Linear Matrix Equations From an Inverse Problem of Vibration Theory

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ABSTRACT

The symmetric, positive semidefinite, and positive definite real solutions of matrix equations $A^T X A = D$ and $(A^T X A, X A - Y A D) = (D, 0)$ are considered. Necessary and sufficient conditions for the existence of such solutions and their general forms are derived using the singular value decomposition. The theory is motivated and illustrated with a problem of vibration theory.

1. INTRODUCTION

Let $\mathcal{O}_n$, $\mathcal{S}_n$, $\mathcal{SP}_n$ denote the set of all orthogonal matrices in $\mathbb{R}^{n \times n}$, the set of all symmetric matrices in $\mathbb{R}^{n \times n}$, and the set of all symmetric positive semidefinite (definite) matrices in $\mathbb{R}^{n \times n}$. Throughout the paper definite and semidefinite matrices are assumed to be real and symmetric. The notation $A \geq 0$ ($A > 0$) means that $A$ is positive semidef-
nite (definite). A generalized inverse of $A$, denoted by $A^-$, is a matrix which satisfies the equation $AA^-A = A$, and $A^+$ stands for the Moore-Penrose generalized inverse of the matrix $A$.

An inverse problem [3, 4, 7, 10, 16, 17] arising in structural modification of the dynamic behaviour of a structure calls for solution of the matrix equations

\begin{align*}
ATXA &= D, \quad (1.1) \\
ATXA &= D, \quad XA = YAD, \quad (1.2)
\end{align*}

where $A \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times m}$, and the unknowns $X$ and $Y$ are required to be real and symmetric, and positive semidefinite or possibly definite. No assumption is made about the relative sizes of $m$ and $n$, and it is assumed throughout that $A \neq 0$ and $D \neq 0$.

Equation (1.2) arises in the following way. Let $X$ and $Y$ be the stiffness and mass matrices, respectively, for a vibrating system with $n$ degrees of freedom. Thus, $X \geq 0$ and $Y > 0$ are generally satisfied, and natural frequencies are determined by eigenvalues $d_1, d_2, \ldots, d_n$ of the pencil $\lambda Y - X$. The corresponding real eigenvectors $a_1, \ldots, a_n$ determine modes of vibration. Given spectral data $d_1, \ldots, d_m$ and $a_1, \ldots, a_m (m \leq n)$, form the matrices

\begin{align*}
D &= \text{diag}[d_1, \ldots, d_m], \quad A = [a_1, a_2, \ldots, a_m]. \quad (1.3)
\end{align*}

Then we have $YAD =XA$ [as in (1.2)], and the eigenvectors can be normalized so that $ATYA = I$, in which case $ATXA = D$, the second equation of (1.2). An inverse problem of some practical interest is now: Given the spectral data summarized in matrices $D$ and $A$, find matrices $X \geq 0, Y > 0$ satisfying the Equations (1.2).

The Equation (1.1) is a special case of the matrix equation

\begin{align*}
AXB &= C. \quad (1.4)
\end{align*}

Consistency conditions for Equation (1.4) were given by Penrose [15] (see also [2]). When the equation is consistent, a solution can be obtained using generalized inverses. Khatri and Mitra [12] gave necessary and sufficient conditions for the existence of symmetric and positive semidefinite solutions as well as explicit formulae using generalized inverses. Solutions of the single equation $XA = YAD$ with $X^T = X$ and $Y > 0$ were considered by Sun in the case that $A$ has full rank (see Problem SIEP of [16]). In [5, 6, 8] solvability conditions for symmetric and positive definite solutions and general solutions of Equation (1.4) were obtained through the use of the generalized singular value decomposition [11, 14].
The normalization condition $A^T Y A = I_m$ may seem more natural than Equation (1.1). However, $A^T Y A = I_m$ would imply that rank $A = m$ ($A$ has "full rank"), a condition that we wish to avoid. When (1.1) is consistent, it is clear that the data of (1.3) must satisfy rank $D \leq$ rank $A$. Thus, redundancy in the columns of $A$ must be accommodated by zero eigenvalues of $D$. Note also that if $D > 0$ and rank $A = m$, then the two normalizations are equivalent.

In Section 2, the singular value decomposition (SVD) will be used to investigate Equation (1.1), and a simple and clear exposition, in terms of solvability conditions for symmetric, positive semidefinite, and positive definite solutions, will be given. In Section 3 the least squares approach to Equation (1.1) is considered.

In Section 4 we consider common solution pairs $(X, Y)$ of Equation (1.2) that are symmetric, positive semidefinite, or positive definite. We give necessary and sufficient conditions for the existence of such solutions and their general forms using the SVD. Section 5 contains a simple special case concerning vibrating systems. It is shown how a system can be formed for which some eigenvalues and just a few eigenvectors are prescribed, i.e., $m < n$.

2. SOLUTIONS OF $A^T X A = D$

To study the solvability of Equation (1.1) with symmetric, positive semidefinite, and positive definite conditions on the solution, we decompose the given matrix $A$ by the SVD:

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$  

(2.1)

where $U = [U_1, U_2] \in \mathbb{O} \mathbb{R}^{n \times n}$, $V = [V_1, V_2] \in \mathbb{O} \mathbb{R}^{m \times m}$, $\Sigma = \text{diag} [\sigma_1, \ldots, \sigma_r]$, $\sigma_i > 0 \ (i = 1, \ldots, r)$, $r = \text{rank} A$, $U_1 \in \mathbb{R}^{n \times r}$, $V_1 \in \mathbb{R}^{m \times r}$.

Notice that a necessary condition for existence of a solution of $A^T X A = D$ is that $Ax = 0$ implies $Dx = 0$, i.e., that the kernel, or nullspace, of $A$ is contained in that of $D$. Since the kernel of $A$ is just the column space of $V_2$, this necessary condition is concisely expressed as $DV_2 = 0$. It will be convenient to write

$$X_0 = \Sigma^{-1} V_1^T D V_1 \Sigma^{-1}.$$  

(2.2)

**Theorem 2.1.** Let a SVD of the matrix $A$ be (2.1). Equation (1.1) has a symmetric solution if and only if

$$D^T = D, \quad DV_2 = 0,$$  

(2.3)
in which case the general symmetric solution is
\begin{equation}
X = U \begin{bmatrix}
X_0 & X_{12} \\
X_{12}^T & X_{22}
\end{bmatrix} U^T,
\end{equation}
where $X_{12}$ is an arbitrary $r \times (n - r)$ matrix, and $X_{22}$ is an arbitrary $(n - r) \times (n - r)$ symmetric matrix.

**Proof.** Let $X$ be a symmetric solution of Equation (1.1). Then $D = A^T X A$ is clearly symmetric. The necessity of the condition $D U = 0$ has been noted above.

Conversely, observe first that Equation (1.1) is equivalent to
\begin{equation}
\begin{bmatrix}
\Sigma & 0 \\
0 & 0
\end{bmatrix} U^T X U \begin{bmatrix}
\Sigma & 0 \\
0 & 0
\end{bmatrix} = V^T D V.
\end{equation}
Given the conditions (2.3), we see that
\[V^T D V = \begin{bmatrix}
V_1^T D V_1 & 0 \\
0 & 0
\end{bmatrix},\]
and it is readily verified that any matrix $X$ of the form (2.4) is a solution of (2.5).

To see that all solutions have this form, note first that the number of free parameters in (2.4) is
\[r(n - r) + \frac{1}{2}(n - r)(n - r + 1) = \frac{1}{2}(n - r)(n + r + 1). \quad (2.6)\]
Then, since $A$ has rank $r$, the equation $A^T X A = D$ represents $\frac{1}{2}r(r + 1)$ independent linear equations in the $\frac{1}{2}n(n + 1)$ unknowns of the symmetric matrix $X$. Thus, the number of parameters in a general solution is
\[\frac{1}{2}n(n + 1) - \frac{1}{2}r(r + 1) = \frac{1}{2}(n - r)(n + r + 1).\]
Comparing with (2.5), we deduce that (2.4) does indeed give the general solution.

In searching among matrices $X$ of (2.4) for (semi)definite matrices we will need:

**Lemma 2.1** [1]. **Suppose that a real symmetric matrix is partitioned as**
\[
\begin{bmatrix}
E & F \\
F^T & G
\end{bmatrix},
\]
(a) This matrix is symmetric positive semidefinite if and only if
\[ E \geq 0, \quad G - F^T E^+ F \geq 0, \quad EE^+ F = F. \tag{2.7} \]

(b) This matrix is symmetric positive definite if and only if
\[ E > 0, \quad G - F^T E^{-1} F > 0. \tag{2.8} \]

**Lemma 2.2 [2].** Let \( E \in \mathbb{R}^{m \times n}, F \in \mathbb{R}^{p \times q}, \) and \( G \in \mathbb{R}^{m \times q}. \) Then the matrix equation \( E X F = G \) is consistent if and only if for some \( E^- \) and \( F^- \), we have \( EE^- GF^- F = G. \) In this case the general solution of \( E X F = G \) is \( X = E^- GF^- + Z - E^- EZFF^- \) for arbitrary \( Z \in \mathbb{R}^{m \times p}. \)

Let us introduce the notation
\[ E := I - X_0 X_0^+ = I - X_0^+ X_0, \tag{2.9} \]
where \( X_0 \) is defined in Equation (2.2). Recall that \( X_0^+ \) is the Moore-Penrose generalized inverse of \( X_0 \), so that \( X_0 X_0^+ \) is the orthogonal projection onto \( \text{Im} \ X_0 \) and \( E \) is the orthogonal projection onto \( (\text{Im} \ X_0)^\perp = \text{Ker} \ X_0. \) Also, \( E^- E \) is a projection along \( \text{Ker} \ E = \text{Im} \ X_0, \) so that
\[ P := I - E^- E \tag{2.10} \]
is a (possible skew) projection along \( \text{Ker} \ E = \text{Im} \ X_0. \)

**Theorem 2.2.** Let a SVD of the matrix \( A \) be (2.1).

(a) Equation (1.1) has a symmetric positive semidefinite solution if and only if
\[ D \geq 0, \quad DV_2 = 0, \tag{2.11} \]
in which case the general symmetric positive semidefinite solution is
\[ X = U \begin{bmatrix} X_0 & PZ_{12} \\ (PZ_{12})^T & Z_{12}^T P^T X_0^+ PZ_{12} + Z_{22} \end{bmatrix} U^T, \tag{2.12} \]
where \( Z_{12} \in \mathbb{R}^{r \times (n-r)} \) and \( Z_{22} \in \mathbb{SPR}^{(n-r) \times (n-r)} \) are arbitrary.

(b) Equation (1.1) has a symmetric positive definite solution if and only if
\[ D^T = D, \quad V_1^T D V_1 = D_{11} > 0, \quad DV_2 = 0, \tag{2.13} \]
in which case the general symmetric positive definite solution is

\[ X = U \begin{bmatrix} X_0 & X_{12} \\ X_{12}^T & X_{12}^T X_0^{-1} X_{12} + Z_{22} \end{bmatrix} U^T, \tag{2.14} \]

where \( X_{12} \in \mathbb{R}^{r \times (n-r)} \) and \( Z_{22} \in \mathbb{S}^{(n-r) \times (n-r)}_+ \) are arbitrary.

**Proof.**

(a) The necessity of the conditions (2.11) is apparent. For sufficiency, we construct a symmetric positive semidefinite solution of Equation (1.1). From \( D \geq 0 \) it follows that \( X_0 \geq 0 \). Since \( X \) is required to be symmetric positive semidefinite, it follows from Lemma 2.1(a) and Equation (2.4) that \( X_{12} \) and \( X_{22} \) must satisfy

\[ X_{22} - X_{12}^T X_0^+ X_{12} \in \mathbb{S}^{(n-r) \times (n-r)}_+, \tag{2.15} \]

\[ X_0 X_0^+ X_{12} = X_{12}. \]

From Lemma 2.2 we obtain the general solution \( X_{12} \) of the last equation in the form

\[ X_{12} = (I - E \ E) Z_{12} = P Z_{12}, \tag{2.16} \]

where \( Z_{12} \) is an arbitrary \( r \times (n - r) \) real matrix.

Letting

\[ X_{22} - X_{12}^T X_0^+ X_{12} = Z_{22}, \tag{2.17} \]

where \( Z_{22} \) is an arbitrary \( (n-r) \times (n-r) \) symmetric positive semidefinite matrix, we have

\[ X_{22} = Z_{12}^T P^T X_0^+ P Z_{12} + Z_{22}. \tag{2.18} \]

Substituting (2.16) and (2.18) in (2.4), we get the general symmetric positive semidefinite solution (2.12) of Equation (1.1).

(b) If \( X \) is a symmetric positive definite solution of Equation (1.1), then \( D = A^T X A \) is symmetric. Using (2.1), we have \( D_{11} = V_1^T D V_1 = \Sigma U_1^T X U_1 \Sigma > 0 \) and \( D V_2 = 0 \). Hence (2.13) holds.

Conversely, from \( D_{11} > 0 \) we obtain \( X_{11} = X_0 > 0 \). Since \( X \) is required to be symmetric positive definite, it follows from Lemma 2.1(b) and Equation (2.4) that \( X_{12} \) and \( X_{22} \) must satisfy

\[ X_{22} - X_{12}^T X_0^{-1} X_{12} > 0. \tag{2.19} \]
Letting \( X_{22} - X_{12}^T X_0^{-1} X_{12} = Z_{22} \) where \( Z_{22} \in \mathbb{S}_{\mathbb{R}^{(n-r) \times (n-r)}} \) is arbitrary, we obtain the general symmetric positive definite solution (2.14) of Equation (1.1).

3. THE INCONSISTENT EQUATION \( A^T X A = D \)

If the consistency and symmetry conditions (2.3) are not satisfied, a least squares approach can be used, i.e., minimize \( \| A^T X A - D \|_F \) with respect to \( X \) subject to the linear constraint \( X^T = X \). Write \( U^T X U \) in the partitioned form \( X_{ij} \) with \( X_{11} \) of size \( r \times r \), and let \( D_{ij} = V_i^T D V_j \), \( i,j = 1,2 \). Then it follows that

$$
\| A^T X A - D \|_F = \left\| \begin{bmatrix} \Sigma X_{11} & -D_{12} \\ -D_{21} & -D_{22} \end{bmatrix} \right\|_F.
$$

(3.1)

It is well known that the nearest symmetric matrix to any given matrix \( P \) in the \( F \)-norm is \( \frac{1}{2}(P^T + P) \), and it follows that \( \| A^T X A - D \|_F \) is minimized with respect to symmetric matrices \( X \) by taking

$$
X_{11} = \frac{1}{2} \Sigma^{-1} (D_{11}^T + D_{11}) \Sigma^{-1}.
$$

(3.2)

It follows that the general solution to this constrained minimization problem is given by

$$
X = U \begin{bmatrix} \frac{1}{2} \Sigma^{-1} (D_{11}^T + D_{11}) \Sigma^{-1} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} U^T,
$$

(3.3)

where \( X_{12} \in \mathbb{R}^{r \times (n-r)} \) and \( X_{22} \in \mathbb{S}_{\mathbb{R}^{(n-r) \times (n-r)}} \) are arbitrary.

Clearly the minimizing symmetric \( \hat{X}_0 \) of least \( F \)-norm is obtained by putting \( X_{12} = 0 \) and \( X_{22} = 0 \) in (3.3). When \( D^T = D \) this is confirmed by writing \( \hat{X}_0 = (A^T)^+ DA^+ \) (see [9], for example) and using the fact that

$$
A^+ = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T.
$$

(3.4)

Now let us consider the best approximate solution of \( A^T X A = D \) when \( X \) is constrained to be positive semidefinite. We take advantage of a result of Higham [13]:

**Lemma 3.1.** Let \( P \in \mathbb{R}^{n \times n} \), \( S = \frac{1}{2}(P^T + P) \), and let \( S = QH \) be a polar decomposition \( (Q \in \mathbb{O}_{\mathbb{R}^{n \times n}}, H \geq 0) \). Then \( \frac{1}{2}(S + H) \) is the nearest positive semidefinite matrix to \( P \) in the \( F \)-norm.
It follows from this result and Equation (3.1) that \(\|A^TXA - D\|_F\) is minimized over positive semidefinite matrices \(X\) by taking

\[X_{11} = \frac{1}{2}\Sigma^{-1}(S_1 + H_1)\Sigma^{-1},\]  

(3.4)

where \(S_1 = \frac{1}{2}(D_{11}^T + D_{11})\) and \(S_1 = Q_1H_1\) is a polar decomposition.

As in the proof of Theorem 2.2(a), we obtain the general least squares symmetric positive semidefinite solution of the inconsistent equation \(A^TXA = D\) in the form

\[X = U \begin{bmatrix} X_{11} & P_1Z_{12} \\ (P_1Z_{12})^T & Z_{12}^TP_1^TX_{11}^+P_1Z_{12} + Z_{22} \end{bmatrix} U^T,\]  

(3.5)

where \(X_{11}\) is given by (3.4), \(E_1 = I - X_{11}X_{11}^+\), \(P_1 = I - E_1^+E_1\), and \(Z_{12} \in \mathbb{R}^{r \times (n-r)}\), \(Z_{22} \in \mathbb{SPR}^{(n-r) \times (n-r)}\) are arbitrary.

Generally speaking, there is no nearest symmetric positive definite matrix in \(F\)-norm to an arbitrary real matrix. So we do not consider the least squares symmetric positive definite solution of the inconsistent equation \(A^TXA = D\).

4. SOLUTIONS OF \((A^TXA, XA - YAD) = (D, 0)\)

We now consider the problem of finding solution pairs \((X, Y)\) for consistent equations of the form (1.2), where \(X\) and \(Y\) are real symmetric positive semidefinite or positive definite. Let us first introduce some more notation. Consider first a spectral decomposition of the \(r \times r\) matrix \(D_{11} = V_1^TDV_1\).

We write

\[D_{11} = QAQ^T,\]  

(4.1)

where \(Q\) is an \(r \times r\) orthogonal matrix, \(\Lambda = \text{diag}[\Lambda_1, 0]\), and \(\Lambda_1\) is an \(m_1 \times m_1\) nonsingular diagonal matrix \((m_1 \leq r)\). Then define \(r \times r\) matrices

\[Y_0 := \Sigma^{-1}Q \text{diag}[I_{m_1}, C] Q^T \Sigma^{-1},\]  

(4.2)

where \(C\) is any symmetric matrix of size \(m_2 = r - m_1\).

In terms of \(Y_0\) define the orthogonal projection

\[F = I - Y_0Y_0^+ = I - Y_0^+Y_0\]  

(4.3)

onto Ker \(Y_0\) and then a (generally skew) projection

\[P = I - EE^-\]  

(4.4)
which is onto some complementary subspace of \( \text{Ker} \ Y_0 \) in \( \mathbb{R}^r \). (In particular, if \( D_{11} \) is nonsingular, or if \( D_{11} \) is singular and \( C \) is nonsingular, then \( E = 0 \) and \( P = I \).)

**Lemma 4.1.** The equation \( XD_{11} = D_{11} \) has symmetric solutions \( X \) and the general solution can be written in the form

\[
X = Q \text{diag}[I_{m_1}, C] Q^T,
\]

where \( Q \) is given by (4.1), and \( C \) is an arbitrary symmetric matrix of size \( m_2 = r - m_1 \).

**Proof.** This is a special case of Theorem 1 of [6] using the spectral decomposition (4.1).

**Theorem 4.1.** Let a SVD of the matrix \( A \) be (2.1). The Equations (1.2) have solution pairs \((X = X^T, Y = Y^T), (X = X^T, Y \geq 0), \) and \((X = X^T, Y > 0)\) if and only if

\[
D^T = D, \quad DV_2 = 0.
\]

If the conditions (4.6) are satisfied, then:

(a) The general solution pairs \((X = X^T, Y = Y^T)\) are

\[
X = U \begin{bmatrix} X_0 & (Y_{21} \Sigma^2 X_0)^T \\ Y_{21} \Sigma^2 X_0 & X_{22} \end{bmatrix} U^T,
\]

\[
Y = U \begin{bmatrix} Y_0 & Y_{21}^T \\ Y_{21} & Y_{22} \end{bmatrix} U^T,
\]

where \( X_{22}, Y_{22} \in \mathbb{S}_{n-r} \times (n-r), C \in \mathbb{S}_{m_2 \times m_2} \) (in (4.2)), and \( Y_{21} \in \mathbb{R}^{(n-r) \times r} \) are arbitrary.

(b) The general solution pairs \((X = X^T, Y \geq 0)\) are

\[
X = U \begin{bmatrix} X_0 & (Z_{21} P \Sigma^2 X_0)^T \\ Z_{21} P \Sigma^2 X_0 & X_{22} \end{bmatrix} U^T,
\]

\[
Y = U \begin{bmatrix} Y_0 & (Z_{21} P)^T \\ Z_{21} P & Z_{21} P Y_0^+ P^T Z_{21}^T + M_{22} \end{bmatrix} U^T,
\]

where \( X_{22} \in \mathbb{S}_{n-r} \times (n-r), M_{22} \in \mathbb{S}_{(n-r) \times (n-r)}, C \in \mathbb{S}_{m_2 \times m_2} \) (in (4.2)), and \( Y_{21} \in \mathbb{R}^{(n-r) \times r} \) are arbitrary.
(c) The general solution pairs \((X = X^T, Y > 0)\) are

\[
X = U \begin{bmatrix} X_0 & (Y_{21} \Sigma^2 X_0)^T \\ Y_{21} \Sigma^2 X_0 & X_{22} \end{bmatrix} U^T,
\]

\[
Y = U \begin{bmatrix} Y_0 & Y_{21}^T \\ Y_{21} & Y_{21} Y_0^{-1} Y_{21}^T + M_{22} \end{bmatrix} U^T,
\] (4.9)

where \(X_{22} \in \mathbb{S}_+(n-r) \times (n-r), M_{22} \in \mathbb{S}_{n-r} \times (n-r), C \in \mathbb{S}_{m_2 \times m_2}^+(\text{in } (4.2)), \) and \(Y_{21} \in \mathbb{R}^{(n-r) \times r}\) are arbitrary.

Proof. The proof of solvability under the conditions (4.6) is similar to that of Theorem 2.1.

(a) We first construct solution pairs \((X = X^T, Y = Y^T)\) of Equation (1.2). Let

\[
U^T X U = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \quad U^T Y U = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix},
\] (4.10)

where \(X_{11}, Y_{11} \in \mathbb{R}^{r \times r}\). Equation (1.2) is equivalent to

\[
\begin{bmatrix} \Sigma X_{11} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},
\]

\[
\begin{bmatrix} X_{11} \Sigma & 0 \\ X_{21} \Sigma & 0 \end{bmatrix} = \begin{bmatrix} Y_{11} \Sigma D_{11} & Y_{11} \Sigma D_{12} \\ Y_{21} \Sigma D_{11} & Y_{21} \Sigma D_{12} \end{bmatrix}.
\] (4.11)

Since \(D^T = D, DV_2 = 0\), then \(D_{11} - V_1^T D V_1 \in \mathbb{S}_+^{r \times r}, D_{12} = V_1^T D V_2 = 0, \) and \(D_{21} = D_{21}^T = 0, \) and \(D_{22} = V_2^T D V_2 = 0\). The Equations (4.11) are then equivalent to the following matrix equations:

\[
\Sigma X_{11} \Sigma = D_{11},
\]

\[
X_{11} \Sigma = Y_{11} \Sigma D_{11},
\]

\[
X_{21} \Sigma = Y_{21} \Sigma D_{11}.
\] (4.12) (4.13) (4.14)

Then we have \(X_{11} = \Sigma^{-1} D_{11} \Sigma^{-1} = X_0\) and using this in (4.13) we obtain

\[
\Sigma Y_{11} \Sigma D_{11} = D_{11}.
\] (4.15)
Letting $\Sigma Y_{11} \Sigma = \hat{Y}_{11}$, we have $\hat{Y}_{11} D_{11} = D_{11}$. It follows from Lemma 4.1 that this equation has a symmetric solution $\hat{Y}_{11}$ and the general symmetric solution has the form

$$\hat{Y}_{11} = Q \text{diag}[I, C] Q^T,$$

where $C \in \mathbb{S}^{m_2 \times m_2}$ is arbitrary. Then [see the Definition (4.2)]

$$Y_{11} = \Sigma^{-1} Q \text{diag}[I, C] Q^T \Sigma^{-1} = Y_0. \quad (4.16)$$

Let $Y_{21}$ be an arbitrary $(n-r) \times r$ real matrix. From (4.14), we have

$$X_{21} = Y_{21} \Sigma D_{11} \Sigma^{-1} = Y_{21} \Sigma^2 X_0. \quad (4.17)$$

Since $X$ and $Y$ are required to be symmetric, then

$$X_{12} = X_{21}^T, \quad Y_{12} = Y_{21}^T, \quad (4.18)$$

and $X_{22}, Y_{22}$ may be arbitrary $(n-r) \times (n-r)$ symmetric matrices. Substituting $X_{11} = X_0$ and (4.16)-(4.18) in (4.10), we obtain the general solution pairs $(X = X^T, Y = Y^T)$ of (4.7) for Equation (1.2).

(b) If $Y$ is required to be symmetric positive semidefinite, then it follows from Lemma 2.1(a) that $Y_{ij} (i, j = 1, 2)$ must satisfy

$$Y_{11} \geq 0, \quad Y_{12} = Y_{21}^T, \quad Y_{22} - Y_{21} Y_{11} Y_{21}^T \geq 0, \quad Y_{11} Y_{12} Y_{12} = Y_{12}. \quad (4.19)$$

From (4.16), (4.18), and the definition of the Moore-Penrose generalized inverse, $Y_{11} Y_{12} Y_{12} = Y_{12}$ is equivalent to

$$Y_{21} E = 0. \quad (4.20)$$

From Lemma 2.2 and (4.4) we obtain the general solution of (4.20) in the form

$$Y_{21} = Z_{21} (I - EE^-) = Z_{21} P, \quad (4.21)$$

where $Z_{21} \in \mathbb{R}^{(n-r) \times r}$ is arbitrary.

Letting $C \in \mathbb{S}^{m_2 \times m_2}$ and $Y_{22} - Y_{21} Y_{11} Y_{21}^T = M_{22}$, where $M_{22}$ is an arbitrary $(n-r) \times (n-r)$ symmetric positive semidefinite matrix, we have

$$Y_{11} \geq 0, \quad (4.22)$$

$$Y_{22} = Z_{21} P Y_{0}^+ P^T Z_{21}^T + M_{22}. \quad (4.22)$$
Substituting $X_{11} = X_0$, (4.16)-(4.18), (4.21), and (4.22) in (4.10), we obtain the general solution pairs $(X = X^T, Y \geq 0)$ of Equation (4.8) for Equation (1.2).

c) Using Lemma 2.1(b), we may easily prove that the general solution pairs $(X = X^T, Y > 0)$ of Equation (1.2) have the form (4.9).

**Theorem 4.2.** Let a SVD of the matrix $A$ be (2.1). Equation (1.2) has solution pairs $(X \geq 0, Y = Y^T)$, $(X \geq 0, Y \geq 0)$, and $(X \geq 0, Y > 0)$, if and only if

$$D \geq 0, \quad DV_2 = 0. \quad (4.23)$$

If the conditions (4.23) are satisfied, then:

(a) The general solution pairs $(X \geq 0, Y = Y^T)$ are

$$X = U \begin{bmatrix} X_0 & (Y_{21} \Sigma^2 X_0)^T \\ Y_{21} \Sigma^2 X_0 & Y_{21} \Sigma^2 X_0 \Sigma^2 Y_{21}^T + K_{22} \end{bmatrix} U^T,$$

$$Y = U \begin{bmatrix} Y_0 & Y_{21}^T \\ Y_{21} & Y_{22} \end{bmatrix} U^T,$$

where $K_{22} \in \mathbb{SPR}^{(n-r) \times (n-r)}$, $Y_{22} \in \mathbb{SR}^{(n-r) \times (n-r)}$, $C \in \mathbb{SPR}^{m_2 \times m_2}$ (in (4.2)), and $Y_{21} \in \mathbb{R}^{(n-r) \times r}$ are arbitrary.

(b) The general solution pairs $(X \geq 0, Y \geq 0)$ are

$$X = U \begin{bmatrix} X_0 & (Z_{21} P \Sigma^2 X_0)^T \\ Z_{21} P \Sigma^2 X_0 & Z_{21} P \Sigma^2 X_0 \Sigma^2 P^T Z_{21}^T + K_{22} \end{bmatrix} U^T,$$

$$Y = U \begin{bmatrix} Y_0 & (Z_{21} P)^T \\ Z_{21} P & Z_{21} P Y_0^+ P^T Z_{21}^T + M_{22} \end{bmatrix} U^T,$$

where $K_{22}, M_{22} \in \mathbb{SPR}^{(n-r) \times (n-r)}$, $C \in \mathbb{SPR}^{m_2 \times m_2}$ (in (4.2)), and $Z_{21} \in \mathbb{R}^{(n-r) \times r}$ are arbitrary.

(c) The general solution pairs $(X \geq 0, Y > 0)$ are

$$X = U \begin{bmatrix} X_0 & (Y_{21} \Sigma^2 X_0)^T \\ Y_{21} \Sigma^2 X_0 & Y_{21} \Sigma^2 X_0 \Sigma^2 Y_{21}^T + K_{22} \end{bmatrix} U^T,$$

$$Y = U \begin{bmatrix} Y_0 & Y_{21}^T \\ Y_{21} & Y_{21} Y_0^{-1} Y_{21}^T + M_{22} \end{bmatrix} U^T,$$

where $K_{22} \in \mathbb{SPR}^{(n-r) \times (n-r)}$, $M_{22} \in \mathbb{SPR}^{(n-r) \times (n-r)}$, $C \in \mathbb{SPR}_+^{m_2 \times m_2}$ (in (4.2)), and $Y_{21} \in \mathbb{R}^{(n-r) \times r}$ are arbitrary.
Proof. The proof of the solvability conditions (4.23) is similar to that of Theorem 2.2(a).

We now look for solution pairs \((X \geq 0, Y = Y^T)\) of Equation (1.2). Since \(X\) is required to be symmetric positive semidefinite, it follows from Lemma 2.1(a) that \(X_{ij} (i, j = 1, 2)\) in (4.10) must satisfy

\[
X_{11} \geq 0, \quad X_{12} = X_{22}^T, \quad X_{22} - X_{21}X_{11}^+X_{21} \geq 0, \quad X_{11}X_{11}^+X_{12} = X_{12}
\]

(4.27)

From (4.12) we obtain \(X_{11} = X_0 \geq 0\), and (4.14) implies that

\[
X_{11}X_{11}^+X_{12} = X_0X_0^+(Y_{21}^T\Sigma^2X_0)^T
\]

\[
= X_0\Sigma^2Y_{21}^T = X_{21}^T = X_{12}
\]

holds always for arbitrary \((n - r) \times r\) matrices \(Y_{21}\).

Letting \(X_{22} - X_{21}X_{11}^+X_{21}^T = K_{22}\), where \(K_{22} \in \mathbb{SP}^{(n-r)\times (n-r)}\) is arbitrary, we have

\[
X_{22} = Y_{21}^T\Sigma^2X_0X_0^+X_0\Sigma Y_{21}^T + K_{22}
\]

(4.28)

Substituting (4.16)–(4.18) and (4.28) in (4.10), we obtain general solution pairs \((X \geq 0, Y = Y^T)\) of the form (4.24) for Equation (1.2).

As in the proof of Theorem 4.1(b), (c), it follows from Lemma 2.1 and (4.24) that solution pairs \((X \geq 0, Y \geq 0)\) and \((X \geq 0, Y > 0)\) of Equation (1.2) are (4.25) and (4.26), respectively. □

**Theorem 4.3.** Let a SVD of the matrix \(A\) be (2.1). Equation (1.2) has solution pairs \((X \geq 0, Y = Y^T), (X > 0, Y \geq 0), \) and \((X > 0, Y > 0)\) if and only if

\[
D^T = D, \quad D_{11} = V_1^TDV_1 > 0, \quad DV_2 = 0.
\]

(4.29)

If the conditions (4.29) are satisfied, then:

(a) The general solution pairs \((X > 0, Y = Y^T)\) are

\[
X = U \begin{bmatrix}
X_0 & (Y_{21}\Sigma^2X_0)^T \\
Y_{21}\Sigma^2X_0 & Y_{21}\Sigma^2X_0\Sigma^2Y_{21}^T + K_{22}
\end{bmatrix} U^T,
\]

(4.30)

\[
Y = U \begin{bmatrix}
\Sigma^{-2} & Y_{21}^T \\
Y_{21} & Y_{22}
\end{bmatrix} U^T,
\]
where \( K_{22} \in \mathbb{SPR}_{+}^{(n-r)\times(n-r)} \), \( Y_{22} \in \mathbb{SR}^{(n-r)\times(n-r)} \), and \( Y_{21} \in \mathbb{R}^{(n-r)\times r} \) are arbitrary.

(b) The general solution pairs \((X > 0, Y \geq 0)\) are

\[
X = U \begin{bmatrix} X_0 & (Y_{21} \Sigma^2 X_0)^T \\ Y_{21} \Sigma^2 X_0 & Y_{21} \Sigma^2 X_0 \Sigma^2 Y_{21}^T + K_{22} \end{bmatrix} U^T,
\]

\[
Y = U \begin{bmatrix} \Sigma^{-2} & Y_{21}^T \\ Y_{21} \Sigma^2 Y_{21}^T + M_{22} \\ Y_{21} \Sigma^2 Y_{21}^T + M_{22} \end{bmatrix} U^T,
\]

where \( K_{22} \in \mathbb{SPR}_{+}^{(n-r)\times(n-r)} \), \( M_{22} \in \mathbb{SR}^{(n-r)\times(n-r)} \), and \( Y_{21} \in \mathbb{R}^{(n-r)\times r} \) are arbitrary.

(c) The general solution pairs \((X > 0, Y > 0)\) are

\[
X = U \begin{bmatrix} X_0 & (Y_{21} \Sigma^2 X_0)^T \\ Y_{21} \Sigma^2 X_0 & Y_{21} \Sigma^2 X_0 \Sigma^2 Y_{21}^T + K_{22} \end{bmatrix} U^T,
\]

\[
Y = U \begin{bmatrix} \Sigma^{-2} & Y_{21}^T \\ Y_{21} \Sigma^2 Y_{21}^T + M_{22} \\ Y_{21} \Sigma^2 Y_{21}^T + M_{22} \end{bmatrix} U^T,
\]

where \( K_{22}, M_{22} \in \mathbb{SPR}_{+}^{(n-r)\times(n-r)} \) and \( Y_{21} \in \mathbb{R}^{(n-r)\times r} \) are arbitrary.

**Proof.** The proof of the solvability conditions (4.29) is similar to that of Theorem 2.2(b).

If the conditions (4.29) are satisfied, then Equations (4.12) and (4.13) have the respective unique symmetric positive definite solutions \( X_{11} = \Sigma^{-1} D_{11} \Sigma^{-1} \in \mathbb{SPR}_{+}^{r\times r} \), \( Y_{11} = \Sigma^{-2} \in \mathbb{SPR}_{+}^{r\times r} \). It follows from (4.10), (4.17), (4.18), Lemma 2.1, and these expressions that the solution pairs \((X > 0, Y = Y^T)\), \((X > 0, Y \geq 0)\), and \((X > 0, Y > 0)\) of Equation (1.2) have the forms (4.30), (4.31), and (4.32), respectively.

5. ILLUSTRATIVE EXAMPLES

**Example A.** Consider the case of a vibrating system, as described in the introduction, so that the conditions \( X \geq 0 \) and \( Y > 0 \) are to be satisfied. This case is treated here in part (c) of Theorem 4.2. Let us suppose that \( m \) linearly independent eigenvectors are specified and make up the columns of the \( n \times m \) matrix \( A \) along with \( m \) eigenvalues \( d_m \geq d_{m-1} \geq \cdots \geq d_1 > 0 \),
as in the Equations (1.3), and that \( m < n \). Then in the SVD (2.1) we have

\[
A = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T
\]

(and the matrices \( U_2, V_2 \) do not appear), and the consistency conditions (4.23) are satisfied.

In Equation (4.1) we have \( D_{11} = V^T D V \), and this is already a spectral decomposition, so we may take \( Q = V^T \). In Equation (4.2) we have \( Y_0 = \Sigma^{-2} \), and the general solution pairs \((X \geq 0, Y > 0)\) from (4.26) are

\[
X = U \begin{bmatrix} \Sigma^{-1} V^T D V \Sigma^{-1} \\ Y_{21} \Sigma V^T D V \Sigma^{-1} \\ Y_{21} \Sigma V^T D V Y_{21}^T + K_{22} \end{bmatrix} U^T, \tag{5.1}
\]

\[
Y = U \begin{bmatrix} \Sigma^{-2} \\ Y_{21}^T \\ Y_{21} \Sigma^2 Y_{21}^T + M_{22} \end{bmatrix} U^T,
\]

where \( K_{22} \in \mathbb{S}^R((n-m) \times (n-m)) \) and \( M_{22} \in \mathbb{S}^R((n-m) \times (n-m)) \) are arbitrary.

Let us make the simple choice \( Y_{21} = 0 \). Then we obtain solutions

\[
\hat{X} = U \begin{bmatrix} \Sigma^{-2} \\ 0 \\ K_{22} \end{bmatrix} U^T, \quad \hat{Y} = U \begin{bmatrix} \Sigma^{-2} \\ 0 \\ M_{22} \end{bmatrix} U^T, \tag{5.2}
\]

where \( K_{22} \geq 0 \) and \( M_{22} > 0 \) are arbitrary.

Now the eigenvalues of the pencil \( \lambda \hat{Y} - \hat{X} \) are easily seen to be those of \( D \) (as required by our construction) together with those of \( \lambda M_{22} - K_{22} \), which can obviously be assigned as any \( n - m \) nonnegative numbers. Thus, we have a procedure for solving the inverse problem for a vibrating system in which \( n_1 \) eigenvalues and \( m \leq n_1 \) eigenvectors are predetermined.

**Example B.** To illustrate the case \( \text{rank } A = r < m \), suppose that \( A = [a_1 \ a_2] \), where \( \|a_1\| = 1 \) and \( a_2 = \alpha a_1 \). Thus, \( m = 2, \ r = 1 \). An SVD of \( A \) has the form

\[
A = \begin{bmatrix} a_1 & b_1 & \cdots & b_{n-1} \end{bmatrix} \begin{bmatrix} \sqrt{1+\alpha^2} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ \alpha & -1 \end{bmatrix} \frac{1}{\sqrt{1+\alpha^2}}
\]

where \( U = [a_1 \ b_1 \cdots b_{n-1}] \) is an orthogonal matrix. The rank of \( D \) cannot exceed the rank of \( A \) (in the consistent case), and so \( D = \text{diag}[d_1, d_2] \) with
$d_1 d_2 = 0$. In the case $Y > 0$ it is easily seen that the relation $XA = YAD$ implies both $d_1 = 0$ and $d_2 = 0$, i.e., $D = 0$.

From (2.2) and (4.2) we find that $X_0$ and $Y_0$ are both of size one and $X_0 = 0, Y_0 = c$, an arbitrary real number. For general solutions $X \geq 0$ and $Y > 0$ Equation (4.26) now give,

$$X = U \begin{bmatrix} 0 & 0 \\ 0 & K_{22} \end{bmatrix} U^T, \quad Y = U \begin{bmatrix} c & y^T \\ y & c^{-1}yy^T + M_{22} \end{bmatrix} U^T,$$

where $c > 0$, $y \in \mathbb{R}^{n-1}$ is arbitrary, $K_{22}$ is an arbitrary semidefinite matrix, and $M_{22}$ is an arbitrary positive definite matrix.

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REFERENCES


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