Sharpened bounds for corner singularities and boundary layers in a simple convection–diffusion problem

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Abstract

A singularly perturbed convection–diffusion problem posed on the unit square is considered. Its solution may have exponential and parabolic boundary layers, and corner singularities may also be present. Sharpened pointwise bounds on the solution and its derivatives are derived. The bounds improve bounds near an outflow corner of the problem that were derived in an earlier paper of the authors. Application is made to an error analysis of a finite element method for the problem.

Keywords: Singular perturbations; Corner singularities; Convection diffusion

1. The main result

In this work we sharpen pointwise bounds on the derivatives of the solution to a singularly perturbed convection diffusion problem that were obtained recently in [3]. This is achieved with improved bounds near an “outflow corner” of the problem. These improved bounds are given in Lemma 2 below. The desirability of sharpened bounds arises from a numerical analysis of the problem in [1]. It is an example of the needs of numerical analysis informing investigations in partial differential equations.

Let $Q = (0, 1) \times (0, 1)$ denote the unit square. The problem under consideration is:

\begin{align*}
Lu & := -\varepsilon \Delta u + p u_x + q u = f \quad \text{in } Q, \quad (1a) \\
u(x, 0) & = g_s(x), \quad u(x, 1) = g_n(x) \quad \text{for } 0 < x < 1, \quad (1b) \\
u(0, y) & = g_w(y), \quad u(1, y) = g_e(y) \quad \text{for } 0 < y < 1. \quad (1c)
\end{align*}
The coefficients $p$ and $q$ are positive constants while the parameter $\varepsilon$ lies in $(0, 1]$. The functions $f, g_w, g_e, g_s, g_n$ are assumed to satisfy, for some non-negative integer $\ell$ and $\alpha \in (0, 1)$,

$$f \in C^{2\ell,\alpha}(\bar{Q}), \quad g_w, g_e, g_s, g_n \in C^{2\ell,\alpha}([0, 1]).$$

But the data of the problem is not assumed to be compatible with the differential equation at the four corners of $Q$, and in particular, the boundary data is not assumed to be continuous at the corners. The amount of compatibility at the vertex $(\lambda, \mu)$ is denoted by an integer $\nu_{\lambda \mu}$. The integer $\nu_0$ is defined as follows. If $g_s(0) \neq g_w(0)$, then $\nu_0 = -1$. If $g_s(0) = g_w(0)$ but $f(0, 0) \neq -\varepsilon g''_s(0) - \varepsilon g''_w(0) + pg'_s(0) + qg_s(0)$, then $\nu_0 = 0$. The conditions for higher order compatibility are obtained in a similar manner (see [2] for more details), and integers $\nu_{10}$, $\nu_{01}$ and $\nu_{11}$ denoting the amount of compatibility at the other three corners are defined in the same way. The compatibility indicator $\nu_{\lambda \mu}$ involves derivatives of the boundary data of order $2\nu_{\lambda \mu}$ and derivatives of order $2\nu_{\lambda \mu} - 2$, so we must have $\nu_{\lambda \mu} \leq \ell$.

**Notation.** For various sets $\Omega$ we shall use the Sobolev space $W^{m,\infty}(\Omega)$ with norm $\|\cdot\|_{m,\infty,\Omega}$. If $m = 0$ we write $\|\cdot\|_{\infty,\Omega}$. Let $\Pi_x$ denote the half-plane $\{(x, y) : x > 0\}$.

The analysis in [3] proceeds by decomposing the solution $u$ in the form (see [3, (5.11)])

$$u = S + E + z_00 + z_{01} + z_{10} + z_{11} + u^{(2)},$$

where each of the functions on the right-hand side of (3) represents a component in the asymptotic structure of $u$ and also satisfies a boundary value problem associated with the operator $L$ that is simpler than (1). Bounds for the derivatives of each of these terms are obtained, and the derivative bounds for the solution $u$ are obtained by summing these bounds.

In (3), $S$ is the “smooth” component of $u$, the function $E$ is an outflow boundary layer, and $z_00$ and $z_{01}$ represent the effects of the inflow corner singularities at $(0, 0)$ and $(0, 1)$ and their associated parabolic boundary layers. The remainder term $u^{(2)}$ and its derivatives are exponentially small and can be ignored here. The terms $z_{10}$ and $z_{11}$, which are defined as solutions to certain boundary value problems in a quarter-plane, represent the effects of the outflow corner points $(1, 0)$ and $(1, 1)$ respectively. It is for this last pair of functions that we shall improve the derivative bounds of [3].

First, consider $S$. Let $f^*$ be a smooth extension of $f$ from $Q$ to the half-plane $x > 0$. Also, let $g^*_w$ and $g^*_e$ be smooth extensions of $g_w$ and $g_e$ from $[0, 1]$ to $(-\infty, \infty)$, and let $g^n_s$ and $g^n_n$ be smooth extensions of $g_s$ and $g_n$ from $[0, 1]$ to $[0, \infty)$. Then $S$ is defined to be the solution of the incoming half-plane problem

$$LS = f^* \quad \text{on } \Pi_x, \quad S(0, y) = g^*_w(y) \quad \text{for } -\infty < y < \infty.$$  \hspace{1cm} (4)

From [3, Theorem 3.1] applied to the function $S(x, y) - g^*_w(y)$, it follows that

$$\|D_x^m D_y^n S\|_{\infty,\Pi_x} \leq C(\|f^*\|_{m+n,\infty,\Pi_x} + \|g^*_w\|_{n+2,\infty,}\mathbb{R}).$$  \hspace{1cm} (5)

The function $E$, which models the exponential outflow boundary layer in the solution of (1), is defined to be the solution to the outgoing half-plane problem

$$LE = 0 \quad \text{for } x < 1, \quad E(1, y) = g^*_e(y) - S(1, y) \quad \text{for } -\infty < y < \infty.$$  \hspace{1cm} (6)

On setting $W(x, y) = E(1 - x, y)$ for $(x, y) \in \Pi_x$, sharp derivative bounds for $E$ are given by [3, Theorem 3.4]:

$$|D_x^m D_y^n E(x, y)| \leq C\|E(1, \cdot)\|_{m+n,\infty,\mathbb{R}} e^{-m\varepsilon e^{-p(1-x)/\varepsilon}} \quad \text{for } m \leq 2\ell,$$

where $\bar{m}$ is the smallest even integer satisfying $\bar{m} \geq m$.

We shall need a further bound on the derivatives of $E$. Set $L^*w = -\varepsilon \Delta w - pw_x + qw$ for all sufficiently regular functions $w$.

**Lemma 1.** For $m \leq 2\ell - 4$ we have

$$|D_x^m (e^{p\varepsilon/\ell} E(1 - x, y))| \leq C(\|f^*\|_{m+2,\infty,\Pi_x} + \|g^*_w\|_{m+4,\infty,\mathbb{R}} + \|g^*_e\|_{m+2,\infty,\mathbb{R}}) \quad \text{on } \Pi_x.$$
Proof. The definitions of $E$ and $W$ imply that
\[ L^*W = 0 \] on $\Pi_x$, \quad $W(0, y) = g^*_e(y) - S(1, y)$ \quad for $-\infty < y < \infty$.

Set $W_1(x, y) = e^{p x / \epsilon} W(x, y)$. Then
\[ L^*W = L^*(e^{-p x / \epsilon} W_1) = e^{-p x / \epsilon} \{ -\epsilon \Delta W_1 + p(W_1)_x + qW_1 \}, \]
so $W_1$ satisfies the boundary value problem
\[ L W_1 = 0 \quad \text{on} \quad \Pi_x, \quad W(0, y) = g^*_e(y) - S(1, y) \quad \text{for} \quad -\infty < y < \infty. \]

Set $W_2(x, y) = W_1(x, y) - g^*_e(y) + S(1, y)$. Then
\[ L W_2(x, y) = \epsilon (g^*_e(y) - S(1, y))' - q(g^*_e(y) - S(1, y)) \quad \text{on} \quad \Pi_x, \quad W_2(0, y) = 0 \quad \text{for} \quad -\infty < y < \infty. \]

Applying [3, Theorem 3.1] to this incoming half-plane problem, it follows that
\[ \| D^m W_2 \|_{\Pi_x} \leq C \| L W_2 \|_{m, \infty, \Pi_x} \leq C(\| f^* \|_{m+2, \infty, \Pi_x} + \| g^*_w \|_{m+4, \infty, \mathbb{R}} + \| g^*_e \|_{m+2, \infty, \mathbb{R}}) \]
by (5). But $e^{p x / \epsilon} E(1 - x, y) = W_1(x, y) = W_2(x, y) + g^*_e(y) - S(1, y)$, so the proof is complete. \hfill \blacksquare

Remark 1. The regularity restrictions imposed on $f$, $g_e$ and $g_w$ can possibly be weakened.

The function $z_{10}$ is defined in [3, (5.7)] to be the solution to the boundary value problem
\[ L z_{10} = 0 \quad \text{for} \quad x < 1, \quad 0 < y, \]
\[ z_{10}(x, 0) = -\chi(x)E(x, 0) \quad \text{for} \quad x < 1, \]
\[ z_{10}(1, y) = -\chi(1 - y)z_{00}(1, y) \quad \text{for} \quad 0 < y. \]

The function $z_{11}$ has a similar definition. The functions $\chi$ and $z_{00}$ are given in [3] and their properties are known. Recapitulating these properties, $\chi$ is a smooth function on $\mathbb{R}$ that satisfies $\chi(t) = 0$ for $t \leq 1/3$ and $\chi(t) = 1$ for $t \geq 2/3$, and the function $z_{00}(1, y)$ satisfies
\[ |D^n_y z_{00}(1, y)| \leq C \epsilon^{-n/2} e^{-\beta y/(2\sqrt{\epsilon})} \quad \text{for} \quad n \leq 2\ell, \]
where $\beta \in (0, \sqrt{q})$ is a positive constant.

Let $r_{1\mu} = \sqrt{(x - \lambda)^2 + (y - \mu)^2}$ for $\lambda, \mu = 0, 1$. That is, $r_{1\mu}$ denotes the distance from $(x, y)$ to the vertex $(\lambda, \mu)$ of $Q$. The following lemma gives the asserted improvement in the bounds for $z_{1\mu}$.

Lemma 2. For $m + n \leq 2\ell - 4$ and $\mu = 0, 1$, the functions $z_{1\mu}$ satisfy
\[ |D^m_x D^n_y z_{1\mu}(x, y)| \leq C [\epsilon^{-m-n/2} + e^{v_{1\mu}+1-m-n}] \quad \text{for} \quad m + n < 2v_{1\mu} + 2 \text{ and } r_{1\mu} < \epsilon. \]
\[ |D^m_x D^n_y z_{1\mu}(x, y)| \leq C [\epsilon^{-m-n/2} + e^{-v_{1\mu}-1} \ln r_{1\mu}] \quad \text{for} \quad m + n = 2v_{1\mu} + 2 \text{ and } r_{1\mu} < \epsilon. \]
\[ |D^m_x D^n_y z_{1\mu}(x, y)| \leq C [\epsilon^{-m-n/2} + e^{-v_{1\mu}-1} r_{1\mu}^{1/2} + 2-m-n] \quad \text{for} \quad m + n > 2v_{1\mu} + 2 \text{ and } r_{1\mu} < \epsilon. \]
\[ |D^m_x D^n_y z_{10}(x, y)| \leq C \epsilon^{-m-n/2} [1 + r_{10}^{1/2} + 1-n/2] e^{-\beta r_{10} / (2\sqrt{\epsilon})} \quad \text{for} \quad \epsilon \leq r_{10} \leq 1, \]
\[ |D^m_x D^n_y z_{11}(x, y)| \leq C \epsilon^{-m-n/2} [1 + r_{11}^{1/2} + 1-n/2] e^{-\beta r_{11} / (2\sqrt{\epsilon})} \quad \text{for} \quad \epsilon \leq r_{11} \leq 1. \]

Proof. The proof is given for the case $\mu = 0$; the case $\mu = 1$ is similar. Let $z_{10}(x, y) = e^{-\beta (1-x)/\epsilon} \tilde{v}(x, y)$. Then
\[ 0 = L z_{10}(x, y) = e^{-\beta (1-x)/\epsilon} L^* \tilde{v}(x, y), \]
so $L^* \tilde{v}(x, y) = 0$ for $x < 1$ and $y > 0$. Set $v(x, y) = \tilde{v}(1 - x, y)$. Then $L v(x, y) = L^* \tilde{v}(x, y) = 0$, so $v$ satisfies the boundary value problem
\[ -\epsilon \Delta v + pv_x + q v = 0 \quad \text{on} \quad (0, \infty) \times (0, \infty), \]
\[ v(x, 0) = g_1(x) \quad \text{for} \quad x > 0, \]
\[ v(0, y) = -\chi(1 - y)z_{00}(1, y) \quad \text{for} \quad y > 0, \]
where \( g_1(x) := -\chi(1-x)e^{px}\) satisfies
\[
|g_1^{(m)}(x)| \leq C \quad \text{for } m \leq 2\ell - 4
\] (10)
by Lemma 1. The function \( v \) has the same compatibility \( v_{10} \) at \((0,0)\) as the function \( z_{10} \) had at \((1,0)\). We now apply [3, Theorem 4.2] to \( v \) with, in the notation of that theorem, \( v = v_{10}, \ell = m + n/2 \) (so that \( 2m + n \leq 2\ell \) and the theorem can be applied) and \( \bar{G}_\ell = C \) by (10), \( H_\ell = C \) by (8). This yields
\[
|D_x^m D_y^n v(x,y)| \leq C[e^{-n/2} + e^{v_{10}+1-m-n}] \quad \text{for } m + n < 2v_{10} + 2 \text{ and } r_{10} < \epsilon, \quad (11a)
\]
\[
|D_x^m D_y^n v(x,y)| \leq C[e^{-n/2} + e^{v_{10}-1}|\ln r_{10}|] \quad \text{for } m + n = 2v_{10} + 2 \text{ and } r_{10} < \epsilon, \quad (11b)
\]
\[
|D_x^m D_y^n v(x,y)| \leq C[e^{-n/2} + e^{-v_{10}-1}r_{10}^{2v_{10}+2-m-n}] \quad \text{for } m + n > 2v_{10} + 2 \text{ and } r_{10} < \epsilon, \quad (11c)
\]
\[
|D_x^m D_y^n v(x,y)| \leq C[e^{-n/2}[1 + v_{10}^{1+1-m-n/2}]e^{-\beta(\sqrt{r_{10}})}e^{-\beta(\sqrt{r_{10}})/(2\sqrt{2})}] \quad \text{for } \epsilon \leq r_{10} \leq 1. \quad (11d)
\]
But
\[
|D_x^m D_y^n z_{10}(x,y)| \leq C \sum_{i+j=m} |D_x^i(e^{-\rho(1-x)\epsilon})D_y^j v(1-x,y)| \leq Ce^{-\rho(1-x)/\epsilon} \sum_{i+j=m} \epsilon^{-i} |D_x^i D_y^j v(1-x,y)|,
\]
and substitution of the bounds (11) gives the desired result. ■

We now sharpen the main result (Theorem 5.1) of [3]. Each term \( T_{\lambda,\mu} \) below describes the behavior induced in the solution by the vertex at \((\lambda,\mu)\); the terms \( T_{00}, \mu \) also include the effect of the parabolic boundary layers along \( y = \mu \). The term \( T_E \) describes the effect of the exponential outflow layer at \( x = 1 \).

**Theorem 1.** Let \( m, n \) be non-negative integers satisfying \( 2m + n \leq 2\ell \) and \( m + n \leq 2\ell - 4 \). Then for \( (x,y) \in Q \), the solution \( u \) of (1) satisfies
\[
|D_x^m D_y^n u(x,y)| \leq C(1 + T_{00} + T_{01} + T_{10} + T_{11} + T_E)
\]
with \( T_E = \epsilon^{-m}e^{-\rho(1-x)/\epsilon}, \) where for \( \mu = 0, 1 \), one has
\[
T_{00} = \epsilon^{-n/2} + \epsilon^{v_{00}+1-m-n} \quad \text{for } m + n < 2v_{00} + 2 \text{ and } r_{00} < \epsilon,
\]
\[
T_{0\mu} = \epsilon^{-n/2} + \epsilon^{v_{0\mu}-1}|\ln r_{0\mu}| \quad \text{for } m + n = 2v_{0\mu} + 2 \text{ and } r_{0\mu} < \epsilon,
\]
\[
T_{\mu 0} = \epsilon^{-n/2} + \epsilon^{v_{\mu 0}-1}1^{2v_{\mu 0}+2-m-n} \quad \text{for } m + n > 2v_{0\mu} + 2 \text{ and } r_{0\mu} < \epsilon,
\]
\[
T_{0\mu} = \epsilon^{-n/2}[1 + v_{0\mu}^{1+1-m-n/2}]e^{-\beta(y)/\sqrt{r_{0\mu}}} \quad \text{for } \epsilon \leq r_{0\mu} \leq 1,
\]
\[
T_{\mu 0} = \epsilon^{-n/2}[1 + v_{\mu 0}^{1+1-m-n/2}]e^{-\beta(y)/\sqrt{r_{\mu 0}}} \quad \text{for } \epsilon \leq r_{0\mu} \leq 1,
\]
and
\[
T_{1\mu} = \epsilon^{-n/2} + \epsilon^{v_{1\mu}+1-m-n} \quad \text{for } m + n < 2v_{1\mu} + 2 \text{ and } r_{1\mu} < \epsilon,
\]
\[
T_{1\mu} = \epsilon^{-n/2} + \epsilon^{-v_{1\mu}-1}|\ln r_{1\mu}| \quad \text{for } m + n = 2v_{1\mu} + 2 \text{ and } r_{1\mu} < \epsilon,
\]
\[
T_{1\mu} = \epsilon^{-n/2} + \epsilon^{-v_{1\mu}-1}2^{2v_{1\mu}+2-m-n} \quad \text{for } m + n > 2v_{1\mu} + 2 \text{ and } r_{1\mu} < \epsilon,
\]
\[
T_{10} = \epsilon^{-n/2}[1 + 1^{1+1-m-n/2}]e^{-\beta(\sqrt{r_{1\mu}})}e^{-\beta(\sqrt{r_{1\mu}})/(2\sqrt{2})} \quad \text{for } \epsilon \leq r_{10} \leq 1,
\]
\[
T_{11} = \epsilon^{-n/2}[1 + 1^{1+1-m-n/2}]e^{-\beta(\sqrt{r_{11}})/(2\sqrt{2})}e^{-\beta(\sqrt{r_{11}})/(2\sqrt{2})} \quad \text{for } \epsilon \leq r_{11} \leq 1.
\]
The constants \( C \) and \( c \) depend on \( m, n \) and \( \ell \).

**Proof.** The proof is the same as the proof of [3, Theorem 5.1], but using Lemma 2 instead of [3, Corollary 4.2]. ■

In Theorem 1, \( T_{\lambda,\mu} \) is a bound on \( |z_{\lambda,\mu}(x,y)| \) for \( \lambda, \mu = 0, 1 \); see [3].
2. An application

We now discuss the implications of Theorem 1 for a recent analysis by Franz et al. [1] of a streamline diffusion finite element method for (1) that assumes certain unproven bounds [1, Assumption 1] on the derivatives of terms in a decomposition of the solution $u$ in order to prove a convergence result on a Shishkin mesh. In the case where the differential operator has constant coefficients, we shall show that the analysis of Section 1 readily yields sufficient conditions – which are verifiable in practice – for the conclusions of [1].

The decomposition in [1, Assumption 1] is essentially the same as in (3) above; then plausible pointwise bounds are assumed on the derivatives up to order 2 of each term in this decomposition, and analogous $L_2$-norm bounds are assumed on all third-order derivatives of each of these terms. But an inspection of the arguments in [1] and in [4, Lemma 4.4], which [1] invokes, reveals that in fact for the third-order derivatives one needs precisely the following:

$$
\| D_x^m D_y^n S \|_{L_2(Q)} \leq C \quad \text{for } m + n = 3, \tag{12a}
$$

$$
\| D_x^2 D_y E \|_{L_2(Q)} \leq C \varepsilon^{-3/2}, \tag{12b}
$$

$$
\| D_x^1 D_y^2 E \|_{L_2(Q)} \leq C \varepsilon^{-1/2}, \tag{12c}
$$

$$
\| D_x D_y^2 z_{00} \|_{L_2(Q)} + \| D_x D_y^2 z_{01} \|_{L_2(Q)} \leq C \varepsilon^{-3/4}, \tag{12d}
$$

$$
\| D_x^3 z_{00} \|_{L_2(Q)} + \| D_x^3 z_{01} \|_{L_2(Q)} \leq C \varepsilon^{-3/4}, \tag{12e}
$$

$$
\| D_x^2 D_y z_{00} \|_{L_2(Q)} + \| D_x^2 D_y z_{01} \|_{L_2(Q)} \leq C \varepsilon^{-1/2}, \tag{12f}
$$

$$
\| D_x^2 D_y z_{10} \|_{L_2(Q)} + \| D_x^2 D_y z_{11} \|_{L_2(Q)} \leq C \varepsilon^{-7/4}, \tag{12g}
$$

$$
\| D_x D_y^2 z_{10} \|_{L_2(Q)} + \| D_x D_y^2 z_{11} \|_{L_2(Q)} \leq C \varepsilon^{-5/4}. \tag{12h}
$$

In particular the bound (12f) is weaker than the corresponding bound of $C \varepsilon^{-1/4}$ that is assumed in [1].

Assume that the data of (1) satisfies $\ell = 4$ and

$$
v_{00} = v_{01} = v_{10} = v_{11} = 1. \tag{13}
$$

Then for $0 \leq m + n \leq 3$ one has $2m + n \leq 2\ell$ and $m + n \leq 2\ell - 4$, so Theorem 1 can be invoked to yield

$$
| D_x^m D_y^n u(x, y) | \leq C (1 + T_{00} + T_{01} + T_{10} + T_{11} + T_E)
$$

where for $\mu = 0, 1$ one has

$$
T_{0\mu} = e^{-n/2} + e^{2-m-n} \quad \text{for } r_{0\mu} < \varepsilon, \\
T_{00} = e^{-n/2} [1 + r_{00}^{2-m-n/2}] e^{-c\varepsilon/\sqrt{\varepsilon}} \quad \text{for } \varepsilon \leq r_{00} \leq 1, \\
T_{01} = e^{-n/2} [1 + r_{01}^{2-m-n/2}] e^{-c(1-x)/\sqrt{\varepsilon}} \quad \text{for } \varepsilon \leq r_{01} \leq 1,
$$

and

$$
T_{1\mu} = e^{-m-n/2} + e^{2-m-n} \quad \text{for } r_{1\mu} < \varepsilon, \\
T_{10} = e^{-m-n/2} [1 + r_{10}^{2-n/2}] e^{-\beta_{y/2} \varepsilon} e^{p(1-x)/\varepsilon} \quad \text{for } \varepsilon \leq r_{10} \leq 1, \\
T_{11} = e^{-m-n/2} [1 + r_{11}^{2-n/2}] e^{-\beta_{(1-y)/2} \varepsilon} e^{p(1-x)/\varepsilon} \quad \text{for } \varepsilon \leq r_{11} \leq 1.
$$

Here $T_{\lambda\mu}$ is a bound on $| z_{\lambda\mu}(x, y) |$ for $\lambda, \mu = 0, 1$. For $m + n \leq 2$, these formulas easily imply the bounds assumed in [1, Assumption 1], since (5) and (6) bound $S$ and $E$ respectively. Likewise, (12a) follows from (5), while (6) implies (12b) and (12c), and the bounds (12d) and (12e) follow from the formulas for $T_{0\mu}$. The bound (12f) is the most difficult: to derive it, consider separately the cases $r_{0\mu} \leq \varepsilon$, $\varepsilon \leq r_{0\mu} \leq \sqrt{\varepsilon}$ and $\sqrt{\varepsilon} \leq r_{0\mu} \leq \sqrt{2}$ and invoke the formulas for the $T_{0\mu}$. Finally, (12g) and (12h) follow from the formulas for the $T_{1\mu}$.

The compatibility conditions (13) can easily be expressed explicitly for the problem considered in [1], where all the boundary data is homogeneous. Thus zero-order compatibility (continuity of the boundary data) is automatically
satisfied at each corner and consequently to obtain (13) we need only assume that
\[ f(0, 0) = f(0, 1) = f(1, 0) = f(1, 1) = 0 \] 
(14)
(see [2] for a derivation of such conditions).

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References