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APPLICATION OF LAGRANGIAN RELAXATION TO COMPUTER NETWORK CONTROL

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Abstract—The problem of routing and flow control in a computer communication network is formulated as a mixed integer nonlinear programming problem. The Lagrangian relaxation method is used to decompose the problem into subproblems that are relatively easier to solve than the original problem. An algorithm is then developed which can obtain an optimal solution to the problem.

1. INTRODUCTION

Computer communication networks have developed out of the need to interconnect computer systems in order to share resources [1]. However, there are several problems that arise when computer systems are interconnected. These problems relate to network control, particularly routing and flow control. The routing problem is concerned with how to forward messages units, called packets, from one node to another through the network, for each origin–destination (OD) pair. For each OD pair, routes are usually chosen to minimize some cost function. The most commonly used cost function is delay. The flow control problem is concerned with regulating the amount of traffic entering and transiting the network so as not to overload the network. The result of poor routing and flow control is network congestion which in the worst case can lead to deadlock. It has been noted that the routing problem and the flow control problem are inter-related [2, 3]. Therefore, it has been found necessary to formulate a joint routing and flow control policy.

A computer network can comprise from a few to hundreds of nodes. Therefore, the task of choosing a route or routes for each pair of communicating nodes and regulating traffic on these routes can be a tedious one. There are two approaches to solving this problem: in one case, a central node is in charge of doing all computations to determine the appropriate amount of traffic to be admitted at each source node, and the path (or paths) over which this traffic may be routed; in the other case, distributed control is used. Here, each source node performs the necessary computation to determine its input traffic rate and the path (or paths) over which the traffic may be routed.

There are several advantages to using distributed control rather than centralized control. These include the fact that in the centralized scheme the central node represents a single point of failure. Its failure causes the network to break down. However, a network with centralized control is more likely to provide an optimal solution to the routing and flow control problem since the controller has a perfect knowledge of the state of the network. In a network with distributed control, each node makes its decisions on the basis of partial knowledge of the state of the network. Neighboring nodes periodically exchange information on their knowledge of the network state. And with this new information, each node updates its table of the status of the network and makes new routing and flow control decisions.

There is one complication to the routing and flow control problem in some networks. In these networks, all traffic between each OD pair is required to be routed over one path. This is called *virtual circuit switching*. As we mentioned earlier, the usual criterion for route selection is minimum delay. That is, the paths are chosen to minimize the total average delay in the network. The delay on each path is the sum of the delays on the links on the path. Since the delay on each link is some nonlinear function of the traffic on that link, the routing and flow control problem becomes a mixed integer non-linear programming problem when virtual circuit switching is practiced. This class of problems can be efficiently solved by the Lagrangian relaxation method [4–6]. The application of the Lagrangian relaxation method to the solution of the routing and flow control problem in virtual circuit-switched computer networks is the subject of this paper.

This paper is organized as follows. In Section 2 we present a formulation of the problem. In Section 2 the Lagrangian relaxation method is used to decompose the problem into two subproblems. An algorithm for solving the problem is presented in Section 4.

2. FORMULATION OF THE PROBLEM

The following notation will be used in the remainder of the paper:

L = the set of links in the network
l = a typical link
W = the set of OD pairs in the network
w = a typical OD pair
P_w = the set of paths available to OD pair w
p = a typical path
f_{wp} = the traffic on path p belonging to OD pair w
F_l = the total traffic on link l
r_w = the rate at which the OD pair w can transmit
C_l = the capacity of link l.

Since the traffic on each path must be non-negative, we must have that

$$f_{wp} \geqslant 0, \quad \forall p \in P_w, \quad \forall w \in W. \tag{1}$$

Also, the relationship between F_l and f_{wp} is the following:

$$F_l = \sum_{w} \sum_{p \in P_w} f_{wp} \delta_{lp}, \tag{2}$$

where δ_{lp} is defined by

$$\delta_{lp} = \begin{cases} 1, & \text{if } l \in p, \\ 0, & \text{otherwise.} \end{cases}$$

Since virtual circuit switching is used, we have the following constraints:

$$\sum_{p \in P_w} f_{wp} x_p = r_w, \qquad \forall w \in W, \tag{3}$$

$$\sum_{p \in P_w} x_p = 1, \qquad \forall w \in W, \tag{4}$$

$$x_p = 0, 1, \quad \forall p \in P_w, \quad \forall w \in W. \tag{5}$$

The delay on each link is usually a function of the total traffic on that link. If we assume that messages arrive at each node in a Poisson manner and that the transmission times of messages are exponentially distributed, then by the *independence assumption* [7] the traffic on each link can be modeled as an $M/M/1$ queue. It is well-known that for such a queue the delay becomes unbounded as the total traffic, F_l, approaches the link capacity. Therefore, we define the delay function, $D_l(F_l)$, for each link $l \in L$ to be a convex increasing and twice differentiable function of F_l. A typical expression for $D_l(F_l)$ is given by Ref. [7]:

$$D_l(F_l) = \frac{F_l}{C_l - F_l}. \tag{6}$$

Flow control is effected by varying the input rate r_w at each source in accordance with the state of the network [3]. Specifically, when the network is lightly loaded each source will be permitted to increase its input rate; and when the network becomes heavily loaded, each source will be forced to reduce its transmission rate. This goal is achieved by defining for each OD pair w a flow control function $B_w(r_w)$ to be a convex non-increasing and twice differentiable function of the input rate r_w. It is assumed that there is some maximum rate r_{max} at which any source can transmit. A typical expression for $B_w(r_w)$ is given by Ref. [8]:

$$B_w(r_w) = K(1/r_w - 1/r_{max}), \tag{7}$$

where K is some positive normalizing constant. And the constraint on r_w is

$$0 < r_w \leqslant r_{max}. \tag{8}$$

We now state the routing and flow problem as the following mixed integer non-linear programming problem:

Minimize

$$J = \sum_{l \in L} D_l(F_l) + \sum_{w \in W} B_w(r_w), \tag{P}$$

subject to conditions (1)–(5), (8).

3. LAGRANGIAN RELAXATION OF THE PROBLEM

The Lagrangian relaxation method is a computationally efficient method for solving problems which have a set of complicating constraints [4–6]. In this method, the complicating constraints are taken up into the objective function by means of the Lagrangian multipliers. The relaxed problem is usually easier to solve than the original problem, and its solution provides a lower bound on the solution to the original problem.

Let $\lambda = \{\lambda_w\}$ be a set of non-negative multipliers. Then relaxing the problem (P) relative to constraint (3) we obtain the following problem:

Minimize

$$J_1 = J + \sum_{w \in W} \lambda_w \left\{ r_w - \sum_{p \in P_w} f_{wp} x_p \right\}, \tag{P1}$$

subject to conditions (1), (2), (4), (5), (8).

The relaxed problem (P1) can be partitioned into two subproblems as follows:

Minimize

$$J_{11} = \sum_{l \in L} D_l(F_l) - \sum_{w \in W} \sum_{p \in P_w} \lambda_w f_{wp} x_p, \tag{S1}$$

subject to $f_{wp} \geqslant 0$

$$F_l = \sum_{w \in W} \sum_{p \in P_w} f_{wp} \delta_{lp}$$

$$\sum_{p \in P_w} = 1, \quad \forall w \in W,$$

$$x_p = 0.1, \quad \forall p \in P_w, \quad \forall w \in W.$$

Minimize

$$J_{12} = \sum_{w \in W} \{B_w(r_w) + \lambda_w r_w\}, \tag{S2}$$

subject to $0 < r_w \leqslant r_{max}$.

The following theorems establish the necessary and sufficient conditions for minimizing J_{11} and J_{12} for given values of λ.

Theorem 1

For given feasible values of x_p, the necessary and sufficient conditions on the flows f^*_{wp} to minimize subproblem (S1) subject to conditions (1), (2), (4) and (5) are

$$\sum_{l \in L} D'_l(F^*_l)\delta_{lp} - \lambda_w x_p \begin{cases} = 0, & \text{if } f^*_{wp} > 0, \\ \geqslant 0, & \text{if } f^*_{wp} = 0. \end{cases} \tag{9}$$

Theorem 2

The necessary and sufficient conditions on the input rates r^*_w to minimize subproblem (S2) subject to condition (8) are

$$B'_w(r_w) + \lambda_w \begin{cases} = 0, & \text{if } 0 < r^*_w < r_{max}, \\ \leqslant 0, & \text{if } r_{max}. \end{cases} \tag{10}$$

The proofs of these theorems can be found in any standard text on optimization, such as Ref. [9], and so is omitted here. Let $\beta \in P_w$ be a path that achieves the minimum

$$Z_\beta = \max_{p \in P_w} \lambda_w f_{wp}.$$

Then the x_p which solve subproblem (S1) are obtained as follows:

$$x_p = \begin{cases} 1, & \text{if } p = \beta, \\ 0, & \text{if } p \neq \beta. \end{cases} \tag{11}$$

The optimality conditions (10) suggest that λ_w is a distance metric. Therefore, if we define Λ_l as the length of link l, where

$$\Lambda_l = D'_l(F_l), \tag{12}$$

then one possible way to choose the initial value of λ_w is

$$\lambda^0_w = \min_{p \in P_w} \sum_l \Lambda^0_l \delta_{lp}. \tag{13}$$

The solution to the relaxed problem (P1), which is the sum of the solutions to the subproblems (S1) and (S2), is a function of λ. As stated earlier, this solution provides a lower bound on the solution to the original problem (P). Therefore, a good method of choosing λ is to find the one that provides the greatest lower bound. Thus, λ is the optimal solution to the dual problem [5]

$$Q = \max_{\lambda \geqslant 0} J_1(\lambda). \tag{D}$$

The dual problem is usually solved by means of the subgradient optimization method [10, 11]. The subgradient η^k_w at the kth iteration is defined as follows:

$$\eta^k_w = r^k_w - \sum_{p \in P_w} f^k_{wp} x^k_p. \tag{14}$$

The subgradient optimization generates the solutions $\{\lambda^k_w\}$ by the following rule [5]:

$$\lambda^k_w = \max[0, \lambda^{k-1}_w + \theta_{k-1}\eta^{k-1}_w], \quad k = 1, 2, \ldots, \tag{15}$$

θ_k is a positive step size which is given by

$$\theta_k = \frac{\alpha_k[\bar{Q} - - J_1(\lambda^k)]}{\|\eta^k\|^2}, \tag{16}$$

where $\bar{Q}$ is an upper bound on J_1, $\|\cdot\|$ is the Euclidean norm, and $0 < \alpha_k \leqslant 2$. The fundamental theoretical result on the convergence of the subgradient method is that $J_1 \to Q$ if $\theta_k \to 0$ and $\Sigma^k_{i=0}\theta_i \to \infty$[11]. Finally, the sequence α_k is usually determined by setting $\alpha_0 = 2$, and halving α_k whenever $J_1(\lambda)$ fails to increase in some fixed number of iterations [6].

4. THE ALGORITHM FOR SOLVING THE PROBLEM

The technique used to solve the routing and flow control problem consists of using the guidelines provided by the optimality conditions (9)–(11) to make incremental adjustments to the flows f_{wp} and input rates r_w, and setting the values of x_p. That is, the adjustments and path selection are made in a manner that moves the solution toward the optimal value. Thus, after obtaining λ_w^k via the subgradient optimization algorithm, we use the information provided by conditions (9) and (10) to compute f_{wp}^k and r_w^k, respectively, and obtain x_p^k by condition (11). the detailed algorithm is as follows.

A. Initialization. For each $w \in W$.

1. Set $r_w^0 = r_{\max}$ and choose any path $\beta \in P_w$ so that $x_\beta^0 = 1$, and $x_p^0 = 0$ for $p \neq \beta$.
2. Set $f_{wp}^0 = r_w^0$, and

$$F_l^0 = \sum_w \sum_{p \in P_w} f_{wp}^0 \delta_{lp}.$$

3. Set

$$\lambda_w^0 = \min_{p \in P_w} \sum_l \Lambda_l^0 \delta_{lp},$$

where Λ_l is given by condition (12).

4. Define $J_0 = 0$ and $\bar{Q} = M$, where M is a large number.

B. Iteration k. $k \geqslant 1$.

5. Compute the multipliers as follows:

$$\lambda_w^k = \max[0, \lambda_w^{k-1} + \theta_{k-1}\eta_w^{k-1}], \quad \forall w \in W,$$

where η_w is given by condition (14), and θ_k by condition (16).

6. Compute the "path lengths" as follows:

$$\sum_{p \in P_w} = \Lambda_{wp}^k = \sum_l \Lambda_l^{k-1} \delta_{lp}.$$

7. Let $\pi \in P_w$ be the path that achieves the following minimum:

$$\min_{p \in P_w} \lambda_w^{k-1} f_{wp}^{k-1}.$$

Then set $x_\pi^k = 1$ and $x_p^k = 0$ for $p \neq \pi$.

8. Make incremental adjustments to the routing and flow control variables f_{wp} and r_w as follows:

$$f_{w\pi}^k = \max[0, f_{w\beta}^{k-1} - \delta(\Lambda_{w\pi}^k - \lambda_w^k)]$$

$$r_w^k = \min[\sqrt{K/\lambda_w^k}, r_{\max}]$$

$$f_{wp}^k = 0, \quad \text{if} \quad p \neq \pi,$$

where δ is a positive scale factor, and $\beta \in P_w$ is the optimal path in iteration $k - 1$.

9. Obtain $J_1^k(\lambda^k)$ using the above results.
10. If $J_1^k(\lambda^k) > J_0$, then set $J_0 = J_1^k(\lambda^k)$.
11. If $u_{wp}^k = \{f_{wp}^k, r_w^k, x_{pj}^k\}$ is feasible in the original problem (P), then compute J^k. (Note that because the objective function of problem (P) is convex, the equality constraint (3) can be replaced by the less-than-or-equal-to inequality without loss of generality.)
12. If $J^k < \bar{Q}$, then set $\bar{Q} = J^k$.
13. If $\bar{Q} - J_0 \leqslant \epsilon$, stop; where ϵ is some predefined positive number.
14. If $J_1^k - J_1^{k-1} < \Delta$, set α_k to $\alpha_k/2$. Δ is another predefined positive number.
15. Obtain the subgradient η_w^k by condition (14).
16. Update θ_k by condition (16).
17. Set k to $k + 1$ and go to step 5.

Generally for very small step size, δ, the algorithm is guaranteed to converge, but the rate of convergence is small. For large values of δ, the rate of convergence increases but with the increased danger of no convergence. Computational results obtained in Refs [12–15] indicate that when λ^* is chosen in the manner described above, the algorithm always converges. Therefore, the choice of δ will determine the performance of the algorithm. We are currently testing this algorithm on some prototype networks.

5. SUMMARY

We have presented the computer communication network control problem as a viable candidate for the application of the Lagrangian relaxation method. We have developed an algorithm for solving this problem via the Lagrangian relaxation method. We are currently investigating the performance of the algorithm, and the results of the investigation will be reported later.

It must be mentioned that the dual problem (D) may not always be solved optimally; thus, a duality gap may exist. However, in several problems where the Lagrangian relaxation method has been used, the solutions have been reported to be optimal [12–15]. Therefore, we believe that the algorithm we have developed will generate an optimal solution to problem (P).

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