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Color degree and alternating cycles in edge-colored graphs[☆]

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ABSTRACT

Let G be an edge-colored graph. An alternating cycle of G is a cycle of G in which any two consecutive edges have distinct colors. Let $d^c(v)$, the color degree of a vertex v, be defined as the maximum number of edges incident with v that have distinct colors. In this paper, we study color degree conditions for the existence of alternating cycles of prescribed length.

1. Introduction

Let $G = (V, E)$ be a graph. An *edge-coloring* of G is a function $C : E \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. If G is assigned such a coloring C, then we say that G is an *edge-colored graph*. Let $C(e)$ denote the color of the edge $e \in E$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$ and $c(H) = |C(H)|$. For a color $i \in C(H)$, let $i_H = |\{e : C(e) = i \text{ and } e \in E(H)\}|$ and say that *color i appears i_H times in H*. For an edge-colored graph G, if $c(G) = c$, we call it a *c-edge-colored graph*.

For a vertex v in edge-colored graph G, a *color neighborhood* of v is defined as a set $T \subseteq N(v)$ such that the colors of the edges joining v and vertices of T are pairwise distinct. A *maximum color neighborhood* $N^c(v)$ of v is a color neighborhood of v with maximum size. Let $d^c(v) = |N^c(v)|$ and call it the *color degree* of v.

If $P = v_1v_2\cdots v_p$ is a path, let $P[v_i, v_j]$ denote the subpath $v_iv_{i+1}\cdots v_j$, and $P^-[v_i, v_j]$ denote the subpath $v_jv_{j-1}\cdots v_i$. The length of a path is the number of its edges.

A path or cycle in an edge-colored graph is called *alternating* if any two consecutive edges have distinct colors. Note that this definition of alternating paths or cycles differs from the usual one, which is with respect to only two colors. Besides a number of applications in graph theory and algorithms, the concept of alternating paths and cycles, appears in various other fields: Genetics [7–9], social sciences [6], etc. A good resource on alternating paths and cycles is the survey paper [2] by Bang-Jensen and Gutin.

Grossman and Häggkvist were the first to study the problem of the existence of alternating cycles in c-edge-colored graphs. They proved Theorem 1.1 below in the case $c = 2$. The case $c \geq 3$ was proved by Yeo [13]. Let v be a cut-vertex in an edge-colored graph G. We say that v *separates colors* if no component of $G - v$ is joined to v by at least two edges of different colors.

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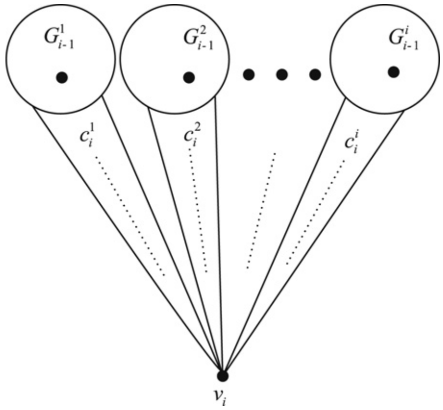

Fig. 1. An edge-colored graph G_i.

Theorem 1.1 (*Grossman and Häggkvist [10], and Yeo [13]*). *Let G be a c-edge-colored graph, for $c \geq 2$, such that every vertex of G is incident with at least two edges of different colors. Then either G has a cut-vertex separating colors, or G has an alternating cycle.*

We use the notation K_n^c to denote a complete graph on n vertices, each edge of which is colored by a color from the set $\{1, 2, \ldots, c\}$. Let $\Delta(K_n^c)$ be the maximum number of edges of the same color adjacent to a vertex of K_n^c. We have the following conjecture due to Bollobás and Erdős [3].

Conjecture 1.2 (*Bollobás and Erdős [3]*). *If $\Delta(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains an alternating Hamiltonian cycle.*

Bollobás and Erdős managed to prove that $\Delta(K_n^c) < \frac{n}{69}$ implies the existence of an alternating Hamiltonian cycle in K_n^c. This result was improved by Chen and Daykin [5] to $\Delta(K_n^c) < \frac{n}{17}$ and by Shearer [12] to $\Delta(K_n^c) < \frac{n}{7}$. So far the best asymptotic estimate was obtained by Alon and Gutin [1].

Theorem 1.3 (*Alon and Gutin [1]*). *For every $\epsilon > 0$, there exists an $n_0 = n_0(\epsilon)$ so that for every $n > n_0$, K_n^c satisfying $\Delta(K_n^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)n$ has an alternating Hamiltonian cycle.*

In [11], Li, Wang and Zhou proved the following result.

Theorem 1.4 (*Li, Wang and Zhou [11]*). *If $\Delta(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains an alternating cycle of length at least $\lceil \frac{n+2}{3} \rceil + 1$.*

2. Main results

We begin with a study of the existence of an alternating cycle with a prescribed property and prove the following theorem.

Theorem 2.1. *Let G be an edge-colored graph of order n, for $n \geq 3$. If for each vertex v of G, $d^c(v) > \frac{n+1}{3}$, then G has an alternating cycle AC such that each color in $C(AC)$ appears at most two times in AC.*

Moreover, for the existence of an alternating cycle, we have the following proposition.

Proposition 2.2. *For any positive integer i, there exists an edge-colored graph G_i such that for each vertex v of G_i, $d^c(v) \geq i$, and G_i contains no alternating cycle.*

Proof. The graph G_i are constructed inductively for $i \geq 1$. We let G_1 be the graph K_1 with color $C(K_1) = 1$, and, having constructed $G_{i-1} (i \geq 2)$, obtain G_i as follows (see Fig. 1). Let $G^1_{i-1}, G^2_{i-1}, \ldots, G^i_{i-1}$ denote i vertex disjoint copies of G_{i-1} and $\{c^1_i, c^2_i, \ldots, c^i_i\}$ be the colors such that $\{c^1_i, c^2_i, \ldots, c^i_i\} \cap C(G_{i-1}) = \phi$. Now G_i is obtained by adding the edges between a new vertex v_i and each vertex in G^j_{i-1} and coloring these edges by c^j_i, for $j = 1, \ldots, i$. Clearly G_i is an edge-colored graph such that for each vertex v of G_i, $d^c(v) \geq i$, and G_i contains no alternating cycle. □

For the short alternating cycle, we prove the following theorem.

Theorem 2.3. *Let G be an edge-colored graph of order n, for $n \geq 3$. If for each vertex v of G, $d^c(v) \geq \frac{37n-17}{75}$, then G contains at least one alternating triangle or one alternating quadrilateral.*

We also consider long alternating paths and cycles in edge-colored graphs and prove the following result.

Theorem 2.4. *Let G be an edge-colored graph. If for each vertex v of G, $d^c(v) \geq d \geq 2$, then either G has an alternating path of length at least 2d, or G has an alternating cycle of length at least $\lceil \frac{2d}{3} \rceil + 1$.*

Theorem 2.4 implies the following corollary.

Corollary 2.5. *Let G be an edge-colored graph of order n, for $n \geq 4$. If for each vertex v of G, $d^c(v) \geq \frac{n}{2}$, then G has an alternating cycle of length at least $\lceil \frac{n}{3} \rceil + 1$.*

We believe that the bound in the above corollary may be improved and propose the following conjecture.

Conjecture 2.6. *Let G be an edge-colored graph of order n, for $n \geq 3$. If for each vertex v of G, $d^c(v) \geq \frac{n}{2}$, then G has an alternating Hamiltonian cycle.*

If the above conjecture is true, it would be best possible. We present the following example to prove this. For any integer m, let K_m, K'_{m+1} be two properly edge-colored complete graphs with orders m and $m+1$, respectively. Choose a vertex $u' \in K'_{m+1}$. For every vertex $u \in K_m$, add the edge uu' and let $C(uu') = c_0$, where $c_0 \notin C(K_m)$. The new edge-colored graph is denoted by B. Clearly, $|V(B)| = n = 2m+1$. Moreover, for every vertex v of B, it holds that $d^c(v) \geq m = \frac{n-1}{2}$, and B contains no alternating Hamiltonian cycle.

Since the proof of Theorem 2.1 is analogous to that of Theorem 2.3 and simpler than it, we omit this proof. The proofs of Theorems 2.3 and 2.4 will be given in Section 3.

3. Proofs of the main results

3.1. Proof of Theorem 2.3

If $n = 3, 4$, clearly Theorem 2.3 holds. So assume that $n \geq 5$. Suppose that Theorem 2.3 is not true. Let G be a counterexample. For an edge uv, let $N_1^c(u), N_1^c(v)$ denote the maximum color neighborhood of u, v, respectively, such that $v \in N_1^c(u)$, $u \in N_1^c(v)$ and $|N_1^c(u) \cup N_1^c(v)|$ is maximum. Let $N^c(u, v)$ denote $N_1^c(u) \cup N_1^c(v)$. Choose an edge $uv \in E(G)$ such that $|N^c(u, v)|$ is maximum.

Without loss of generality, assume that $N_1^c(u) = \{v, u_1, u_2, \ldots, u_s\}$ and $N_1^c(v) \setminus N_1^c(u) = \{u, v_1, v_2, \ldots, v_t\}$, in which $s = d^c(u) - 1$. Let $X = \{u_1, \ldots, u_s, v_1, \ldots, v_t\}$. Note that $|N^c(u, v)| = s + t + 2$. Consider graph $G[X]$, and we have the following claim.

Claim 1. *Suppose $e \in E(G[X])$,*

(i) *if $e = u_iu_j$ $(1 \leq i, j \leq s)$, then $C(e) \in \{C(uu_i), C(uu_j)\}$;*
(ii) *if $e = v_iv_j$ $(1 \leq i, j \leq t)$, then $C(e) \in \{C(vv_i), C(vv_j)\}$;*
(iii) *if $e = u_iv_j$ $(1 \leq i \leq s, 1 \leq j \leq t)$ and $C(uu_i) \neq C(vv_j)$, then $C(e) \in \{C(uu_i), C(vv_j)\}$.*

Proof. Clearly (i) and (ii) hold, otherwise G contains an alternating triangle, which is a contradiction.

If (iii) does not hold, then there exists an edge $e = u_iv_j$ $(1 \leq i \leq s, 1 \leq j \leq t)$ such that $C(uu_i) \neq C(vv_j)$ and $C(e) \notin \{C(uu_i), C(vv_j)\}$. Since $v, u_i \in N_1^c(u)$, $C(uu_i) \neq C(uv)$. Similarly, it holds that $C(vv_j) \neq C(uv)$. Now uvv_ju_iu is an alternating quadrilateral, which is a contradiction. □

Given graph $G[X]$, let D_1 denote the digraph obtained by the following operations.

(1) Remove any edge u_iv_j from $G[X]$ if $C(uu_i) = C(vv_j)$, $1 \leq i \leq s$ and $1 \leq j \leq t$. Note that if $C(uu_i) = C(vv_j)$ and $u_iv_j \in E(G[X])$, then $C(u_iv_j) = C(uu_i) = C(vv_j)$).

(2) Orient the rest edges by the rule: For an edge $xy \in E(G_1[X])$, if $C(xy) = C(uy)$ or $C(xy) = C(vy)$, then the orientation of xy is from x to y; otherwise, by Claim 1, $C(xy) = C(ux)$ or $C(xy) = C(vx)$, then the orientation of xy is from y to x.

For any vertex $w \in V(D_1)$, let $N_{D_1}^+(w)$ denote the outneighbors of w in D_1 and $d_{D_1}^+(w) = |N_{D_1}^+(w)|$. Let $G_0 = G[X \cup \{u, v\}]$.

Lemma 3.1. *Every simple digraph with minimum outdegree at least 1 has a directed cycle.*

Claim 2. *There exists a directed cycle in D_1.*

Proof. Otherwise, by Lemma 3.1, there is a vertex w such that $d_{D_1}^+(w) = 0$. Without loss of generality, assume that $w \in N_1^c(u)$. If $N^c(w)$ is a maximum color neighbor of w in G, then it holds that $|N^c(w) \setminus (X \cup \{u, v\})| \geq d^c(w) - 1$. By the choice of the edge uv, it follows that $d^c(w) - 1 \leq t$. Thus

$$
\begin{aligned}
n &\geq |X| + |u| + |v| + |N^c(w) \setminus (X \cup \{u, v\})| \\
&\geq d^c(u) + t - 1 + 2 + d^c(w) - 1 \\
&\geq d^c(u) + 2d^c(w) - 1 \\
&\geq 3\left(\frac{37n - 17}{75}\right) - 1 \\
&> 3\left(\frac{n+1}{3}\right) - 1 = n.
\end{aligned}
$$

This contradiction finishes the proof. □

Claim 3. *If $\overrightarrow{C_p}$ is a directed cycle in D_1, then C_p is an alternating cycle in G, and each color in $C(C_p)$ appears at most two times in C_p.*

Proof. First, we will prove that C_p is an alternating cycle of G. Assume that xy and yz are adjacent edges of C_p, and furthermore, in $\overrightarrow{C_p}$, the orientations of xy and yz are from x to y, from y to z, respectively. By the orientation rule, we conclude that $C(xy) = C(uy)$ or $C(xy) = C(vy)$, and $C(yz) = C(uz)$ or $C(yz) = C(vz)$.

If $C(xy) = C(uy)$ and $C(yz) = C(uz)$, or $C(xy) = C(vy)$ and $C(yz) = C(vz)$, then by the definition of the maximum color neighborhood, it holds that $C(uy) \neq C(uz)$ and $C(vy) \neq C(vz)$. Thus we have that $C(xy) \neq C(yz)$.

Otherwise, without loss of generality, assume that $C(xy) = C(uy)$ and $C(yz) = C(vz)$. By (1) and Claim 1(iii), we have that $C(uy) \neq C(vz)$. It follows that $C(xy) \neq C(yz)$.

Thus C_p is an alternating cycle of G. Moreover, by the definition of $N^c(u, v)$, we can conclude that each color in $C(C_p)$ appears at most two times in C_p. □

The *girth* of a digraph D containing directed cycles is the length of the smallest directed cycle in D. Since G has neither alternating triangle nor alternating quadrilateral, it follows that the girth of D_1 is at least 5.

Lemma 3.2 (*Häggkvist [4]*). *Let D be a digraph on m vertices with girth at least 5, then $\delta^+ < \frac{9(m-1)}{28}$.*

Let $\alpha = \frac{9}{28}$. By Lemma 3.2, there is a vertex w of D_1 such that $d^+_{D_1}(w) < \alpha(|V(D_1)|-1) = \alpha(s+t-1) = \alpha(d^c(u)+t-2)$. Without loss of generality, assume that $w \in N^c_1(u)$. If $N^c(w)$ is a maximum color neighborhood of w in G and $N^c_{G_0}(w)$ is a maximum color neighborhood of w in G_0, then $|N^c_{G_0}(w)| = |N^+_{D_1}(w)| + |u| = d^+_{D_1}(w) + 1$. It follows that

$$|N^c(w) \setminus (X \cup \{u, v\})| \geq d^c(w) - |N^c_{G_0}(w)| > d^c(w) - \alpha(d^c(u) + t - 2) - 1.$$

If $d^c(w) - \alpha(d^c(u) + t - 2) - 1 > t$, then consider the edge uw and it holds that

$$\begin{aligned} |N^c(u, w)| &\geq |\{v, u_1, u_2, \ldots, u_s\}| + |N^c(w) \setminus (X \cup \{u, v\})| + |u| \\ &> s + t + 2 \\ &= |N^c(u, v)|, \end{aligned}$$

which contradicts with the choice of uv.

Now $d^c(w) - \alpha(d^c(u) + t - 2) - 1 \leq t$, that is $t \geq \frac{d^c(w)}{1+\alpha} - \frac{\alpha d^c(u)}{1+\alpha} + \frac{2\alpha-1}{1+\alpha}$. It follows that

$$\begin{aligned} n &\geq |X| + |u| + |v| + |N^c(w) \setminus (X \cup \{u, v\})| \\ &> d^c(u) + t - 1 + 2 + d^c(w) - \alpha(d^c(u) + t - 2) - 1 \\ &\geq \frac{1-\alpha}{1+\alpha} d^c(u) + \frac{2}{1+\alpha} d^c(w) + \frac{5\alpha - 1}{1+\alpha}. \end{aligned}$$

Since for each vertex v of G, $d^c(v) \geq \frac{37n-17}{75}$, and $\alpha = \frac{9}{28}$, the above inequality is

$$n > \frac{3-\alpha}{1+\alpha} \frac{37n-17}{75} + \frac{5\alpha-1}{1+\alpha} \geq n.$$

This contradiction completes the proof of Theorem 2.3. □

3.2. *Proof of* Theorem 2.4

If $d = 2$, Theorem 2.4 holds clearly. So assume that $d \geq 3$. Suppose that Theorem 2.4 is false. Let G be a counterexample. Without loss of generality, we assume that $P_l = v_1 v_2 \cdots v_l$ is a longest alternating path of G. Clearly $l \leq 2d$. Choose a maximum color neighborhood $N^c(v_1)$ of v_1 such that $v_2 \in N^c(v_1)$. By the choice of l, it holds that $N^c(v_1) \subseteq V(P_l)$. Since $|N^c(v_1)| = d^c(v) \geq d$, it follows that $l \geq d + 1$.

Choose integer s such that

(R_1) $v_s \in N^c(v_1)$;
(R_2) $s \geq \lceil \frac{2d}{3} \rceil + 1$;
(R_3) subject to (R_1), (R_2), s is minimum.

Since $d \geq 3$, it holds that $d - \lceil \frac{2d}{3} \rceil \geq \lfloor \frac{d}{3} \rfloor$, then $s < l$.

Claim 1. (1.1) *$s < l$;*
(1.2) *If $v_i \in N^c(v_1)$ and $s \leq i \leq l - 1$, then $C(v_i v_{i+1}) \neq C(v_1 v_i)$.*

Proof. If $s = \lceil \frac{2d}{3} \rceil + 1$, then $s = \lceil \frac{2d}{3} \rceil + 1 < d + 1 \leq l$. If $s > \lceil \frac{2d}{3} \rceil + 1$, then $v_{\lceil \frac{2d}{3} \rceil + 1} \notin N^c(v_1)$. Thus $N^c(v_1) \cap V(P[v_1, v_{\lceil \frac{2d}{3} \rceil + 1}]) \leq \lceil \frac{2d}{3} \rceil - 1$. By the minimality of s, $N^c(v_1) \cap V(P[v_1, v_s]) \leq \lceil \frac{2d}{3} \rceil$. Since $|N^c(v_1)| \geq d > \lceil \frac{2d}{3} \rceil$, $s < l$.

If (1.2) is false, then there exists $s \leq i \leq l$ such that $C(v_i v_{i+1}) = C(v_1 v_i)$. Since P_l is an alternating path, $C(v_{i-1} v_i) \neq C(v_i v_{i+1})$. Thus $P[v_1, v_i] v_i v_1$ is an alternating cycle of length $i \geq s \geq \lceil \frac{2d}{3} \rceil + 1$, which is a contradiction. □

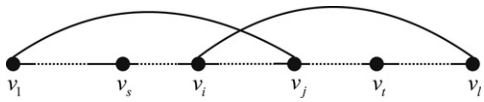

Fig. 2. An alternating cycle : $v_1v_jP[v_j, v_l]v_lv_iP^-[v_i, v_1]$.

We choose a maximum color neighborhood of $N^c(v_l)$ of v_l such that $v_{l-1} \in N^c(v_l)$. Similarly, $N^c(v_l) \subseteq V(P_l)$.

Choose integer t satisfying

(R'_1) $v_t \in N^c(v_l)$;
(R'_2) $t \le l - \lceil \frac{2d}{3} \rceil$;
(R'_3) subject to (R'_1), (R'_2), t is maximum.

Similarly, we have the following claim, here we omit the proof.

Claim 2. (2.1) $t > 1$;
(2.2) *If* $v_i \in N^c(v_l)$ *and* $2 \le i \le t$, *then* $C(v_{i-1}v_i) \ne C(v_iv_l)$.

Claim 3. $s < t$.

Proof. Otherwise, it holds that $s \ge t$. If $s > t$, then $AC^0 = v_1v_sP[v_s, v_l]v_lv_tP^-[v_t, v_1]$ is an alternating cycle. Also, $|AC^0| = |V(P[v_s, v_l])| + |V(P[v_1, v_t])| \ge 2(d - \lceil \frac{2d}{3} \rceil + 1) = 2(\lfloor \frac{d}{3} \rfloor + 1) = 2\lfloor \frac{d}{3} \rfloor + 2 \ge \lceil \frac{2d}{3} \rceil + 1$, which is a contradiction.

So $s = t$. If there exists $v_j \in N^c(v_1)$ such that $s + 1 \le j \le l - 1$, then there is an alternating cycle $AC^1 = v_1v_jP[v_j, v_l]v_lv_sP^-[v_s, v_1]$ of length $|AC^1| \ge 2 + |V(P[v_1, v_s])| \ge 3 + \lceil \frac{2d}{3} \rceil$, which gives a contradiction. Similarly, if there exists $v_j \in N^c(v_l)$ such that $2 \le j \le s - 1$, then $v_1v_sP[v_s, v_l]v_lv_jP^-[v_j, v_1]$ is an alternating cycle of length $3 + \lceil \frac{2d}{3} \rceil$. We also obtain a contradiction.

Thus we can conclude that $v_j \notin N^c(v_1)$ if $s + 1 \le j \le l - 1$ and $v_j \notin N^c(v_l)$ if $2 \le j \le s - 1$. On the other hand, by (R_3) there are at least $d - \lceil \frac{2d}{3} \rceil = \lfloor \frac{d}{3} \rfloor \ge 1$ vertices in $V(P[v_{s+1}, v_l]) \cap N^c(v_1)$. Clearly $v_l \in N^c(v_1)$. Similarly, $v_1 \in N^c(v_l)$. Now $C(v_1v_l) \ne C(v_1v_2)$ and $C(v_1v_l) \ne C(v_{l-1}v_l)$. So $P[v_1, v_l]v_lv_1$ is an alternating cycle of length $l \ge d + 1 > \lceil \frac{2d}{3} \rceil + 1$, which is a contradiction. □

Claim 4. (4.1) *For* $2 \le j \le s - 1$, $v_j \notin N^c(v_l)$;
(4.2) *For* $t + 1 \le j \le l - 1$, $v_j \notin N^c(v_1)$.

Proof. By symmetry, we only need to prove (4.1). Assume that (4.1) is not true, then there exists $v_j \in N^c(v_l)$ such that $2 \le j \le s - 1$. Clearly, $j \le t$, thus by Claim 2(2.2), $C(v_{j-1}v_j) \ne C(v_jv_l)$. Now $AC^2 = v_1v_sP[v_s, v_l]v_lv_jP^-[v_j, v_1]$ is an alternating cycle and $|AC^2| \ge |V(P[v_s, v_l])| + 2 \ge \lfloor \frac{2d}{3} \rfloor + 2 \ge \lceil \frac{2d}{3} \rceil + 1$, which is a contradiction. □

Let $A = N^c(v_1) \cap V(P[v_s, v_t])$ and $B = N^c(v_l) \cap V(P[v_s, v_t])$.

Claim 5. $|A| + |B| \ge 2\lfloor \frac{d}{3} \rfloor + 2$.

Proof. By (R_3), the number of vertices in $N^c(v_1) \cap V(P[v_s, v_l])$ is at least $d - (|P[v_1, v_{s-1}]| - 1) \ge d - (\lceil \frac{2d}{3} \rceil - 1) = \lfloor \frac{d}{3} \rfloor + 1$. By Claim 4(4.2), $N^c(v_1) \cap V(P[v_s, v_l]) = N^c(v_1) \cap (V(P[v_s, v_t]) \cup \{v_l\}) = A \cup (N^c(v_1) \cap \{v_l\})$. It follows that $|A| \ge \lfloor \frac{d}{3} \rfloor + 1 - |N^c(v_1) \cap \{v_l\}|$. Similarly, $|B| \ge \lfloor \frac{d}{3} \rfloor + 1 - |N^c(v_l) \cap \{v_1\}|$. Now $|A| + |B| \ge 2\lfloor \frac{d}{3} \rfloor + 2 - (|N^c(v_1) \cap \{v_l\}| + |N^c(v_l) \cap \{v_1\}|)$.

We will prove that $|N^c(v_1) \cap \{v_l\}| + |N^c(v_l) \cap \{v_1\}| = 0$, then Claim 5 holds. Otherwise, suppose that $|N^c(v_1) \cap \{v_l\}| + |N^c(v_l) \cap \{v_1\}| \ge 1$. By symmetry, we assume that $v_l \in N^c(v_1)$. If $C(v_{l-1}v_l) \ne C(v_lv_l)$, then $P[v_1, v_l]v_lv_1$ is an alternating cycle of length $l \ge d + 1$, which is a contradiction. Thus $C(v_{l-1}v_l) = C(v_lv_l)$. Now $v_1v_lv_lv_tP^-[v_t, v_1]$ is an alternating cycle of length at least $s + 2 \ge \lceil \frac{2d}{3} \rceil + 3$, which is a contradiction. □

Now we finish the proof of Theorem 2.4. It holds that $|V(P[v_s, v_t])| \le l - |V(P[v_1, v_{s-1}])| - |V(P[v_{t+1}, v_l])| \le l - \lceil \frac{2d}{3} \rceil - \lceil \frac{2d}{3} \rceil \le 2d - 2\lceil \frac{2d}{3} \rceil \le 2\lfloor \frac{d}{3} \rfloor$. By Claim 5, $|A| + |B| \ge 2\lfloor \frac{d}{3} \rfloor + 2$. Now there exist $v_j \in N^c(v_1)$ and $v_i \in N^c(v_l)$ such that $s \le i < j \le t$. By Claim 1(1.2) and Claim 2(2.2), $C(v_1v_j) \ne C(v_jv_{j+1})$ and $C(v_lv_i) \ne C(v_iv_{i-1})$. It follows that $v_1v_jP[v_j, v_l]v_lv_iP^-[v_i, v_1]$ is an alternating cycle of length at least $s + 1 \ge \lceil \frac{2d}{3} \rceil + 2$ (see Fig. 2), which is contradiction. This completes the proof. □

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