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O_2-convergence in posets

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Abstract

In this paper, o_2-convergence in posets is further studied. Some properties of o_2-convergence are obtained. Especially, a sufficient condition and a necessary condition for o_2-convergence to be topological are given respectively.

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Let P be a poset. The Birkhoff–Frink–McShane definition of o-convergence in P is defined as follows (see [1,2,5]): A net $(x_i)_{i\in I}$ in P is said to o-converge to $y \in P$ if there exist subsets D and F of P such that

(1) D is up-directed and F is down-directed,
(2) $y = \sup D = \inf F$, and
(3) for each $a \in D$ and $b \in F$, there exists $k \in I$ such that $a \leqslant x_i \leqslant b$ holds for all $i \geqslant k$.

As is pointed out in [7], in general, o-convergence is not topological. In [8], a sufficient condition for o-convergence to be topological is given. The o_2-convergence is studied in [4,6]. In fact, the o_2-convergence is a generalization of o-convergence, and o_2-convergence is also not topological generally. In this paper, we give a sufficient condition and a necessary condition for o_2-convergence to be topological respectively. In addition, we obtain some properties of o_2-convergence.

Definition 1. (See [4,6].) Let P be a poset, a net $(x_i)_{i\in I}$ in P is said to o_2-converge to $y \in P$ if there exist subsets M and N of P such that

(1) $y = \sup M = \inf N$,
(2) for each $a \in M$ and $b \in N$, there exists $k \in I$ such that $a \leqslant x_i \leqslant b$ holds for all $i \geqslant k$.

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Definition 2. Let P be a complete lattice. A net $(x_i)_{i\in I}$ in P is said to o_2-converge to $y \in P$ if $y = \liminf x_i = \limsup x_i$, where $\liminf x_i = \sup_i \inf_{j\geqslant i} x_j$ and $\limsup x_i = \inf_i \sup_{j\geqslant i} x_j$.

Remark 3. (1) In a complete lattice the alternative definitions of o_2-convergence in the cases of a complete lattice and a poset agree.

(2) The o_2-convergent point of a net $(x_i)_{i\in I}$ in a poset, if it exists, is unique.

Proof. Suppose an o_2-convergent point of $(x_i)_{i\in I}$ exists. Let $A = \{y \in P\colon\ y \text{ is an eventual lower bound of } (x_i)_{i\in I}\}$ and $B = \{z \in P\colon\ z \text{ is an eventual upper bound of } (x_i)_{i\in I}\}$. Suppose that x_1, x_2 are o_2-convergent points of the net $(x_i)_{i\in I}$, then there exist subsets M_1, M_2 of A and subsets N_1, N_2 of B, satisfying the conditions of Definition 1. For each $y \in A$ and $z \in B$, we have $y \leqslant z$. Thus $\sup M_i \leqslant \sup A \leqslant \inf B \leqslant \inf N_i$ $(i = 1, 2)$, whence $\sup M_i = \sup A = \inf B = \inf N_i = x_i$ $(i = 1, 2)$. Since for given net $(x_i)_{i\in I}$, A and B are uniquely decided, $\sup A$ and $\inf B$ are unique too. Thus $x_1 = x_2$. Hence the o_2-convergent point of a net $(x_i)_{i\in I}$ is unique. □

(3) Any constant net $(x_i)_{i\in I}$ in a poset P with value x o_2-converges to x.

(4) [6] If $(x_i)_{i\in I}$ o-converges to x, then it o_2-converges to x.

The converse is not necessarily true.

Let $P = \{d_1, d_2, \ldots\} \cup \{c\} \cup \{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\}$, the order $\leqslant$ on P is defined as follows:

(i) $a_i \leqslant c$, $b_i \leqslant c$, $c \leqslant d_i$ for all $i = 1, 2, 3, \ldots$;
(ii) if $k \geqslant i$, then $a_k \geqslant b_i$.

By definition, if $i \neq j$, then a_i and a_j are incomparable, b_i and b_j are incomparable and d_i and d_j are incomparable. Let $M = \{b_1, b_2, \ldots\}$ and $N = \{d_1, d_2, \ldots\}$, then $\sup M = \inf N = c$. Since for each $b_i \in M$, $d_j \in N$, $b_i \leqslant a_n \leqslant d_j$ hold whenever $n \geqslant i$, the net $(a_i)_{i\in N}$ o_2-converges to c. However the net does not o-converge to c.

(5) Let D and F be directed and filtered subsets of a poset P respectively, in addition, $\sup D$ and $\inf F$ exist, then the net $(x_d)_{d\in D}$ with $x_d = d$ and the net $(y_a)_{a\in F^{op}}$ with $y_a = a$ o_2-converge to $\sup D$ and $\inf F$ respectively.

(6) In a lattice o_2-convergence is equivalent to o-convergence.

Definition 4. Let P be a poset. let $x, y, z \in P$, define $x \ll_\alpha y$ if for every net $(x_i)_{i\in I}$ in P which o_2-converges to $y \in P$, $x_i \geqslant x$ holds eventually; $z \rhd_\alpha y$ if for every net $(x_i)_{i\in I}$ in P which o_2-converges to $y \in P$, $x_i \leqslant z$ holds eventually.

It follows from Definition 4 that $x \ll_\alpha y \Longrightarrow x \leqslant y$ and $z \rhd_\alpha y \Longrightarrow z \geqslant y$.

Definition 5. A poset P is called an α-double continuous poset if $a = \sup\{x \in P\colon\ x \ll_\alpha a\} = \inf\{y \in P\colon\ y \rhd_\alpha a\}$ for every $a \in P$.

Example 6. (1) Every finite lattice is α-double continuous.

(2) Every chain is α-double continuous.

Definition 7. For $x, y, z \in P$, a poset, define $x \ll_{\alpha^*} y$ if for every net$(x_i)_{i\in I}$ in P which o_2-converges to some $w \in P$ with $y \leqslant w$, we have $x \leqslant x_i$ eventually. We define the order dual relation $z \rhd_{\alpha^*} y$ if $z \ll_{\alpha^*} y$ in P^{op}, where P^{op} denotes P endowed with the reverse order.

Definition 8. A poset P is called an α^*-double continuous poset if $a = \sup\{x \in P\colon\ x \ll_{\alpha^*} a\} = \inf\{y \in P\colon\ y \rhd_{\alpha^*} a\}$ for every $a \in P$.

Condition (∗). Let P be a poset and $x, y, z \in P$ with $x \ll_\alpha y \leqslant z$, then $x \ll_\alpha z$. Let $w, s, t \in P$ with $s \rhd_\alpha t \geqslant w$, then $s \rhd_\alpha w$.

It is easy to see that if $y \leqslant x \Longrightarrow y \ll_\alpha x$ for any $x, y \in P$ and $z \geqslant w \Longrightarrow z \rhd_\alpha w$ for any $z, w \in P$, then P satisfies condition (∗).

Example 9. (1) Let $P_1 = \{\top\} \cup \{b, c\} \cup \{a_1, a_2, a_3, \ldots\}$. The order $\leqslant$ on P_1 is defined as follows: $b \leqslant c \leqslant \top$; $a_i \leqslant \top$ for all $i \in N$; $a_i \leqslant a_j$, whenever $i \leqslant j$, for all $i, j \in N$.

By definition of $\leqslant$, for all $i \in N$, a_i and c are incomparable, a_i and b are incomparable. Suppose $(x_i)_{i\in I}$ is a net that o_2-converges to b. Then there exist subsets M and N of P_1 such that $b = \sup M = \inf N$ and for each $m \in M$, $n \in N$, $m \leqslant x_i \leqslant n$ holds eventually. From $b = \sup M$, we have $M = \{b\}$ or $\{b, c\}$, so we must have $b \in M$. For $b \in M$, $b \leqslant x_i$ holds eventually. Since $c \leqslant b$, $c \leqslant x_i$ holds eventually. Thus $c \ll_\alpha b$. It is easy to see that $(a_i)_{i\in N}$ o_2-converges to $\top$, but for each $i \in N$, $c \leqslant a_i$ does not hold. So $c \ll_\alpha \top$ does not hold. Hence P_1 does not satisfy condition $(*)$.

(2) Let $P_2 = \{a_1, a_2, a_2, \ldots\} \cup \{c\} \cup \{b_1, b_2, b_3, \ldots\}$. The order $\leqslant$ on P_2 is defined as follows: $a_i \leqslant c \leqslant b_j$ for all $i, j \in N$.

In what follows, we will prove that $y \leqslant x \Longrightarrow y \ll_\alpha x$ for any $x, y \in P_2$ and $z \geqslant w \Longrightarrow z \rhd_\alpha w$ for any $z, w \in P_2$. Suppose $(x_i)_{i\in I}$ is a net that o_2-converges to c. Then there exist subsets M and N of P_2 such that $c = \sup M = \inf N$ and for each $m \in M$, $n \in N$, $m \leqslant x_i \leqslant n$ holds eventually. From $c = \sup M$, we have (i) $c \in M$ or (ii) $M \subseteq \{a_i\colon i \in N\}$ and M is a nonsingle set. If (i) holds, then $c \leqslant x_i$ holds eventually, and if (ii) holds, then there exist $a_i, a_j \in M$, $a_i \leqslant x_i$ holds eventually and $a_j \leqslant x_i$ holds eventually. Since $a_i \vee a_j = c$, $c \leqslant x_i$ holds eventually. Since $a_i \leqslant c$ for all $i \in N$, $a_i \leqslant x_i$ holds eventually. Hence $a_i \ll_\alpha c$ for all $i \in N$. Similarly we can prove $a_i \ll_\alpha b_j$ for all $i, j \in N$, and $a_i \ll_\alpha a_i$ and $a_i \ll_\alpha c$ for all $i \in N$, and $c \ll_\alpha c$, $c \ll_\alpha b_j$ and $b_j \ll_\alpha b_j$ for all $j \in N$. Similarly we can prove the case of $\rhd_\alpha$. Thus P_2 satisfies condition $(*)$.

Remark 10. (1) If P is an α^*-double continuous poset, then for each $a \in P$, $a = \sup\{x \in P\colon \exists z \in P,\ x \ll_{\alpha^*} z \ll_{\alpha^*} a\} = \inf\{y \in P\colon \exists w \in P,\ y \rhd_{\alpha^*} w \rhd_{\alpha^*} a\}$. This is because $a = \sup\{x \in P\colon x \ll_{\alpha^*} a\}$ and for each $x \ll_{\alpha^*} a$, $x = \sup\{z \in P\colon z \ll_{\alpha^*} x\}$ and $a = \inf\{y \in P\colon y \rhd_{\alpha^*} a\}$ and for each $y \rhd_{\alpha^*} a$, $y = \inf\{w \in P\colon w \rhd_{\alpha^*} y\}$.

(2) Let P be a poset, $x, y, z \in P$. If $x \ll_{\alpha^*} y$, then $x \ll_\alpha y$. Similarly $z \rhd_{\alpha^*} y$ implies $z \rhd_\alpha y$. Thus an α^*-double continuous poset implies that it is α-double continuous.

(3) An α-double continuous poset satisfying condition $(*)$ implies that it is α^*-double continuous.

Let $\mathcal{L}$ be the class consisting of all the pairs $((x_i)_{i\in I}, x))$ of a net $(x_i)_{i\in I}$ and an element x in a poset P with $(x_i)_{i\in I}$ o_2-converging to x. The class $\mathcal{L}$ is called topological if there is a topology τ on P such that $((x_i)_{i\in I}, x) \in \mathcal{L}$ if and only if the net $(x_i)_{i\in I}$ converges to x with respect to the topology τ. By Kelley [3], $\mathcal{L}$ is topological if and only if it satisfies the following four conditions:

(CONSTANTS) If $(x_i)_{i\in I}$ is a constant net with $x_i = x$ for each $i \in I$, then $((x_i)_{i\in I}, x) \in \mathcal{L}$.
(SUBNETS) If $((x_i)_{i\in I}, x) \in \mathcal{L}$ and $(y_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$, then $((y_j)_{j\in J}, x) \in \mathcal{L}$.
(DIVERGENCE) If $((x_i)_{i\in I}, x)$ is not in $\mathcal{L}$, then there exists a subnet $(y_j)_{j\in J}$ of $(x_i)_{i\in I}$ which has no subnet $(z_k)_{k\in K}$ so that $((z_k)_{k\in K}, x)$ belongs to $\mathcal{L}$.
(ITERATED LIMITS) If $((x_i)_{i\in I}, x) \in \mathcal{L}$ and if $((x_{i,j})_{j\in J_i}, x_i) \in \mathcal{L}$ for all $i \in I$, then $((x_{i,f(i)})_{(i,f)\in I\times M}, x) \in \mathcal{L}$, where $M = \prod_{i\in I} J_i$ is a product of directed sets.

Lemma 11. *Let P be a poset, $x, y, z \in P$, then $x \ll_\alpha y$, $z \rhd_\alpha y$ if and only if for every net $(x_i)_{i\in I}$ in P which o_2-converges to y, $x \leqslant x_i \leqslant z$ holds eventually.*

Proof. Suppose $x \ll_\alpha y$, $z \rhd_\alpha y$ and $(x_i)_{i\in I}$ o_2-converges to y. Then $x \leqslant x_i$ holds eventually and $x_i \leqslant z$ holds eventually. Thus there exist $i_1, i_2 \in I$ such that $x \leqslant x_i$ hold for all $i \geqslant i_1$ and $x_i \leqslant z$ hold for all $i \geqslant i_2$. Take $i_0 \in I$ with $i_0 \geqslant i_1, i_2$, then $x \leqslant x_i \leqslant z$ hold for all $i \geqslant i_0$. It follows that $x \leqslant x_i \leqslant z$ holds eventually. Thus the necessity has been proved. □

The sufficience is easily proved.

Proposition 12. *If P is a poset such that the class $\mathcal{L}$ satisfies the axiom (ITERATED LIMITS), then P is α-double continuous.*

Proof. For any $a \in P$, consider the collection $\{(x_{i,j})_{j\in J_i}\colon i \in I\}$ of nets $(x_{i,j})_{j\in J_i}$ that o_2-converges to a. Let $(x_i)_{i\in I}$ be constant net with $x_i = a$ for each $i \in I$. So for each $i \in I$, $(x_{i,j})_{j\in J_i}$ o_2-converges to x_i. Thus by assumption the net

$(x_{i,f(i)})_{(i,f)\in I\times M}$ o_2-converges to a, where $M=\prod_{i\in I} J_i$ and I is equipped with the largest pseudo order that $k\leqslant i$ hold for any $k,i\in I$. Thus there are subsets A, B of P such that $a=\sup A=\inf B$ and $y\leqslant x_{i,f(i)}\leqslant z$ holds eventually for any $y\in A$ and $z\in B$. For $y\in A$ and $z\in B$, there exists $(i_0,f_0)\in I\times M$ such that when $(i,f)\in I\times M$ and $(i,f)\geqslant(i_0,f_0)$, $y\leqslant x_{i,f(i)}\leqslant z$ holds. Take $i_1\in I$, by the largest pseudo order on I, we have $i_1\geqslant i_0$, let $j_0=f_0(i_1)$. If $j\geqslant j_0$, define $h_j\in M$ such that $h_j(i_1)=j$ for $i=i_1$ and $h_j(i)=f_0(i)$ for $i\neq i_1$. Obviously $h_j\geqslant f_0$ holds for each $j\geqslant j_0$, thus $(i_1,h_j)\geqslant(i_0,f_0)$. So $y\leqslant x_{i_1,h_j(i_1)}=x_{i_1,j}\leqslant z$ holds for all $j\geqslant j_0$. Since i_1 are arbitrary, $y\ll_\alpha a$, $z\rhd_\alpha a$ hold. It follows that $A\subseteq\{x\in P\colon x\ll_\alpha a\}$ and $B\subseteq\{w\in P\colon w\rhd_\alpha a\}$. Thus $a=\sup A\leqslant\sup\{x\in P\colon x\ll_\alpha a\}\leqslant a$ and $a=\inf B\geqslant\inf\{w\in P\colon w\rhd_\alpha a\}\geqslant a$, which implies $a=\sup\{x\in P\colon x\ll_\alpha a\}=\inf\{w\in P\colon w\rhd_\alpha a\}$. Hence P is α-double continuous. □

Theorem 13. *For any poset P, if o_2-convergence is topological, then P is α-double continuous.*

Lemma 14. *For any poset P, the class $\mathcal{L}$ satisfied the axioms* (*CONSTANTS*) *and* (*SUBNETS*)*.*

Proof. By Remark 3(3), the axiom (CONSTANTS) is satisfied.

Suppose $((x_i)_{i\in I},x)\in\mathcal{L}$ and M, N are subsets of P such that for each $a\in M$ and $b\in N$, there exists $k\in K$ such that $a\leqslant x_i\leqslant b$ for all $i\geqslant k$ and $\sup M=\inf N=x$. Let $(y_j)_{j\in J}$ be any subnet of $(x_i)_{i\in I}$, then there exists $f:J\to I$ such that for each $j\in J$, $x_{f(j)}=y_j$ and for each $i\in I$, there exists $j_1\in J$ such that $f(j)\geqslant i$ for all $j\geqslant j_1$. Take $a\in M, b\in N$, then there exists $i_0\in I$ such that $a\leqslant x_i\leqslant b$ for all $i\geqslant i_0$. For i_0, there exists $j_0\in J$ such that $f(j)\geqslant i_0$ for all $j\geqslant j_0$. Thus $a\leqslant x_{f(j)}=y_j\leqslant b$ for all $j\geqslant j_0$. Therefore $((y_j)_{j\in J},x)\in\mathcal{L}$. Hence the axiom (SUBNETS) is satisfied. □

Let P be a poset. we say a set $A\subseteq P$ is closed if for every net $(x_i)_{i\in I}$ in A which o_2-converges to some x, it follows that $x\in A$.

Let $\mathcal{F}_{o_2}$ be the class of all the closed sets of a poset P. From lemma 14, we can prove that $\mathcal{F}_{o_2}$ is a topology of closed sets. We call $\mathcal{F}_{o_2}$ o_2-topology on P.

Lemma 15. *If P is an α^*-double continuous poset, $y\in P$, then the sets $\Uparrow_{\alpha^*}y=\{x\in P\colon y\ll_{\alpha^*}x\}$ and $\triangle_{\alpha^*}y=\{z\in P\colon y\rhd_{\alpha^*}z\}$ are open in the o_2-topology.*

Proof. We show that the complement of $\Uparrow_{\alpha^*}y$ is closed. Let $(x_i)_{i\in I}$ be a net in the complement o_2-converging to x. If x is not in the complement, that is, $y\ll_{\alpha^*}x$, then there is an $i_0\in I$ such that $y\leqslant x_i$ for $i\geqslant i_0$. Now each x_i is not in $\Uparrow_{\alpha^*}y$, so there exists a net $(x_{i,j})_{j\in J_i}$ o_2-converging to x_i such that $y\nleqslant x_{i,j}$ for all j (actually the inequality only holds cofinally, but passing to this cofinal subset,we get the desired net). By Remark 10(1), $x=\sup\{y\in P\colon \exists z\in P,\ y\ll_{\alpha^*}z\ll_{\alpha^*}x\}=\inf\{a\in P\colon \exists w\in P,\ a\rhd_{\alpha^*}w\rhd_{\alpha^*}x\}$. If $y\ll_{\alpha^*}z\ll_{\alpha^*}x$ and $a\rhd_{\alpha^*}w\rhd_{\alpha^*}x$, from the fact that $(x_i)_{i\in I}$ o_2-converges to x, we know there exists $i_0'\in I$ such that $z\leqslant x_i\leqslant w$ for all $i\geqslant i_0'$. Then $y\ll_{\alpha^*}z\leqslant x_i$ and $a\rhd_{\alpha^*}w\geqslant x_i$ hold for all $i\geqslant i_0'$.

Again $(x_{i,j})_{j\in J_i}$ o_2-converges to x_i, so for each $i\geqslant i_0'$, there exists $g(i)\in J_i$ such that if $j\in J_i$ and $j\geqslant g(i)$, then $y\leqslant x_{i,j}\leqslant a$. Define $h\in\prod_{i\in i}J_i$ such that $h(i)=g(i)$ if $i\geqslant i_0'$ and $h(i)$ is any element in J_i otherwise. Now if $(i,f)\in I\times M$ and $(i,j)\geqslant(i_0',h)$, where $M=\prod_{i\in i}J_i$, then $y\leqslant x_{i,f(i)}\leqslant a$. So $(x_{i,f(i)})_{(i,f)\in I\times M}$ o_2-converges to x. Since $y\ll_{\alpha^*}x$, $(x_{i,f(i)})_{(i,f)\in I\times M}$ must eventually be in $\uparrow y$. But this contradicts the fact they were all chosen outside of $\uparrow y$. Hence x is in the complement of $\Uparrow_{\alpha^*}y$. So $\Uparrow_{\alpha^*}y$ is open in the o_2-topology. □

Dually, we can prove the case of $\triangle_{\alpha^*}y$.

Theorem 16. *If P is an α^*-double continuous poset, then o_2-convergence is topological.*

Proof. Let $(x_i)_{i\in I}$ be a net in P which converges to x in the o_2-topology. Since P is α^*-double continuous, $x=\sup\{y\in P\colon y\ll_{\alpha^*}x\}=\inf\{z\in P\colon z\rhd_{\alpha^*}x\}$. If $y\ll_{\alpha^*}x$ and $z\rhd_{\alpha^*}x$, then $x\in\Uparrow_{\alpha^*}y$ and $x\in\triangle_{\alpha^*}z$. By Lemma 15, $\Uparrow_{\alpha^*}y$ and $\triangle_{\alpha^*}z$ are open in the o_2-topology. So there exists $i_0\in I$ such that $x_i\in\Uparrow_{\alpha^*}y\cap\triangle_{\alpha^*}z$ for all $i\geqslant i_0$, that is, $y\ll_{\alpha^*}x_i$ and $z\rhd_{\alpha^*}x_i$ for all $i\geqslant i_0$. So $y\leqslant x_i\leqslant z$ for all $i\geqslant i_0$. Hence $(x_i)_{i\in I}$ o_2-converges to x.

Suppose that $(x_i)_{i\in I}$ is a net in P which o_2-converges to x, U is an open set in the o_2-topology which contains x. If $(x_i)_{i\in I}$ is not in U eventually, we can get a subnet $(y_j)_{j\in J}$ of $(x_i)_{i\in I}$ such that $(y_j)_{j\in J}$ is in the complement of U. From Lemma 14, $(y_j)_{j\in J}$ o_2-converges to x, so x is in the complement of U. This contradicts the fact that x is in U. Hence $(x_i)_{i\in I}$ converges to x in the o_2-topology. □

Therefore o_2-convergence is topological.

Corollary 17. *For any poset P which satisfies condition $(*)$, o_2-convergence is topological if and only if P is α-double continuous.*

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