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Weak bases and quasi-pseudo-metrization of bispaces

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ABSTRACT

In this paper it is obtained a quasi-pseudo-metrization theorem which provides a certain unification in the treatment of the biquasi-metrization problem when it is considered via sequences of neighborhoods of each point satisfying certain properties. In particular, the well-known theorems of Fox, Raghavan, Künzi, and Raghavan and Reilly are deduced from our results. We also obtain some quasi-metrization theorems in terms of pairwise locally symmetric bifunctions.

1. Introduction

Throughout this paper the letters $\mathbb{N}$ and $\mathbb{R}^+$ will denote the set of all positive integer numbers and the set of all nonnegative real numbers, respectively. If A is a subset of X and τ is a topology on X, then $\tau - cl\,A$ will denote the closure of A in the space (X, τ).

The importance of Williams' metrization theorem [33] and its antecedents (theorem of Niemytzki [23], Frink [8]) in the solution of the metrization problem is well known. However, as Lindgren and Fletcher have noted [19] the Williams theorem can be deduced from the famous Frink metrization theorem. Frink, in his turn, uses the Alexandroff–Uryshon theorem in the proof of his result. The interest of Frink's theorem has been remarked, among others, by Martin [21,22].

Since Kelly [15] began a systematized study of bitopological spaces one of the main problems in this area has been to obtain necessary and sufficient conditions for quasi-metrizability of bispaces. This problem was considered in [25,18, 32,24,2,3,7,17,27,29,30,28,31]. In Section 2 of this paper we obtain a quasi-pseudo-metrization theorem in the style of Frink's metrization theorem, by using weak bases. From this result we deduce the Fox–Künzi theorem (which is the biquasi-metric generalization of William's theorem), the bitopological extension of the "double sequence" theorem of Nagata (proved independently, by Fox [7], and Raghavan [27] and, in a slightly different form, by Raghavan and Reilly [28]) and one other related result.

Let us recall that the notion of a weak base was introduced by Arhangelskii in [1]; it was recently shown in [20] that it provides a useful tool to the study of topological properties of semicones. In [14,9,10] a cone on $\mathbb{R}^+$ is defined as a

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semicone $(X, +, \cdot)$ such that $(X, +)$ is an abelian monoid, but there are some structures in computer science that are not abelian. For instance, it is obvious that the set of all finite and infinite words on a nonempty alphabet Σ [26] is a monoid when is equipped with the operation $\circ$ of concatenation. However, it is neither abelian nor cancellative, in general. In fact, if $\Sigma = \{0, 1\}$, and put $a = 10$, and $b = 01$, it is evident that $a \circ b = 1001 \neq b \circ a = 0110$. Now, if $c = 1010$ and $d = 101010\ldots$, then $a \circ d = c \circ d$ but $a \neq c$.

Finally, Section 3 is devoted to obtain some quasi-metrization theorems for pairwise stratifiable and pairwise developable spaces weakening certain conditions considered by Collins and Roscoe in [5].

Let us recall that a quasi-pseudo-metric on a set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $d(x, x) = 0$,
(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

If d is a quasi-pseudo-metric satisfying the condition:

(i′) $d(x, y) = d(y, x) = 0$ if, and only if, $x = y$,

we say that d is a quasi-metric.

We say that a function $d : X \times X \to \mathbb{R}^+$ is a symmetric if for all $x, y \in X$:

(i″) $d(x, y) = 0$ if, and only if, $x = y$,
(iii) $d(x, y) = d(y, x)$.

If d is a quasi-(pseudo-)metric on X the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ is called the conjugate quasi-(pseudo-)metric of d on X. If d is a quasi-(pseudo-)metric then the function d^s defined by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$ is a (pseudo-)metric on X.

Each quasi-pseudo-metric d on a set X induces a topology $\tau(d)$ on X where for all $x \in X$ and all $r > 0$, $B_d(x, r) = \{y \in X: d(x, y) < r\}$ is an open d-ball and the family of open d-balls $\{B_d(x, r): x \in X, r > 0\}$ is a base for the topology $\tau(d)$.

A bispace (a bitopological space in [15]) is a triple (X, τ_1, τ_2) where X is a nonempty set and τ_1 and τ_2 are two topologies on X. A bispace (X, τ_1, τ_2) is quasi-(pseudo-)metrizable if there exists a quasi-(pseudo-)metric d on X such that $\tau(d) = \tau_1$ and $\tau(d^{-1}) = \tau_2$.

A local quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that:

(i) for each $U \in \mathcal{U}$, $\Delta = \{(x, x) \in X\} \subseteq U$,
(ii) for each $U \in \mathcal{U}$ and each $x \in X$ there exists some $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$ where, $V^2 = V \circ V$ and $U(x) = \{y \in X: (x, y) \in U\}$.

A quasi-uniformity on a set X is a local quasi-uniformity $\mathcal{U}$ on X satisfying:

(ii′) for each $U \in \mathcal{U}$ there exists some $V \in \mathcal{U}$ such that $V^2 \subseteq U$.

A local uniformity on a set X is a local quasi-uniformity $\mathcal{U}$ on X satisfying:

(iii) for each $U \in \mathcal{U}$ there exists some $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.

Each local quasi-uniformity on a set X induces a topology $\tau(\mathcal{U})$ on X where a set $A \subseteq X$ is $\tau(\mathcal{U})$-open if for each $x \in A$ there is a $U \in \mathcal{U}$ such that

$$x \in U(x) = \bigl\{y \in X: (x, y) \in U\bigr\} \subseteq A.$$

We say that a bispace (X, τ_1, τ_2) has a compatible local quasi-uniformity if there exists a local quasi-uniformity $\mathcal{U}$ on X, such that $\mathcal{U}^{-1}$ is a local quasi-uniformity on X, and $\tau(\mathcal{U}) = \tau_1$ and $\tau(\mathcal{U}^{-1}) = \tau_2$.

Let (X, τ) be a topological space. A family $\mathcal{B}$ of subset of X is a weak base [1] for the topology τ if for each $x \in X$, there is a subfamily $\mathcal{B}_x$ of $\mathcal{B}$, such that:

(a) $x \in B$, for each $B \in \mathcal{B}_x$,
(b) if $A, B \in \mathcal{B}_x$, there is a $C \in \mathcal{B}_x$ such that $C \subseteq A \cap B$,
(c) a subset $U \subseteq X$ is τ-open if, and only if, for each $x \in U$ there exists a subset $B \in \mathcal{B}_x$ such that $B \subseteq U$.

The family $\mathcal{B}_x$ is called a local weak base at x.

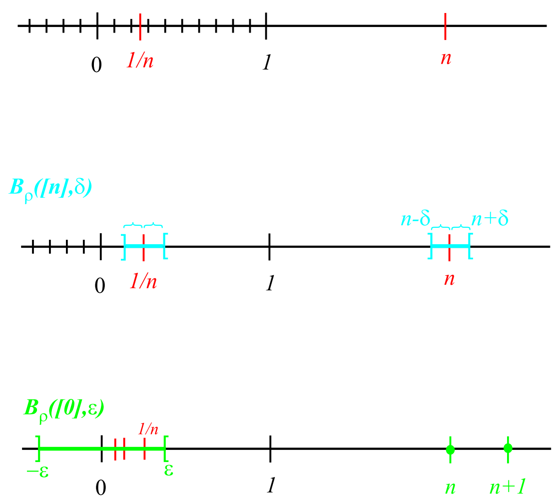

Fig. 1.

Example. ([11]) Let ρ be the usual metric on $\mathbb{R}$, and denote by S the following equivalence relation: for $x, y \in \mathbb{R}$, xSy if, and only if, $x = y$ or there exists $n \in \mathbb{N}$ such that $x = n$ and $y = 1/n$. Let $f : \mathbb{R} \to X = \mathbb{R}/S$, and denote the elements of X by $[x]$, then the function d given by

$$d\big([x],[y]\big) = \rho\big(f^{-1}[x], f^{-1}[y]\big)$$

is a symmetric on X. The family $\{B_d([0],\varepsilon):\ \varepsilon > 0\}$ is a local weak base at $[0]$ but it is not a neighborhood base (see Fig. 1).

2. Quasi-pseudo-metrization theorems

We shall use the quasi-uniform analogue [6, Theorem 4] of the Alexandroff–Uryshon metrization theorem, namely, a bispace is quasi-pseudo-metrizable if, and only if, it has a compatible quasi-uniformity with a countable base, to obtain an analogous to Frink's metrization theorem, in terms of weak bases.

Lemma 1. *Let X be a nonempty set such that for each $x \in X$ there exist two decreasing sequences $\{g_i(n,x):\ n \in \mathbb{N}\}$, $i = 1, 2$, of subsets of X with $x \in \bigcap_{n=1}^{\infty}(g_i(n,x) \cap g_j(n,x))$, $i \neq j$, for all $x \in X$. Suppose that, given $x \in X$ and $n \in \mathbb{N}$ there exists $m = m(n,x) > n$ satisfying for $i, j = 1, 2$, $i \neq j$:*

$$g_i(m,x) \cap g_j(m,y) \neq \emptyset \quad \Rightarrow \quad g_i(m,y) \subset g_i(n,x).$$

Then, there exists a quasi-pseudo-metric d on X such that, for all $x \in X$, $\{g_1(n,x):\ n \in \mathbb{N}\}$ is base of $\tau(d)$-neighborhoods and $\{g_2(n,x):\ n \in \mathbb{N}\}$ is base of $\tau(d^{-1})$-neighborhoods of x.

Proof. For each $x \in X$, put $n_1(x) = 1$ and $n_2(x) = m(1,x)$. Following this process, let $n_k(x) = m(n_{k-1}(x), x)$ for all $k > 1$. Now, we define, for all $n \in \mathbb{N}$,

$$U_{1,k}(x) = \bigcup\big\{g_1\big(n_k(p), p\big):\ x \in g_2\big(n_k(p), p\big)\big\},$$

and

$$U_{2,k}(x) = \bigcup\big\{g_2\big(n_k(p), p\big):\ x \in g_1\big(n_k(p), p\big)\big\},$$

and for $i = 1, 2$,

$$U_{i,k} = \big\{(x,y):\ x \in X \text{ and } y \in U_{i,k}(x)\big\}.$$

It is easy to prove that $\{U_{1,k}:\ k \in \mathbb{N}\}$ is a base for a quasi-uniformity $\mathcal{U}$ on X. Similarly, $\{U_{2,k}:\ k \in \mathbb{N}\}$ is a base for a quasi-uniformity $\mathcal{V}$ on X. Since $(x,y) \in U_{1,k}$ if, and only if, $(y,x) \in U_{2,k}$, we deduce $\mathcal{U}^{-1} = \mathcal{V}$, so by [6, Theorem 4], there exists a quasi-pseudo-metric d on X satisfying $\tau(d) = \tau(\mathcal{U})$ and $\tau(d^{-1}) = \tau(\mathcal{U}^{-1})$.

On the other hand, for each $x \in X$, we have $U_{i,m(n,x)}(x) \subseteq g_i(n,x)$, which implies that the family $\{g_1(n,x):\ n \in \mathbb{N}\}$ is a base of $\tau(d)$-neighborhoods of x and the family $\{g_2(n,x):\ n \in \mathbb{N}\}$ is a base of $\tau(d^{-1})$-neighborhoods of x. This concludes the proof. $\square$

Theorem 1. *A bispace (X, τ_1, τ_2) is quasi-pseudo-metrizable if, and only if, for each $x \in X$ there exist two decreasing sequences*

$$\mathcal{B}_{i,x} = \{V_i(n,x)\colon\ n \in \mathbb{N}\}$$

with $i = 1, 2$ of subsets of X such that $\mathcal{B}_i = \bigcup\{\mathcal{B}_{i,x}\colon\ x \in X\}$ is a weak base for (X, τ_i) and such that given $x \in X$ and $n \in \mathbb{N}$, there exists $m = m(n,x) > n$ satisfying

$$V_i(m,x) \cap V_j(m,y) \neq \emptyset \quad \Rightarrow \quad V_i(m,y) \subseteq V_i(n,x)$$

for $i, j = 1, 2$, and $i \neq j$.

Proof. Necessity. Let d be a quasi-pseudo-metric on X such that $\tau(d) = \tau_1$ and $\tau(d^{-1}) = \tau_2$. It is enough to put, for all $x \in X$ and all $n \in \mathbb{N}$,

$$V_1(n,x) = B_d(x, 2^{-n}) \quad \text{and} \quad V_2(n,x) = B_{d^{-1}}(x, 2^{-n}).$$

The sufficiency follows from Lemma 1 with $g_i(n,x) = V_i(n,x)$ for $i = 1, 2$, $x \in X$ and $n \in \mathbb{N}$. $\square$

Theorem 2. *A bispace (X, τ_1, τ_2) is quasi-pseudo-metrizable if, and only if, for each $x \in X$ there exist four decreasing sequences*

$$\{U_i(n,x)\colon\ n \in \mathbb{N}\} \quad \textit{and} \quad \{S_i(n,x)\colon\ n \in \mathbb{N}\},$$

$i = 1, 2$, of supersets of x such that if $i, j = 1, 2$, and $i \neq j$:

(i) *For each $x \in X$, $\{U_i(n,x)\colon\ n \in \mathbb{N}\}$ is a local weak base for (X, τ_i) at x.*
(ii) *If $S_i(n,x) \cap S_j(n,y) \neq \emptyset$ then $S_i(n,y) \subseteq U_i(n,x)$.*
(iii) *Given $S_i(n,x)$ there exists $m \in \mathbb{N}$ such that $U_i(m,x) \subseteq S_i(n,x)$.*

Proof. Sufficiency. For each $x \in X$ and each $n \in \mathbb{N}$ define $V_i(n,x) = \bigcap_{r=1}^{n} S_i(r,x)$, with $i = 1, 2$. It is easily seen that $\{V_i(n,x)\colon\ n \in \mathbb{N}\}$ is a τ_i-local weak base at x. The quasi-pseudo-metrizability of (X, τ_1, τ_2) follows from Theorem 1. We omit the easy proof of the necessity. $\square$

From the preceding theorem we deduce the following two versions for bispaces of the "double sequence" Nagata metrization theorem.

Corollary 1. *([7,27]) A bispace (X, τ_1, τ_2) is quasi-pseudo-metrizable if, and only if, for each $x \in X$ there exist four decreasing sequences of τ_i-neighborhoods of x,*

$$\{U_i(n,x)\colon\ n \in \mathbb{N}\} \quad \textit{and} \quad \{S_i(n,x)\colon\ n \in \mathbb{N}\}$$

$i = 1, 2$, such that if $i, j = 1, 2$, and $i \neq j$:

(i) *For each $x \in X$, $\{U_i(n,x)\colon\ n \in \mathbb{N}\}$ is a base of τ_i-neighborhoods of x.*
(ii) *If $S_i(n,x) \cap S_j(n,y) \neq \emptyset$ then $y \in U_i(n,x)$.*
(iii) *If $y \in S_i(n,x)$ then $S_i(n,y) \subseteq U_i(n,x)$.*

Corollary 2. *([28]) A bispace (X, τ_1, τ_2) is quasi-pseudo-metrizable if, and only if, for each $x \in X$ there exist two countable bases of τ_i-neighborhoods of x,*

$$\{S_i(n,x)\colon\ n \in \mathbb{N}\}$$

$i = 1, 2$, such that if $i, j = 1, 2$, and $i \neq j$:

(i) *If $S_i(n,x) \cap S_j(n,y) \neq \emptyset$ then $y \in S_i(n-1,x)$.*
(ii) *If $y \in S_i(n,x)$ then $S_i(n,y) \subseteq S_i(n-1,x)$.*

The next result is deduced in a similar way to Theorem 2.

Corollary 3. *([7,17]) A bispace (X, τ_1, τ_2) is quasi-pseudo-metrizable if, and only if, it has a compatible local quasi-uniformity with a countable base.*

Proof. Sufficiency. Let $\mathcal{U}$ be a local quasi-uniformity on X with a decreasing countable base $\{U_n\colon\ n \in \mathbb{N}\}$ such that $\mathcal{U}^{-1}$ is a local quasi-uniformity, and $\tau(\mathcal{U}) = \tau_1$ and $\tau(\mathcal{U}^{-1}) = \tau_2$. For each $x \in X$ and each $n \in \mathbb{N}$, we deduce, similarly to Theorem 2, that there exist $k, m \in \mathbb{N}$ such that $k > m > n$ and $U_k^3(x) \subseteq U_m^2(x) \subseteq U_n(x)$, and $(U_k^{-1})^3(x) \subseteq (U_m^{-1})^2(x) \subseteq U_n^{-1}(x)$. If we define $V_1(n,x) = U_n(x)$ and $V_2(n,x) = U_n^{-1}(x)$, then it is clear that condition $V_i(k,x) \cap V_j(k,y) \neq \emptyset$ implies $V_i(k,y) \subseteq V_i(n,x)$ for $i, j = 1, 2$, $i \neq j$. $\square$

3. Pairwise local symmetry and quasi-metrization

The conditions (i) and (ii) given in Corollary 1 provide a natural extension for bispaces of the notion of a Nagata space [4]. On the other hand, it is well known that a topological space is a Nagata space if, and only if, it is a stratifiable and a first countable space. Next we recall the definitions of a pairwise stratifiable space, which may be found in [7,12,16].

Definition 1. A bispace (X,τ_1,τ_2) is called pairwise stratifiable if to each τ_i-closed set $H \subseteq X$ one can assign a sequence $\{H_n: n \in \mathbb{N}\}$ of τ_j-open set such that for $i,j=1,2$, $i \neq j$:

(i) $H = \bigcap_{n=1}^{\infty} H_n$,
(ii) if $G \subseteq H$ are τ_i-closed sets then $G_n \subseteq H_n$,
(iii) $H = \bigcap_{n=1}^{\infty} \tau_i\, cl(H_n)$.

Theorem 3. *For a bispace (X,τ_1,τ_2) the following are equivalent*:

(a) *(X,τ_1,τ_2) is pairwise stratifiable and (X,τ_i) are first countable spaces $i=1,2$.*
(b) *For each $x \in X$ there exist four sequences of τ_i-neighborhoods of x, $\{U_i(n,x): n \in \mathbb{N}\}$ and $\{S_i(n,x): n \in \mathbb{N}\}$, $i=1,2$, such that if $i,j=1,2$, and $i \neq j$:*
  (i) *For each $x \in X$, $\{U_i(n,x): n \in \mathbb{N}\}$ is a base of τ_i-neighborhoods of x.*
  (ii) *If $S_i(n,x) \cap S_j(n,y) \neq \emptyset$ then $y \in U_i(n,x)$.*
(c) *For each $x \in X$ there exist two functions $g_i : \mathbb{N} \times X \to \tau_i$ such that $x \in \bigcap_{n=1}^{\infty} g_i(n,x)$, $i=1,2$, and if $g_i(n,x) \cap g_j(n,x_n) \neq \emptyset$ then the sequence $\{x_n: n \in \mathbb{N}\}$ τ_i-converges to x, for $i,j=1,2$, $i \neq j$.*

Definition 2. A topological space (X,τ) is a γ-space [13] if there is a function $g:\mathbb{N} \times X \to \tau$ such that $x \in \bigcap_{n=1}^{\infty} g(n,x)$, and if $x_n \in g(n,x)$ and $y_n \in g(n,x_n)$ then the point x is a cluster point for the sequence $\{y_n: n \in \mathbb{N}\}$. In this case, we say that g is a γ-function for (X,τ).

It is clear that every quasi-pseudo-metrizable topological space is a γ-space and that every γ-space is a first countable space. On the other hand, it is well known [19] that a topological space (X,τ) is a γ-space if, and only if, for each $x \in X$ there exist two sequences of neighborhoods of x satisfying the conditions (i) and (iii) of Corollary 1. This observation jointly with Theorem 3 permits us to deduce from Corollary 1 the following essentially known result.

Theorem 4. *A bispace (X,τ_1,τ_2) is quasi-pseudo-metrizable if, and only if, it is pairwise stratifiable and (X,τ_i) is a γ-space for $i=1,2$.*

Definition 3. A pair open cover in a bispace (X,τ_1,τ_2) is a family of pairs

$$(\mathcal{G}_1,\mathcal{G}_2) = \{(G_{1,\alpha},G_{2,\alpha}): \alpha \in I\}$$

such that:

(i) for each $\alpha \in I$, $G_{i,\alpha} \in \tau_i$, for each $i=1,2$,
(ii) $\mathcal{G}_i = \{G_{i,\alpha}: \alpha \in I\}$ is a cover of X, for each $i=1,2$,
(iii) for each $x \in X$ there is an $\alpha \in I$ such that $x \in G_{1,\alpha} \cap G_{2,\alpha}$.

Let $(\mathcal{G}_1,\mathcal{G}_2)$ and $(\mathcal{G}'_1,\mathcal{G}'_2)$ be pair open covers of (X,τ_1,τ_2). We say that $(\mathcal{G}'_1,\mathcal{G}'_2)$ refines $(\mathcal{G}_1,\mathcal{G}_2)$ (that is $(\mathcal{G}'_1,\mathcal{G}'_2) < (\mathcal{G}_1,\mathcal{G}_2)$) if for each pair $(G'_{1,\alpha},G'_{2,\alpha}) \in (\mathcal{G}'_1,\mathcal{G}'_2)$ there is a pair $(G_{1,\beta},G_{2,\beta}) \in (\mathcal{G}_1,\mathcal{G}_2)$ such that $G'_{i,\alpha} \subseteq G_{i,\beta}$ for each $i=1,2$.

Let $(\mathcal{G}_1,\mathcal{G}_2)$ be a pair open cover of (X,τ_1,τ_2). Let A be a nonempty subset of X, we define for each $i,j=1,2$ and $i \neq j$,

$$St(A,\mathcal{G}_i,\mathcal{G}_j) = \bigcup \{G_{i,\alpha} \in \mathcal{G}_i: A \cap G_{j,\alpha} \neq \emptyset\}.$$

If $x \in X$ we define

$$St(x,\mathcal{G}_i,\mathcal{G}_j) = \bigcup \{G_{i,\alpha} \in \mathcal{G}_i: x \in G_{j,\alpha}\}$$

and

$$St^2(x,\mathcal{G}_i,\mathcal{G}_j) = St\big(St(x,\mathcal{G}_i,\mathcal{G}_j),\mathcal{G}_i,\mathcal{G}_j\big).$$

Definition 4. A pair development in a bispace (X,τ_1,τ_2) is a sequence $\{(\mathcal{G}_{1,n},\mathcal{G}_{2,n}): n \in \mathbb{N}\}$ of pair open covers of X such that, for each $x \in X$, $\{St(x,\mathcal{G}_{1,n},\mathcal{G}_{2,n}): n \in \mathbb{N}\}$ is a base of τ_i-neighborhoods of x. A bispace (X,τ_1,τ_2) is pairwise developable if it is a pair development $\{(\mathcal{G}_{1,n},\mathcal{G}_{2,n}): n \in \mathbb{N}\}$ such that $(\mathcal{G}_{1,n+1},\mathcal{G}_{2,n+1}) < (\mathcal{G}_{1,n},\mathcal{G}_{2,n})$ for each $n \in \mathbb{N}$.

The following results are useful characterizations of pairwise stratifiable spaces and pairwise developable spaces (see [3,31]).

Proposition 1. *A bispace (X, τ_1, τ_2) is pairwise stratifiable if, and only if, there exist two functions $g_i : \mathbb{N} \times X \to \tau_i$, $i = 1, 2$ such that:*

(i) *$x \in g_1(n, x) \cap g_2(n, x)$ for each $x \in X$ and each $n \in \mathbb{N}$,*
(ii) *for each τ_i-closed H and each $x \in X - H$ there exists an $n \in \mathbb{N}$ such that $x \in X - \tau_i\, cl(\bigcup\{g_j(n, y)\colon\ y \in H\})$, $i, j = 1, 2$, $i \neq j$.*

The pair (g_1, g_2) is called a pairwise stratifiable bifunction for (X, τ_1, τ_2).

Proposition 2. *A bispace (X, τ_1, τ_2) is pairwise developable if, and only if, there exist two functions $g_i : \mathbb{N} \times X \to \tau_i$, $i = 1, 2$, such that:*

(i) *$x \in g_1(n, x) \cap g_2(n, x)$ for each $x \in X$ and each $n \in \mathbb{N}$,*
(ii) *if $x \in g_i(n, y_n)$ and $x_n \in g_j(n, y_n)$ for all $n \in \mathbb{N}$ then x is a τ_j-cluster point of the sequence $\{x_n\colon\ n \in \mathbb{N}\}$, $i = 1, 2$, $i \neq j$.*

Then we say that the pair (g_1, g_2) is a pairwise developable bifunction for (X, τ_1, τ_2).

Definition 5. Let (X, τ_1, τ_2) be a bispace and let $g_i : \mathbb{N} \times X \to \tau_i$ be two functions verifying for $i = 1, 2$, $i \neq j$:

(i) $x \in g_1(n, x) \cap g_2(n, x)$ for each $x \in X$ and each $n \in \mathbb{N}$,
(ii) for each $x \in X$ and each $n \in \mathbb{N}$ there exists $m = m(n, x) > n$ such that $x \in g_j(n, y)$ whenever $y \in g_i(m, x)$.

Then we say that the pair (g_1, g_2) is a pairwise locally symmetric bifunction for (X, τ_1, τ_2).

The following results provide quasi-pseudo-metrization theorems using pairwise locally symmetric bifunctions.

Theorem 5. *Let (X, τ_1) and (X, τ_2) be two topological spaces. Then the bispace (X, τ_1, τ_2) is quasi-metrizable if, and only if, it has a pairwise locally symmetric pairwise stratifiable bifunction.*

Proof. Let (g_1, g_2) be a pairwise locally symmetric pairwise stratifiable bifunction for (X, τ_1, τ_2). We can suppose, without loss of generality, that for all $x \in X$ and $n \in \mathbb{N}$, $g_i(n+1, x) \subseteq g_i(n, x)$, $i = 1, 2$. It suffices to show that (X, τ_1) and (X, τ_2) are γ-spaces. It is easy to prove that the family $\{g_i(n, x)\colon\ n \in \mathbb{N}\}$ is a base of τ_i-neighborhoods of x, $i = 1, 2$. Define for all $x \in X$ and $n \in \mathbb{N}$, $h_i(n, x) = g_i(m(n, x), x)$, $i = 1, 2$. Suppose that $x_n \in h_i(n, x)$, $y_n \in h_i(n, x_n)$ but x is not a cluster point of $\{y_n\colon\ n \in \mathbb{N}\}$. Then, there exists a τ_i-open neighborhood U of x and a $p \in \mathbb{N}$, such that $y_n \in H = X - U$ for $n \geqslant p$. By Proposition 1 there exists $k \in \mathbb{N}$ such that $g_i(k, x) \cap (\bigcup\{g_j(k, y)\colon\ y \in H\}) = \emptyset$, $i \neq j$. Since $x \in g_j(n, x_n)$ it follows that the sequence $\{x_n\colon\ n \in \mathbb{N}\}$ τ_i-converges to x. Now let $r \geqslant \max\{p, k\}$ such that for all $n \geqslant r$, $x_n \in g_i(k, n)$. Since $x_n \in g_j(n, y_n) \subseteq g_j(k, y_n)$, $n \geqslant r$, we obtain a contradiction. Consequently, (X, τ_i) is a γ-space, $i = 1, 2$. We omit the easy proof of the converse. □

Theorem 6. *Let (X, τ_1) and (X, τ_2) be two topological spaces. Then the bispace (X, τ_1, τ_2) is quasi-metrizable if, and only if, it has a pairwise locally symmetric developable bifunction.*

Proof. Let (g_1, g_2) be a pairwise locally symmetric developable bifunction for (X, τ_1, τ_2). We can suppose, without loss of generality, that for all $x \in X$ and $n \in \mathbb{N}$, $g_i(n+1, x) \subseteq g_i(n, x)$, $i = 1, 2$. Define for all $x \in X$ and $n \in \mathbb{N}$, $h_i(n, x) = g_i(m(n, x), x)$, $i = 1, 2$. Suppose that $x_n \in h_i(n, x)$, $y_n \in h_i(n, x_n)$. Then $x \in g_j(n, x_n)$ and $y_n \in g_i(n, x_n)$, $i \neq j$. By Proposition 2, x is a τ_i-cluster point of the sequence $\{y_n\colon\ n \in \mathbb{N}\}$ and, hence, (X, τ_i) is a γ-space, $i = 1, 2$. It remains to show that (X, τ_1, τ_2) is pairwise stratifiable. To see this take a τ_i-closed set H and $x \in X - H$. Suppose that, for all $n \in \mathbb{N}$, $x \in \tau_i\, cl \bigcup\{h_j(n, y)\colon\ y \in H\}$, $i \neq j$. Then there exists a sequence $\{x_n\colon\ n \in \mathbb{N}\}$ satisfying $x_n \in h_i(n, x) \cap h_j(n, y_n)$ whenever $y_n \in H$ for all $n \in \mathbb{N}$, $i \neq j$. It follows, $x \in g_j(n, x_n)$ and $y_n \in g_i(n, x_n)$. By Proposition 2, x is a τ_i-cluster point of $\{y_n\colon\ n \in \mathbb{N}\}$, a contradiction. We omit the easy proof of the converse. □

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