Convergence in the $p$th-Mean and Some Weak Laws of Large Numbers for Weighted Sums of Random Elements in Separable Normed Linear Spaces

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In this paper, a general result is proved from which the theorem follows.

**Theorem.** Let $X_n, n \geq 1$, be a sequence of tight random elements taking values in a separable Banach space $B$ such that $\|X_n\|, n \geq 1$, is uniformly integrable. Let $a_{nk}, n \geq 1, k \geq 1$, be a double array of real numbers satisfying $\sum_{k \geq 1} |a_{nk}| \leq \Gamma$ for every $n \geq 1$ for some positive constant $\Gamma$. Then $\sum_{k \geq 1} a_{nk} X_k, n \geq 1$, converges to 0 in probability if and only if $\sum_{k \geq 1} a_{nk} f(X_k), n \geq 1$, converges to 0 in probability for every $f$ in the dual space $B^*$.

1. Introduction

Limit theorems for sequences of Banach space valued random variables have engaged the attention of some probabilists over the last few decades. Under some conditions, a stochastic process can be identified as a certain Banach space valued random variable and a limit theorem for a sequence of Banach space valued random variables is virtually a limit theorem for a sequence of stochastic processes. This paper is an attempt to make a contribution in this area.

The general theme followed in this paper can be described as follows. Let $X_n, n \geq 1$, be a sequence of random elements taking values in a separable Banach space $B$ and $a_{nk}, n \geq 1, k \geq 1$, a double array of real numbers satisfying $\sum_{k \geq 1} |a_{nk}| \leq \Gamma$ for every $n \geq 1$ for some positive constant $\Gamma$. We want to examine under what conditions $\sum_{k \geq 1} a_{nk} X_k, n \geq 1$, converges in the $p$th-mean to 0, or, less restrictively, the Weak Law of Large Numbers holds.
for $\sum_{k \geq 1} a_{nk} X_k$, $n \geq 1$, i.e., $\sum_{k \geq 1} a_{nk} X_k$, $n \geq 1$, converges to 0 in probability. A natural approach to establish, for example, a Weak Law of Large Numbers for the sequence $\sum_{k \geq 1} a_{nk} X_k$, $n \geq 1$, is to check the validity of Weak Law of Large Numbers for the sequence $f(\sum_{k \geq 1} a_{nk} X_k)$, $n \geq 1$, of real random variables for every continuous linear functional $f$ from $B$ to the real line, $\mathbb{R}$, and then expect Weak Law of Large Numbers to hold for the sequence $\sum_{k \geq 1} a_{nk} X_k$, $n \geq 1$. This approach may not work all the time. See, for example, Example 5.2.2 of Taylor [12, p. 127]. In this paper, we persist with this approach and show that it indeed works in the case we deal with. This line of reasoning was pursued in the past with some success under each of the following conditions.

(A) $\sup_{n \geq 1} E \|X_n\|^p < \infty$ for some $p > 1$.

(B) $X_n$, $n \geq 1$, is uniformly bounded by a non-negative integrable real random variable $X$, i.e., $P(\|X_n\| > a) \leq P(X > a)$ for every $n \geq 1$ and $a > 0$, where $EX < \infty$.

(Some references are given in Remark 3 following Theorem 2.3.) The condition under which we operate in this paper is the uniform integrability of the sequence $\|X_n\|$, $n \geq 1$. This condition is weaker than each of (A) and (B). The following lemma elaborates this point.

**Lemma 1.1.** Let $X_n$, $n \geq 1$, be a sequence of real random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. In the following, (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d).

(a) $\sup_{n \geq 1} E |X_n|^r < \infty$ for some $r > 0$.

(b) There exists a random variable $X$ on $\Omega$ such that $X_n$, $n \geq 1$, is uniformly bounded by $X$, i.e., $P(\|X_n\| > a) \leq P(|X| > a)$ for every $n \geq 1$ and $a > 0$, and $E |X|^s < \infty$ for every $0 \leq s < r$.

(c) $X_n$, $n \geq 1$, is uniformly bounded by a random variable $X$ and $E |X|^r < \infty$ for some $r > 0$.

(d) $|X_n|^r$, $n \geq 1$, is uniformly integrable.

**Proof.** See Wang and Bhaskara Rao [16].

Theorem 2.3 of Section 2 deals with convergence in probability and convergence in the $p$th-mean for weighted sums of random elements and they are shown to be equivalent under an appropriate uniform integrability condition on the sequence $X_n$, $n \geq 1$. As a consequence of Theorem 2.3, we establish a Weak Law of Large Numbers for sequences of weighted sums of random elements under conditions weaker than those presented in the literature.

For most of the terminology used in this paper, the reader may refer to
Padgett and Taylor [7]. For ease in reference, however, we include the following notations and results.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A sequence $X_n$, $n \geq 1$, of real random variables on $\Omega$ is said to be uniformly absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_A |X_n| dP < \varepsilon$$

for every $n \geq 1$ whenever $A \in \mathcal{F}$ and $P(A) < \delta$. A sequence $X_n$, $n \geq 1$, of real random variables is said to be uniformly integrable if $X_n$, $n \geq 1$, is uniformly absolutely continuous and $\sup_{n \geq 1} E |X_n| < \infty$.

A Banach space $B$ is said to admit a Schauder basis if there exists a sequence $b_n$, $n \geq 1$, in $B$ with the following property: for every $x$ in $B$ there is a unique sequence $t_n$, $n \geq 1$, of real numbers such that $x = \sum_{n \geq 1} t_n b_n$. We can choose $b_n$'s such that $\|b_n\| = 1$ for every $n \geq 1$. For each $n \geq 1$, the coordinate functional $f_n$ is defined by $f_n(x) = t_n$ for $x$ in $B$, the Partial Sum Operator $U_n$ by $U_n(x) = f_1(x) b_1 + f_2(x) b_2 + \cdots + f_n(x) b_n$ for $x$ in $B$ and the Residual Operator $Q_n$ by $Q_n(x) = x - U_n(x)$ for $x$ in $B$. $U_n$, $n \geq 1$, is a sequence of continuous linear operators from $B$ to $B$ such that $\lim_{n \to \infty} U_n(x) = x$ for every $x$ in $B$. $Q_n$, $n \geq 1$, is a sequence of continuous linear operators from $B$ to $B$ such that $\lim_{n \to \infty} Q_n(x) = 0$ for every $x$ in $B$.

For any Normed Linear Space $B$, $B^*$ denotes the dual space of $B$ and consists of all real continuous linear functionals defined on $B$. The weak* topology on $B^*$ is described by defining convergence in $B^*$ as follows: a net $f_\alpha$, $\alpha \in D$, in $B^*$ is said to converge to an $f$ in $B^*$ if $f_\alpha$, $\alpha \in D$, converges to $f$ pointwise, i.e., $\lim_{\alpha \in D} f_\alpha(x) = f(x)$ for every $x$ in $B$. A subset $B_1$ of $B^*$ is said to be total if for some $x$ in $B$, $g(x) = 0$ for every $g$ in $B_1$ then $x = 0$.

Let $B_1$ be a total subset of $B^*$, where $B$ is a separable Banach space. Let $S(B_1)$ be the linear span of $B_1$. Then $S(B_1)$ is weak*-dense in $B^*$. See Kelley and Namioka [5, Theorem 16.5, p. 142]. Using Theorem V.5.1 and the Krein–Smulian Theorem V.5.7 in Dunford and Schwartz [2, pp. 426 and 429, respectively], one can establish the following result.

**Lemma 1.2.** A convex subset of $B^*$ is weak*-closed in $B^*$ if and only if it is sequentially weak*-closed in $B^*$.

See also Exercise 16 on page 437 of Dunford and Schwartz [2].

A sequence $X_n$, $n \geq 1$, of random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in a Normed Linear Space $B$ is said to be uniformly tight if for every $\varepsilon > 0$ there exists a compact $C$ of $B$ such that $P\{X_n \in C\} > 1 - \varepsilon$ for every $n \geq 1$. If $B$ is a separable Banach space and $X_n$, $n \geq 1$, is identically distributed, then $X_n$, $n \geq 1$, is uniformly tight.
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We need some preliminary lemmas before proving the main result.

**Lemma 2.1.** Let $C$ be a compact subset of a Banach space $B$ and $T_n$, $n \geq 1$, a sequence of continuous linear operators from $B$ into a Banach space $F$. Suppose there is a continuous linear operator $T$ from $B$ to $F$ such that $T_n$, $n \geq 1$, converges to $T$ pointwise, i.e.,

$$\lim_{n \to \infty} T_n(x) = T(x) \quad \text{for every } x \in B.$$ 

Then $T_n$, $n \geq 1$, converges to $T$ uniformly on $C$.

Taylor and Wei [15, Lemma 5, p. 154] established the above result in the special case when $B$ is a Banach space admitting a Schauder basis $b_n$, $n \geq 1$, and the sequence $T_n$, $n \geq 1$, is the sequence $Q_t$, $t \geq 1$, of Residual Operators from $B$ to $B$ associated with the basis $b_n$, $n \geq 1$. Their proof can be adapted to prove the above result by using Dini's theorem. See Kelley [4, Problem E, p. 239]. We need the above result in the form stated by Taylor and Wei and also for the case when $F = \mathbb{R}$ in the subsequent results.

**Lemma 2.2.** Let $B$ be a Banach space with a Schauder basis $b_n$, $n \geq 1$. Let $Q_t$, $t \geq 1$, be the sequence of Residual Operators associated with the basis $b_n$, $n \geq 1$. Let $X_n$, $n \geq 1$, be a sequence of uniformly tight random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in $B$ such that $\|X_n\|^p$, $n \geq 1$, is uniformly absolutely continuous for some $p \geq 1$. Then

$$\lim_{t \to \infty} \sup_{n \geq 1} E \| Q_t(X_n) \|^p = 0.$$ 

**Proof.** Let $\varepsilon > 0$. Since $Q_t$, $t \geq 1$, converges to 0 pointwise on $B$, there exists a positive constant $K$ such that $\|Q_t\| \leq K$ for every $t \geq 1$. See Dunford and Schwartz [2, Theorem II.3.6, p. 60]. Since $X_n$, $n \geq 1$, is uniformly tight and $\|X_n\|^p$, $n \geq 1$, is uniformly absolutely continuous, there exists a compact subset $C$ of $B$ such that

$$\int_{\{X_n \in \mathbb{C} \}} \|X_n\|^p \, dP < \varepsilon/(2K)^p \quad (2.1)$$

for every $n \geq 1$. For each $n \geq 1$, let

$$Y_n = \begin{cases} X_n & \text{if } X_n \in C, \\ 0 & \text{if } X_n \in C^c. \end{cases}$$
For this compact set C, by Lemma 2.1, there exists $t_0 > 1$ such that

$$\|Q_t(Y_n)\|^p < \varepsilon/2^p \tag{2.2}$$

for every $n \geq 1$ whenever $t \geq t_0$. Thus, we have by $C_r$-inequality (see Loève [6, p. 155]),

$$E \|Q_t(X_n)\|^p = E \|Q_t(Y_n) + Q_t(X_n - Y_n)\|^p$$

$$\leq 2^{p-1}[E \|Q_t(Y_n)\|^p + E \|Q_t(X_n - Y_n)\|^p]$$

$$< \varepsilon/2 + [\|Q_t\|^p E \|X_n - Y_n\|^p] 2^{p-1}$$

$$< \varepsilon/2 + 2^{p-1}K^p \int \|X_n\|^p dP$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$  \text{by (2.1) and (2.2)},

for every $t \geq t_0$ and $n \geq 1$. This completes the proof.

The following is the main result.

**Theorem 2.3.** Let $X_n$, $n \geq 1$, be a sequence of uniformly tight random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in a separable Banach space $B$ such that $\|X_n\|^p$, $n \geq 1$, is uniformly absolutely continuous for some $p \geq 1$. Let $a_{nk}$, $n \geq 1$, $k \geq 1$, be a double array of real numbers satisfying $\sum_{k \geq 1} |a_{nk}| < \Gamma$ for every $n \geq 1$ for some positive constant $\Gamma$. Let $B_1$ be any total subset of $B^*$. Then the following statements are equivalent.

(i) $E |g(\sum_{k \geq 1} a_{nk}X_k)|^p$, $n \geq 1$, converges to 0 for every $g$ in $B_1$.

(ii) $g(\sum_{k \geq 1} a_{nk}X_k)$, $n \geq 1$, converges to 0 in probability for every $g$ in $B_1$.

(iii) $E |f(\sum_{k \geq 1} a_{nk}X_k)|^p$, $n \geq 1$, converges to 0 for every $f$ in $B^*$.

(iv) $f(\sum_{k \geq 1} a_{nk}X_k)$, $n \geq 1$, converges to 0 in probability for every $f$ in $B^*$.

(v) $E \|\sum_{k \geq 1} a_{nk}X_k\|^p$, $n \geq 1$, converges to 0.

(vi) $\sum_{k \geq 1} a_{nk}X_k$, $n \geq 1$, converges to 0 in probability.

**Proof.** The proof is carried out in the following steps.

1. Observe that if $X_n$, $n \geq 1$, is uniformly tight and $\|X_n\|^p$, $n \geq 1$, is uniformly absolutely continuous, then $\|X_n\|^p$, $n \geq 1$, is uniformly integrable.

2. The almost sure convergence of the series $\sum_{k \geq 1} a_{nk}X_k$ follows from the facts that $\sum_{k \geq 1} |a_{nk}| E \|X_k\| < \infty$ and $B$ is a complete metric space. See Chung [1, (xi), p. 42].
(3) The integrability of $\|\sum_{k \geq 1} a_{nk} X_k \|^p$ can be established as follows. By Fatou's lemma (see Chung [1, (xii), p. 421]) and Minkowski's inequality

$$
\left( E \left\| \sum_{k \geq 1} a_{nk} X_k \right\|^p \right)^{1/p} = \left( E \lim_{m \to \infty} \left\| \sum_{k = 1}^m a_{nk} X_k \right\|^p \right)^{1/p}
$$

$$\leq \left( \liminf_{m \to \infty} E \left\| \sum_{k = 1}^m a_{nk} X_k \right\|^p \right)^{1/p}
$$

$$\leq \liminf_{m \to \infty} \sum_{k = 1}^m |a_{nk}| (E \|X_k\|^p)^{1/p}
$$

$$\leq \sum_{k \geq 1} |a_{nk}| (E \|X_k\|^p)^{1/p} < \infty.
$$

(4) Equivalence of (i) and (ii) can be established if we show that $\left\| g \left( \sum_{k \geq 1} a_{nk} X_k \right) \right\|^p, n \geq 1$, is uniformly integrable. See Chung [1, Theorem 4.5.4, p. 97]. We note that for any $n \geq 1$ and for any $A$ in $\mathcal{B}$

$$
\left\| g \left( \sum_{k \geq 1} a_{nk} X_k \right) \right\|^p \leq \|g\|^p \left\| \sum_{k \geq 1} a_{nk} X_k \right\|^p
$$

and

$$
\left( \int_A \left| g \left( \sum_{k \geq 1} a_{nk} X_k \right) \right|^p dP \right)^{1/p} \leq \|g\| \sum_{k \geq 1} |a_{nk}| \left( \int_A \|X_k\|^p dP \right)^{1/p}.
$$

Since $\|X_k\|^p, k \geq 1$, is uniformly integrable and $\sum_{k \geq 1} |a_{nk}| \leq \Gamma$ for every $n \geq 1$, it follows that $\left\| g \left( \sum_{k \geq 1} a_{nk} X_k \right) \right\|^p, n \geq 1$, is uniformly integrable. The equivalence of (iii) and (iv) and that of (v) and (vi) follows in the same vein.

(5) We prove (i) $\Rightarrow$ (iii). Let $B_2 = \{ f \in B^*; E |f(\sum_{k \geq 1} a_{nk} X_k)|^p, n \geq 1, \text{ converges to 0} \}$. It is obvious that $B_2$ is a convex set and that it contains $S(B_1)$, the linear span of $B_1$. We show that $B_2$ is sequentially weak*-closed in $B^*$. Let $f_n, n \geq 1$, be a sequence in $B_2$ converging to some $f$ in $B^*$ in the weak*-topology of $B^*$. We show that $f \in B_2$. There exists a positive constant $\Gamma_1$ such that $\|f_m\| \leq \Gamma_1$ for every $m \geq 1$ and $\|f\| \leq \Gamma_1$. See Dunford and Schwartz [2, Theorem II.3.6, p. 60]. Let $\epsilon > 0$. Since $X_n, n \geq 1$, is uniformly tight and $\|X_n\|^p, n \geq 1$, is uniformly absolutely continuous, there exists a compact subset $C$ of $B$ such that

$$
\left( \int_{\{X_n \in C\}} \|X_n\|^p dP \right)^{1/p} < \epsilon/4\Gamma_1
$$

for every $n \geq 1$. Assume, without loss of generality, that $0 \in C$. Define for each $n \geq 1$
\[ Y_n = X_n \quad \text{if } X_n \in C, \]
\[ = 0 \quad \text{if } X_n \in C^c, \]
and
\[ Z_n = X_n - Y_n. \]

From (2.3), for every \( n \geq 1, \)
\[
(E \|Z_n\|_p)^{1/p} = \left( \int_{\{X_n \in C\}} \|X_n\|^p dP \right)^{1/p} < \varepsilon/4 \Gamma_1. \tag{2.4}
\]

For this compact set \( C, \) by Lemma 2.1, there exists \( m_0 \geq 1 \) such that
\[
|f - f_m(Y_k(w))| < \varepsilon/4 \Gamma
\]
holds uniformly in \( w \) in \( \Omega, \) for every \( k \geq 1 \) and \( m \geq m_0. \) Also, there exists \( n_0 \geq 1 \) such that
\[
\left( E \left| f_{m_0} \left( \sum_{k \geq 1} a_{nk} X_k \right) \right|^p \right)^{1/p} < \varepsilon/4 \tag{2.6}
\]
whenever \( n \geq n_0. \) So, if \( n \geq n_0, \) by Minkowski's inequality,
\[
\left( E \left| f \left( \sum_{k \geq 1} a_{nk} X_k \right) \right|^p \right)^{1/p}
\]
\[
= \left| E \left( \sum_{k \geq 1} a_{nk} (f - f_{m_0})(X_k) + f_{m_0} \left( \sum_{k \geq 1} a_{nk} X_k \right) \right) \right|^{1/p}
\]
\[
= \left| E \left( \sum_{k \geq 1} a_{nk} (f - f_{m_0})(Y_k) + \sum_{k \geq 1} a_{nk} f(Z_k) \right. \right.
\]
\[
- \left. \left. \sum_{k \geq 1} a_{nk} f_{m_0}(Z_k) + f_{m_0} \left( \sum_{k \geq 1} a_{nk} X_k \right) \right) \right|^{1/p}
\]
\[
\leq \left( E \left( \sum_{k \geq 1} a_{nk} (f - f_{m_0})(Y_k) \right) \right)^{1/p} + \left( E \left( \sum_{k \geq 1} a_{nk} f(Z_k) \right) \right)^{1/p}
\]
\[
+ \left( E \left( \sum_{k \geq 1} a_{nk} f_{m_0}(Z_k) \right) \right)^{1/p} + \left( E \left| f_{m_0} \left( \sum_{k \geq 1} a_{nk} X_k \right) \right| \right)^{1/p}
\]
\[
< \sum_{k \geq 1} |a_{nk}| \left( E \|f - f_{m_0}\|(Y_k)\|^p \right)^{1/p} + \sum_{k \geq 1} |a_{nk}| \|f\| (E \|Z_k\|_p)^{1/p}
\]
\[
+ \sum_{k \geq 1} |a_{nk}| \|f_{m_0}\| (E \|Z_k\|_p)^{1/p} + \varepsilon/4
\]
\[
< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon.
\]
by (2.4), (2.5) and (2.6). Hence $E |f(\sum_{k \geq 1} a_{nk}X_k)|^p$, $n \geq 1$, converges to 0. This proves that $B_2$ is sequentially weak*-closed in $B^*$. By Lemma 1.2, $B_2$ is weak*-closed in $B^*$. Since $S(B_1)$ is weak*-dense in $B^*$ and $B_2 \supset S(B_1)$, it follows that $B_2 = B^*$. Hence the implication (i) $\Rightarrow$ (iii) follows.

(6) We prove (iii) $\Rightarrow$ (v). First, we treat the case when $B$ admits a Schauder basis $b_n$, $n \geq 1$. Let $f_t$, $t \geq 1$, $U_t$, $t \geq 1$, and $Q_t$, $t \geq 1$, be the sequences of coordinate functionals, Partial Sum Operators and Residual Operators, respectively, associated with the basis $b_n$, $n \geq 1$. By Lemma 2.2, there exists $t_0 \geq 1$ such that

$$
(E \|Q_{t_0}(X_k)\|^p)^{1/p} < \varepsilon/2T
$$

(2.7) for every $k \geq 1$. By (iii), there exists $n_0 \geq 1$ such that

$$
\left( E \left| f_t \left( \sum_{k \geq 1} a_{nk}X_k \right) \right|^p \right)^{1/p} < \varepsilon/2t_0
$$

(2.8)

for $1 \leq t \leq t_0$, whenever $n \geq n_0$. So, if $n \geq n_0$,

$$
\left( E \left| \sum_{k \geq 1} a_{nk}X_k \right|^p \right)^{1/p} = \left( E \|U_{t_0} \left( \sum_{k \geq 1} a_{nk}X_k \right) + Q_{t_0} \left( \sum_{k \geq 1} a_{nk}X_k \right)\|^p \right)^{1/p}
$$

$$
\leq \left( E \|U_{t_0} \left( \sum_{k \geq 1} a_{nk}X_k \right)\|^p \right)^{1/p}
$$

$$
+ \left( E \|Q_{t_0} \left( \sum_{k \geq 1} a_{nk}X_k \right)\|^p \right)^{1/p}
$$

$$
\leq \left( E \| \sum_{i=1}^{t_0} f_t \left( \sum_{k \geq 1} a_{nk}X_k \right) b_i \|^p \right)^{1/p}
$$

$$
+ \sum_{k \geq 1} |a_{nk}| \left( E \|Q_{t_0}(X_k)\|^p \right)^{1/p}
$$

$$
< \sum_{i=1}^{t_0} \left( E \left| f_t \left( \sum_{k \geq 1} a_{nk}X_k \right) \right|^p \right)^{1/p} + \varepsilon/2
$$

$$
< \varepsilon/2 + \varepsilon/2 = \varepsilon,
$$

by (2.7) and (2.8).

Now, let $B$ be any general Banach space. Then there exists an isometric isomorphism $h$ from $B$ onto a closed subspace of $C[0, 1]$, the space of all real valued continuous functions defined on the unit interval $[0, 1]$ equipped with the supremum norm. See Semadeni [10, Theorem 8.7.2, p. 157]. $C[0, 1]$ has a Schauder basis. The sequence $h(X_n)$, $n \geq 1$, of random elements taking values in $C[0, 1]$ has the property that
converges to 0 for every \( f \) in the dual space of \( C[0, 1] \), where \( f \circ h \) is the composition of the maps of \( f \) and \( h \). By what has been established above,

\[
E \left| f \left( \sum_{k \geq 1} a_{nk} h(X_k) \right) \right|^p = E \left| f \circ h \left( \sum_{k \geq 1} a_{nk} X_k \right) \right|^p, \quad n \geq 1,\]

But \( \| \sum_{k \geq 1} a_{nk} X_k \| = \| h(\sum_{k \geq 1} a_{nk} X_k) \| = \| \sum_{k \geq 1} a_{nk} h(X_k) \| \) for every \( n \geq 1 \).

(7) The implication (v) \( \Rightarrow \) (i) is obvious.

This completes the proof.

Remarks. (1) The above result can be rephrased in the framework of double arrays \( X_{nk}, n \geq 1, k \geq 1, \) of random elements. The relevant condition is that \( \| X_{nk} \|^p, n \geq 1, k \geq 1, \) is uniformly absolutely continuous for some \( p \geq 1. \)

(2) In the realm of separable Normed Linear Spaces \( B \) and double arrays \( X_{nk}, n \geq 1, k \geq 1, \) of random elements, one can show that (iii), (iv), (v) and (vi) of Theorem 2.3 are equivalent for the sequence \( \sum_{k \geq 1} a_{nk} X_k, n \geq 1, \) under the conditions that (a) \( \| X_{nk} \|^p, n \geq 1, k \geq 1, \) is uniformly absolutely continuous for some \( p \geq 1, \) (b) \( \sum_{k \geq 1} a_{nk} X_k \) converges \( [P] \) a.e. for every \( n \geq 1, \) (c) \( X_{nk}, n \geq 1, k \geq 1, \) is uniformly tight and (d) \( \sum_{k \geq 1} |a_{nk}| \leq \Gamma \) for every \( n \geq 1 \) for some positive constant \( \Gamma. \)

(3) Theorem 3.4 of Howell and Taylor [3, p. 228] follows from Remark (1) above and Lemma 1.1. Theorem 4 of Taylor and Wei [15, p. 153] follows from Remark (2) above and Lemma 1.1. Theorem 2 of Taylor and Padgett [13, p. 438] can be deduced from Theorem 2.3. Theorems 2.3 and 2.4 of Taylor and Padgett [14, pp. 231 and 232] can also be derived using Theorem 2.3 above. Finally, Theorem 4 of Wei and Taylor [17, p. 285] is a special case of Theorem 2.3 above.

Now, we derive a Weak Law of Large Numbers for weighted sums using Theorem 2.3.

**THEOREM 2.4.** Let \( X_n, n \geq 1, \) be a sequence of random elements defined on some probability space \( (\Omega, \mathcal{F}, P) \) taking values in a separable Banach space \( B. \) Let \( a_{nk}, n \geq 1, k \geq 1, \) be a double array of real numbers. Let \( B_1 \) be a total subset of the dual space \( B^* \) of \( B. \) Suppose the following hold.

(a) \( \| X_n \|, n \geq 1, \) is uniformly absolutely continuous.
(b) \( X_n, n \geq 1, \) is uniformly tight.
(c) \( g(X_n), n \geq 1, \) is pairwise independent for every \( g \) in \( B_1. \)
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(d) \( \sum_{k \geq 1} |a_{nk}| \leq \Gamma \) for every \( n \geq 1 \) for some positive constant \( \Gamma \).
(e) \( \max_{k \geq 1} |a_{nk}|, \ n \geq 1 \), converges to 0.

Then \( \sum_{k \geq 1} a_{nk}(X_k - EX_k), \ n \geq 1 \), converges to 0 in the mean and hence in probability.

Proof: First, we draw the attention of the reader to the following result. "If \( Y_n, \ n \geq 1 \), is a sequence of pairwise independent uniformly integrable real random variables and \( a_{nk}, \ n \geq 1, \ k \geq 1 \), is a double array of real numbers satisfying (d) and (e) above, then \( \sum_{k \geq 1} a_{nk}(Y_k - EY_k), \ n \geq 1 \), converges to 0 in the mean." See Wang and Bhaskara Rao [16].

Second, we note that \( X_n - EX_n, \ n \geq 1 \), is uniformly tight. Taylor [12, Lemma 5.2.1, p. 121] established this result under the assumption that \( \sup_{n \geq 1} E \|X_n\|^p < \infty \) for some \( p > 1 \). This conclusion is also valid under the weaker condition that \( \|X_n\|, \ n \geq 1 \), is uniformly absolutely continuous and essentially the same proof works. Uniform integrability of \( \|X_n - EX_n\|, \ n \geq 1 \), follows from (a) above and that \( \{EX_n; n \geq 1\} \) is a totally bounded subset of \( B \). It now follows that (i) of Theorem 2.3 holds for the sequence \( X_n - EX_n, \ n \geq 1 \), and the double array of real numbers \( a_{nk}, \ n \geq 1, \ k \geq 1 \).

Hence \( \sum_{k \geq 1} a_{nk}(X_k - EX_k), \ n \geq 1 \), converges to 0 in the mean.

In the context of separable Normed Linear Spaces, we have the following analogue of Theorem 2.4.

THEOREM 2.5. Let \( X_n, \ n \geq 1 \), be a sequence of random elements defined on a probability space \( (\Omega, \mathcal{F}, P) \) taking values in a separable Normed Linear Space \( B \). Let \( a_{nk}, \ n \geq 1, \ k \geq 1 \), be a double array of real numbers. Suppose, in addition to (a), (b), (d) and (e) of Theorem 2.4, that the following hold.

(f) \( f(X_n), \ n \geq 1 \), is pairwise independent for every \( f \) in \( B^* \).
(g) \( EX_n \) exists for every \( n \geq 1 \).
(h) \( \sum_{k \geq 1} a_{nk}(X_k - EX_k) \) converges \( [P] \) a.e. for every \( n \geq 1 \).

Then \( \sum_{k \geq 1} a_{nk}(X_k - EX_k), \ n \geq 1 \), converges to 0 in the mean and hence in probability.

Remarks. Weak Laws of Large Numbers stated above include some results of Taylor [12, Chap. 5]. Theorem 1 of Rohatgi [9, p. 305] extends to separable Banach spaces if we assume, in addition, that \( X_n, \ n \geq 1 \), is uniformly right. This extension complements the discussion initiated by Padgett and Taylor [8, p. 393] and Taylor [12, p. 114] on Rohatgi's result.
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REFERENCES


