $\overline{T}$ its quotient group.

We now extend v, ϕ to maps

$$\bar{v} : \overline{\mathcal{U}} \to \overline{S} : up^{-1} \mapsto v(u)v(p)^{-1}$$
$$\overline{\phi} : \overline{\mathcal{U}} \to \mathrm{Sym}_n : up^{-1} \mapsto \phi(u)\phi(p)^{-1},$$

where $p \in P$. This is well defined because of (4.4), (4.5) and the fact that it is clear from Lemma 4.4 that every element of $\overline{\mathcal{U}}$ can be written as up^{-1}, $p \in P_0 \subset P$.

LEMMA 5.1. (1) *If $s \in S$ then there exists $t \in S$ such that $ts \in v(P)$, $st \in v(P)$.*

(2) *$\bar{v}$ is a bijection.*

Proof. (1) Assume $t = v(c)$. We have to find $b \in \mathcal{U}$ such that

$$\phi\big(v^{-1}(v(b)v(c))\big) = \phi(b)\phi(c) = \mathrm{id}$$
$$\phi\big(v^{-1}(v(c)v(b))\big) = \phi(c)\phi(b) = \mathrm{id}.$$

It is clear that this is possible since im ϕ is a group.

(2) It is easy to see that $\bar{v}$ is an injection, and from (1) we deduce that it is also a surjection. ∎

One verifies that $\bar{v}$ satisfies (1.1) and it is also clear ker $\overline{\phi}$, im $\overline{\phi}$ have the same properties as ker ϕ, im ϕ (Lemma 4.4). Furthermore ker $\overline{\phi}$ is now actually a group and im $\overline{\phi}$ = im ϕ. We deduce the following slight strengthening of Lemma 4.4 (and generalization of [3]) which is however not needed in the sequel.

PROPOSITION 5.2. *For all* i, $u_i^{n!} \in \ker \phi$.

Proof. Let p be the smallest positive integer such that $u_i^p \in \ker \phi$. Then p divides $|\overline{\mathscr{U}}/\ker \overline{\phi}|$ Now $\overline{\phi}$ defines a bijection (*not* a group homomorphism) between $\overline{\mathscr{U}}/\ker \overline{\phi}$ and $\operatorname{im} \overline{\phi}$. Thus p divides $|\operatorname{im} \overline{\phi}|$ which in turn divides $|\mathrm{Sym}_n| = n!$ ∎

$\overline{S}$ acts on itself by right and left multiplication. If we transport this action to $\mathscr{U}$ through v we obtain *commuting* left and right actions of $\overline{S}$ on $\overline{\mathscr{U}}$ given by the formulas

$$\forall a \in \overline{S}, b \in \overline{\mathscr{U}}, \qquad a \cdot b = \overline{v}^{-1}(a\overline{v}(b)) \tag{5.1}$$

$$\forall a \in \overline{\mathscr{U}}, b \in \overline{S}, \qquad a \cdot b = \overline{v}^{-1}(\overline{v}(a)b). \tag{5.2}$$

In the previous sections we have concentrated on the action (5.1). Now we will say something about the action (5.2).

Using (4.2) we deduce that for $a \in \overline{\mathscr{U}}$, $b \in \overline{S}$,

$$a \cdot b = \overline{\phi}\left(\overline{v}^{-1}(b)\right)^{-1}(a)\overline{v}^{-1}(b).$$

Proof of Theorem 1.6. By permuting the x_i we may and we will assume that $v(u_i) = x_i$. Consider the map

$$\psi : \mathbb{Z}^n \to \overline{\mathscr{U}} : (a_1, \dots, a_n) \mapsto u_1^{a_1} \cdots u_n^{a_n}.$$

For $a \in \mathbb{Z}^n$, $b \in \overline{S}$ we write

$$a \cdot b = \psi^{-1}(\psi(a) \cdot b)$$

and we put $\tilde{\phi}(c) = \phi(c) \circ \psi$, $\tilde{\phi}_i = \tilde{\phi}(u_i)$. We find for $(a_1, \dots, a_n) \in \mathbb{Z}^n$,

$$(a_1, \dots, a_n) \cdot x_i = \left(a_{\tilde{\phi}_i(1)}, \dots, a_{\tilde{\phi}_i(i)} + 1, \dots, a_{\tilde{\phi}_i(n)}\right). \tag{5.3}$$

We conclude that $(x_i)_i$, and hence all of $\overline{S}$ acts on the right of $\mathbb{Z}^n$ by Euclidean transformations. Keeping the formula (5.3) we can extend this action to an action on $\mathbb{R}^n$ and it is then clear that $[0,1[^n$ is a fundamental domain. Furthermore if the action were not free then there would be a fixed point $(a_1, \dots, a_n) \in \mathbb{R}^n$ for some element s of $\overline{S}$. But then $(\lfloor a_1 \rfloor, \dots, \lfloor a_n \rfloor) \in \mathbb{Z}^n$ is also a fixed point for s. This is impossible since by construction the action of $\overline{S}$ on $\mathscr{U}$ and hence on $\mathbb{Z}^n$ is free. ∎

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