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On representing some lattices as lattices of intermediate subfactors of finite index

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Abstract

We prove that the very simple lattices which consist of a largest, a smallest and $2n$ pairwise incomparable elements where n is a positive integer can be realized as the lattices of intermediate subfactors of finite index and finite depth. Using the same techniques, we give a necessary and sufficient condition for subfactors coming from Loop groups of type A at generic levels to be maximal.

Keywords: Subfactors; Conformal field theory

1. Introduction

Let M be a factor and N a subfactor of M which is irreducible, i.e., $N' \cap M = \mathbb{C}$. Let K be an intermediate von Neumann subalgebra for the inclusion $N \subset M$. Note that $K' \cap K \subset N' \cap M = \mathbb{C}$, K is automatically a factor. Hence the set of all intermediate subfactors for $N \subset M$ forms a lattice under two natural operations $\wedge$ and $\vee$ defined by:

$$K_1 \wedge K_2 = K_1 \cap K_2, \qquad K_1 \vee K_2 = (K_1 \cup K_2)''.$$

Let G_1 be a group and G_2 be a subgroup of G_1. An interval sublattice $[G_1/G_2]$ is the lattice formed by all intermediate subgroups K, $G_2 \subseteq K \subseteq G_1$.

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By cross product construction and Galois correspondence, every interval sublattice of finite groups can be realized as intermediate subfactor lattice of finite index. The study of intermediate subfactors has been very active in recent years (cf. [3,18,20,28,39,36] for only a partial list). By a result of S. Popa (cf. [33]), if a subfactor $N \subset M$ is irreducible and has finite index, then the set of intermediate subfactors between N and M is finite. This result was also independently proved by Y. Watatani (cf. [39]). In [39], Y. Watatani investigated the question of which finite lattices can be realized as intermediate subfactor lattices. Related questions were further studied by P. Grossman and V.F.R. Jones in [18] under certain conditions. As emphasized in [18], even for a lattice which shapes like a Hexagon and consists of six elements, it is not clear if it can be realized as intermediate subfactor lattice with finite index. This question has been solved recently by M. Aschbach in [1] among other things. In [1], M. Aschbach constructed a finite group G_1 with a subgroup G_2 such that the interval sublattice $[G_1/G_2]$ is a Hexagon. The lattices that appear in [18,39,1] can all be realized as interval sublattice of finite groups.

It turns out that which finite lattice can be realized as an interval sublattice $[G_1/G_2]$ with G_1 finite is an old problem in finite group theory. See [31] for an excellent review and a list of references.

Most of the attention has been focused on the very simple lattice M_n consisting of a largest, a smallest and n pairwise incomparable elements. For $n = 1, 2, q + 1$ (where q is a prime power), examples of M_n have been found in the finite solvable groups. After the first interesting examples found by W. Feit (cf. [11]), A. Lucchini (cf. [30]) discovered new series of examples for $n = q + 2$ and for $n = \frac{(q^t+1)}{(q+1)} + 1$ where t is an odd prime.

At the present the only values of n for which M_n occurs as an interval sublattice of a finite group are $n = 1, 2, q + 1, q + 2, \frac{(q^t+1)}{(q+1)} + 1$ where t is an odd prime. The smallest undecided case is $n = 16$. In a major contribution to the problem about subgroup lattices of finite groups in [2], R. Baddeley and A. Lucchini have reduced the problem of realizing M_n as interval sublattice of finite groups to a collection of questions about finite simple groups. These questions are still quite hard, but eventually they might be resolved using the classification of finite simple groups. In this paper, the authors are cautious, but their ultimate goal seems to be to show that the list above is complete. In view of the above results about finite groups, it seems an interesting problem to ask if M_{16} can be realized as the lattice of intermediate subfactors with finite index. This problem is the main motivation for our paper. One of the main results of this paper, Theorem 2.40, states that all M_{2n} are realized as the lattice of intermediate subfactors of a pair of hyperfinite type III_1 factors with finite depth. Note that by [36] this implies that M_{2n} can also be realized as the lattice of intermediate subfactors of a pair of hyperfinite type II_1 factors with finite depth. Thus modulo the conjectures of R. Baddeley, A. Lucchini and possibly others we have an infinite series of lattices which can be realized by the lattice of intermediate subfactors with finite index and finite depth but cannot be realized by interval sublattices of finite groups.

The subfactors which realize M_{2n} are “orbifold subfactors” of [10,5,41], and we are motivated to examine these subfactors by the example of lattice of type M_6 in [18] which is closely related to an $\mathbb{Z}_2$ orbifold. To explain their construction, after first two preliminary sections, we will first review the result of A. Wassermann (cf. [21,38]) about Jones–Wassermann subfactors (cf. Remark 2.27) coming from representations of Loop groups of type A in Section 2.5. Section 2.6 is then devoted to a description of “orbifold subfactors” from an induction point of view. Although it is not too hard to show that the subfactor contains $2n$ incomparable intermediate subfactors, the hard part of the proof of Theorem 2.40 is to show that there are no more intermediate subfactors. Here we give a brief explanation of basic ideas behind our proof and describe how the paper is structured. We will use freely notations and concepts that can be found in preliminary sections. Let $\rho(M) \subset M$ be a subfactor and M_1 be an intermediate subfactor. In our examples below all factors are isomorphic to the hyperfinite type III_1 factor, and $\rho\bar{\rho}$ are direct sums of sectors from a set Δ with finitely many irreducible sectors and a nondegenerate braiding. Here we use the endomorphism theory pioneered by R. Longo (cf. [25]). Since M_1 is isomorphic to M, we can choose an isomorphism $c_1 : M \to M_1$. Denoting $c_2 = c_1^{-1}\rho$ we have $\rho = c_1 c_2$ where $c_1, c_2 \in \mathrm{End}(M)$. Note that $c_1\bar{c_1} \prec \rho\bar{\rho}$ is in Δ. Our basic idea to investigate the property of c_1 is to consider the following set $H_{c_1} := \{[a] \mid a \prec \lambda c_1,\ \lambda \in \Delta,\ a \text{ irreducible}\}$. Since Δ has finitely many irreducible sectors, H_{c_1} is a finite set. Moreover since $c_1\bar{c}_1 \in \Delta$, an induction method using braidings as in [42] is available. This induction method was used by the author in [42] to study subfactors from conformal inclusions, and developed further by J. Böckenhauer, D. Evans and J. Böckenhauer, D. Evans and Y. Kawahigashi in [4–9], and leads to strong constraints on the set H_{c_1}. Thus by using $\lambda \in \Delta$ to act from the left on c_1, one may hope to find what c_1 is made of. In the cases of Theorem 2.40 and Corollary 5.23, it turns out that there is a sector c in H_{c_1} with smallest index such that $c_1 \prec \lambda c$, and c is close to be an automorphism (it is an automorphism in the case of Corollary 5.23), and the corresponding subfactors have been well studied as those in [42]. In the simplest case $n = 2$, due to $A - D - E$ classification of graphs with norm less than 2, the above idea can be applied directly to give a rather quick proof of Theorem 2.40. We refer the reader to the paragraph after Theorem 2.40 which illustrates the above idea.

When $n > 2$, the norms of fusion graphs are in general greater than 2, no $A - D - E$ classification is available, and this is the main problem we must resolve to carry out the above idea. To explain our method, we note that S matrix as defined in Eq. (3) has the property

$$\left|\frac{S_{\lambda\mu}}{S_{1\mu}}\right| \leqslant \frac{S_{\lambda 1}}{S_{11}}, \quad \forall\mu$$

and

$$\left|\frac{S_{\lambda\mu}}{S_{1\mu}}\right| = \frac{S_{\lambda 1}}{S_{11}}, \quad \forall\mu$$

iff λ is an automorphism, i.e., λ has the smallest index 1. Our first observation is that for small index (close to 1) sectors c, certain entries of S-matrix like quantities (cf. Definition 3.10, Corollary 3.14) called ψ-matrix attain their extremum just like S-matrices. Hence to detect these small index sectors, we need to have a good estimation of the entries of ψ-matrix. In view of the Verlinde formula (cf. Eq. (4)) relating S-matrix with fusion rules, it is natural to use the known fusion rules to estimate ψ matrix. However, since the definition of ψ involves sectors which are not braided, the above idea does not work unless one can show that certain intertwining operators are central (cf. Theorem 3.8 and Section 5.1 for discussions). Our second observation is that a class of intertwining operators in Definition 3.7 is central (cf. Theorem 3.8). Thanks to a number of known results about representations of Loop groups of type A, we show that the assumption of Theorem 3.8 is verified in our case (cf. Proposition 4.7).

This allows us to show that certain sector with small index does exist (cf. Corollary 3.14), we can indeed find that c_1 is made of known subfactors. After a straightforward calculations involving known fusion rules in Proposition 4.10, we are able to finish the proof of Theorem 2.40 for general n.

In the last section we discuss a few related issues. Conjecture 5.1 is formulated which is equivalent to centrality of certain intertwining operators (cf. Proposition 5.7), and this is motivated by our proof of Theorem 2.40. We show in Proposition 5.11 that these intertwining operators are central on a subspace which is a linear span of products of (cf. Definition 5.9) cups, caps and braiding operators only. These motivate us to make Conjecture 5.12 which claims that the subspace is in fact the whole space. In view of recent development using category theory (cf. [12]), both conjectures can in fact be stated in categorical terms, and we do not know any counter examples in the categorical setting. In Proposition 5.17 we prove that a weaker version of Conjecture 5.12 implies Conjecture 5.1, and from this we are able to prove Conjecture 5.1 for modular tensor category from $SU(n)$ at level k (cf. Corollary 5.18).

In Section 5.2 we give applications of Corollary 5.18. To explain these applications, recall that a subfactor $N \subset M$ is called maximal if M_1 is an intermediate von Neumann algebra between N and M implies that $M_1 = M$ or $M_1 = N$. This notion is due to V.F.R. Jones when he outlined an interesting programme to investigate questions in group theory using subfactors (cf. [22]). In the case when M is the crossed product of N by a finite group G, it is easy to see that $N \subset M$ is maximal iff G is an abelian group of prime order. Hence for most of G the corresponding subfactor is not maximal. Corollary 5.23 gives a necessary and sufficient condition for subfactors from representations λ of $SU(n)$ at level $k \neq n \pm 2, n$ to be maximal: λ is maximal iff λ is not fixed by a nontrivial cyclic automorphism of extended Dynkin diagram (such cyclic automorphisms generate a group isomorphic to $\mathbb{Z}_n$). Hence it follows from Corollary 5.23 that most of such λ are maximal. For an example, if $k \neq n \pm 2, n$, k and n are relatively prime, then all λ are maximal.

Besides propositions and theorems that have been already mentioned, the first two preliminary sections are about sectors, covariant representations, braiding-fusion equations, Yang–Baxter equations, Rehren's S, T matrices. The third preliminary section summarizes properties of an induction method from [42]. These properties have been extensively studied and applied in subsequent work in [4–9] from a different point of view where induction takes place between two different but isomorphic algebras, and we recall a dictionary relating these two as provided in [44]. We think that in this paper it is simpler to take the point of view of [42] when discussing intermediate subfactors, and it is convenient to represent these intermediate subfactors as the range of endomorphisms of one fixed factor, so we do not have to switch between different but isomorphic algebras.

Using the dictionary we translate some properties of relative braidings and local extensions from [7] to our setting (cf. Proposition 2.24). The next two preliminary sections are devoted to subfactors from representations of $SU(n)$ at level k and its extensions. We collect a few properties about fusion rules, S matrices, and we define the subfactor which appears in Theorem 2.40. In Proposition 2.41 we show that this subfactor contains $2n$ incomparable proper intermediate subfactors.

2. Preliminaries

For the convenience of the reader we collect here some basic notions that appear in this paper. This is only a guideline and the reader should look at the references such as preliminary sections of [24] for a more complete treatment.

2.1. Sectors

Let M be a properly infinite factor and $\mathrm{End}(M)$ the semigroup of unit preserving endomorphisms of M. In this paper M will always be the unique hyperfinite III_1 factors. Let $\mathrm{Sect}(M)$

denote the quotient of End(M) modulo unitary equivalence in M. We denote by $[\rho]$ the image of $\rho \in \mathrm{End}(M)$ in Sect(M).

It follows from [25] and [26] that Sect(M), with M a properly infinite von Neumann algebra, is endowed with a natural involution $\theta \to \bar{\theta}$; moreover, Sect(M) is a semiring.

Let $\rho \in \mathrm{End}(M)$ be a normal faithful conditional expectation $\epsilon : M \to \rho(M)$. We define a number d_ϵ (possibly ∞) by:

$$d_\epsilon^{-2} := \mathrm{Max}\bigl\{\lambda \in [0, +\infty) \bigm| \epsilon(m_+) \geqslant \lambda m_+,\ \forall m_+ \in M_+\bigr\}$$

(cf. [32]).

We define

$$d = \mathrm{Min}_\epsilon\{d_\epsilon \mid d_\epsilon < \infty\}.$$

d is called the statistical dimension of ρ and d^2 is called the Jones index of ρ. It is clear from the definition that the statistical dimension of ρ depends only on the unitary equivalence classes of ρ. The properties of the statistical dimension can be found in [25–27].

Denote by Sect$_0(M)$ those elements of Sect(M) with finite statistical dimensions. For $\lambda, \mu \in \mathrm{Sect}_0(M)$, let Hom$(\lambda, \mu)$ denote the space of intertwiners from λ to μ, i.e. $a \in \mathrm{Hom}(\lambda, \mu)$ iff $a\lambda(x) = \mu(x)a$ for any $x \in M$. Hom(λ, μ) is a finite dimensional vector space and we use $\langle\lambda, \mu\rangle$ to denote the dimension of this space. $\langle\lambda, \mu\rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have $\langle\nu\lambda, \mu\rangle = \langle\lambda, \bar{\nu}\mu\rangle$, $\langle\nu\lambda, \mu\rangle = \langle\nu, \mu\bar{\lambda}\rangle$ which follows from Frobenius duality (see [26]). We will also use the following notation: if μ is a subsector of λ, we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$. A sector is said to be irreducible if it has only one subsector.

For any $\rho \in \mathrm{End}(M)$ with finite index, there is a unique standard minimal inverse $\phi_\rho : M \to M$ which satisfies

$$\phi_\rho\bigl(\rho(m)m'\rho(m'')\bigr) = m\phi_\rho(m')m'', \quad m, m', m'' \in M.$$

ϕ_ρ is completely positive. If $t \in \mathrm{Hom}(\rho_1, \rho_2)$ then we have

$$d_{\rho_1}\phi_{\rho_1}(mt) = d_{\rho_2}\phi_{\rho_2}(tm), \quad m \in M. \tag{1}$$

2.2. *Sectors from conformal nets and their representations*

We refer the reader to §3 of [24] for definitions of conformal nets and their representations. Suppose a conformal net $\mathcal{A}$ and a representation λ are given. Fix an open interval I of the circle and let $M := \mathcal{A}(I)$ be a fixed type III_1 factor. Then λ gives rises to an endomorphism still denoted by λ of M. We will recall some of the results of [35] and introduce notations.

Suppose $\{[\lambda]\}$ is a finite set of all equivalence classes of irreducible, covariant, finite-index representations of an irreducible local conformal net $\mathcal{A}$. We will use $\Delta_{\mathcal{A}}$ to denote all finite index representations of net $\mathcal{A}$ and will use the same notation $\Delta_{\mathcal{A}}$ to denote the corresponding sectors of M.[2]

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[2] Many statements in this section and Section 2.3 hold true in general case when the set $\{[\lambda]\}$ is only braided (cf. [8]) and we hope to consider such cases elsewhere.

We will denote the conjugate of $[\lambda]$ by $[\bar{\lambda}]$ and identity sector (corresponding to the vacuum representation) by $[1]$ if no confusion arises, and let $N_{\lambda\mu}^{\nu} = \langle [\lambda][\mu], [\nu] \rangle$. Here $\langle \mu, \nu \rangle$ denotes the dimension of the space of intertwiners from μ to ν (denoted by $\mathrm{Hom}(\mu, \nu)$). We will denote by $\{T_e\}$ a basis of isometries in $\mathrm{Hom}(\nu, \lambda\mu)$. The univalence of λ and the statistical dimension of (cf. §2 of [19]) will be denoted by ω_λ and $d(\lambda)$ (or d_λ)) respectively. The unitary braiding operator $\epsilon(\mu, \lambda)$ (cf. [19]) verifies the following

Proposition 2.1.

(1) *Yang–Baxter-Equation (YBE)*

$$\varepsilon(\mu, \gamma)\mu\big(\varepsilon(\lambda, \gamma)\big)\varepsilon(\lambda, \mu) = \gamma\big(\varepsilon(\lambda, \mu)\big)\varepsilon(\lambda, \gamma)\lambda\big(\varepsilon(\mu, \gamma)\big).$$

(2) *Braiding-Fusion-Equation (BFE)*

For any $w \in \mathrm{Hom}(\mu\gamma, \delta)$

$$\begin{aligned} \varepsilon(\lambda, \delta)\lambda(w) &= w\mu\big(\varepsilon(\lambda, \gamma)\big)\varepsilon(\lambda, \mu), \\ \varepsilon(\delta, \lambda)w &= \lambda(w)\varepsilon(\mu, \lambda)\mu\big(\varepsilon(\gamma, \lambda)\big), \\ \varepsilon(\delta, \lambda)^*\lambda(w) &= w\mu\big(\varepsilon(\gamma, \lambda)^*\big)\varepsilon(\mu, \lambda)^*, \\ \varepsilon(\lambda, \delta)^*\lambda(w) &= w\mu\big(\varepsilon(\gamma, \lambda)^*\big)\varepsilon(\lambda, \mu)^*. \end{aligned}$$

Lemma 2.2. *If* λ, μ *are irreducible, and* $t_\nu \in \mathrm{Hom}(\nu, \lambda\mu)$ *is an isometry, then*

$$t_\nu \varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu)t_\nu^* = \frac{\omega_\nu}{\omega_\lambda \omega_\mu}.$$

By Proposition 2.1, it follows that if $t_i \in \mathrm{Hom}(\mu_i, \lambda)$ is an isometry, then

$$\varepsilon(\mu, \mu_i)\varepsilon(\mu_i, \mu) = t_i^* \varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu)t_i.$$

We shall always identify the center of M with $\mathbb{C}$. Then we have the following

Lemma 2.3. *If*

$$\varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu) \in \mathbb{C},$$

then

$$\varepsilon(\mu, \mu_i)\varepsilon(\mu_i, \mu) \in \mathbb{C}, \quad \forall \mu_i \prec \lambda.$$

Let ϕ_λ be the unique minimal left inverse of λ, define:

$$Y_{\lambda\mu} := d(\lambda)d(\mu)\phi_\mu\big(\epsilon(\mu, \lambda)^*\epsilon(\lambda, \mu)^*\big), \tag{2}$$

where $\epsilon(\mu, \lambda)$ is the unitary braiding operator (cf. [19]).

We list two properties of $Y_{\lambda\mu}$ (cf. (5.13), (5.14) of [35]):

Lemma 2.4.

$$Y_{\lambda\mu} = Y_{\mu\lambda} = Y^*_{\lambda\bar{\mu}} = Y_{\bar{\lambda}\bar{\mu}},$$
$$Y_{\lambda\mu} = \sum_k N^\nu_{\lambda\mu} \frac{\omega_\lambda \omega_\mu}{\omega_\nu} d(\nu).$$

We note that one may take the second equation in the above lemma as the definition of $Y_{\lambda\mu}$.

Define $a := \sum_i d^2_{\rho_i} \omega^{-1}_{\rho_i}$. If the matrix $(Y_{\mu\nu})$ is invertible, by proposition on p. 351 of [35] a satisfies $|a|^2 = \sum_\lambda d(\lambda)^2$.

Definition 2.5. Let $a = |a| \exp(-2\pi i \frac{c_0}{8})$ where $c_0 \in \mathbb{R}$ and c_0 is well defined mod $8\mathbb{Z}$.

Define matrices

$$S := |a|^{-1} Y, \qquad T := C \operatorname{Diag}(\omega_\lambda) \tag{3}$$

where

$$C := \exp\left(-2\pi i \frac{c_0}{24}\right).$$

Then these matrices satisfy (cf. [35]):

Lemma 2.6.

$$SS^\dagger = TT^\dagger = \mathrm{id},$$
$$STS = T^{-1} S T^{-1},$$
$$S^2 = \hat{C},$$
$$T\hat{C} = \hat{C}T,$$

where $\hat{C}_{\lambda\mu} = \delta_{\lambda\bar{\mu}}$ *is the conjugation matrix.*

Moreover

$$N^\nu_{\lambda\mu} = \sum_\delta \frac{S_{\lambda\delta} S_{\mu\delta} S^*_{\nu\delta}}{S_{1\delta}} \tag{4}$$

is known as Verlinde formula. The commutative algebra generated by λ's with structure constants $N^\nu_{\lambda\mu}$ is called *fusion algebra* of $\mathcal{A}$. If Y is invertible, it follows from Lemma 2.6, (4) that any nontrivial irreducible representation of the fusion algebra is of the form $\lambda \to \frac{S_{\lambda\mu}}{S_{1\mu}}$ for some μ.

2.3. Induced endomorphisms

Suppose that $\rho \in \mathrm{End}(M)$ has the property that $\gamma = \rho\bar{\rho} \in \Delta_{\mathcal{A}}$. By §2.7 of [29], we can find two isometries $v_1 \in \mathrm{Hom}(\gamma, \gamma^2)$, $w_1 \in \mathrm{Hom}(1, \gamma)$[3] such that $\bar{\rho}(M)$ and v_1 generate M and

$$v_1^* w_1 = v_1^* \gamma(w_1) = d_\rho^{-1},$$
$$v_1 v_1 = \gamma(v_1) v_1.$$

By Theorem 4.9 of [29], we shall say that ρ is *local* if

$$v_1^* w_1 = v_1^* \gamma(w_1) = d_\rho^{-1}, \tag{5}$$
$$v_1 v_1 = \gamma(v_1) v_1, \tag{6}$$
$$\bar{\rho}\big(\epsilon(\gamma, \gamma)\big) v_1 = v_1. \tag{7}$$

Note that if ρ is local, then

$$\omega_\mu = 1, \quad \forall \mu \prec \rho\bar{\rho}. \tag{8}$$

For each (not necessarily irreducible) $\lambda \in \Delta_{\mathcal{A}}$, let $\varepsilon(\lambda, \gamma) : \lambda\gamma \to \gamma\lambda$ (resp. $\tilde{\varepsilon}(\lambda, \gamma)$), be the positive (resp. negative) braiding operator as defined in Section 1.4 of [42]. Denote $\lambda_\varepsilon \in \mathrm{End}(M)$ which is defined by

$$\lambda_\varepsilon(x) := ad\big(\varepsilon(\lambda, \gamma)\big)\lambda(x) = \varepsilon(\lambda, \gamma)\lambda(x)\varepsilon(\lambda, \gamma)^*,$$
$$\lambda_{\tilde{\varepsilon}}(x) := ad\big(\tilde{\varepsilon}(\lambda, \gamma)\big)\lambda(x) = \tilde{\varepsilon}(\lambda, \gamma)^*\lambda(x)\tilde{\varepsilon}(\lambda, \gamma)^*, \quad \forall x \in M.$$

By (1) of Theorem 3.1 of [42], $\lambda_\varepsilon \rho(M) \subset \rho(M)$, $\lambda_{\tilde{\varepsilon}} \rho(M) \subset \rho(M)$, hence the following definition makes sense.[4]

Definition 2.7. If $\lambda \in \Delta_{\mathcal{A}}$ define two elements of $\mathrm{End}(M)$ by

$$a_\lambda^\rho(m) := \rho^{-1}\big(\lambda_\varepsilon \rho(m)\big), \qquad \tilde{a}_\lambda^\rho(m) := \rho^{-1}\big(\lambda_{\tilde{\varepsilon}} \rho(m)\big), \quad \forall m \in M.$$

a_λ^ρ (resp. $\tilde{a}_\lambda^\rho$) will be referred to as positive (resp. negative) induction of λ with respect to ρ.

Remark 2.8. For simplicity we will use $a_\lambda, \tilde{a}_\lambda$ to denote $a_\lambda^\rho, \tilde{a}_\lambda^\rho$ when it is clear that inductions are with respect to the same ρ.

The endomorphisms a_λ are called braided endomorphisms in [42] due to their braiding properties (cf. (2) of Corollary 3.4 in [42]), and enjoy an interesting set of properties (cf. Section 3

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[3] We use v_1, w_1 instead of v, w here since v, w are used to denote sectors in Section 2.5.

[4] We have changed the notations $a_\lambda, \tilde{a}_\lambda$ of [42] to $\tilde{a}_\lambda, a_\lambda$ of this paper to make some of the formulas such as Eq. (13) simpler.

of [42]). Though [42] focus on the local case[5] which was clearly the most interesting case in terms of producing subfactors, as observed in [4–7] that many of the arguments in [42] can be generalized. These properties are also studied in a slightly different context in [4–6]. In these papers, the induction is between M and a subfactor N of M, while the induction above is on the same algebra. A dictionary between our notations here and these papers has been set up in [44] which simply use an isomorphism between N and M. Here one has a choice to use this isomorphism to translate all endomorphisms of N to endomorphims of M, or equivalently all endomorphims of M to endomorphims of N. In [44] the later choice is made (hence M in [44] will be our N below). Here we make the first choice which makes the dictionary slightly simpler. Our dictionary here is equivalent to that of [44]. Set $N = \bar{\rho}(M)$. In the following the notations from [4] will be given a subscript BE. The formulas are:

$$\rho \restriction N = i_{BE}, \quad \bar{\rho}\rho \restriction N = \bar{i}_{BE} i_{BE}, \tag{9}$$

$$\gamma = \bar{\rho}^{-1}\theta_{BE}\bar{\rho}, \quad \bar{\rho}\rho = \gamma_{BE}, \tag{10}$$

$$\lambda = \bar{\rho}^{-1}\lambda_{BE}\bar{\rho}, \quad \varepsilon(\lambda, \mu) = \bar{\rho}\big(\varepsilon^{+}(\lambda_{BE}, \mu_{BE})\big), \tag{11}$$

$$\tilde{\varepsilon}(\lambda, \mu) = \bar{\rho}\big(\varepsilon^{-}(\lambda_{BE}, \mu_{BE})\big). \tag{12}$$

The dictionary between $a_\lambda \in \mathrm{End}(M)$ in Definition 2.7 and α_λ^- as in Definitions 3.3, 3.5 of [4] is given by:

$$a_\lambda = \alpha^{+}_{\lambda_{BE}}, \qquad \tilde{a}_\lambda = \alpha^{-}_{\lambda_{BE}}. \tag{13}$$

The above formulas will be referred to as our *dictionary* between the notations of [42] and that of [4]. The proof is the same as that of [44]. Using this dictionary one can easily translate results of [42] into the settings of [4–9] and vice versa. First we summarize a few properties from [42] which will be used in this paper (cf. Theorem 3.1, Corollary 3.2 and Theorem 3.3 of [42]):

Proposition 2.9.

(1) *The maps* $[\lambda] \to [a_\lambda]$, $[\lambda] \to [\tilde{a}_\lambda]$ *are ring homomorphisms;*
(2) $a_\lambda \bar{\rho} = \tilde{a}_\lambda \bar{\rho} = \bar{\rho}\lambda$;
(3) *When* $\rho\bar{\rho}$ *is local,* $\langle a_\lambda, a_\mu\rangle = \langle \tilde{a}_\lambda, \tilde{a}_\mu\rangle = \langle a_\lambda\bar{\rho}, a_\mu\bar{\rho}\rangle = \langle \tilde{a}_\lambda\bar{\rho}, \tilde{a}_\mu\bar{\rho}\rangle$;
(4) (3) *remains valid if* a_λ, a_μ *(resp.* $\tilde{a}_\lambda, \tilde{a}_\mu$*) are replaced by their subsectors.*

Definition 2.10. H_ρ is a finite dimensional vector space over $\mathbb{C}$ with orthonormal basis consisting of irreducible sectors of $[\lambda\rho]$, $\forall \lambda \in \Delta_{\mathcal{A}}$.

$[\lambda]$ acts linearly on H_ρ by $[\lambda][a] = \sum_b \langle \lambda a, b\rangle [b]$ where $[b]$ are elements in the basis of H_ρ.[6] By abuse of notation, we use $[\lambda]$ to denote the corresponding matrix relative to the basis of H_ρ.

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[5] As we will see in Proposition 2.24, the induction with respect to nonlocal ρ is closely related to induction with respect to certain local ρ' related to ρ.

[6] By abuse of notation, in this paper we use $\sum_b$ to denote the sum over the basis $[b]$ in H_ρ.

By definition these matrices are normal and commuting, so they can be simultaneously diagonalized. Recall the irreducible representations of the fusion algebra of $\mathcal{A}$ are given by

$$\lambda \to \frac{S_{\lambda\mu}}{S_{1\mu}}.$$

Definition 2.11. Assume $\langle \lambda a, b\rangle = \sum_{\mu,i\in(\mathrm{Exp})} \frac{S_{\lambda\mu}}{S_{1\mu}} \cdot \phi_a^{(\mu,i)}\phi_b^{(\mu,i)*}$ where $\phi_a^{(\mu,i)}$ are normalized orthogonal eigenvectors of $[\lambda]$ with eigenvalue $\frac{S_{\lambda\mu}}{S_{1\mu}}$, Exp is a set of μ, i's and i is an index indicating the multiplicity of μ. Recall if a representation is denoted by 1, it will always be the vacuum representation.

The following lemma is elementary:

Lemma 2.12.

(1)
$$\sum_b d_b^2 = \frac{1}{S_{11}^2}$$

where the sum is over the basis of H_ρ. The vacuum appears once in Exp *and*

$$\phi_a^{(1)} = S_{11} d_a;$$

(2)
$$\sum_i \frac{\phi_a^{(\lambda,i)}\phi_a^{(\lambda,i)*}}{S_{1\lambda}^2} = \sum_\nu \langle \bar{\nu} a, b\rangle \frac{S_{\nu\lambda}}{S_{1\lambda}}$$

where if λ does not appear in Exp *then the right-hand side is zero.*

Proof. Ad (1): By definition we have

$$[a\bar{\rho}] = \sum_\lambda \langle a\bar{\rho}, \lambda\rangle[\lambda] = \sum_\lambda \langle a, \lambda\rho\rangle[\lambda]$$

where in the second = we have used Frobenius reciprocity. Hence

$$d_a d_{\bar{\rho}} = \sum_\lambda \langle a\bar{\rho}, \lambda\rangle d_\lambda$$

and we obtain

$$\sum_\lambda d_\lambda^2 = \sum_{\lambda,a} \langle a\bar{\rho}, \lambda\rangle d_\lambda d_a / d_\rho = \sum_a d_a^2$$

(2) follows from definition and orthogonality of S matrix. □

2.4. Relative braidings

In [42], commutativity among subsectors of $a_\lambda, \tilde{a}_\mu$ was studied. We record these results in the following for later use:

Lemma 2.13.

(1) *Let* $[b]$ *(resp.* $[b']$*) be any subsector of* a_λ *(resp.* $\tilde{a}_\lambda$*). Then*

$$[a_\mu b] = [b a_\mu], \qquad [\tilde{a}_\mu b'] = [b' \tilde{a}_\mu] \quad \forall \mu, \qquad [bb'] = [bb'];$$

(2) *Let* $[b]$ *be a subsector of* $a_\mu \tilde{a}_\lambda$*, then* $[a_\nu b] = [b a_\nu]$, $[\tilde{a}_\nu b] = [b \tilde{a}_\nu]$, $\forall \nu$.

Proof. (1) follows from (1) of Theorem 3.6 and Lemma 3.3 of [42]. (2) follows from the proof of Lemma 3.3 of [42]. Also cf. Lemma 3.20 of [6]. □

In the proof of these commutativity relations in [42], an implicit use of relative braidings was made. These braidings are further studied in [5,6] and let us recall their properties in our setting by using our dictionary (9), (13).

Let $\tilde{\beta}, \delta \in \mathrm{End}(M)$ be subsectors of $\tilde{a}_\lambda$ and a_μ. By Lemma 3.3 of [42], $[\tilde{\beta}]$ and $[\delta]$ commute. Denote $\epsilon_r(\tilde{\beta}, \delta)$ given by:

$$\epsilon_r(\tilde{\beta}, \delta) := s^* a_\mu(t^*) \bar{\rho}(\sigma_{\lambda\mu}) \tilde{a}_\lambda(s) t \in \mathrm{Hom}(\beta\delta, \delta\beta), \tag{14}$$

$$\epsilon_r(\delta, \tilde{\beta}) := \epsilon_r(\tilde{\beta}, \delta)^{-1}, \tag{15}$$

with isometries $t \in \mathrm{Hom}(\tilde{\beta}, \tilde{a}_\lambda)$ and $s \in \mathrm{Hom}(\delta, a_\mu)$. Also

$$\epsilon_r(\tilde{a}_\lambda, a_\mu) = \bar{\rho}(\sigma_{\lambda\mu}), \qquad \epsilon_r(a_\lambda, \tilde{a}_\mu) = \bar{\rho}(\tilde{\sigma}_{\lambda\mu}).$$

Lemma 2.14. *The operator* $\epsilon_r(\beta, \delta)$ *defined above does not depend on* λ, μ *and the isometries* s, t *in the sense that, if there are isometries* $x \in \mathrm{Hom}(\beta, \tilde{a}_\nu)$ *and* $y \in \mathrm{Hom}(\delta, a_{\delta_1})$*, then*

$$\epsilon_r(\beta, \delta) = s^* a_{\delta_1}(t^*) \bar{\rho}(\sigma_{\nu\lambda_1}) \tilde{a}_\nu(y) x.$$

Lemma 2.15. *The system of unitaries of Eq.* (14) *provides a relative braiding between representative endomorphisms of subsectors of* $\tilde{a}_\lambda$ *and* a_μ *in the sense that, if* $\beta, \delta, \omega, \xi$ *are subsectors of* $[\tilde{a}_\lambda]$, $[a_\mu]$, $[\tilde{a}_\nu]$, $[a_{\delta_1}]$*, respectively, then we have initial conditions*

$$\epsilon_r(\mathrm{id}_M, \delta) = \epsilon_r(\beta, \mathrm{id}_M) = 1,$$

compositions rules

$$\epsilon_r(\beta\omega, \delta) = \epsilon_r(\beta, \delta)\beta\big(\epsilon_r(\omega, \delta)\big), \qquad \epsilon_r(\beta, \delta\xi) = \delta\big(\epsilon_r(\beta, \xi)\big)\epsilon_r(\beta, \delta),$$

and naturality

$$\delta(q_+)\epsilon_r(\beta, \delta) = \epsilon_r(\omega, \delta) q_+, q_-, \quad \epsilon_r(\beta, \delta) = \epsilon_r(\beta, \xi)\beta(q_-)$$

whenever $q_+ \in \mathrm{Hom}(\beta, \omega)$ *and* $q_- \in \mathrm{Hom}(\delta, \xi)$.

For the collection of β, δ *such that* $\beta \prec a_\lambda, \beta \prec \tilde{a}_\lambda$ *and* $\delta \prec a_\mu, \delta \prec \tilde{a}_\mu$ *for some (varying)* $\lambda, \mu \in \Delta_\alpha$*, the unitaries* $\varepsilon_r(\beta, \delta), \varepsilon_r(\delta, \beta)$ *define a braiding: i.e., they verify YBE and BFE in Proposition* 2.1.

Lemma 2.16. *Let* $r \in \mathrm{Hom}(\lambda_3, \lambda_1\lambda_2)$. *Then*

$$\bar{\rho}(r) \in \mathrm{Hom}(a_{\lambda_3}, a_{\lambda_1}a_{\lambda_2}) \cap \mathrm{Hom}(\tilde{a}_{\lambda_3}, \tilde{a}_{\lambda_1}\tilde{a}_{\lambda_2}).$$

Proof. When $\rho\bar{\rho}$ is local, the lemma follows from Theorem 3.3 of [42]. Let us prove the general case. Since $a_\lambda\bar{\rho} = \bar{\rho}\lambda$, we have $\bar{\rho}(r) \in \mathrm{Hom}(a_{\lambda_3}\bar{\rho}, a_{\lambda_1\lambda_2}\bar{\rho})$. Since M is generated by $\bar{\rho}(M)$, v_1, to finish the proof we just need to check that

$$\bar{\rho}(r)a_{\lambda_3}(v_1) = a_{\lambda_1\lambda_2}(v_1)\bar{\rho}(r).$$

Since ρ is one-to-one, applying ρ to the above equation it is sufficient to check that

$$\gamma(r)\rho a_{\lambda_3}(v_1) = \rho a_{\lambda_1\lambda_2}(v_1)\gamma(r).$$

Using $\rho a_\lambda = \varepsilon(\lambda, \gamma)\lambda\rho\varepsilon(\lambda, \gamma)^*$, one can check directly that this equation follows from Proposition 2.1. □

The following is Lemma 3.25 of [4] in our setting:

Lemma 2.17. *If* $r \in \mathrm{Hom}(\bar{\rho}\lambda, \bar{\rho}\mu)$, *then*

$$r\bar{\rho}\big(\varepsilon(\mu_1, \lambda)\big) = \bar{\rho}\big(\varepsilon(\mu_1, \lambda)\big)a_{\mu_1}(r), \qquad r\bar{\rho}\big(\tilde{\varepsilon}(\mu_1, \lambda)\big) = \bar{\rho}\big(\tilde{\varepsilon}(\mu_1, \lambda)\big)\tilde{a}_{\mu_1}(r).$$

Following [8] we define

Definition 2.18. For $\lambda, \mu \in \Delta_{\mathcal{A}}$, $Z_{\lambda\mu} := \langle a_\lambda, \tilde{a}_\mu\rangle$.

We can now translate Theorems 5.7 and 6.12 of [8] into our setting:

Proposition 2.19.

(1) μ *appears in* Exp *as defined in Definition* 2.11 *with multiplicity* $Z_{\mu\mu}$*;*
(2) $Z_{\lambda\mu}$ *as a matrix commutes with* S, T *matrices as defined in Eq.* (3).

By Lemma 2.12 and Proposition 2.19 we have the following:

Lemma 2.20. *If*

$$\sum_\nu \langle \bar{\nu}a, b\rangle \frac{S_{\nu\lambda}}{S_{1\lambda}} \neq 0,$$

then $\langle a_\lambda, \tilde{a}_\lambda\rangle \geqslant 1$.

The following follows from Proposition 3.1 of [8]:

Lemma 2.21. *For any* $\lambda \in \Delta_{\mathcal{A}}, b \in H_\rho$ *we have* $\varepsilon(\lambda, b\bar{\rho}) \in \mathrm{Hom}(\lambda b, ba_\lambda)$, $\tilde{\varepsilon}(\lambda, b\bar{\rho}) \in \mathrm{Hom}(\lambda b, b\tilde{a}_\lambda)$.

Later we will consider the following analogue of S-matrix using relative braidings. Suppose that $T_\mu \in \mathrm{Hom}(a_\mu, \tilde{a}_\mu)$, $\forall \mu \in \Delta_{\mathcal{A}}$ (T_μ can be zero).

Definition 2.22. For $\mu \in \Delta_{\mathcal{A}}$, $b \in H_\rho$ irreducible, define

$$\psi_b^{(T_\mu)} := S_{11} d_b d_\mu \phi_\mu\big(\varepsilon(b\bar{\rho}, \mu) b(T_\mu) \varepsilon(\mu, b\bar{\rho})\big).$$

Lemma 2.23.

(1) $\psi_b^{(T_\mu)}$ *depends only on* $[b]$*;*

(2)
$$\sum_b \psi_b^{(T_\mu)^*}[b]$$

is either zero or an eigenvector of $[\lambda]$ *with eigenvalue* $\frac{S_{\lambda\mu}}{S_{1\mu}}$*, and* $\sum_b \psi_b^{(T_\mu)} d_b = 0$ *unless* $[\mu] = [1]$*;*

(3) *If* $T_\mu, T_{\bar{\mu}}$ *are unitaries, and for any irreducible* $\lambda \prec \mu\bar{\mu}$*,* $1 \prec a_\lambda$ *iff* $[\lambda] = [1]$*, then* $|\sum_b \psi_b^{(T_\mu)} \psi_b^{(T_{\bar{\mu}})}| = 1$*;*

(4) *If* T_μ *is unitary then* $|\psi_b^{(T_\mu)}| \leqslant S_{11} d_\mu d_b$*.*

Proof. Ad (1): Suppose that $[b_1] = [b]$ and let $U \in \mathrm{Hom}(b_1, b)$ be a unitary. We have

$$\begin{aligned}
\psi_b^{(T_\mu)} &= S_{11} d_b d_\mu \phi_\mu\big(\varepsilon(b\bar{\rho}, \mu) b(T_\mu) \varepsilon(\mu, b\bar{\rho})\big) \\
&= S_{11} d_b d_\mu \phi_\mu\big(\mu(U^*) \varepsilon(b\bar{\rho}, \mu) b\bar{\rho}(T_\mu) \varepsilon(\mu, b\bar{\rho}) \mu(U)\big) \\
&= S_{11} d_b d_\mu \phi_\mu\big(\varepsilon(b_1\bar{\rho}, \mu) U^* b(T_\mu) U \varepsilon(\mu, b_1\bar{\rho})\big) \\
&= S_{11} d_b d_\mu \phi_\mu\big(\varepsilon(b_1\bar{\rho}, \mu) b_1(T_\mu) \varepsilon(\mu, b_1\bar{\rho})\big) \\
&= \psi_{b_1}^{(T_\mu)}
\end{aligned}$$

where we have used BFE of Proposition 2.1 in the third $=$.

Ad (2): Let $t_{b,i} \in \mathrm{Hom}(b, \bar{\lambda} b')$ be isometries such that $\sum_i t_{b,i} t_{b,i}^* = 1$. Then

$$\sum_b \psi_b^{(T_\mu)} \langle b, \bar{\lambda} b' \rangle = \sum_{b,i} S_{11} d_\mu d_\lambda d_{b'} \phi_{\bar{\lambda}} \phi_\mu\big(\mu(t_{b,i}) \varepsilon(b\bar{\rho}, \mu)^* b(T_\mu) \varepsilon(\mu, b\bar{\rho}) \mu\big(t_{b,i}^*\big)\big)$$

where we have used Eq. (1). By Proposition 2.1 we have

$$\begin{aligned}
&\sum_{b,i} S_{11} d_\mu d_\lambda d_{b'} \phi_{\bar{\lambda}} \phi_\mu\big(\mu(t_{b,i}) \varepsilon(b\bar{\rho}, \mu) b(T_\mu) \varepsilon(\mu, b\bar{\rho}) \mu\big(t_{b,i}^*\big)\big) \\
&\quad = S_{11} d_\mu d_\lambda d_{b'} \phi_{\bar{\lambda}} \phi_\mu\big(\varepsilon(\bar{\lambda} b' \bar{\rho}, \mu) b(T_\mu) \varepsilon(\mu, \bar{\lambda} b' \bar{\rho})\big) \\
&\quad = \frac{S_{\bar{\lambda}\mu}}{S_{1\mu}} \psi_{b'}^{(T_\mu)}.
\end{aligned}$$

Hence

$$\sum_b [\lambda]\psi_b^{(T_\mu)^*}[b] = \sum_{b,b'} \psi_b^{(T_\mu)^*}\langle b, \bar{\lambda} b'\rangle [b'] = \frac{S_{\lambda\mu}}{S_{1\mu}} \sum_{b'} \psi_{b'}^{(T_\mu)^*}[b'].$$

By (1) of Lemma 2.12 we conclude that $\sum_b \psi_b^{(T_\mu)} d_b = 0$ unless $[\mu] = [1]$.

Ad (3): Let $t_{\lambda,i} \in \mathrm{Hom}(\lambda, \mu\bar{\mu})$ be isometries such that $\sum_{\lambda,i} t_{\lambda,i} t_{\lambda,i}^* = 1$. Then

$$\begin{aligned}
\psi_b^{(T_\mu)} \psi_b^{(T_{\bar{\mu}})} &= S_{11} d_b d_\mu \phi_{\bar{\mu}}\big(\psi_b^{(T_\mu)} \varepsilon(b\bar{\rho}, \bar{\mu}) b(T_\mu) \varepsilon(\bar{\mu}, b\bar{\rho})\big) \\
&= S_{11}^2 d_b^2 d_\mu \phi_{\mu\bar{\mu}}\big(\varepsilon(b\bar{\rho}, \mu\bar{\mu}) b\big(T_\mu a_\mu(T_{\bar{\mu}})\big) \varepsilon(\mu\bar{\mu}, b\bar{\rho})\big) \\
&= S_{11}^2 d_b \sum_{\lambda,i} d_b d_\lambda \phi_\lambda\big(\varepsilon(b\bar{\rho}, \lambda) b\big(\bar{\rho}(t_{\lambda,i})^* T_\mu a_\mu(T_{\bar{\mu}}) \bar{\rho}(t_{\lambda,i})\big) \varepsilon(\lambda, b\bar{\rho})\big)
\end{aligned}$$

where we have used Eq. (1) and Lemma 2.21 in the second = and BFE of Proposition 2.1 in the third = . By (2) of Lemma 2.23

$$\sum_b d_b d_b d_\lambda \phi_\lambda\big(\varepsilon(b\bar{\rho}, \lambda) b\big(\bar{\rho}(t_{\lambda,i})^* T_\mu a_\mu(T_{\bar{\mu}}) \bar{\rho}(t_{\lambda,i})\big) \varepsilon(\lambda, b\bar{\rho})\big) = 0$$

unless $[\lambda] = [1]$. Denote by $t_1 \in \mathrm{Hom}(1, \mu\bar{\mu})$ the unique (up to scalar) isometry. Then we have (recall we always identify the center of M with $\mathbb{C}$)

$$\sum_b \psi_b^{(T_\mu)} \psi_b^{(T_{\bar{\mu}})} = \bar{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \bar{\rho}(t_1).$$

On the other hand since $T_\mu, T_{\bar{\mu}}$ are unitaries, we have

$$\sum_{\lambda,i} \bar{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \bar{\rho}(t_{\lambda,i}) \bar{\rho}(t_{\lambda,i})^* a_\mu(T_{\bar{\mu}})^* T_\mu^* \bar{\rho}(t_1) = 1.$$

Since $\bar{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \bar{\rho}(t_{\lambda,i}) \in \mathrm{Hom}(a_\lambda, 1)$, by assumption it is 0 unless $[\lambda] = [1]$. We conclude that $|\bar{\rho}(t_1)^* T_\mu a_\mu(T_{\bar{\mu}}) \bar{\rho}(t_1)| = 1$ and (3) is proved. (4) follows since ϕ_μ is completely positive. □

Using Eqs. (9), (13), the following is a translation of Proposition 3.2 and Theorem 4.7 of [7] into our setting:

Proposition 2.24. *Suppose that $\rho\bar{\rho} \in \Delta$. Then:*

(1) *ρ is local iff $\langle 1, a_\mu\rangle = \langle \rho\bar{\rho}, \mu\rangle$, $\forall \mu \in \Delta_{\mathcal{A}}$;*

(2)
$$\rho = \rho'\rho'' = \tilde{\rho}'\tilde{\rho}''$$

where $\rho', \rho'', \tilde{\rho}', \tilde{\rho}'' \in \mathrm{End}(M)$, and $\rho', \tilde{\rho}'$ are local which verifies

$$\langle \rho'\bar{\rho}', \mu\rangle = \langle 1, a_\mu\rangle = \bigl\langle 1, a_\mu^{\rho'}\bigr\rangle,$$
$$\langle \tilde{\rho}'\overline{\tilde{\rho}'}, \mu\rangle = \langle 1, \tilde{a}_\mu\rangle = \bigl\langle 1, \tilde{a}_\mu^{\tilde{\rho}'}\bigr\rangle$$

$\forall \mu \in \Delta_{\mathcal{A}}$.

The following lemma is Proposition 3.23 of [4] (the proof was also implicitly contained in the proof of Lemma 3.2 of [42]):

Lemma 2.25. *If $\rho\bar{\rho}$ is local, then $[a_\lambda] = [\tilde{a}_\lambda]$ iff $\varepsilon(\lambda, \rho\bar{\rho})\varepsilon(\rho\bar{\rho}, \lambda) = 1$.*

2.5. Jones–Wassermann subfactors from representation of Loop groups

Let $G = SU(n)$. We denote by LG the group of smooth maps $f : S^1 \mapsto G$ under pointwise multiplication. The diffeomorphism group of the circle $\mathrm{Diff}\, S^1$ is naturally a subgroup of $\mathrm{Aut}(LG)$ with the action given by reparametrization. In particular the group of rotations $\mathrm{Rot}\, S^1 \simeq U(1)$ acts on LG. We will be interested in the projective unitary representations $\pi : LG \to U(H)$ that are both irreducible and have positive energy. This means that π should extend to $LG \rtimes \mathrm{Rot}\, S^1$ so that $H = \bigoplus_{n\geqslant 0} H(n)$, where the $H(n)$ is the eigenspace for the action of $\mathrm{Rot}\, S^1$, i.e., $r_\theta \xi = \exp(in\theta)$ for $\theta \in H(n)$ and $\dim H(n) < \infty$ with $H(0) \neq 0$. It follows from [34] that for fixed level k which is a positive integer, there are only finite number of such irreducible representations indexed by the finite set

$$P_{++}^k = \left\{ \lambda \in P \,\middle|\, \lambda = \sum_{i=1,\ldots,n-1} \lambda_i \Lambda_i,\ \lambda_i \geqslant 0,\ \sum_{i=1,\ldots,n-1} \lambda_i \leqslant k \right\}$$

where P is the weight lattice of $SU(n)$ and Λ_i are the fundamental weights. We will write $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$, $\lambda_0 = k - \sum_{1\leqslant i\leqslant n-1} \lambda_i$ and refer to $\lambda_0, \ldots, \lambda_{n-1}$ as components of λ.

We will use Λ_0 or simply 1 to denote the trivial representation of $SU(n)$. For $\lambda, \mu, \nu \in P_{++}^k$, define $N_{\lambda\mu}^\nu = \sum_{\delta\in P_{++}^k} S_\lambda^{(\delta)} S_\mu^{(\delta)} S_\nu^{(\delta*)}/S_{\Lambda_0}^{(\delta}$ where $S_\lambda^{(\delta)}$ is given by the Kac–Peterson formula (cf. Eq. (17) below for an equivalent formula):

$$S_\lambda^{(\delta)} = c \sum_{w\in S_n} \varepsilon_w \exp\bigl(i w(\delta) \cdot \lambda 2\pi/n\bigr)$$

where $\varepsilon_w = \det(w)$ and c is a normalization constant fixed by the requirement that $S_\mu^{(\delta)}$ is an orthonormal system. It is shown in [23, p. 288] that $N_{\lambda\mu}^\nu$ are nonnegative integers. Moreover, define $Gr(C_k)$ to be the ring whose basis are elements of P_{++}^k with structure constants $N_{\lambda\mu}^\nu$. The natural involution $*$ on P_{++}^k is defined by $\lambda \mapsto \lambda^* =$ the conjugate of λ as representation of $SU(n)$.

We shall also denote $S_{\Lambda_0}^{(\Lambda)}$ by $S_1^{(\Lambda)}$. Define $d_\lambda = \frac{S_1^{(\lambda)}}{S_1^{(\Lambda_0)}}$. We shall call $(S_\nu^{(\delta)})$ the S-matrix of $LSU(n)$ at level k.

We shall encounter the $\mathbb{Z}_n$ group of automorphisms of this set of weights, generated by

$$\sigma : \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \to \sigma(\lambda) = (k - 1 - \lambda_1 - \cdots \lambda_{n-1}, \lambda_1, \ldots, \lambda_{n-2}).$$

Define $\mathrm{col}(\lambda) = \Sigma_i(\lambda_i - 1)i$. The central element $\exp\frac{2\pi i}{n}$ of $SU(n)$ acts on representation of $SU(n)$ labeled by λ as $\exp(\frac{2\pi i\,\mathrm{col}(\lambda)}{n})$. The irreducible positive energy representations of $LSU(n)$ at level k give rise to an irreducible conformal net $\mathcal{A}$ (cf. [24]) and its covariant representations. We will use $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ to denote irreducible representations of $\mathcal{A}$ and also the corresponding endomorphism of $M = \mathcal{A}(I)$.

All the sectors $[\lambda]$ with λ irreducible generate the fusion ring of $\mathcal{A}$.

For λ irreducible, the univalence ω_λ is given by an explicit formula (cf. 9.4 of [34]). Let us first define $h_\lambda = \frac{c_2(\lambda)}{k+n}$ where $c_2(\lambda)$ is the value of Casimir operator on representation of $SU(n)$ labeled by dominant weight λ. h_λ is usually called the conformal dimension. Then we have: $\omega_\lambda = \exp(2\pi i h_\lambda)$. The conformal dimension of $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ is given by

$$h_\lambda = \frac{1}{2n(k+n)} \sum_{1\leqslant i\leqslant n-1} i(n-i)\lambda_i^2 + \frac{1}{n(k+n)} \sum_{1\leqslant j\leqslant i\leqslant n-1} j(n-i)\lambda_j\lambda_i + \frac{1}{2(k+n)} \sum_{1\leqslant j\leqslant n-1} j(n-j)\lambda_j. \tag{16}$$

The following form of Kac–Peterson formula for S matrix will be used later:

$$\frac{S_{\lambda\mu}}{S_{1\mu}} = Ch_{\lambda'}(x_1, \ldots, x_{n-1}, 1) \tag{17}$$

Where $Ch_{\lambda'}$ is the character associated with finite irreducible representation of $SU(n)$ labeled by λ, and $x_i = \exp(-2\pi i \frac{\mu_i'}{k+n})$, $\mu_i' = \sum_{i\leqslant j\leqslant n-1}(\mu_j + 1)$, $1 \leqslant i \leqslant n-1$.

It follows that S matrix verifies:

$$S_{\lambda\omega^i(\mu)} = \exp\left(\frac{2\pi i\,\mathrm{col}(\lambda)}{n}\right) S_{\lambda\mu}. \tag{18}$$

The following result is proved in [38] (see Corollary 1 of Chapter V in [38]).

Theorem 2.26. *Each $\lambda \in P_{++}^{(k)}$ has finite index with index value d_λ^2. The fusion ring generated by all $\lambda \in P_{++}^{(k)}$ is isomorphic to $Gr(C_k)$.*

Remark 2.27. The subfactors in the above theorem are called Jones–Wassermann subfactors after the authors who first studied them (cf. [21,38]).

Definition 2.28. $v := (1, 0, \ldots, 0)$, $v_0 := (1, 0, \ldots, 0, 1)$, $\omega^i = k\Lambda_i$, $0 \leqslant i \leqslant n-1$.

The following is observed in [16]:

Lemma 2.29. *Let $(0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith $(1 \leqslant i \leqslant n-1)$ fundamental weight. Then $[(0, \ldots, 0, 1, 0, \ldots, 0)\lambda]$ are determined as follows: $\mu \prec (0, \ldots, 0, 1, 0, \ldots, 0)\lambda$ iff when the Young diagram of μ can be obtained from Young diagram of λ by adding i boxes on i different rows of λ, and such μ appears in $[(0, \ldots, 0, 1, 0, \ldots, 0)\lambda]$ only once.*

Lemma 2.30.

(1) *If* $[\lambda] \neq \omega^i$ *for some* $0 \leqslant i \leqslant n-1$, *then* $v_0 \prec \lambda\bar{\lambda}$;
(2) *If* $\lambda_1\lambda_2$ *is irreducible, then either* λ_1 *or* $\lambda_2 = \omega^i$ *for some* $0 \leqslant i \leqslant n-1$.

Proof. By Lemma 2.29 we have that

$$\langle v\lambda, v\lambda\rangle = 1$$

iff $\lambda = \omega^i$ for some $0 \leqslant i \leqslant n-1$. By Frobenius reciprocity

$$\langle v\lambda, v\lambda\rangle = \langle 1 + v_0, \lambda\bar{\lambda}\rangle = 1 + \langle v_0, \lambda\bar{\lambda}\rangle.$$

Hence

$$\langle v_0, \lambda\bar{\lambda}\rangle = 0$$

iff $\lambda = \omega^i$ for some $0 \leqslant i \leqslant n-1$. If $\lambda_1\lambda_2$ is irreducible, then by Frobenius reciprocity again we have

$$\langle \lambda_1\bar{\lambda}_1, \lambda_2\bar{\lambda}_2\rangle = 1 \geqslant 1 + \langle v_0, \lambda_1\bar{\lambda}_1\rangle\langle v_0, \lambda_2\bar{\lambda}_2\rangle.$$

Hence either

$$\langle v_0, \lambda_1\bar{\lambda}_1\rangle = 0$$

or

$$\langle v_0, \lambda_2\bar{\lambda}_2\rangle = 0$$

and the lemma follows. □

Lemma 2.31. *Suppose* $\lambda \in \Delta_{\mathcal{A}}$ *and* λ *is not necessarily irreducible. Then*

$$\varepsilon(\lambda, v)\varepsilon(v, \lambda) \in \mathbb{C}$$

iff $[\lambda] = \sum_j [\omega^j]$ *where the summation is over a finite set.*

Proof. By Proposition 2.1 we have that

$$\varepsilon\big(v^m, \lambda\big)\varepsilon\big(\lambda, v^m\big) \in \mathbb{C}$$

for all $m \geqslant 0$. Since any irreducible μ is a subsector of v^m for some $m \geqslant 0$, by Lemma 2.3 we have that $\varepsilon(\mu, \lambda_1)\varepsilon(\lambda_1, \mu) \in \mathbb{C}$, $\forall \mu, \lambda_1 \prec \lambda$. By definition of S matrix we have $|S_{\mu\lambda_1}|^2 = |S_{1\lambda_1} d_\mu|^2$. Summing over μ we have $d_{\lambda_1} = 1$, i.e., λ_1 is an automorphism, and this implies that $v\lambda_1$ is irreducible. The lemma now follows from Lemma 2.30. □

Lemma 2.32. *For any* $m \geqslant 1$, $\mathrm{Hom}(v^m, v^m)$ *is generated as an algebra by* $1, v^i(\varepsilon(v, v))$, $1 \leqslant i \leqslant m-1$.

Proof. This is (3) of Lemma 3.1.1 in [44] and is essentially contained in [40]. □

Now let $\rho\bar{\rho} \in \Delta_{\mathcal{A}}$ where $\mathcal{A}$ is the conformal net associated with $SU(n)$ at level k, and consider induction with respect to ρ as defined in Definition 2.7. We have

Lemma 2.33.

(1) *$a_v, \tilde{a}_v$ are always irreducible;*
(2) *$d_{v_0} = 1$ iff $k = n = 2$;*
(3) *If $k \neq n \pm 2, n$, then $a_{v_0}, \tilde{a}_{v_0}$ are irreducible.*

Proof. It is enough to prove the lemma for positive induction. The negative induction case is similar. Assume that $\rho = \rho'\rho''$ as in Proposition 2.24, since $\langle a_\lambda, 1\rangle = \langle \rho'\bar{\rho}', \lambda\rangle = \langle a_\lambda^{\rho'}, 1\rangle$, $\forall\lambda$, it is enough to prove the lemma for induction with respect to ρ'. Hence we may assume that ρ is local. By (3) of Proposition 2.9 we have

$$\langle a_v, a_v\rangle = \langle \rho\bar{\rho}, v\bar{v}\rangle = 1 + \langle \rho\bar{\rho}, v_0\rangle.$$

Since $\omega_{v_0} = \exp(\frac{2\pi i n}{k+n}) \neq 1$, by Eq. (8) we conclude that $\langle \rho\bar{\rho}, v_0\rangle = 0$ and (1) is proved. (2) follows from Eq. (17).

Ad (3) By Lemma 2.29 we have

$$\begin{aligned}\big[v_0^2\big] = [1] + 2[v_0] + \big[(2,0,\dots,0,2)\big] + \big[(0,1,0,\dots,1,0)\big] + \big[(0,1,0,\dots,0,2)\big] \\ + \big[(2,0,\dots,0,1,0)\big].\end{aligned}$$

By computing the conformal dimensions of the descendants of v_0^2 using Eq. (16) we have

$$h_{(2,0,\dots,0,2)} = \frac{2+2n}{k+n}, \qquad h_{(0,1,\dots,0,2)} = h_{(2,0,\dots,1,0)} = \frac{2n}{k+n}, \qquad h_{(0,1,\dots,1,0)} = \frac{2n-2}{k+n}.$$

By Eq. (8) we conclude that if $k \neq n \pm 2, n$, then $\langle v_0^2, \rho\bar{\rho}\rangle = 1$ and (3) is proved. □

2.6. *Induced subfactors from simple current extensions*

In this section we assume that the level $k = n'n$ where $n' \geqslant 3$, and n' is an even integer if n is even. This last condition comes from [41]. For such level it is shown in §3 of [5] that there is $\rho_o \in \mathrm{End}(M)$ such that $[\rho_o\bar{\rho}_o] = \sum_{0\leqslant i\leqslant n-1}[\omega^i]$ and $\rho_o\bar{\rho}_o$ is local. It also follows from definitions that one can choose $\bar{\rho}_o\rho_o = \sum_{0\leqslant i\leqslant n-1}[g^i]$ where $[g^n] = [1]$ and $[\tilde{a}_v] = [a_v g]$ (cf. §6.1 of [24]). Also note that $[a_{\omega^i}] = [1]$, $\forall i$. The following is a consequence of Lemma 2.12 and Proposition 2.9:

Lemma 2.34. *There exists an orthonormal basis $\sum_a \phi_a^\mu [a]$ where $\mathrm{col}(\mu) = 0 \bmod n$ and the sum is over all irreducible subsectors of a_λ, $\forall\lambda$, such that*

$$\langle a_\lambda a, b\rangle = \sum_{\mu, i, \mathrm{col}(\mu)=0 \bmod n} \frac{S_{\lambda\mu}}{S_{1\mu}} \phi_a^{(\mu,i)} \phi_b^{(\mu,i)*}.$$

The following follows from Corollary 4.9 of [24]:

Lemma 2.35.

(1) *Let* λ *be irreducible and suppose* l *is the smallest positive integer with* $[\omega^l\lambda] = [\lambda]$. *Then* $[a_\lambda] = \sum_{1\leqslant i\leqslant l'}[x_i]$ *where* $l'l = n$ *and* $[g^i x_1 g^{-i}] = [x_i]$, $1 \leqslant i \leqslant l'$, $[x_i] \neq [x_j]$ *if* $i \neq j$. *Similar statements hold true for* $\tilde{a}_\lambda$;

(2) $\langle a_\lambda, a_\mu\rangle \neq 0$ *iff* $[\lambda] = [\omega^j(\mu)]$ *for some* $1 \leqslant j \leqslant n$ *iff* $[a_\lambda] = [a_\mu]$. *Similar statements hold true for* $\tilde{a}_\lambda, \tilde{a}_\mu$.

Later we will use the following analogue of Lemma 2.31:

Lemma 2.36. *If* $\varepsilon(v_0, \lambda)\varepsilon(\lambda, v_0) \in \mathbb{C}$, *then* $[\lambda] = \sum_j \omega^j$ *where the sum is over a finite set of positive integers.*

Proof. By Proposition 2.1 and Lemma 2.3 we have that $\varepsilon(v_0^m, \lambda_1)\varepsilon(\lambda_1, v_0^m) \in \mathbb{C}$ for all $m \geqslant 0$, $\lambda_1 \prec \lambda$. By Lemma 2.3 again we have $\varepsilon(\mu, \lambda_1)\varepsilon(\lambda_1, \mu) \in \mathbb{C}$ for all $\mu \prec v_0^m, \lambda_1 \prec \lambda$. Since by Lemma 2.29 any μ with $\mathrm{col}(\mu) = 0 \bmod n$ is a subsector of v_0^m for some $m \geqslant 0$, we conclude that $\varepsilon(\mu, \lambda_1)\varepsilon(\lambda_1, \mu) \in \mathbb{C}$ for all $\mu, \mathrm{col}(\mu) = 0 \bmod n$, $\lambda_1 \prec \lambda$. By the definition of S matrix we have

$$|S_{\mu\lambda_1}| = d_{\lambda_1}|S_{\mu 1}|, \quad \forall\mu, \mathrm{col}(\mu) = 0 \mod n.$$

Setting $[a] = [b] = [1]$ in Lemma 2.34 we have

$$\langle a_{\lambda_1}, a_{\lambda_1}\rangle = \sum_{\mu,i,\mathrm{col}(\mu)=0 \bmod n} d_{\lambda_1}^2 \phi_1^{(\mu,i)} \phi_1^{(\mu,i)*} = d_{\lambda_1}^2.$$

By Lemma 2.35 we have

$$d_{\lambda_1} \geqslant \langle a_{\lambda_1}, a_{\lambda_1}\rangle$$

and we conclude that $d_{\lambda_1} = 1$, and in particular $v\lambda_1$ is irreducible. The lemma now follows from Lemma 2.30. □

The subfactors $a_\lambda(M) \subset M$ are type *III* analogue of “orbifold subfactors” studied in [10] and [41].

Lemma 2.37. *If* $x \prec a_\lambda$, λ *irreducible and* $d_x = 1$, *then* $[\lambda] = [\omega^i]$, $1 \leqslant i \leqslant n$ *and* $[x] = [1]$.

Proof. If $[\lambda] \neq [\omega^i]$, $\forall i$, then by Lemma 2.30 $\lambda\bar{\lambda} \succ v_0$, and by Lemma 2.33 we have $a_\lambda a_{\bar{\lambda}} \succ a_{v_0}$. Since $x \prec a_\lambda$, $d_x = 1$, by Lemma 2.35 we conclude that $d_{a_{v_0}} = d_{v_0} = 1$. This is impossible by Lemma 2.33 and our assumption $k = n'n$, $n' \geqslant 3$. □

Let $(n', n', \ldots, n')$ be the unique fixed representation under the action of $\mathbb{Z}_n$. By Lemma 2.35

$$[a_{(n',n',\ldots,n')}] = \sum_{1\leqslant i\leqslant n} [b_i], \left[g^i b_1 g^{-i}\right] = [b_{i+1}], \quad 0 \leqslant i \leqslant n-1.$$

Definition 2.38. Denote $u := (n'+1, n', n', \dots, n')$.

Note that by Lemma 2.35 a_u is irreducible.

Lemma 2.39.

(1) $S_{uv_0} \neq 0$;
(2) *Let* $\Lambda = (n, 0, \dots, 0)$. *Then* $\langle a_\Lambda, \tilde{a}_{\bar{\Lambda}} \rangle = 0$, *and* $S_{u\Lambda} \neq 0$.

Proof. Ad (1) Since $n[a_u] = [a_v b_i]$, by Lemma 2.34

$$\frac{S_{uv_0}}{S_{1v_0}} = \frac{S_{vv_0}}{nS_{1v_0}} \frac{S_{(n',\dots,n')v_0}}{S_{1v_0}}.$$

Direct computation using Eq. (17) shows that $\frac{S_{vv_0}}{S_{1v_0}} \neq 0$. Note that by Eq. (18)

$$\frac{S_{(n',\dots,n')v}}{S_{1v}} = 0$$

since $\mathrm{col}(v) = 1$, hence

$$\frac{S_{(n',\dots,n')v_0}}{S_{1(n',\dots,n')}} = -1$$

and this implies that $S_{(n',\dots,n')v_0} \neq 0$ and (1) is proved.

Ad (2) Since $k = n'n \geqslant 3n$, it follows that $\langle \omega^j \Lambda, \bar{\Lambda} \rangle = 0$, $\forall 1 \leqslant j \leqslant n$. By Lemma 2.35 $\langle a_\Lambda, \tilde{a}_{\bar{\Lambda}} \rangle = 0$. Since $[a_v a_{(n',n',\dots,n')}] = n[a_u]$, by Lemma 2.34 we have

$$n\frac{S_{u\Lambda}}{S_{1\Lambda}} = \frac{S_{v\Lambda}}{S_{1\Lambda}} \frac{S_{(n',\dots,n')\Lambda}}{S_{1\Lambda}}.$$

Hence to finish the proof we just have to check that $S_{v\Lambda} \neq 0$, $S_{(n',\dots,n')\Lambda} \neq 0$. Since $Ch_{v'}(x_1, \dots, x_n) = \sum_{1\leqslant i\leqslant n} x_i$, by Eq. (17) up to a nonzero constant $S_{v\Lambda}$ is equal to

$$\exp\bigl(-2\pi i(2n-1)/(k+n)\bigr) + \sum_{0\leqslant j\leqslant n-2} \exp\bigl(-2\pi i j/(k+n)\bigr).$$

This sum is equal to 0 iff $n = k = 2$. Note that $Ch_{\Lambda'}(x_1, \dots, x_n)$ is a complete symmetric polynomial of degree n. $S_{v\Lambda} \neq 0$ now follows directly from Eq. (17) (cf. (2.7a) of [14] for more general statement). □

The main theorem of this section is:

Theorem 2.40. *The lattice of intermediate subfactors of* $a_u(M) \subset M$ *is* M_{2n}.

The proof will be given in Section 4. Let us first show that the subfactor in Theorem 2.40 contains $2n$ incomparable intermediate subfactors. By fusion rule with v in Lemma 2.29 we have

$$[a_u] = [a_v b_i] = [b_i a_v], \quad \forall 1 \leqslant i \leqslant n.$$

Therefore we can assume that

$$a_u = U_i a_v b_i U_i^* = V_i b_i a_v V_i^*, \quad 1 \leqslant i \leqslant n,$$

where U_i, V_i are unitaries.

Proposition 2.41.

(1) *As von Neumann algebras*

$$U_i a_v(M) U_i^* = U_j a_v(M) U_j^*, \qquad V_i b_i(M) V_i^* = V_j b_j(M) V_j^*$$

iff $i = j$;

(2) $U_i a_v(M) U_i^*$ *is not an intermediate subfactor in* $V_j b_j(M) V_j^* \subset M$;

(3) $V_j b_j(M) V_j^*$ *is not an intermediate subfactor in* $U_i a_v(M) U_i^* \subset M$.

Proof. Ad (1): If $U_i a_v(M) U_i^* = U_j a_v(M) U_j^*$, then $U_i a_v(m) U_i^* = U_j a_v(\theta(m)) U_j^*$, $\forall m \in M$, where θ is an automorphism of M. By Frobenius reciprocity we have $[\theta] \prec [a_v a_{\bar{v}}]$. By Lemma 2.37 we conclude that $[\theta] = [1]$ and hence

$$U_i a_v(m) U_i^* = U_j a_v(U) a_v(m) a_v(U)^* U_j^*, \quad \forall m \in M,$$

for some unitary $U \in M$. Hence

$$Ad_{U_i} a_v b_i = Ad_{U_j a_v(U)} a_v b_i = Ad_{U_j} a_v b_j$$

and we conclude that $[b_i] = [b_j]$, hence $i = j$. The second statement in (1) is proved similarly.

Ad (2): If $U_i a_v(M) U_i^*$ is an intermediate subfactor in $V_j b_j(M) V_j^* \subset M$, then $Ad_{V_j} b_j = Ad_{U_i} a_v C$ for some $C \in \operatorname{End}(M)$, and it follows that $[b_j \bar{b}_j] \succ [a_v \bar{a}_v] \succ [a_{v_0}]$. Hence

$$\langle a_v b_j, a_v b_j \rangle = \langle b_j \bar{b}_j, a_v \bar{a}_v \rangle \geqslant 2$$

contradicting the irreducibility of $[a_u] = [a_v b_j]$.

Ad (3): If $V_j b_j(M) V_j^*$ is an intermediate subfactor in $U_i a_v(M) U_i^* \subset M$, then there is $C' \in \operatorname{End}(M)$ such that $[b_j C'] = [a_v]$. Since $[a_v \bar{a}_v] = [1] + [a_{v_0}]$ and a_{v_0} is irreducible by Lemma 2.33, we must have $[b_j \bar{b}_j] = [a_v \bar{a}_v]$ and therefore $d_{C'} = 1$. By Frobenius reciprocity $C' \prec [\bar{b}_j a_v]$, but $[\bar{b}_j a_v]$ is irreducible since a_u is irreducible, a contradiction. □

Here we give a quick proof of Theorem 2.40 for $n = 2$ and $k \neq 10, 28$ to illustrate some ideas behind the proof. Suppose that M_1 is an intermediate subfactor of $a_u(M) \subset M$. Since all factors in this paper are isomorphic to hyperfinite type III_1 factor, we can find $c_1, c_2 \in \operatorname{End}(M)$ such that $a_u = c_1c_2$ and $c_1(M) = M_1$. Let $\rho = \rho_0 c_1$, and enumerate the basis of H_ρ by irreducible sectors a. Note that a must be of the form $\rho_0 c$ with c irreducible, and so $d_a \geqslant d_{\rho_0} = \sqrt{2}$.

Consider the fusion graph associated with the action of v on H_ρ: the vertices of this graph are irreducible sectors a, and vertices a and b are connected by $\langle va, b\rangle$ edges. By Lemma 2.12 this graph is connected and has norm $2\cos(\frac{\pi}{k+2})$, and hence it must be $A-D-E$ graph (cf. Chapter 1 of [17]). Since $k \neq 10, 28$ it must be A or D graph. By Lemma 2.12 we have $\sum_a d_a^2 = \frac{1}{S_{11}^2} = \frac{1}{\frac{1}{k+2}\sin^2(\frac{\pi}{k+2})}$. Since $d_a \geqslant d_{\rho_0} = \sqrt{2}$ are the entries of Perron–Frobenius eigenvector for the graph (such eigenvector is unique up to a positive scalar), compare with the eigenvectors of $A-D-E$ graphs listed for example in Chapter 1 of [17]) we conclude that the graph is D graph and there is a sector c with $d_c = 1$ and $c_1 \prec a_\mu c$ for some $\mu \in \Delta$. We conclude that either $[c_1] = [a_\mu c]$, or $[c_1] = [b_i c]$, $1 \leqslant i \leqslant 2$. In the former case $[c_2] = [c^{-1}a_\lambda]$ or $[c_2] = [c^{-1}b_j]$, $1 \leqslant j \leqslant 2$. But if $[c_2] = [c^{-1}a_\lambda]$ then $[a_u] = [a_\mu a_\lambda]$ is irreducible, and by Lemma 2.30 $[a_\mu] = [a_u]$ or $[a_\mu] = [1]$, which implies that M_1 is either $a_u(M)$ or M. If $[c_2] = [c^{-1}b_j]$, $1 \leqslant j \leqslant 2$, then $[a_u] = [a_\mu b_j]$ and by computing the index and note that the colors of u and b_j are 1 mod 2, 0 mod 2 respectively we have $a_\mu = a_v$, and we conclude that M_1 must be one of the intermediate subfactors given in Proposition 2.41. The case of $[c_1] = [b_i c]$, $1 \leqslant i \leqslant 2$ is treated similarly. By Proposition 2.41 we have proved Theorem 2.40 for $n = 2$, $k \neq 10, 28$. The same idea as presented above can be used to give a complete list of all intermediate subfactors of Goodman–Harpe–Jones subfactors. We hope to discuss this and related problems elsewhere.

3. Centrality of a class of intertwiners and its consequences

We preserve the setup of Section 2.5.

Assume that $\rho\bar{\rho} \in \Delta_{\mathcal{A}}$. We will investigate a class of inductions which are motivated by finding a proof of Theorem 2.40.

In this section we assume that $[a_v] = [h\tilde{a}_v], [h^n] = [1]$, a_{v_0} is irreducible, and if $\mu \prec v_0^2$, $1 \prec a_\mu$, then $[\mu] = [1]$.

Choose a unitary $T \in \mathrm{Hom}(a_v, h\tilde{a}_v)$. Such T is unique up to scalar since a_v is irreducible. By Lemma 2.13 we have $[h\tilde{a}_v] = [\tilde{a}_v h]$. Choose a unitary $T_1 \in \mathrm{Hom}(\tilde{a}_v h, h\tilde{a}_v)$. Note that T_1 is unique up to scalar since $h\tilde{a}_v$ is irreducible.

Definition 3.1. Denote $U_n := Ta_v(T)a_v^2(T)\ldots a_v^{n-1}(T) \in \mathrm{Hom}(a_v^n, (h\tilde{a}_v)^n)$.

Denote $T_i := T_1\tilde{a}_v(T_1)\ldots\tilde{a}_v^{i-1}(T_1) \in \mathrm{Hom}(\tilde{a}_v^i h, h\tilde{a}_v^i)$, $1 \leqslant i \leqslant n-1$.

Choose $T' \in \mathrm{Hom}(h^n, 1)$ (T' is unique up to scalar).

Definition 3.2. Set $w = v^n$ and define $u_w := T'h^{n-1}(T_{n-1})h^{n-2}(T_{n-2})\ldots h(T_1)U_n \in \mathrm{Hom}(a_v^n, \tilde{a}_v^n)$.

For example when $n = 3$, $u_w = T'h^2(T_1)h^2(\tilde{a}_v(T_1))h(T_1)Ta_v(T)a_v^2(T)$. The reader is encouraged to give a diagrammatic representation of u_w as in [42].

Lemma 3.3. *Suppose that* x, y *are sectors such that*

$$[x] = \sum_{1\leqslant i\leqslant m} [x_i], \qquad [y] = \sum_{1\leqslant i\leqslant m} [y_i], \quad d_{x_i} < d_{x_j},\ d_{y_i} < d_{y_j}$$

if $i < j$, and x_i, y_i are irreducible. Let $T_{x,i} \in \mathrm{Hom}(x_i, x), T_{y,i} \in \mathrm{Hom}(y_i, y), i = 1, \ldots, m$ be isometries.

If $U \in \mathrm{Hom}(x, y)$ is unitary then $UT_{x,i}T^*_{x,i}U^* = T_{y,i}T^*_{y,i}, i = 1, \ldots, m$.

Proof. By assumption $\mathrm{Hom}(x, x), \mathrm{Hom}(y, y)$ are finite dimensional abelian algebras, and so for each $1 \leqslant i \leqslant m$ we have $UT_{x,i}T^*_{x,i}U* = T_{y,j}T^*_{y,j}$ for some j.

By Eq. (1) we have

$$d_y\phi_y(UT_{x,i}T^*_{x,i}U^*) = d_x\phi_x(T_{x,i}T^*_{x,i}) = d_{x_i}.$$

Hence $d_{x_i} = d_{y_j}$. By assumption it follows that $i = j$, $1 \leqslant i \leqslant m$. □

Lemma 3.4. *Let $U \in \mathrm{Hom}(a_v^2h^j, h^i\tilde{a}_v^2), i, j \geqslant 0$ be a unitary. Then $h^i(\bar{\rho}(\varepsilon(v, v)))U = U\bar{\rho}(\varepsilon(v, v))$.*

Proof. Since a_{v_0} is irreducible, we have $\langle a_va_v, a_va_v\rangle = \langle a_v\bar{a}_v, a_v\bar{a}_v\rangle = 2$. We note that $[a_va_v] = [a_{(2,0,\ldots,0)}] + [a_{(0,1,0,\ldots,0)}]$ and $\frac{d_{a_{(2,0,\ldots,0)}}}{d_{a_{(2,0,\ldots,0)}}} = \frac{\sin(\frac{(n+1)\pi}{k+n})}{\sin(\frac{(n-1)\pi}{k+n})} > 1$ and so the assumption of Lemma 3.3 is verified. Denote by $P_1, P_2 \in \mathrm{Hom}(v^2, v^2)$ the two different minimal projections corresponding to $(2, 0, \ldots, 0), (0, 1, \ldots, 0)$ respectively. Note that $\bar{\rho}(P_l), h^i(\bar{\rho}(P_l)), l = 1, 2$, are minimal projections in $\mathrm{Hom}(a_v^2h^j, a_v^2h^j)$, $\mathrm{Hom}(h^i\tilde{a}_v^2, h^i\tilde{a}_v^2)$ respectively and by Lemma 3.3 we have $U^*h^i(\bar{\rho}(P_l))U = \bar{\rho}(P_l), l = 1, 2$.

Assume that $\varepsilon(v, v) = z_1P_1 + z_2P_2$ where $z_1, z_2 \in \mathbb{C}$ (cf. Lemma 3.1.1 in [44] for explicit formulas for z_1, z_2). Then $h^i(\bar{\rho}(\varepsilon(v, v))) = z_1h^i(\bar{\rho}(P_1)) + z_2h^i(\bar{\rho}(P_2))$ and the lemma follows. □

Lemma 3.5. $\tilde{a}_v^i(\bar{\rho}(\varepsilon(v, v)))u_w = u_wa_v^i(\bar{\rho}(\varepsilon(v, v))$, $0 \leqslant i \leqslant n - 2$.

Proof. By Definition 3.2 we can write $u_w = V_1'V_2'V_3'$ where

$$V_3' = a_v^{i+2}(V_3), \quad V_3 = h^{n-i-3}(T_{n-i-3})\ldots h^2(T_2)h(T_1) \in \mathrm{Hom}\big(a_v^{n-i-2}, h^{n-i-2}\tilde{a}_v^{n-i-2}\big).$$

$$V_2' = a_v^i(V_2), \quad V_2 = h^{n-i-1}(T_2)\ldots h^2(T_2)h(T_1)Ta_v(T) \in \mathrm{Hom}\big(a_v^2h^{n-i-2}, h^{n-i}\tilde{a}_v^2\big)$$

and

$$\begin{aligned}V_1' &= T'h^{n-1}(T_i)\ldots h^i(T_i)h^{i-1}(T_{i-1})h^{i-2}(T_{i-2})\ldots h(T_1)Ta_v(T)\ldots a_v^{i-1}(T)\\ &\in \mathrm{Hom}\big(a_v^ih^{n-i}, \tilde{a}_v^i\big).\end{aligned}$$

Although the complicated but explicit formulas of V_1', V_2, V_3 are given above, we only use their intertwining properties in what follows.

Hence

$$\begin{aligned}\tilde{a}_v^i\big(\bar{\rho}\big(\varepsilon(v, v)\big)\big)u_w &= \tilde{a}_v^i\big(\bar{\rho}\big(\varepsilon(v, v)\big)\big)V_1'a_v^i(V_2)a_v^{i+2}(V_3)\\ &= V_1'a_v^i\big(h^{n-i}\big(\bar{\rho}\big(\varepsilon(v, v)\big)\big)V_2\big)a_v^{i+2}(V_3)\\ &= V_1'a_v^i\big(V_2\bar{\rho}\big(\varepsilon(v, v)\big)\big)a_v^{i+2}(V_3)\end{aligned}$$

$$
\begin{aligned}
&= V_1' a_v^i(V_2) a_v^i\big(\bar{\rho}\big(\varepsilon(v,v)\big) a_v^2(V_3)\big)\\
&= V_1' a_v^i(V_2) a_v^{i+2}(V_3) a_v^i\big(\bar{\rho}\big(\varepsilon(v,v)\big)\big)\\
&= u_w a_v^i\big(\bar{\rho}\big(\varepsilon(v,v)\big)\big)
\end{aligned}
$$

where in the third $=$ we have used Lemma 3.4. □

Lemma 3.6. $\tilde{a}_v^{n-1}(\bar{\rho}(\varepsilon(v,v)))u_w a_w(u_w) = u_w a_w(u_w) a_v^{n-1}(\bar{\rho}(\varepsilon(v,v)))$.

Proof. By Definition 3.2 we can write $u_w a_w(u_w) = W_1' W_2' W_3'$ where $W_3' = a_v^{n+1}(W_3)$, $W_3 = h^{n-2}(T_{n-2}) \ldots h(T_2) h(T_1) T a_v(T) \ldots a_v^{n-2}(T) \in \mathrm{Hom}(a_v^{n-1}, h^{n-1}\tilde{a}_v^{n-1})$, $W_2' = a_v^{n-1}(W_2)$, $W_2 = T T' h^{n-1}(T_1) \ldots h(T_1) a_v(T) \in \mathrm{Hom}(a_v^2 h^{n-1}, h\tilde{a}_v^2)$ and $W_1' = T' h^{n-1}(T_n) h^{n-2}(T_{n-2}) \ldots h(T_1) T a_v(T) \ldots a_v^{n-2}(T) \in \mathrm{Hom}(a_v^{n-1}h, \tilde{a}_v^{n-1})$.

As in the proof of Lemma 3.5, even though explicit formulas of W_2, W_3, W_1' are given as above, what we need in the following is their intertwining properties.

Hence

$$
\begin{aligned}
\tilde{a}_v^{n-1}\big(\bar{\rho}\big(\varepsilon(v,v)\big)\big) u_w \tilde{a}_w(u_w) &= \tilde{a}_v^{n-1}\big(\bar{\rho}\big(\varepsilon(v,v)\big)\big) W_1' a_v^{n-1}(W_2) a_v^{n+1}(W_3)\\
&= W_1' a_v^{n-1}\big(h\big(\bar{\rho}\big(\varepsilon(v,v)\big)\big) W_2\big) a_v^{n+1}(W_3)\\
&= W_1' a_v^{n-1}(W_2) a_v^{n-1}\big(\bar{\rho}\big(\varepsilon(v,v)\big) a_v^2(W_3)\big)\\
&= W_1' a_v^{n-1}(W_2) a_v^{n+1}(W_3) a_v^{n-1}\big(\bar{\rho}\big(\varepsilon(v,v)\big)\big)\\
&= u_w a_w(u_w) a_v^{n-1}\big(\bar{\rho}\big(\varepsilon(v,v)\big)\big)
\end{aligned}
$$

where in the third $=$ we have used Lemma 3.4. □

Definition 3.7. For each integer $m \geqslant 1$, $u_{w^m} := u_w a_w(u_w) \ldots a_w^{m-1}(u_w) \in \mathrm{Hom}(a_{w^m}, \tilde{a}_{w^m})$.

Theorem 3.8. *Let $m \geqslant 1$ be any integer and $R \in \mathrm{Hom}(w^m, w^m)$. Then*

$$\bar{\rho}(R) u_{w^m} = u_{w^m} \bar{\rho}(R).$$

Proof. By Lemma 2.32 it is sufficient to prove the theorem for $R = v^{m'}(\bar{\rho}(\varepsilon(v,v)))$, $1 \leqslant m' \leqslant m-1$. When $nn_1 < m' < n(n_1+1)$, $n_1 \in \mathbb{Z}$ we can write

$$u_w a_w(u_w) \ldots a_w^{m-1}(u_w) = U_1' a_w^{n_1}(u_w) U_2'$$

where $U_1' \in \mathrm{Hom}(a_w^{n_1}, a_w^{n_1})$, $U_2' \in a_w^{n_1+1}(M)$ and the theorem follows from Lemma 3.5. Similarly when $m' = nn_1$, $n_1 \in \mathbb{Z}$ we can write

$$u_w a_w(u_w) \ldots a_w^{m-1}(u_w) = U_1'' a_w^{n_1-1}\big(u_w a_w(u_w)\big) U_2''$$

with $U_1'' \in \mathrm{Hom}(a_w^{n_1-1}, a_w^{n_1-1})$, $U_2'' \in a_w^{n_1+2}(M)$ and the theorem follows from Lemma 3.6. □

Lemma 3.9. *Suppose that $\mu \prec w^m$ are irreducible and let $t_{\mu,i} \in \mathrm{Hom}(\mu, w^m)$, $m \geqslant 1$ be a set of isometries such that $\sum_{\mu,i} t_{\mu,i} t_{\mu,i}^* = 1$. Then*

(1) *For each fixed* μ, $\bar\rho(t_{\mu,i})^* u_{w^m}\bar\rho(t_{\mu,i}) \in \mathrm{Hom}(a_\mu, \tilde a_\mu)$ *is independent of choices of* $t_{\mu,i}$*;*
(2) $\bar\rho(t_{\mu,i})^* u_{w^m}\bar\rho(t_{\mu,i}) \in \mathrm{Hom}(a_\mu, \tilde a_\mu)$ *is unitary.*

Proof. (1) follows immediately from Theorem 3.8. To prove (2), note that for each fixed μ, i

$$1 = \sum_{\lambda,j} \bar\rho(t_{\mu,i})^* u_{w^m}\bar\rho(t_{\lambda,j})\bar\rho(t_{\lambda,j})^* u^*_{w^m}\bar\rho(t_{\mu,i})^* = \bar\rho(t_{\mu,i})^* u_{w^m}\bar\rho(t_{\mu,i})\bar\rho(t_{\mu,i})^* u^*_{w^m}\bar\rho(t_{\mu,i})$$

where in the second $=$ we have used Theorem 3.8. Similarly

$$1 = \bar\rho(t_{\mu,i})^* u^*_{w^m}\bar\rho(t_{\mu,i})\bar\rho(t_{\mu,i})^* u_{w^m}\bar\rho(t_{\mu,i})$$

and the proposition is proved. □

The unitary in (2) of Proposition 3.9 will be denoted by u_μ (it may depend on m) in the following.

Definition 3.10. Let $\mu \in \Delta_{\mathcal{A}}$ and $b \in H_\rho$ be irreducible. Define

$$\psi_b^{(w)} := S_{11} d_b d_w \phi_w\big(\varepsilon(b\bar\rho, w) b(u_w)\varepsilon(w, b\bar\rho)\big), \quad b \in H_\rho.$$

Lemma 3.11. *Let* $m \geqslant 1$ $t_{\mu,i}$ *be as in Proposition* 3.9. *Then*

$$\left|\sum_b d_b^2 \left(\frac{\psi_b^{(w)}}{d_b S_{11}}\right)^m\right| = \frac{1}{S_{11}^2}\langle w^m, 1\rangle, \quad \forall m \geqslant 1.$$

Proof.

$$\begin{aligned}
\left(\frac{\psi_b^{(w)}}{d_b S_{11}}\right)^m &= d_w^m \phi_w^m\big(\varepsilon\big(b\bar\rho, w^m\big) b(u_{w^m})\varepsilon\big(w^m, b\bar\rho\big)\big)\\
&= \sum_{\mu,i} d_\mu \phi_\mu\big(t^*_{\mu,i}\varepsilon\big(b, w^m\big) b(u_{w^m})\varepsilon\big(w^m, b\bar\rho\big)t_{\mu,i}\big)\\
&= \sum_{\mu,i} d_\mu \phi_\mu\big(\varepsilon(b\bar\rho, \mu) b\big(\bar\rho(t_{\mu,i})^* u_{w^m}\bar\rho(t_{\mu,i})\big)\varepsilon(\mu, b\bar\rho)\big)\\
&= \sum_\mu \langle \mu, w^m\rangle d_\mu \phi_\mu\big(\varepsilon(b\bar\rho, \mu) u_\mu \varepsilon(\mu, b\bar\rho)\big)
\end{aligned}$$

where we have used definition of minimal left inverse in the first $=$, Eq. (1) in the second $=$, Proposition 2.1 in the third $=$, and Lemma 3.9 in the last $=$.

It follows that

$$\sum_b d_b^2 \left(\frac{\psi_b^{(w)}}{d_b S_{11}}\right)^m = \sum_{b,\mu} \langle \mu, w^m\rangle d_b^2 d_\mu \phi_\mu\big(\varepsilon(b\bar\rho, \mu) u_\mu \varepsilon(\mu, b\bar\rho)\big)$$

$$
\begin{aligned}
&= \sum_{\mu} \langle \mu, w^m \rangle d_\mu \sum_b d_b d_b \phi_\mu\big(\varepsilon(b\bar\rho, \mu) b(u_\mu) \varepsilon(\mu, b\bar\rho)\big) \\
&= \sum_b d_b^2 \phi_1(u_1) \langle 1, w^m \rangle
\end{aligned}
$$

where we have used Lemma 2.23 in the third $=$. Since $u_1 \in \mathrm{Hom}(1,1)$ is unitary by Proposition 3.9, $|\phi_1(u_1)| = 1$ and we have proved that

$$\left| \sum_b d_b^2 \left(\frac{\psi_b^{(w)}}{d_b S_{11}} \right)^m \right| = \frac{1}{S_{11}^2} \langle w^m, 1 \rangle.$$

Proposition 3.12. *There is a sector $c \in H_\rho$ such that $|\frac{\psi_c^{(w)}}{S_{11}}| = d_c d_w$.*

Proof. By Lemma 3.11 we have

$$\left| \sum_b d_b^2 \left(\frac{\psi_b^{(w)}}{d_b S_{11}} \right)^m \right| = \frac{1}{S_{11}^2} \langle w^m, 1 \rangle, \quad \forall m \geqslant 1.$$

By repeatedly using Verlinde formula we have

$$\langle w^m, 1 \rangle = \sum_\mu \frac{1}{S_{1\mu}^2} \left(\frac{S_{v\mu}}{S_{1\mu}} \right)^{nm}.$$

By Lemma 2.31, when m goes to infinity, the leading order of $|\sum_b d_b^2 (\frac{\psi_b^{(w)}}{d_b S_{11}})^m|$ must be $n d_w^m$. Note that by Lemma 2.23 $|\frac{\psi_b^{(w)}}{d_b S_{11}}| \leqslant d_w$. It follows that there is a sector $c \in H_\rho$ such that $|\frac{\psi_c^{(w)}}{S_{11}}| = d_c d_w$. □

Choose $m = 1$ and let $t_{\mu,i}$ be isometries as in Lemma 3.9.

Definition 3.13. Assume that $\mu \in \Delta_{\mathcal{A}}$ and $[b] \in H_\rho$ is irreducible. Define

$$\frac{\psi_b^{(\mu)}}{S_{11}} := d_b d_\mu \phi_\mu\Big(\big(\varepsilon(b\bar\rho, \mu) b\big(\bar\rho(t_{\mu,i})^* u_w \bar\rho(t_{\mu,i})\big) \varepsilon(\mu, b\bar\rho)\big)\Big).$$

Note that by Lemma 3.9 $\psi_b^{(\mu)}$ is independent of the choice of i.

Corollary 3.14. *Assume that $[a_v] = [h\tilde a_v]$, $[h^n] = [1]$, a_{v_0} is irreducible, and if $\mu \prec v_0^2$, $1 \prec a_\mu$, then $[\mu] = [1]$. Then there is $[c] \in H_\rho$ such that $|\frac{\psi_c^{(\lambda)}}{S_{11}}| = d_c d_\lambda$, $\forall \lambda, \mathrm{col}(\lambda) = 0 \bmod n$ and $[c\bar c] = \sum_{1 \leqslant i_2 \leqslant \frac{n}{i_1}} [\omega^{i_2 i_1}]$ where i_1 is a divisor of n.*

Proof. Choose $m = 1$ and let $t_{\mu,i}$ be isometries as in Lemma 3.9. By Eq. (1) we have

$$\frac{\psi_c^{(w)}}{S_{11}} = \sum_\mu \langle \mu, w\rangle \frac{\psi_c^{(\mu)}}{S_{11}}.$$

By Lemma 2.23 we have

$$\left|\frac{\psi_c^{(\mu)}}{S_{11}}\right| \leqslant d_c d_\mu.$$

By Proposition 3.12 we conclude that

$$\left|\frac{\psi_c^{(\mu)}}{S_{11}}\right| = d_c d_\mu, \quad \forall \mu \prec w.$$

In particular $|\frac{\psi_c^{(v_0)}}{S_{11}}| = d_c d_{v_0}$. By Lemma 2.23 we know that $\sum_b \psi_b^{(v_0)^*}[b]$ is a nonzero eigenvector of the action of $[\lambda]$ on H_ρ. Since $\langle a_{v_0}, \tilde{a}_{v_0}\rangle = 1$, by Proposition 2.19 we must have $\psi_b^{(v_0)} = z\phi_b^{(v_0)}$, for some constant z independent of b. Since $[\bar{v}_0] = [v_0]$, $\sum_b \phi_b^{(v_0)} b$ is also an eigenvector of the action of $[\lambda]$ with eigenvalue $\frac{S_{\lambda v_0}}{S_{1v_0}}$, it follows that $\phi_b^{(v_0)^*} = z'\phi_b^{(v_0)}$, for some constant $|z'| = 1$ independent of b. Hence

$$\sum_b \psi_b^{(v_0)^2} = \sum_b z^2 \bar{z}' \phi_b^{(v_0)} \phi_b^{(v_0)^*} = z^2 \bar{z}'.$$

By (3) of Lemma 2.23 and our assumption we conclude that $|z| = 1$, and so by Lemma 2.12 we have

$$d_c^2 = \left|\frac{\psi_c^{(v_0)}}{S_{1v_0}}\right|^2 = \left|\frac{\phi_c^{(v_0)}}{S_{1v_0}}\right|^2 = \sum_\mu \langle c\bar{c}, \mu\rangle \frac{S_{\mu v_0}}{S_{1v_0}}.$$

Since $\frac{S_{\mu v_0}}{S_{1v_0}} \leqslant d_\mu$, we must have $\frac{S_{\mu v_0}}{S_{1v_0}} = d_\mu$, $\forall \mu \prec c\bar{c}$.

By Lemma 2.36 we conclude that if $\mu \prec c\bar{c}$, then $\mu = \omega^i$ for some $1 \leqslant i \leqslant n$. Let $1 \leqslant i_1 \leqslant n$ be the smallest positive integer such that $[\omega^{i_1} c] = [c]$. Then it is clear that $[c\bar{c}] = \sum_{1 \leqslant i_2 \leqslant \frac{n}{i_1}} [\omega^{i_2 i_1}]$ where i_1 is a divisor of n. □

4. Proof of Theorem 2.40

In this section we preserve the setting of Section 2.6. Let $c_1, c_2 \in \mathrm{End}(M)$ such that $a_u = c_1 c_2$, $c_1(M) = M_1$, $M_1 \neq a_u(M)$, M. By Proposition 2.41 to prove Theorem 2.40 it is enough to show that M_1 is one of the intermediate subfactors in Proposition 2.41.

4.1. Local consideration

Suppose c is a sector such that $c\bar{c} \prec a_\mu^{\rho_o}$ where $\mu \in \Delta_{\mathcal{A}}$ is a direct sum of irreducible sectors with colors divisible by n. Recall from Section 2.6 that if $\lambda = 0 \bmod n$, then $[a_\lambda^{\rho_o}] = [\tilde{a}_\lambda^{\rho_o}]$, and we can apply induction of $a_\lambda^{\rho_o}$ with respect to c. The following lemma is proved by a translation of the proof of (3) of Lemma 3.3 in [43] into our setting:

Lemma 4.1. *If* $\lambda = 0 \bmod n$*, then* $[a^c_{a_\lambda^{\rho_o}}] = [a_\lambda^{\rho_o c}]$.

By Proposition 2.24 we have $c_1 = c_1' c_1''$. Let $c_2' = c_1'' c_2$ so that $a_u = c_1' c_2'$. Consider induction with respect to $\rho_o c_1'$.

We have

Lemma 4.2. $[c_1' \bar{c}_1'] = [1]$.

Proof. Applying Lemma 2.12 to $a = \rho_0 c_1'$, $b = \rho_0 \bar{c}_2'$ we have

$$\sum_i \frac{\phi_a^{(\lambda,i)} \phi_b^{(\lambda,i)*}}{S_{1\lambda}^2} = \sum_\nu \langle \rho_0 c_1' c_2' \bar{\rho}_0, \nu \rangle \frac{S_{\nu\lambda}}{S_{1\lambda}} = \sum_\nu \langle u \rho_0 \bar{\rho}_0, \nu \rangle \frac{S_{\nu\lambda}}{S_{1\lambda}} = \sum_{1 \leqslant i \leqslant n} \exp\left(\frac{2\pi i \operatorname{col}(\lambda)}{n}\right) \frac{S_{u\lambda}}{S_{1\lambda}}.$$

Choosing $\lambda = v_0$ and using Lemma 2.39 we have

$$\sum_i \frac{\phi_a^{(\lambda,i)} \phi_b^{(\lambda,i)*}}{S_{1\lambda}^2} \neq 0.$$

Hence by Lemma 2.20 we obtain $\langle a_{v_0}^{\rho_0 c_1'}, \tilde{a}_{v_0}^{\rho_o c_1'} \rangle \geqslant 1$. For any $\mu \in \Delta_{\mathcal{A}}$, since $\rho_0 c_1' \bar{c}_1' \bar{\rho}_o \prec \rho_o a_{u\bar{u}} \bar{\rho}_o$ and each irreducible sector of $[\rho_o a_{u\bar{u}} \bar{\rho}_o] = [\rho_0 \bar{\rho}_o u \bar{u}]$ has color divisible by n, it follows that if $\operatorname{col}(\mu) \neq 0 \bmod n$, then $\langle \mu, \rho_0 c_1' \bar{c}_1' \bar{\rho}_o \rangle = 0$. On the other hand if $\operatorname{col}(\mu) = 0 \bmod n$, by Lemma 4.1 and Proposition 2.9 we have

$$\left\langle a_\mu^{\rho_o c_1'}, 1 \right\rangle = \left\langle a_\mu^{\rho_o}, c_1' \bar{c}_1' \right\rangle = \left\langle \mu, \rho_0 c_1' \bar{c}_1' \bar{\rho}_o \right\rangle.$$

By (1) of Proposition 2.24 it follows that $\rho_o c_1'$ is local.

By Lemma 2.33 we have $[a_{v_0}^{\rho_0 c_1'}] = [\tilde{a}_{v_0}^{\rho_0 c_1'}]$, and by Lemmas 2.25 and 2.36 we conclude that $[\rho_o c_1' \bar{c}_1' \bar{\rho}_o] = \sum_j [\omega^j]$ where the sum is over a finite set of positive integers. Since $\rho_0 c_1'$ is irreducible and $[\rho_0 \bar{\rho}_0] = \sum_{1 \leqslant j \leqslant n} [\omega^j]$ we conclude that $[\rho_0 c_1' \bar{c}_1' \bar{\rho}_0] = \sum_{1 \leqslant j \leqslant n} [\omega^j]$. Hence $d_{c_1'} = 1$ and $[c_1' \bar{c}_1'] = [1]$. □

By Proposition 2.24 we have proved

Corollary 4.3. *If* $\lambda \in \Delta_{\mathcal{A}}$ *is irreducible, then* $\langle 1, a_\lambda^{\rho_o c_1} \rangle \geqslant 1$ *iff* $\lambda = \omega^i$, $1 \leqslant i \leqslant n$.

4.2. *Verifying assumptions of Corollary 3.14*

Set $\rho = \rho_0 c_1$ and all inductions in the rest of this section are with respect to ρ.

Lemma 4.4. *a_λ is irreducible for all irreducible descendants of $v^2\bar v^2$, $v\bar v^3$.*

Proof. By Lemma 2.29 and Proposition 2.9 we have for $n \geqslant 3$

$$[a_{v\bar v}a_{v\bar v}] = 2[1] + 4[a_{v_0}] + [a_{(2,0,\dots,0,2)}] + [a_{(0,1,0,\dots,1,0)}] + [a_{(0,1,0,\dots,0,2)}] + [a_{(2,0,\dots,0,1,0)}].$$

Note that by Corollary 4.3 we have

$$\langle a_\lambda, a_\mu\rangle = \langle 1, a_{\bar\lambda\mu}\rangle \geqslant 2$$

iff $[\omega^j(\lambda)] = [\mu]$ for some $1 \leqslant j \leqslant n-1$. It is easy to check with the explicit formulas above that a_λ is irreducible for all irreducible descendants of $v^2\bar v^2$. $n = 2$ case is simpler, and similarly one can check directly that a_λ is irreducible for all irreducible descendants of $v\bar v\bar v^3$. $\square$

Lemma 4.5. *For all λ with* $\mathrm{col}(\lambda) = 0$, $[a_\lambda] = [\tilde a_\lambda]$.

Proof. By (2) of Proposition 2.19 and Theorem 2.1 of [13] all $Z_{\lambda\mu}$ with $Z_{1,\lambda} \neq 0$ iff $\lambda = \omega^i$, $1 \leqslant i \leqslant n$ are classified. Using Corollary 4.3, it follows by inspection of Theorem 2.1 of [13] that for all λ with $\mathrm{col}(\lambda) = 0$, $Z_{\lambda\lambda} = \langle a_\lambda, \tilde a_\lambda\rangle \neq 0$ or $Z_{\lambda\lambda} = \langle a_{\bar\lambda}, \tilde a_\lambda\rangle \neq 0$, $\forall\lambda$. In the latter case by Proposition 2.19 we conclude that λ appears in Exp iff $\langle a_\lambda, a_{\bar\lambda}\rangle \neq 0$. Choose $\lambda = (n, 0, \dots, 0) = \Lambda$ as in Lemma 2.39. It follows from Lemma 2.39 and Corollary 2.20 that $\Lambda \in \mathrm{Exp}$, but $\langle a_\Lambda, \bar a_\Lambda\rangle = 0$, contradiction. Hence $\langle a_\lambda, \tilde a_\lambda\rangle \neq 0$, $\forall\lambda, \mathrm{col}(\lambda) = 0 \bmod n$, and by Lemma 2.35 we conclude that for all λ with $\mathrm{col}(\lambda) = 0$, $[a_\lambda] = [\tilde a_\lambda]$. $\square$

Lemma 4.6. *Suppose that $x_i \prec a_{\lambda_i}\tilde a_{\mu_i}$, $i = 1, 2$ and x_1x_2 is a direct sum of a_ν with a_ν irreducible. Then $[x_1x_2] = [x_2x_1]$.*

Proof. By assumption it is enough to check that

$$\langle x_1x_2, a_\nu\rangle = \langle x_2x_1, a_\nu\rangle.$$

By Lemma 2.13 we have $[a_\nu\bar x_2] = [\bar x_2 a_\nu]$, together with Frobenius reciprocity we obtain

$$\langle x_1x_2, a_\nu\rangle = \langle x_1, a_\nu\bar x_2\rangle = \langle x_1, \bar x_2 a_\nu\rangle = \langle x_2x_1, a_\nu\rangle. \qquad \square$$

Proposition 4.7. *There exists $h \in \mathrm{End}(M)$ such that $[\tilde a_v] = [ha_v]$, $[h^n] = [1]$.*

Proof. First suppose that there is no automorphism h such that $[\tilde a_v] = [ha_v]$ or $[\tilde a_{\bar v}] = [ha_v]$. By Lemma 4.5 $[a_v a_{\bar v}] = [\tilde a_v\tilde a_{\bar v}] = [1] + [a_{v_0}]$. By Lemma 2.33 a_{v_0} is irreducible, it follows that there are sectors x_i, y_i with $d_{x_i} > 1$, $d_{y_i} > 1$ such that

$$[a_v\tilde a_v] = [x_1] + [x_2], \qquad [a_{\bar v}\tilde a_v] = [y_1] + [y_2].$$

We compute

$$[a_v\tilde{a}_v\tilde{a}_{\bar{v}}] = [x_1\tilde{a}_{\bar{v}}] + [x_2\tilde{a}_{\bar{v}}] = [a_v a_v a_{\bar{v}}] = 2[a_v] + [a_{(2,0,\dots,0,1)}] + [a_{(0,1,0,\dots,0,1)}]$$

where we have used Lemma. 4.5 in the second $=$. By assumption $d_{x_i} > 1$, $i = 1, 2$, we have $x_i\tilde{a}_{\bar{v}} \succ a_v$, but $[x_i\tilde{a}_{\bar{v}}] \neq [a_v]$, $i = 1, 2$. Hence we can assume that

$$[x_1\tilde{a}_{\bar{v}}] = [a_v] + [a_{(2,0,\dots,0,1)}], \qquad [x_2\tilde{a}_{\bar{v}}] = [a_v] + [a_{(0,1,0,\dots,0,1)}].$$

Hence

$$\langle a_{\bar{v}}x_i, a_{\bar{v}}x_i\rangle = \langle x_i a_{\bar{v}}, x_i a_{\bar{v}}\rangle = \langle x_i, x_i a_{\bar{v}v}\rangle = \langle x_i, x_i\tilde{a}_{\bar{v}v}\rangle = 2$$

where we have used Lemma 2.13 in the first $=$ and Lemma. 4.5 in the third $=$. We can assume that

$$[a_{\bar{v}}x_i] = [\tilde{a}_v] + [u_i], \quad i = 1, 2,$$

where u_i, $i = 1, 2$, is irreducible and we may have $[u_1] = [u_2]$. Note that $[a_{\bar{v}}x_1] + [a_{\bar{v}}x_2] = [a_v y_1] + [a_v y_2] = [a_{\bar{v}}a_v\tilde{a}_v]$.

The same argument applies to y_i, $i = 1, 2$, and we may choose y_i such that

$$[a_{\bar{v}}x_i] = [a_v y_i], \quad i = 1, 2.$$

Consider now

$$\begin{aligned}\left[a_{v\bar{v}}^2\right] &= [x_1\bar{x}_1] + [x_2\bar{x}_2] + [x_1\bar{x}_2] + [x_2\bar{x}_1]\\ &= 2[1] + 4[a_{v_0}] + [a_{(2,0,\dots,0,2)}] + [a_{(0,1,0,\dots,1,0)}] + [a_{(0,1,0,\dots,0,2)}] + [a_{(2,0,\dots,0,1,0)}].\end{aligned}$$

Note that $x_i\bar{x}_i \succ a_{v\bar{v}}$, and $[x_i\bar{x}_j] = [\bar{x}_j x_i]$ by Lemmas 4.4 and 4.6. Hence

$$\langle x_2\bar{x}_1, x_2\bar{x}_1\rangle = \langle x_2\bar{x}_2, x_1\bar{x}_1\rangle \geqslant 2.$$

By computing the index of sectors we conclude that

$$\begin{aligned}[x_1\bar{x}_1] &= [a_{v\bar{v}}] + [a_{(2,0,\dots,0,2)}], \qquad & [x_1\bar{x}_2] &= [a_{v_0}] + [a_{(0,1,\dots,0,2)}],\\ [x_2\bar{x}_2] &= [a_{v\bar{v}}] + [a_{(0,1,0,\dots,1,0)}], \qquad & [x_2\bar{x}_1] &= [a_{v_0}] + [a_{(2,0,\dots,0,1,0)}].\end{aligned}$$

Similarly we obtain

$$\begin{aligned}[y_1\bar{y}_1] &= [a_{v\bar{v}}] + [a_{(2,0,\dots,0,2)}], \qquad & [y_1\bar{y}_2] &= [a_{v_0}] + [a_{(0,1,\dots,0,2)}],\\ [y_2\bar{y}_2] &= [a_{v\bar{v}}] + [a_{(0,1,0,\dots,1,0)}], \qquad & [y_2\bar{y}_1] &= [a_{v_0}] + [a_{(2,0,\dots,0,1,0)}].\end{aligned}$$

Next compute

$$[a_{\bar{v}^2}a_{v\bar{v}}] = [a_{\bar{v}}\tilde{a}_v a_{\bar{v}}\tilde{a}_{\bar{v}}] = [y_1\bar{x}_1] + [y_1\bar{x}_2] + [y_2\bar{x}_1] + [y_2\bar{x}_2].$$

Note that

$$\langle y_2\bar{y}_1, x_2\bar{x}_1\rangle = \langle \bar{x}_2 y_2, \bar{x}_1 y_1\rangle = 2,$$
$$2 = \langle a_{\bar{v}} x_i, a_v y_i\rangle = \left\langle a_{\bar{v}}^2, y_i\bar{x}_i\right\rangle$$

and

$$\langle y_i\bar{x}_i, y_i\bar{x}_i\rangle = \langle y_i\bar{y}_i, x_i\bar{x}_i\rangle = 3,$$
$$\langle y_1\bar{x}_2, y_1\bar{x}_2\rangle = \langle y_1\bar{y}_2, x_2\bar{x}_2\rangle = 2$$

where we have also used Lemma 4.6. From these equations we conclude that

$$[y_1\bar{x}_1] = \left[a_{\bar{v}}^2\right] + [a_{(1,0,\dots,0,3)}]$$

or

$$[y_1\bar{x}_1] = \left[a_{\bar{v}}^2\right] + [a_{(1,0,\dots,0,1,0,0)}].$$

From $[a_{\bar{v}} x_1] = [a_v y_1]$ we obtain

$$[a_{\bar{v}} x_1\bar{x}_1] = [a_v y_1\bar{x}_1].$$

Using the formulas for $x_1\bar{x}_1$, $y_1\bar{x}_1$ we obtain

$$[a_{\bar{v}} a_{(2,0,\dots,0,2)}] = [a_v a_{(1,0,\dots,0,1,0,0)}]$$

or

$$[a_{\bar{v}} a_{(2,0,\dots,0,2)}] = [a_v a_{(1,0,\dots,0,3)}].$$

Both identities are incompatible with Lemmas 2.29 and 4.5.

Therefore there is an automorphism h such that $[\tilde{a}_v] = [ha_v]$ or $[\tilde{a}_{\bar{v}}] = [ha_v]$. Hence $h^n \prec [\tilde{a}_{\bar{v}^n} a_{v^n}] = [a_{\bar{v}^n} a_{v^n}]$ or $h^n \prec [\tilde{a}_{v^n} a_{v^n}] = [a_{v^n} a_{v^n}]$ by Lemma 4.5. Assume that $h^n \prec a_\mu$ for some μ, $\operatorname{col}(\mu) = 0 \bmod n$. Since $\rho = \rho_0 c_1$, by Lemma 4.1 there is a sector x of $a_\mu^{\rho_0}$ such that $[a_x^{c_1}] = [h^n]$. Since $d_x = 1$, by Lemma 2.37 we conclude that $[x] = [1]$ and $[h^n] = [1]$.

If $[\tilde{a}_{\bar{v}}] = [ha_v]$, use $[h^n] = [1]$ we have $[a_{v^n}] = [a_{\bar{v}^n}]$. Hence $\omega^j(n, 0, \dots, 0) \prec \bar{v}^n$ for some $1 \leqslant j \leqslant n$ which is incompatible with fusion rules in Lemma 2.29 since $k = n'n \geqslant 3n$. □

4.3. Properties of sectors related to a_u

Lemma 4.8. *If $\varepsilon(\omega^l, \lambda)\varepsilon(\lambda, \omega^l) = 1$, then $n | l \operatorname{col}(\lambda)$.*

Proof. By monodromy equation $\varepsilon(\omega^l, \lambda)\varepsilon(\lambda, \omega^l) = \exp(\frac{2\pi i l \operatorname{col}(\lambda)}{n})$ and the lemma follows. □

Lemma 4.9. *If* $[v\lambda] = \sum_{1\leqslant j\leqslant k_1-1}[\omega^{l_1 j}w]$ *where* $k_1 l_1 = n$, $[\omega^{jl_1}w] = [\omega^{j'l_1}w]$ *iff* $j = j' \bmod k_1$, *and* $\sum_{1\leqslant i\leqslant n-1}\lambda_i \leqslant k-1$. *Then* $\lambda = (0,\dots,0,k/k_1,0,\dots,0,k/k_1,\dots,0)$ *where* $(0,\dots,0,k/k_1)$ *(with* l_1-1 0*'s) appears* k_1-1 *times, and the last* l_1-1 *entries are* 0*'s, and* $\mathrm{col}(\lambda) = 0 \bmod n$.

Proof. Since $[\omega^{l_1}\lambda] = [\lambda]$, in the components of λ, $(\lambda_0,\dots,\lambda_{l_1-1})$ appears k_1 times. By assumption $v\lambda$ is a sum of k_1 distinct irreducible subsectors, it follows from Lemma 2.29 that λ has only k_1 nonzero components. Since $\lambda_0 \neq 0$, and $\mathrm{col}(\lambda) = \frac{kl_1(k_1-1)}{2}$, the lemma follows. □

Proposition 4.10. *If* $[a_u] = [x_1y_1]$, $1 < d_{x_1} < d_u$ *where* $x_1 \prec a_{\lambda_1}$, $y_1 \prec a_{\lambda_2}$, *then either* $[x_1] = [a_v], [y_1] = [b_i]$ *or* $[y_1] = [a_v]$, $[x_1] = [b_i]$, $1 \leqslant i \leqslant n$.

Proof. By using the action of ω if necessary, we may assume that the zero-th components of λ_1, λ_2 are positive. By Lemma 2.35 we can assume that

$$[a_{\lambda_1}] = \sum_{1\leqslant i\leqslant k_1}[x_i], \qquad \left[\omega^{l_1}\lambda_1\right] = [\lambda_1], \qquad \left[g^i x_1 g^{-i}\right] = [x_i], \quad 0 \leqslant i \leqslant k_1-1,\ k_1l_1 = n,$$

$$[a_{\lambda_2}] = \sum_{1\leqslant i\leqslant k_2}[y_i], \qquad \left[\omega^{l_2}\lambda_2\right] = [\lambda_2], \qquad \left[g^i x_1 g^{-i}\right] = [x_i], \quad 0 \leqslant i \leqslant k_2-1,\ k_2l_2 = n.$$

Since $a_u \prec a_{\lambda_1\lambda_2}$, $\mathrm{col}(\lambda_1) + \mathrm{col}\,\lambda_2 = \mathrm{col}(u) = 1 \bmod n$. By Lemma 4.8 $k_i|\mathrm{col}(\lambda_i)$, $i = 1,2$. Hence $(k_1,k_2) = 1$.

Since x_1y_1, a_{v_0} are irreducible, we may assume that $\langle \bar{x}_1x_1, a_{v_0}\rangle = 0$, i.e., a_vx_1 is irreducible. Let $w \prec v\lambda_1$. Since $\omega^{l_1}[\lambda_1] = [\lambda_1]$, $\omega^{l_1}w \prec v\lambda_1$. Let $t_1|k_1$ be the least positive integer such that $[\omega^{l_1t_1}w] = [w]$. By Lemma 4.8 $n|l_1t_1\,\mathrm{col}(w)$. But $\mathrm{col}(w) = 1 + \mathrm{col}\,\lambda_1 \bmod n$ with $k_1|\mathrm{col}(\lambda_1)$. We conclude that $t_1 = k_1$ and

$$[v\lambda_1] \prec \sum_{0\leqslant j\leqslant k_1-1}\left[\omega^{l_1j}w\right].$$

Since $a_w \prec a_{v\lambda_1} = \sum_{1\leqslant j\leqslant k_1}[a_vx_j]$ and each a_vx_j is irreducible, $d_{a_w} = d_w \geqslant d_vd_{x_1} = d_vd_{\lambda_1}/n$. Hence

$$[v\lambda_1] = \sum_{0\leqslant j\leqslant k_1-1}\left[\omega^{l_1j}w\right].$$

By Lemma 4.9 we have $\mathrm{col}(\lambda_1) = 0 \bmod n$. Hence $\mathrm{col}(\lambda_2) = 1 \bmod n$ and $k_2 = 1$. If $l_1 = 1$, then $\lambda_1 = (n',\dots,n')$, and $d_{\lambda_2} = d_v$. By proposition on p. 10 of [15] λ_2 must be in the orbit of v or $\bar{v}$ under the action of ω. But $\mathrm{col}(\lambda_2) = 1 \bmod n$, so $[a_{\lambda_2}] = [a_v]$ and proposition is proved. In the following we assume that $l_1 \geqslant 2$ to reach a contradiction.

Note that $[a_{\lambda_1\lambda_2}] = k_1[a_u]$, hence $[\lambda_1\lambda_2] = \sum_{0\leqslant i\leqslant k_1-1}[\omega^{l_1i}u]$. By Lemma 2.30 $k_1 \geqslant 2$. We have

$$\langle\lambda_1\lambda_2, \lambda_1\lambda_2\rangle = k_1 \geqslant 1 + \langle\lambda_1\bar{\lambda}_1, v_0\rangle\langle\lambda_2\bar{\lambda}_2, v_0\rangle = 1 + (k_1-1)\langle\lambda_2\bar{\lambda}_2, v_0\rangle.$$

Hence $\langle v\lambda_2, v\lambda_2\rangle = 2$.

On the other hand since $n = k_1 l_1 \geqslant 4$, by Lemma 2.29 we have $\langle \lambda_1\bar{\lambda}_1, (0,1,0,\dots,0)(0,0,\dots,1,0)\rangle \geqslant k_1+1$, $[(0,1,0,\dots,0)(0,0,\dots,1,0)] = [v\bar{v}] + [(0,1,0,\dots,0,1,0)]$ and we conclude that

$$\big\langle \lambda_1\bar{\lambda}_1, (0,1,0,\dots,0,1,0)\big\rangle \geqslant 1.$$

We must have

$$\big\langle (0,1,0,\dots,0)\lambda_2, (0,1,0,\dots,0)\lambda_2\big\rangle = 2.$$

Hence by Lemma 2.29 $\lambda_2 = (m,0,\dots,0)$ or $\lambda_2 = (0,\dots,0,m)$.

Note that $[(2,0,\dots,0)] + [(0,1,0,\dots,0)] = [v^2]$. If $m > 1$ then by fusion rules

$$\big[(2,0,\dots,0)(0,0,\dots,2)\big] = [v\bar{v}] + \big[(2,0,\dots,2)\big], \qquad \big\langle (2,0,\dots,0)\lambda_2, (2,0,\dots 0)\lambda_2\big\rangle = 3.$$

We obtain $\langle (2,0,\dots,2), \lambda_2\bar{\lambda}_2\rangle = 1$. Similarly we obtain that $\langle (2,0,\dots,2), \lambda_1\bar{\lambda}_1\rangle \geqslant 1$, hence $\langle \lambda_1\lambda_2, \lambda_1\lambda_2\rangle = k_1 \geqslant k_1+1$, a contradiction. Therefore $\lambda_2 = v$ or $\bar{v}$. But $\mathrm{col}(\lambda_2) = 1 \bmod n$ we have $\lambda_2 = v$.

From $[\lambda_1 v] = [\lambda_1\lambda_2] = \sum_{0\leqslant i\leqslant k_1-1}[\omega^{l_1 i}u]$ and Lemma 4.9 we conclude that $\lambda_1 = (n', n', \dots, n')$. Hence $l_1 = 1$ contradicting our assumption $l_1 > 1$. □

4.4. *The proof of Theorem 2.40*

By Lemma 2.33, Corollary 4.3 and Proposition 4.7, the assumptions of Corollary 3.14 are verified. We can find $\rho_o c \in H_\rho$ as in Corollary 3.14. Since $[\rho_o\bar{\rho}_o] = \sum_{1\leqslant i\leqslant n}[\omega^i]$, it follows that $d_c = 1$, and we conclude that $\rho_o c_1 \prec \lambda\rho_o c$ for some λ, and by Proposition 2.9 we have

$$1 \leqslant \langle \rho_o c_1, \rho_o a_\lambda c\rangle = \langle c_1, \bar{\rho}_o\rho_o a_\lambda c\rangle = \langle c_1, a_\lambda\bar{\rho}_o\rho_o c\rangle.$$

It follows that $c_1 \prec a_\lambda g^i c$ for some $1 \leqslant i \leqslant n$. Since $c_1(g^i c)^{-1}(M) = c_1(M)$ as a set, replacing c_1 by $c_1(g^i c)^{-1}$ if necessary, we may assume that $[g^i c] = [1]$, and $c_1 \prec a_\lambda$. Since $a_u = c_1c_2$ it follows that $c_2 \prec a_\mu$ for some μ. By Proposition 4.10 we conclude that $[c_1] = [a_v]$, $[c_2] = [b_i]$, or $[c_1] = [b_i]$, $[c_2] = [a_v]$, $1 \leqslant i \leqslant n$. Assume first that $c_1 = Ua_vU^*$, $c_2 = U'b_iU'^*$ with U, U' unitary. Then we have $a_u = ad_{Ua_v(U')}a_vb_i = ad_{U_i}a_vb_i$. Since a_vb_i is irreducible we have $Ua_v(U')U_i^* \in \mathbb{C}$, and this implies that the intermediate subfactor $c_1(M) = ad_{U_i}a_v(M)$, i.e., it is one of the subfactors in Proposition 2.41. The case when $[c_1] = [b_i]$, $[c_2] = [a_v]$ $1 \leqslant i \leqslant n$ is treated similarly. By Proposition 2.41 Theorem 2.40 is proved.

5. Related issues

5.1. *Centrality of a class of intertwiners*

We preserve the general setup of Section 2.3. If $\rho = \mu c$, $\mu \in \Delta_{\mathcal{A}}$, $d_c = 1$ it follows from Definition 2.7 that $[a_\lambda] = [\tilde{a}_\lambda] = [c^{-1}\lambda c]$, $\forall\lambda$, hence $Z_{\lambda\lambda_1} = \delta_{\lambda,\lambda_1}$. Motivated by our proof of Theorem 2.40 we make the following:

Conjecture 5.1. *If $Z_{\lambda\lambda_1} = \delta_{\lambda,\lambda_1}$, then $\rho = \mu c$, $\mu \in \Delta_{\mathcal{A}}$, $d_c = 1$.*

We will prove that Conjecture 5.1 is equivalent to the centrality of a class of intertwiners. Assume that $Z_{\lambda\lambda_1} = \delta_{\lambda,\lambda_1}$. Then for each irreducible λ there is (up to scalar) a unique unitary $u_\lambda \in \mathrm{Hom}(a_\lambda, \tilde{a}_\lambda)$.

Similar to Definition 3.7 we define:

Definition 5.2. $u_{\lambda_1\lambda_2\ldots\lambda_m} := u_{\lambda_1} a_{\lambda_1}(u_{\lambda_2}) \ldots a_{\lambda_1\lambda_2\ldots\lambda_{n-1}}(u_{\lambda_n}) \in \mathrm{Hom}(a_{\lambda_1\lambda_2\ldots\lambda_m}, \tilde{a}_{\lambda_1\lambda_2\ldots\lambda_m})$.

If $\rho = \mu c$, $\mu \in \Delta_{\mathcal{A}}$, $d_c = 1$, then it follows from definition (2.7) that we can choose u_λ such that $u_\lambda = c^{-1}(\tilde{\varepsilon}(\lambda, \bar{\mu})\tilde{\varepsilon}(\bar{\mu}, \lambda))$. Using BFE in Proposition 2.1 we have

$$u_{\lambda_1\lambda_2\ldots\lambda_m} = c^{-1}\big(\tilde{\varepsilon}(\lambda_1\lambda_2\ldots\lambda_m, \bar{\mu})\tilde{\varepsilon}(\bar{\mu}, \lambda_1\lambda_2\ldots\lambda_m)\big) \in \mathrm{Hom}(a_{\lambda_1\lambda_2\ldots\lambda_m}, \tilde{a}_{\lambda_1\lambda_2\ldots\lambda_m}),$$
$$\mathrm{Hom}(a_{\lambda_1\lambda_2\ldots\lambda_m}, \tilde{a}_{\lambda_1\lambda_2\ldots\lambda_m}) = c^{-1}\big(\mathrm{Hom}(\bar{\mu}\lambda_1\lambda_2\ldots\lambda_m, \bar{\mu}\lambda_1\lambda_2\ldots\lambda_m)\big).$$

By using BFE in Proposition 2.1 again we have proved the following:

Lemma 5.3. *If $\rho = \mu c$, $\mu \in \Delta_{\mathcal{A}}$, $d_c = 1$, then $u_{\lambda_1\lambda_2\ldots\lambda_m} T u^*_{\lambda_1\lambda_2\ldots\lambda_m} = T$, $\forall T \in \mathrm{Hom}(a_{\lambda_1\lambda_2\ldots\lambda_m}, a_{\lambda_1\lambda_2\ldots\lambda_m})$.*

Using u_λ we define:

Definition 5.4. For any irreducible $[b] \in H_\rho$, $\lambda \in \Delta_{\mathcal{A}}$,

$$\psi_b^{(\lambda)} := S_{11} d_b d_\lambda \phi_\lambda\big(\varepsilon(b\bar{\rho}, \lambda) b(u_\lambda) \varepsilon(\lambda, b\bar{\rho})\big).$$

Lemma 5.5. *For any irreducible $[b] \in H_\rho$, $\psi_b^{(\lambda)} = c_\lambda \phi_b^{(\lambda)}$, $|c_\lambda c_{\bar{\lambda}}| = 1$ where c_λ are complex numbers independent of b.*

Proof. Since by Lemma 2.23 $\sum_b \psi_b^{(\lambda)*}[b]$ is an eigenvector of the action of μ with eigenvalue $\frac{S_{\mu\lambda}}{S_{1\lambda}}$, and by Proposition 2.19 there is up to scalar a unique such eigenvector, it follows that there is a complex number c_λ independent of b such that $\psi_b^{(\lambda)} = c_\lambda \phi_b^{(\lambda)}$, $\forall b$. Similarly since $\sum_b \phi_b^{(\lambda)*}[b]$ is an orthogonal eigenvector of the action of μ with eigenvalue $\frac{S_{\mu\bar{\lambda}}}{S_{1\bar{\lambda}}}$, we have $\phi_b^{(\bar{\lambda})} = c'_\lambda \phi_b^{(\lambda)*}$, $|c'_\lambda| = 1$, $\forall b$. We have $\phi_b^{(\bar{\lambda})} = c_{\bar{\lambda}} c'_\lambda \phi_b^{(\lambda)*}$, $\forall b$, $|c'_\lambda| = 1$. By Lemma 2.23 $\sum_b \psi_b^{(\lambda)} \psi_b^{(\bar{\lambda})}$ has absolute value 1, and it follows that $|c_\lambda c_{\bar{\lambda}}| = 1$. □

The following lemma is proved in the same way as Lemma 3.9:

Lemma 5.6. *If $u_{\lambda_1\lambda_2\ldots\lambda_m}$ is central, then for fixed μ, if $t_\mu \in \mathrm{Hom}(\mu, \lambda_1\lambda_2\ldots\lambda_m)$ is an isometry, then $\bar{\rho}(t_\mu)^* u_{\lambda_1\lambda_2\ldots\lambda_m} \bar{\rho}(t_\mu) \in \mathrm{Hom}(a_\mu, \tilde{a}_\mu)$ is a unitary independent of the choice of t_μ, and is a scalar multiple of u_μ.*

Proposition 5.7. *Conjecture 5.1 is equivalent to the following statement: if $Z_{\lambda\lambda_1} = \delta_{\lambda,\lambda_1}$, then $u_{\lambda_1\lambda_2\ldots\lambda_m}$ is central for all $\lambda_1, \ldots, \lambda_m$, $\forall m$.*

Proof. Suppose that Conjecture 5.1 is true. Then it follows from Lemma 5.3 that if $Z_{\lambda\lambda_1} = \delta_{\lambda,\lambda_1}$, then $u_{\lambda_1\lambda_2\ldots\lambda_m}$ is central for all $\lambda_1, \ldots, \lambda_m$, $\forall m$. Suppose now that $u_{\lambda_1\lambda_2\ldots\lambda_m}$ is central for all $\lambda_1, \ldots, \lambda_m$, $\forall m$. As in the proof of Lemma 3.11 by using centrality $u_{\lambda_1\lambda_2\ldots\lambda_m}$ we calculate

$$\frac{\psi_b^{(\lambda_1)}}{\psi_b^{(1)}}\frac{\psi_b^{(\lambda_2)}}{\psi_b^{(1)}}\cdots\frac{\psi_b^{(\lambda_m)}}{\psi_b^{(1)}} = \sum_\mu \langle \mu, \lambda_1\ldots\lambda_m\rangle d_\mu \phi_\mu\big(\varepsilon(b\bar\rho,\mu) b(u_\mu)\varepsilon(\mu,b\bar\rho)\big)c_\mu$$

where $|c_\mu| = 1$. Hence using Lemma 2.23 as in the proof of Lemma 3.11 we have

$$\sum_b \left| d_b^2 \frac{\psi_b^{(\lambda_1)}}{\psi_b^{(\lambda_1)}} \frac{\psi_b^{(\lambda_2)}}{\psi_b^{(1)}}\cdots\frac{\psi_b^{(\lambda_m)}}{\psi_b^{(1)}}\right| = \langle 1,\lambda_1\ldots\lambda_m\rangle \sum_b d_b^2 = \sum_\lambda \frac{S_{\lambda_1\lambda}}{S_{1\lambda}}\frac{S_{\lambda_2\lambda}}{S_{1\lambda}}\cdots\frac{S_{\lambda_m\lambda}}{S_{1\lambda}} d_\lambda^2.$$

Now choose $m = 2m_1$ and $\lambda_{i+m_1} = \bar\lambda_i$, $1 \leqslant i \leqslant m_1$, summing over $\lambda_1, \ldots, \lambda_{m_1}$ and using Lemma 5.5 we obtain

$$\sum_b \frac{1}{d_b^{m-2}} = \sum_\lambda \frac{1}{d_\lambda^{m-2}}.$$

Letting $m = 2m_1$ go to infinity and noticing that $d_b \geqslant 1$ we conclude that there must exist a sector c such that $d_c = 1$ and $\rho = \mu c$ for some $\mu \in \Delta_{\mathcal{A}}$. □

For each irreducible $\lambda \in \Delta_{\mathcal{A}}$ we choose $R_{\lambda\bar\lambda}$ so that $R^*_{\lambda\bar\lambda} R_{\lambda\bar\lambda} = d_\lambda, \lambda(R^*_{\bar\lambda\lambda})R_{\lambda\bar\lambda} = 1$. These operators are unique up to scalars.

Lemma 5.8.

(1) *We can choose* u_λ *such that*

$$\bar\rho\big(R^*_{\lambda\bar\lambda}\big)u_{\lambda\bar\lambda} = \bar\rho\big(R^*_{\lambda\bar\lambda}\big), \qquad u_{\lambda\bar\lambda}\bar\rho(R_{\lambda\bar\lambda}) = \bar\rho(R_{\lambda\bar\lambda}), \quad \forall\lambda;$$

(2) *The relative braiding as defined in Lemma* 2.15 *among* a_λ*'s (resp.* $\tilde a_\lambda$*'s) is a braiding and* $\varepsilon(a_\lambda, a_\mu) = \varepsilon(\tilde a_\lambda, \tilde a_\mu) = \bar\rho(\varepsilon(\lambda,\mu))$, $\forall\lambda, \mu \in \Delta_{\mathcal{A}}$.

Proof. Ad (1): Note that $\bar\rho(R^*_{\lambda\bar\lambda})u_{\lambda\bar\lambda}$ is equal to $\bar\rho(R^*_{\lambda\bar\lambda})$ up to a constant of absolute value 1, hence we can choose to multiply $u_\lambda, u_{\bar\lambda}$ by suitable constants of absolute value 1 so that

$$\bar\rho\big(R^*_{\lambda\bar\lambda}\big)u_{\lambda\bar\lambda} = \bar\rho\big(R^*_{\lambda\bar\lambda}\big).$$

If

$$u_{\lambda\bar\lambda}\bar\rho(R_{\lambda\bar\lambda}) = c_\lambda\bar\rho(R_{\lambda\bar\lambda}), \quad \forall\lambda,$$

multipling both sides on the left by $\bar\rho(R_{\lambda\bar\lambda})^*$ we conclude that $c_\lambda = 1$, $\forall\lambda$.

Ad (2): The relative braidings are braidings since $[a_\lambda] = [\tilde a_\lambda]$ by assumption and Lemma 2.15. By definition we have

$$\varepsilon(a_\lambda, a_\mu) = u^*_\mu\bar\rho\big(\varepsilon(\lambda,\mu)\big)a_\lambda(u_\mu) = u^*_\mu u_\mu\bar\rho\big(\varepsilon(\lambda,\mu)\big) = \bar\rho\big(\varepsilon(\lambda,\mu)\big)$$

where we have used Lemma 2.17 in the second $=$ since $u_\mu \in \mathrm{Hom}(a_\mu, \tilde{a}_\mu) \subset \mathrm{Hom}(\bar{\rho}\mu, \bar{\rho}\mu)$. The other case is proved similarly. □

Definition 5.9. An operator is a cap (resp. cup) operator if it is $\mu(R_{\lambda\bar{\lambda}})$ (resp. $\mu(R_{\lambda\bar{\lambda}})^*$) for some $\mu, \lambda \in \Delta_{\mathcal{A}}$. It is a braiding operator if it is $\mu(\varepsilon(\lambda, \nu))$ or $\mu(\tilde{\varepsilon}(\lambda, \nu))$ for some $\nu, \mu, \lambda \in \Delta_{\mathcal{A}}$.

Definition 5.10. Denote by $B_{\lambda_1\lambda_2\ldots\lambda_m}$ the subspace of $\mathrm{Hom}(\lambda_1\lambda_2\ldots\lambda_m, \lambda_1\lambda_2\ldots\lambda_m)$ which is linearly spanned by operators in $\mathrm{Hom}(\lambda_1\lambda_2\ldots\lambda_m, \lambda_1\lambda_2\ldots\lambda_m)$ consisting of products of only caps, cups and braiding operators.

Proposition 5.11. *For any* $T \in \bar{\rho}(B_{\lambda_1\lambda_2\ldots\lambda_m})$, $u_{\lambda_1\ldots\lambda_m} T = T u_{\lambda_1\ldots\lambda_m}$.

Proof. It is enough to check for an operator T which consists of products of only caps, cups and braiding operators. Note that the statement of proposition is independent of choices of u_λ, and we can choose our u_λ so that they verify (1) of Lemma 5.8. It is useful to think of T as an tangle connecting top m strings labeled by $a_{\lambda_1}, \ldots, a_{\lambda_m}$ to the bottom m strings labeled by $a_{\lambda_1}, \ldots, a_{\lambda_m}$ as in Chapter 2 of [37], where in the tangle only cups, caps and braidings are allowed. Then by Proposition 2.1, uTu^* will be represented by the same tangle, except the top and bottom m strings are now labeled by $\tilde{a}_{\lambda_1}, \ldots, \tilde{a}_{\lambda_m}$. For each closed string in uTu^* labeled by a_μ, by inserting u_μ we can change the label a_μ to $\tilde{a}_\mu$ using Proposition 2.1 without changing the operator since we have a closed string. Therefore uTu^* is represented by the same tangle T with all labels changed from the original labels a_μ of T to $\tilde{a}_\mu$. Since T consists of products of only caps, cups and braiding operators, proposition follows from Lemma 5.8. □

Conjecture 5.12. $B_{\lambda_1\lambda_2\ldots\lambda_m} = \mathrm{Hom}(\lambda_1\lambda_2\ldots\lambda_m, \lambda_1\lambda_2\ldots\lambda_m)$, $\forall \lambda_1, \ldots \lambda_m, m \geqslant 1$.

By Propositions 5.11 and 5.7 we have proved the following:

Proposition 5.13. *Conjecture* 5.12 *implies Conjecture* 5.1 .

By examining the proof of Proposition 5.7, we can formulate a weaker version of Conjecture 5.12.

Definition 5.14. We say that λ is a generator for $\Delta_{\mathcal{A}}$ if for any irreducible $\mu \in \Delta_{\mathcal{A}}$, there is a positive integer m such that $\mu \prec \lambda^m$.

Conjecture 5.15. *For some generator* λ *of* $\Delta_{\mathcal{A}}$, $B_{\lambda\lambda\ldots\lambda} = \mathrm{Hom}(\lambda^m, \lambda^m)$, $\forall m \geqslant 1$, *where m is the number of* λ *that appears in the definition of* $B_{\lambda\lambda\ldots\lambda}$.

Lemma 5.16. *Assume that* λ *is a generator for* $\Delta_{\mathcal{A}}$. *Then the set* $\{[\mu] \mid |\frac{S_{\lambda\mu}}{S_{1\mu}}| = d_\lambda\}$ *is a finite abelian group.*

Proof. Note that by definition $|\frac{S_{\lambda\mu}}{S_{1\mu}}| = d_\lambda$ implies that $\varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu) \in \mathbb{C}$. By Proposition 2.1 this implies that $\varepsilon(\mu, \lambda_1)\varepsilon(\lambda_1, \mu) \in \mathbb{C}$ if $\lambda_1 \prec \lambda^m$, $m \geqslant 1$. Since λ a generator, it follows that $\varepsilon(\mu, \lambda_1)\varepsilon(\lambda_1, \mu) \in \mathbb{C}$, $\forall \lambda_1 \in \Delta_{\mathcal{A}}$. Hence $|\frac{S_{\mu\lambda_1}}{S_{1\lambda_1}}| = d_\mu$, $\forall \lambda_1 \in \Delta_{\mathcal{A}}$. By properties of S matrix this implies that $d_\mu = 1$. On the other hand if $d_\mu = 1$ then $|\frac{S_{\lambda\mu}}{S_{1\mu}}| = d_\lambda$ since $\mu\lambda$ is irreducible. It follows that the set $\{[\mu] \mid |\frac{S_{\lambda\mu}}{S_{1\mu}}| = d_\lambda\}$ is a finite abelian group. $\square$

Proposition 5.17. *Conjecture* 5.15 *implies Conjecture* 5.1.

Proof. Assume Conjecture 5.15 is true. Then by Proposition 5.11 we know that u_{λ^m} is central. As in the proof of Proposition 5.7, replacing λ_i by λ in the summation we have

$$\left|\sum_a \left(\frac{\psi_a^{(\lambda)}}{\psi_a^{(1)}}\right)^m d_a^2\right| = \sum_\mu \left(\frac{S_{\lambda\mu}}{S_{1\mu}}\right)^m S_{1\mu}^2.$$

Choose m to be divisible by the order of the finite abelian group in Lemma 5.16 and let m go to infinity, the RHS of the above equation has leading order (up to multiplication by a positive number) d_λ^m. It follows that there is a sector c such that $|\frac{\psi_c^{(\lambda)}}{\psi_c^{(1)}}| = d_\lambda$. For any $\mu \prec \lambda^l, l \geqslant 1$. Using the centrality of u_{λ^l} we have

$$\left(\frac{\psi_c^{(\lambda)}}{\psi_c^{(1)}}\right)^l = \sum_\mu \langle \mu, \lambda^l \rangle \frac{\psi_c^{(\mu)}}{\psi_c^{(1)}} c_\mu$$

where $|c_\mu| = 1$. So we have $\sum_{\mu \prec \lambda^l} |\frac{\psi_c^{(\mu)}}{\psi_c^{(1)}}| \geqslant d_\lambda^l$. Since $|\frac{\psi_c^{(\mu)}}{\psi_c^{(1)}}| \leqslant d_\mu$ and $\sum_\mu \langle \mu, \lambda^l \rangle d_\mu = d_\lambda^l$, we conclude that $|\frac{\psi_c^{(\mu)}}{\psi_c^{(1)}}| = d_\mu$, $\forall \mu \prec \lambda^l$. Since λ is a generator, we conclude that $|\frac{\psi_c^{(\mu)}}{\psi_c^{(1)}}| = d_\mu$, $\forall \mu$. By Lemma 5.5 we conclude that $|\frac{\phi_c^{(\mu)}}{\phi_c^{(1)}}|^2 = d_\mu^2$. Summing over μ on both sides we conclude that $d_c = 1$, and the proposition is proved. $\square$

By Proposition 5.17 and Lemma 2.32 we have proved the following:

Corollary 5.18. *Conjecture* 5.1 *is true for* $\Delta_{\mathcal{A}}$ *where* $\mathcal{A}$ *is the net associated with* $SU(n)_k$.

5.2. Maximal subfactors

In this section we give an application of Corollary 5.18.

The following notion is due to V.F.R. Jones:

Definition 5.19. A subfactor $N \subset M$ is called maximal if M_1 is a von Neumann algebra such that $N \subset M_1 \subset M$ implies $M_1 = M$ or $M_1 = N$.

We preserve the setting of Section 2.5. We will say that λ is maximal if $\lambda(M) \subset M$ is a maximal subfactor.

Lemma 5.20. *Suppose* $Z_{1\lambda} = \delta_{1\lambda}$, $Z_{\omega^i\omega^i} = 1$. *Then* $Z_{\lambda\mu} = \delta_{\lambda\mu}$.

Proof. By Proposition 3.2 of [9], from $Z_{1\lambda} = \delta_{1\lambda}$ we have $[a_\lambda] = [\tilde{a}_{\tau(\lambda)}]$ where $\lambda \to \tau(\lambda)$ is an automorphism of fusion ring. Such automorphisms are classified in [15]. By the theorem in

Section 2 of [15] there is an integer $0 \leqslant i \leqslant n-1$ such that $\tau(\lambda) = \omega^j \lambda$ or $\tau(\lambda) = \omega^{-j}\bar{\lambda}$ (exact formulas for j are given in [15] but we will not use them). From $Z_{\omega^i \omega^i} = 1$ we conclude that either $j + i = i \bmod n$, $\forall i$ or $-j - i = i \bmod n$, $\forall i$, and hence $\tau(\lambda) = \lambda$, $\forall \lambda$. □

Proposition 5.21. *If* $S_{v\lambda} \neq 0$, *then* λ *is maximal.*

Proof. Let M_1 be an intermediate subfactor between $\lambda(M)$ and M. Suppose that $\lambda = c_1 c_2$ and $c_1 = c_1' c_1''$ as in Proposition 2.24. Since $S_{v\lambda} \neq 0$, applying Lemmas 2.20 and 2.25 to induction with respect to c_1', we conclude that $\varepsilon(v, c_1'\bar{c}_1')\varepsilon(c_1'\bar{c}_1', v) \in \mathbb{C}$. By Lemma 2.31 we conclude that $[c_1'\bar{c}_1'] = [1]$. By Proposition 2.24 we must have $Z_{\lambda 1}^{c_1} = \delta_{\lambda 1}$. Since $S_{\lambda\omega^i} \neq 0$, by Lemma 2.20 we conclude that $Z_{\omega^i \omega^i}^{c_1} = 1$. By Lemma 5.20 and Proposition 5.18 we conclude that $c_1 = \mu c$, $\mu \in \Delta_{\mathcal{A}}$, $d_c = 1$. Replacing c_1 by $c_1 c^{-1}$ if necessary we may assume that $c_1 = \mu$. It follows that $c_2 = \mu_2$ for some $\mu_2 \in \Delta_{\mathcal{A}}$. By Lemma 2.30 we conclude that $[\mu] = [\lambda]$ or $[\mu] = [\omega^i]$, $1 \leqslant i \leqslant n$, hence $M_1 = \lambda(M)$ or $M_1 = M$. □

Corollary 5.22. *If* $k + n = p^l$ *where* p *is a prime number, and* $(k, n) \neq (2, 2)$, *then* λ *is maximal iff there is no* $1 \leqslant i \leqslant n-1$ *such that* $[\omega^i \lambda] = [\lambda]$.

Proof. By Theorem 5 of [14] when $k + n = p^l$ where p is a prime number, $S_{v\alpha} = 0$ iff $[\omega^i \lambda] = [\lambda]$ for some $1 \leqslant i \leqslant n-1$. Let $i_1 | i$ be the smallest positive integer such that $[\omega^{i_1}\lambda] = [\lambda]$. Then $[\omega^i \lambda] = [\lambda]$ for some $1 \leqslant i \leqslant n-1$, then $[\lambda\bar{\lambda}] \prec \sum_{1\leqslant j\leqslant n/i_1}[\omega^{j i_1}]$ and by [20] and our assumption that λ is maximal it follows that $[\lambda\bar{\lambda}] = \sum_{1\leqslant j\leqslant n/i_1}[\omega^{j i_1}]$. By Lemmas 2.30 and 2.33 this is only possible if $k = n = 2$. The corollary now follows from Proposition 5.21. □

Corollary 5.23. *Suppose that* $k \neq n-2, n+2, n$. *Then* λ *is maximal iff there is no* $1 \leqslant i \leqslant n-1$ *such that* $[\omega^i \lambda] = [\lambda]$.

Proof. When $k = 1$ the corollary is obvious. By Lemma 2.33 we can assume that $k \geqslant 2$ and $d_{v_0} > 1$. As in the proof of Corollary 5.22, λ is maximal implies that there is no $1 \leqslant i \leqslant n-1$ such that $[\omega^i \lambda] = [\lambda]$. Now suppose that there is no $1 \leqslant i \leqslant n-1$ such that $[\omega^i \lambda] = [\lambda]$. If $S_{v\lambda} \neq 0$, then λ is maximal by Corollary 5.21. Suppose that $S_{v\lambda} = 0$. Since $[v\bar{v}] = [1] + [v_0]$ we have $S_{v_0\lambda} = -S_{1\lambda} \neq 0$. Assume that M_1 is an intermediate subfactor between $\lambda(M)$ and M, and $\lambda = c_1 c_2$ with $c_1(M) = M_1$ and $c_1 = c_1' c_1''$ as in Proposition 2.24. Apply Lemma 2.20 we have $\langle a_{v_0}^{c_1'}, \tilde{a}_{v_0}^{c_1'} \rangle \geqslant 1$. By Lemma 2.33 we must have $[a_{v_0}^{c_1'}] = [\tilde{a}_{v_0}^{c_1'}]$ and by Lemma 2.36 $[c_1'\bar{c}_1'] = \sum_{1\leqslant j\leqslant n/j_1}[\omega^{j j_1}]$. By Frobenius reciprocity we have $[\omega^{j_1} c_1'] = [c_1']$. Since $\lambda = c_1' c_1'' c_2$, $[\omega^{j_1}\lambda] = [\lambda]$, and by assumption $j_1 = n$ and $[c_1'\bar{c}_1'] = [1]$. The rest of the proof now follows in exactly the same way as in the proof of Proposition 5.21. □

Example 5.24. When $n = 2$ we have Jones subfactors and their reduced subfactors. In the case $k = n = 2$ there are three irreducible subfactors and they are maximal. Let $n = 2$, $k \neq 2$. Then λ can be labeled by an integer $1 \leqslant i \leqslant k$. Corollary 5.23 implies that i is maximal iff $i \neq k/2$ (when $k = 4$ this can be easily checked directly). This can also be proved directly using the same argument at the end of Section 2.6.

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