



Generalized spin Sutherland systems revisited

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Abstract

We present generalizations of the spin Sutherland systems obtained earlier by Blom and Langmann and by Polychronakos in two different ways: from $SU(n)$ Yang–Mills theory on the cylinder and by constraining geodesic motion on the N -fold direct product of $SU(n)$ with itself, for any $N > 1$. Our systems are in correspondence with the Dynkin diagram automorphisms of arbitrary connected and simply connected compact simple Lie groups. We give a finite-dimensional as well as an infinite-dimensional derivation and shed light on the mechanism whereby they lead to the same classical integrable systems. The infinite-dimensional approach, based on twisted current algebras (alias Yang–Mills with twisted boundary conditions), was inspired by the derivation of the spinless Sutherland model due to Gorsky and Nekrasov. The finite-dimensional method relies on Hamiltonian reduction under twisted conjugations of N -fold direct product groups, linking the quantum mechanics of the reduced systems to representation theory similarly as was explored previously in the $N = 1$ case.

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1. Introduction

The Calogero–Moser–Sutherland type many-body systems [1–3] and their generalizations continuously attract attention since they are ubiquitous in physical applications and are related

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to intriguing mathematical structures. See, for instance, the reviews [4–7]. A powerful approach to these integrable systems consists in viewing them as Hamiltonian reductions of higher dimensional free systems. This method first came to light in the papers [8,9]. In particular, Kazhdan, Kostant and Sternberg [9] obtained the classical Sutherland Hamiltonian

$$\mathcal{H}(q, p) = \frac{1}{2} \sum_{k=1}^n p_k^2 + \sum_{i \neq j} \frac{v^2}{\sin^2(q_i - q_j)} \quad (1.1)$$

by reducing the kinetic energy of the geodesic motion on the group $SU(n)$ using the symmetry defined by the conjugation action of $SU(n)$ on itself. Their reduction required fixing the Noether charges of the symmetry in a very special manner. Later it turned out that more general choices lead to extensions of the Sutherland system by ‘spin’ degrees of freedom that belong to reductions of coadjoint orbits of $SU(n)$ by the action of the maximal toral subgroup. Spin Sutherland systems of this kind appear for every simple Lie group [5,10–12]. Moreover, not only the choice of the constraints, alias the value of the momentum map, but also the underlying symmetry can be generalized in several ways. For example, we obtained spin Sutherland systems by reducing the free motion on simple compact Lie groups utilizing twisted conjugations [13]. These reductions were analyzed [14] at the quantum mechanical level, too, which yields a bridge from harmonic analysis to integrable systems.

The derivation of the Calogero and Sutherland systems given in [9] represents a paradigm for the reduction approach to integrable systems. This framework has since been expanded by the introduction of several new parent systems as master integrable systems. We mention only the influential work of Gorsky and Nekrasov [15,16], where the Sutherland Hamiltonian was re-derived from a free Hamiltonian on an infinite-dimensional phase space built on $su(n)$ -valued currents on the circle. As discussed also in [5,17], this connects the Sutherland system to $SU(n)$ Yang–Mills theory on the cylinder.

Our present work was motivated mainly by questions stemming from the papers of Blom and Langmann [18,19] and Polychronakos [20], where a family of generalized spin Sutherland systems was derived by means of two different methods. In these systems the particle positions are coupled to spin variables living on N arbitrary coadjoint orbits of $SU(n)$, and the Hamiltonian also involves N arbitrary scalar parameters, for any integers $n > 1$ and $N > 1$. Blom and Langmann obtained their systems from $SU(n)$ Yang–Mills theory on the cylinder, placing non-dynamical ‘color charges’ at N arbitrary locations on the circle. Thus their derivation fits in the Gorsky–Nekrasov framework. On the other hand, Polychronakos proceeded by constraining the geodesic motion on the N -fold direct product of $SU(n)$ with itself, using N arbitrary scale factors in the definition of the bi-invariant Riemannian metric. Although it was not stressed in [20], this amounts to Hamiltonian symmetry reduction with respect to twisted conjugations on the direct product group, defined with the aid of the cyclic permutation of the N -factors. Based on direct comparison of the Hamiltonians, both Blom–Langmann [19] and Polychronakos [20] pointed out that their respective systems coincide, but they did not provide any conceptual explanation of this remarkable fact. It is natural to ask for a better understanding of the mechanism behind this coincidence.

In this paper we describe two derivations of group theoretic generalizations of the systems of [18,20] and shed light on the mechanism whereby these two derivations lead to the same outcome. The first approach relies on symplectic reduction of geodesic motion on the N -fold direct product $G \times \cdots \times G$ for any compact simple Lie group G . The underlying symmetry is built from twisted conjugations involving the cyclic permutation of the N factors and, for simply laced

groups, also a Dynkin diagram automorphism of G . For $N = 1$ these are the reductions analyzed in [13,14]. The second method utilizes a free Hamiltonian on the cotangent bundle of an infinite-dimensional configuration space consisting of Lie algebra valued (quasi-)periodic currents. The reduction is based on the natural action of the corresponding (twisted) loop group, where the twisting can be non-trivial in the simply laced cases. This second derivation fits in the framework of Gorsky–Nekrasov [15,16,5] and Blom–Langmann [18,19]. In the infinite-dimensional reduction a partial gauge fixing associated with a finite-dimensional gauge slice will be exhibited that will be explicitly mapped into a gauge slice of a corresponding partial gauge fixing that arises in the finite-dimensional reduction. The mapping between these gauge slices will be seen to be a bijection that engenders an isomorphism of the respective reduced systems. When $G = SU(n)$ equipped with the identity diagram automorphism, then the reduced systems are precisely those found in [18,20].

The organization of the paper is as follows. Section 2 is devoted to group theoretic preliminaries regarding the relevant twisted conjugations of the direct product groups. No new results are claimed in this section, even though we could not find references for the statements of the two technical lemmas that we prove. Section 3 contains the analysis of the finite-dimensional Hamiltonian reduction under twisted conjugations. The structure of the reduced system is given by Proposition 3, which represents our first new result. The formula of the reduced free Hamiltonian displayed by Eq. (3.23) is spelled out in detail in Subsection 3.2 in the case of the identity diagram automorphism for any group G . Proposition 4 implies that this reproduces the spin Sutherland systems of [18,20] for $G = SU(n)$. In Section 4 we present an alternative derivation in the general case based on (twisted) current algebras. Our second result is that we here explain the equivalence of the reduced systems that result from the finite-dimensional and from the infinite-dimensional derivations. We conclude in Section 5 by outlining the quantum mechanical analogue of the finite-dimensional classical Hamiltonian reduction of Section 3. There is also Appendix A, where a useful technical result is described.

2. Twisted conjugations and the monodromy matrix

In this section we collect the group theoretic results that will be needed later.

Let G be a connected and simply-connected compact simple Lie group. Choose an automorphism γ of G that corresponds to a diagram automorphism, denoted by γ' , of the Lie algebra \mathcal{G} of G with respect to the Cartan subalgebra $\mathcal{T} < \mathcal{G}$. If \mathcal{G} is not simply laced then γ is the identity automorphism, otherwise it can have order 1, 2 or 3; the latter occurs only for $G = \text{Spin}(8)$. The group G acts on itself by the twisted conjugations C_η^γ that for each $\eta \in G$ operate as

$$C_\eta^\gamma: G \rightarrow G, \quad g \mapsto \gamma(\eta)g\eta^{-1}. \quad (2.1)$$

The space of the corresponding G -orbits can be identified [21] with the space of the orbits of the so-called twisted Weyl group acting on the fixed point set \mathbb{T}^γ of γ in the maximal torus $\mathbb{T} < G$ associated to \mathcal{T} . For non-trivial γ , the G -orbits in question are termed twisted conjugacy classes. The maximal (also called principal) orbits with respect to the C^γ action fill a dense open subset $G' \subset G$, and the space of these orbits can be parametrized by the interior of a convex polytope, $\check{\mathcal{T}}^\gamma \subset \mathcal{T}^\gamma$. This means that every C^γ orbit of G in G' passes through a unique element of the form e^μ for some $\mu \in \check{\mathcal{T}}^\gamma$. Moreover, the isotropy subgroup of these elements equals \mathbb{T}^γ , that is, if $\mu \in \check{\mathcal{T}}^\gamma$ then $C_\eta^\gamma(e^\mu) = e^\mu$ holds if and only if $\eta \in \mathbb{T}^\gamma$. The closure of $\check{\mathcal{T}}^\gamma$ is often called ‘Weyl alcove’, since it is a fundamental domain of the (twisted) affine Weyl group acting on \mathcal{T}^γ . Its explicit form can be found for example in [22] (see also [14]).

Now pick a positive integer, N , and consider the direct product

$$\vec{G} := \overbrace{G \times \cdots \times G}^{N \text{ times}}. \tag{2.2}$$

Adopting an admittedly unusual (but convenient) notation, the elements of \vec{G} will be encoded as N -component ‘vectors’

$$\vec{g} = g_1 \oplus \cdots \oplus g_N, \quad g_k \in G, \quad \forall k = 1, \dots, N. \tag{2.3}$$

The group operation is of course given by componentwise multiplication. We denote the Lie algebra of \vec{G} as

$$\vec{\mathcal{G}} = \overbrace{\mathcal{G} \oplus \cdots \oplus \mathcal{G}}^{N \text{ times}} \tag{2.4}$$

with elements

$$\vec{X} = X_1 \oplus \cdots \oplus X_N, \quad X_k \in \mathcal{G}, \quad \forall k = 1, \dots, N. \tag{2.5}$$

By combining $\gamma \in \text{Aut}(G)$ with the cyclic permutation automorphism of \vec{G} we introduce $\Gamma \in \text{Aut}(\vec{G})$ by

$$\Gamma(\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_N) := \gamma(\eta_N) \oplus \eta_1 \oplus \cdots \oplus \eta_{N-1} \tag{2.6}$$

and let Γ' denote the corresponding automorphism of $\vec{\mathcal{G}}$. We then define an action of \vec{G} on itself by the Γ -twisted conjugations $C_{\vec{\eta}}^\Gamma$ that for each $\vec{\eta} \in \vec{G}$ operate on \vec{g} as follows:

$$C_{\vec{\eta}}^\Gamma(\vec{g}) := \Gamma(\vec{\eta})\vec{g}\vec{\eta}^{-1} = \gamma(\eta_N)g_1\eta_1^{-1} \oplus \eta_1g_2\eta_2^{-1} \oplus \cdots \oplus \eta_{N-1}g_N\eta_{N-1}^{-1}. \tag{2.7}$$

The Γ -twisted conjugations on \vec{G} and the γ -twisted conjugations on G are related by the ‘monodromy map’ $M: \vec{G} \rightarrow G$ given by

$$M(\vec{g}) := g_1g_2 \cdots g_N. \tag{2.8}$$

It is easy to check the equivariance property

$$M(C_{\vec{\eta}}^\Gamma(\vec{g})) = C_{\eta_N}^\gamma(M(\vec{g})). \tag{2.9}$$

This implies immediately that if \vec{g} and \vec{g}' lie on the same orbit with respect to the C^Γ action of \vec{G} , then their ‘monodromy matrices’ lie on the same orbit of G under the C^γ action. The next lemma states that the converse is also true.

Lemma 1. *The elements \vec{g} and \vec{g}' lie on the same orbit of \vec{G} under Γ -twisted conjugations if and only if $M(\vec{g})$ and $M(\vec{g}')$ lie on the same G -orbit under γ -twisted conjugations. Furthermore, the corresponding isotropy subgroups of \vec{g} in \vec{G} and $M(\vec{g})$ in G are isomorphic.*

Proof. Suppose that $\vec{g}, \vec{g}' \in \vec{G}$ verify $M(\vec{g}') = C_{\eta_N}^\gamma(M(\vec{g}))$ for some $\eta_N \in G$. After fixing such an element η_N , the requirement $\vec{g}' = C_{\vec{\eta}}^\Gamma(\vec{g})$, which can be spelled out as

$$g'_1 = \gamma(\eta_N)g_1\eta_1^{-1}, \quad g'_2 = \eta_1g_2\eta_2^{-1}, \dots, g'_{N-1} = \eta_{N-2}g_{N-1}\eta_{N-1}^{-1}, \quad g'_N = \eta_{N-1}g_N\eta_{N-1}^{-1}, \tag{2.10}$$

permits us to uniquely determine $\eta_1, \dots, \eta_{N-1}$. In fact, we simply solve the last $(N - 1)$ of these equations for $\eta_{N-1}, \dots, \eta_1$, and then the first equation is already guaranteed by $M(\vec{\xi}') = C'_{\eta_N}(M(\vec{\xi}))$. The same calculation for $\vec{\xi}' = \vec{\xi}$ shows that $M(\vec{\xi}) = C'_{\eta_N}(M(\vec{\xi}))$ if and only if $\vec{\xi} = C^{\perp}_{\vec{\eta}}(\vec{\xi})$ with

$$\begin{aligned} \eta_{N-1} &= g_N \eta_N g_N^{-1}, \quad \eta_{N-2} = (g_{N-1} g_N) \eta_N (g_{N-1} g_N)^{-1}, \dots, \\ \eta_1 &= (g_2 \cdots g_N) \eta_N (g_2 \cdots g_N)^{-1}, \end{aligned} \tag{2.11}$$

which explicitly establishes the isomorphism claimed by the lemma. \square

To continue, fix positive numbers λ_k ($k = 1, \dots, N$) and define an invariant scalar product on $\vec{\mathcal{G}}$ by

$$\langle \vec{X}, \vec{Y} \rangle_{\vec{\lambda}} := \sum_{k=1}^N \lambda_k \langle X_k, Y_k \rangle, \tag{2.12}$$

where $\langle \cdot, \cdot \rangle$ is a multiple of the Killing form of \mathcal{G} , i.e., $\langle X, Y \rangle = -C \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$ with a constant $C > 0$. For later convenience, we assume that

$$\sum_{k=1}^N \frac{1}{\lambda_k} = 1. \tag{2.13}$$

Let Γ'^T be the transpose of Γ' with respect to this scalar product. The defining property

$$\langle \vec{X}, \Gamma'(\vec{Y}) \rangle_{\vec{\lambda}} = \langle \Gamma'^T(\vec{X}), \vec{Y} \rangle_{\vec{\lambda}}, \quad \forall \vec{X}, \vec{Y} \in \vec{\mathcal{G}}, \tag{2.14}$$

entails that

$$\Gamma'^T : X_1 \oplus \cdots \oplus X_{N-1} \oplus X_N \mapsto \frac{\lambda_2}{\lambda_1} X_2 \oplus \cdots \oplus \frac{\lambda_N}{\lambda_{N-1}} X_N \oplus \frac{\lambda_1}{\lambda_N} (\gamma')^{-1}(X_1), \tag{2.15}$$

where we used that γ' preserves the Killing form of \mathcal{G} .

We now introduce two Abelian subalgebras \mathcal{K} and \mathcal{Q} of $\vec{\mathcal{G}}$, which are isomorphic to \mathcal{T}^γ :

$$\mathcal{K} := \mathcal{T}^\gamma_{\text{diag}} = \{ \vec{X} \mid X_1 = X_2 = \cdots = X_N \in \mathcal{T}^\gamma \}, \tag{2.16}$$

$$\mathcal{Q} := \{ \vec{q} \mid q_k = \frac{q}{\lambda_k}, \quad q \in \mathcal{T}^\gamma \quad (\forall k = 1, \dots, N) \}. \tag{2.17}$$

We also define

$$\check{\mathcal{Q}} := \{ \vec{q} \mid q_k = \frac{q}{\lambda_k}, \quad q \in \check{\mathcal{T}}^\gamma \quad (\forall k = 1, \dots, N) \}. \tag{2.18}$$

It is straightforward to verify that

$$\mathcal{Q} \subseteq \operatorname{Ker}(\Gamma'^T - e^{-\operatorname{ad}_{\vec{q}}}) \quad \text{and} \quad \operatorname{Im}(\Gamma'^T - e^{-\operatorname{ad}_{\vec{q}}}) \subseteq \mathcal{K}^\perp, \quad \forall q \in \mathcal{T}^\gamma, \tag{2.19}$$

where ‘perp’ refers to orthogonal complement with respect to the scalar product (2.12). The following strengthening of these properties will be crucial in our considerations.

Lemma 2. For any $q \in \check{\mathcal{T}}^\gamma$, parametrizing \vec{q} as in (2.18), we have

$$\text{Ker}(\Gamma'^T - e^{-\text{ad}_{\vec{q}}}) = \mathcal{Q} \quad \text{and} \quad \text{Im}(\Gamma'^T - e^{-\text{ad}_{\vec{q}}}) = \mathcal{K}^\perp. \tag{2.20}$$

Consequently, the map

$$\mathcal{Z}(q) := (\Gamma'^T - e^{-\text{ad}_{\vec{q}}})|_{\mathcal{Q}^\perp}: \mathcal{Q}^\perp \rightarrow \mathcal{K}^\perp \tag{2.21}$$

is a linear bijection for any $q \in \check{\mathcal{T}}^\gamma$.

Proof. On account of the first relation in (2.19) and standard linear algebraic facts, using also that $\dim(\mathcal{Q}) = \dim(\mathcal{K})$, it is sufficient to prove that

$$\text{Ker}(\Gamma' - e^{\text{ad}_{\vec{q}}}) = \mathcal{K}, \quad \forall q \in \check{\mathcal{T}}^\gamma. \tag{2.22}$$

Suppose that $\vec{X} \in \vec{\mathcal{G}}$ belongs to this kernel, which means that

$$\Gamma'(\vec{X}) = e^{\text{ad}_{\vec{q}}}(\vec{X}). \tag{2.23}$$

By taking exponential in the group \vec{G} , we conclude that

$$\Gamma(e^{t\vec{X}}) = e^{t\Gamma'(\vec{X})} = e^{\vec{q}} e^{t\vec{X}} e^{-\vec{q}}, \quad \forall t \in \mathbb{R}. \tag{2.24}$$

This in turn means that $e^{t\vec{X}}$ belongs to the C^Γ isotropy group of $e^{\vec{q}}$.

To finish the proof, note that $M(e^{\vec{q}}) = e^q$ holds for all $q \in \mathcal{T}^\gamma$. Therefore we see from (the proof of) Lemma 1 that if $q \in \check{\mathcal{T}}^\gamma$, then the isotropy subgroup of $e^{\vec{q}}$ with respect to the C^Γ action is precisely the connected subgroup K of \vec{G} associated with the Lie algebra \mathcal{K} (alias the diagonal embedding of \mathbb{T}^γ into \vec{G}).

By combining the above, we have shown that (2.23) with $q \in \check{\mathcal{T}}^\gamma$ implies that $\vec{X} \in \mathcal{K}$, i.e., the relation (2.22) is valid. \square

Remark. For $q \in \check{\mathcal{T}}^\gamma$, consider the transpose of $\mathcal{Z}(q)$:

$$\mathcal{Z}(q)^T = (\Gamma' - e^{\text{ad}_{\vec{q}}})|_{\mathcal{K}^\perp}: \mathcal{K}^\perp \rightarrow \mathcal{Q}^\perp. \tag{2.25}$$

Translated into geometric terms, the equality $\text{Im}(\mathcal{Z}(q)^T) = \mathcal{Q}^\perp$ says that

$$T_{e^{\vec{q}}} \vec{C}_G^\Gamma(e^{\vec{q}}) = \mathcal{Q}^\perp e^{\vec{q}}. \tag{2.26}$$

Here $C_G^\Gamma(e^{\vec{q}})$ is the \vec{G} -orbit through $e^{\vec{q}}$ and on the right-hand-side we notationwise ‘pretend’ that we are dealing with a matrix Lie group. Therefore we have the direct sum decomposition

$$T_{e^{\vec{q}}} \vec{G} = \vec{\mathcal{G}} e^{\vec{q}} = \mathcal{Q} e^{\vec{q}} \oplus \mathcal{Q}^\perp e^{\vec{q}} = T_{e^{\vec{q}}} \mathcal{Q} \oplus T_{e^{\vec{q}}} C_G^\Gamma(e^{\vec{q}}), \quad \forall \vec{q} \in \check{\mathcal{Q}}, \tag{2.27}$$

where \mathcal{Q} is the connected subgroup of \vec{G} with Lie algebra \mathcal{Q} . This is an orthogonal decomposition with respect to the bi-invariant Riemannian metric on \vec{G} induced by the scalar product (2.12) on $\vec{\mathcal{G}}$. It is also worth noting that the map $\mathcal{Z}(q)$ is equivariant under the actions of

$$K = \mathbb{T}_{\text{diag}}^\gamma \tag{2.28}$$

given by the corresponding restrictions of the adjoint action of \vec{G} on $\vec{\mathcal{G}}$.

3. Hamiltonian reduction based on twisted conjugations

We are interested in reductions of the free particle moving on the group manifold \vec{G} equipped with the bi-invariant Riemannian metric belonging to the scalar product (2.12) on \vec{G} . This metric is clearly invariant also under the twisted conjugations $C_{\vec{\eta}}^{\Gamma}$ (2.7). We shall characterize the reductions with respect to this symmetry using the standard framework of symplectic geometry [23].

3.1. General description for any $\gamma \in \text{Aut}(G)$

We begin by noting that the phase space of the geodesic motion is the cotangent bundle of \vec{G} . We adopt the identification $T^*\vec{G} \simeq \vec{G} \times \vec{G}$ set up utilizing right-translations and taking \vec{G} as the model of \vec{G}^* by the scalar product $\langle \cdot, \cdot \rangle_{\vec{\lambda}}$. We also choose a coadjoint orbit of \vec{G} ,

$$\vec{O} := \mathcal{O}_1 \oplus \cdots \oplus \mathcal{O}_N, \tag{3.1}$$

where the \mathcal{O}_k are coadjoint orbits of G . We then consider the extended phase space

$$P := T^*\vec{G} \times \vec{O} \simeq \vec{G} \times \vec{G} \times \vec{O} = \{(\vec{g}, \vec{J}, \vec{\xi})\}. \tag{3.2}$$

Like in Section 2, we have $\vec{J} = J_1 \oplus \cdots \oplus J_N$ and $\vec{\xi} = \xi_1 \oplus \cdots \oplus \xi_N$. The phase space P carries the symplectic form

$$\Omega = d\langle \vec{J}, d\vec{g}\vec{g}^{-1} \rangle_{\vec{\lambda}} + \omega \quad \text{with} \quad \omega = \sum_{k=1}^N \omega_k, \tag{3.3}$$

where ω_k is the natural symplectic form of \mathcal{O}_k . The orbital part ω implies the Poisson brackets

$$\{\langle \vec{\xi}, \vec{X} \rangle_{\vec{\lambda}}, \langle \vec{\xi}, \vec{Y} \rangle_{\vec{\lambda}}\} = \langle \vec{\xi}, [\vec{X}, \vec{Y}] \rangle_{\vec{\lambda}}, \tag{3.4}$$

for any constants $\vec{X}, \vec{Y} \in \vec{G}$. The other non-trivial Poisson brackets between the fundamental variables can be displayed as

$$\{\vec{g}, \langle \vec{J}, \vec{X} \rangle_{\vec{\lambda}}\} = \vec{X}\vec{g}, \quad \{\langle \vec{J}, \vec{X} \rangle_{\vec{\lambda}}, \langle \vec{J}, \vec{Y} \rangle_{\vec{\lambda}}\} = \langle \vec{J}, [\vec{X}, \vec{Y}] \rangle_{\vec{\lambda}}. \tag{3.5}$$

Here, we have $\vec{X}\vec{g} = X_1g_1 \oplus \cdots \oplus X_Ng_N$. The Hamiltonian H of the free particle, trivially extended to P , reads

$$H(\vec{g}, \vec{J}, \vec{\xi}) = \frac{1}{2} \langle \vec{J}, \vec{J} \rangle_{\vec{\lambda}}. \tag{3.6}$$

The free flow generated by H through the initial value $(\vec{g}, \vec{J}, \vec{\xi})$ is

$$(\vec{g}(t), \vec{J}(t), \vec{\xi}(t)) = (e^{t\vec{J}}\vec{g}, \vec{J}, \vec{\xi}). \tag{3.7}$$

The twisted conjugations $C_{\vec{\eta}}^{\Gamma}$ lift to the Hamiltonian action \hat{C}^{Γ} of \vec{G} on P :

$$\hat{C}_{\vec{\eta}}^{\Gamma}(\vec{g}, \vec{J}, \vec{\xi}) = (C_{\vec{\eta}}^{\Gamma}(\vec{g}), \text{Ad}_{\Gamma(\vec{\eta})}(\vec{J}), \text{Ad}_{\vec{\eta}}(\vec{\xi})). \tag{3.8}$$

This action is generated by the equivariant momentum map $\Psi: P \rightarrow \mathcal{G}$ given by

$$\Psi(\vec{g}, \vec{J}, \vec{\xi}) = \Gamma^T(\vec{J}) - \text{Ad}_{\vec{g}^{-1}}(\vec{J}) + \vec{\xi}. \tag{3.9}$$

Indeed, the Hamiltonian vector field of the function $\Psi_{\vec{X}} = \langle \Psi, \vec{X} \rangle_{\vec{\lambda}}$ coincides with the infinitesimal generator of the \vec{G} -action (3.8).

According to the standard shifting trick [23], we can represent any reduction of the free motion with respect to the twisted conjugation symmetry as symplectic reduction of the Hamiltonian system (P, Ω, H) at the zero value of the momentum map; in association with some $\vec{\mathcal{O}}$. The resulting reduced Hamiltonian system lives on the space of \vec{G} -orbits

$$P_{\text{red}} = P_{\Psi=0}/\vec{G}, \tag{3.10}$$

which inherits a reduced symplectic structure, Ω_{red} , and Hamiltonian, H_{red} , from Ω and H on P . To be more precise, one should note that P_{red} is not always a smooth manifold, but is in general a so-called stratified symplectic space, i.e., a disjoint union of symplectic manifolds of various dimensions [23]. The detailed structure depends on the choice of the orbit $\vec{\mathcal{O}}$. We here will not dwell on this issue, but shall instead focus on a dense open subset P' of the phase space P whose reduction can be characterized rather directly.

Remember that every element of \vec{G} can be transformed by twisted conjugation into a unique element of the form $e^{\vec{q}}$, parametrized according to (2.18) by q from the closure of the generalized Weyl alcove $\vec{\mathcal{T}}^\gamma$. From now on we restrict our attention to the subset P' consisting of triples $(\vec{g}, \vec{J}, \vec{\xi})$, where the twisted conjugates of \vec{g} can be parametrized by elements q of the open alcove $\vec{\mathcal{T}}^\gamma$. It is then clear that every \vec{G} -orbit in $P'_{\Psi=0}$ intersects the subset $S \subset P$ defined by

$$S := \{ (e^{\vec{q}}, \vec{J}, \vec{\xi}) \mid \Psi(e^{\vec{q}}, \vec{J}, \vec{\xi}) = 0, \quad q \in \vec{\mathcal{T}}^\gamma \}. \tag{3.11}$$

Observe also from Lemma 1 that any \vec{G} -orbit in $P'_{\Psi=0}$ intersects S in an orbit of the subgroup $K < \vec{G}$ displayed in (2.28). In other words, S is the ‘gauge slice’ of a partial gauge fixing of the \vec{G} -action on $P'_{\Psi=0}$, for which the ‘residual gauge transformations’ belong precisely to the subgroup K . In this way we obtain the identification

$$P'_{\text{red}} := P'_{\Psi=0}/\vec{G} \simeq S/K. \tag{3.12}$$

Consequently, the \vec{G} -invariant smooth functions on $P'_{\Psi=0}$, which descend to smooth functions on P'_{red} , are equivalent to the K -invariant smooth functions on S . The Poisson brackets of these functions can be determined with the aid of the restriction of the symplectic form Ω to S .

In order to make the above concrete, we proceed to solve the restriction of the momentum map constraint on S . This amounts to the following requirement:

$$(\Gamma'^T - e^{-\text{ad}_{\vec{q}}})(\vec{J}) + \vec{\xi} = 0. \tag{3.13}$$

To solve this equation we use the decompositions

$$\mathcal{K} + \mathcal{K}^\perp = \vec{\mathcal{G}} = \mathcal{Q} + \mathcal{Q}^\perp \tag{3.14}$$

and write accordingly

$$\vec{\xi} = \vec{\xi}_{\mathcal{K}} + \vec{\xi}_{\mathcal{K}^\perp} \quad \text{and} \quad \vec{J} = \vec{J}_{\mathcal{Q}} + \vec{J}_{\mathcal{Q}^\perp}. \tag{3.15}$$

Here, recall the definitions of \mathcal{K} and \mathcal{Q} from Eqs. (2.16) and (2.17). Referring to Lemma 2, the \mathcal{K} -component of the momentum map constraint (3.13) enforces that $\vec{\xi}_{\mathcal{K}} = 0$, while its \mathcal{K}^\perp -component is completely solved by

$$\vec{J}_{\mathcal{Q}^\perp} = -\mathcal{Z}(q)^{-1}(\vec{\xi}_{\mathcal{K}^\perp}) \tag{3.16}$$

using the operator $\mathcal{Z}(q)$ defined in (2.21). The element $q \in \vec{\mathcal{T}}^\gamma$ can be chosen arbitrarily, and $\vec{J}_{\mathcal{Q}}$ is also a free variable, which we parametrize as

$$\vec{J}_{\mathcal{Q}} = \vec{p} = \frac{p}{\lambda_1} \oplus \frac{p}{\lambda_2} \oplus \dots \oplus \frac{p}{\lambda_N}, \quad p \in \mathcal{T}^\gamma. \tag{3.17}$$

Since the variables q, p and $\vec{\xi}_{\mathcal{K}^\perp}$ uniquely label the points of S , we obtain the identification of manifolds

$$S \simeq \check{\mathcal{T}}^\gamma \times \mathcal{T}^\gamma \times (\vec{\mathcal{O}} \cap \mathcal{K}^\perp) = \{(q, p, \vec{\xi}_{\mathcal{K}^\perp})\}. \tag{3.18}$$

According to the general principles of Hamiltonian reduction, the reduced system on P'_{red} is determined by the pull-backs of the symplectic form Ω and the Hamiltonian H to S . Let us denote these pull-backs by Ω_S and H_S . In terms of the model (3.18) we obtain

$$\Omega_S = d\langle p, dq \rangle + \omega_{\vec{\mathcal{O}} \cap \mathcal{K}^\perp}. \tag{3.19}$$

Here $\omega_{\vec{\mathcal{O}} \cap \mathcal{K}^\perp}$ is the contribution of the orbital form ω from (3.3). The first term is the natural symplectic form of the cotangent bundle

$$T^*\check{\mathcal{T}}^\gamma \simeq \check{\mathcal{T}}^\gamma \times \mathcal{T}^\gamma = \{(q, p)\}. \tag{3.20}$$

Since the residual gauge transformations by $K = \mathbb{T}^\gamma_{\text{diag}}$ act only on $\vec{\xi}_{\mathcal{K}^\perp}$, the reduced phase space can be decomposed according to

$$P'_{\text{red}} \simeq T^*\check{\mathcal{T}}^\gamma \times (\vec{\mathcal{O}} \cap \mathcal{K}^\perp)/K. \tag{3.21}$$

This is a symplectic identification in the following sense. The natural action of the group $K < \vec{\mathcal{G}}$ on $\vec{\mathcal{O}}$ is generated by the momentum map $\vec{\xi} \mapsto \vec{\xi}_K$, and

$$\vec{\mathcal{O}}_{\text{red}} = (\vec{\mathcal{O}} \cap \mathcal{K}^\perp)/K \tag{3.22}$$

is the reduced orbit obtained by setting this momentum map to zero. We denote the (in general stratified) symplectic structure inherited by $\vec{\mathcal{O}}_{\text{red}}$ as ω_{red} . This is carried by the second factor in the Cartesian product (3.21). Regarding the reduced Hamiltonian of the free motion, we already mentioned that it is determined by H_S . By means of the model (3.18), H_S takes the form

$$H_S = \frac{1}{2}\langle p, p \rangle + \frac{1}{2}\langle \mathcal{Z}(q)^{-1}(\vec{\xi}_{\mathcal{K}^\perp}), \mathcal{Z}(q)^{-1}(\vec{\xi}_{\mathcal{K}^\perp}) \rangle_{\vec{\lambda}}. \tag{3.23}$$

The first term is the kinetic energy of the free motion on $\check{\mathcal{T}}^\gamma$ with respect to the flat metric, while the second (also K -invariant) term describes the interaction of the spatial degrees of freedom q, p with the orbital variables given by the K -orbits in (3.22). The outcome of the foregoing discussion can be summarized as follows.

Proposition 3. *A model of the open subset of the reduced phase space (3.10) associated with generic monodromy matrices $M(\vec{g})$, parametrized by $q \in \check{\mathcal{T}}^\gamma$, is provided by $T^*\check{\mathcal{T}}^\gamma \times \vec{\mathcal{O}}_{\text{red}}$. Here $T^*\check{\mathcal{T}}^\gamma$ carries its Darboux form and $\vec{\mathcal{O}}_{\text{red}}$ (3.22) is equipped with its own (stratified) symplectic structure arising by reduction via the K -action. The reduced free Hamiltonian on this subset $P'_{\text{red}} = S/K$ is encoded by the K -invariant function H_S (3.23).*

We finish this subsection with a few comments. First, we remark that the Poisson bracket, $\{, \}_{\text{red}}$, that is induced on the smooth functions on P'_{red} is encoded by

$$\{ \langle q, X \rangle, \langle p, Y \rangle \}_{\text{red}} = \langle X, Y \rangle, \quad \forall X, Y \in \mathcal{T}^\gamma, \tag{3.24}$$

together with the restriction of the Poisson brackets of the K -invariant functions on $\vec{\mathcal{O}}$ to $\vec{\mathcal{O}} \cap \mathcal{K}^\perp$, which can be determined by means of the Lie–Poisson brackets (3.4).

Second, we recall that the $N = 1$ special case of our classical Hamiltonian reduction was previously studied in [13] and the corresponding quantum Hamiltonian reduction was investigated in [14]. As will be discussed in Section 5, the quantum Hamiltonian reduction can be described similarly in the general case. To facilitate the comparison with these earlier works, we now recast the inverse of $\mathcal{Z}(q)$ (2.21) in the form

$$\mathcal{Z}(q)^{-1} = -e^{\text{ad}_{\vec{\gamma}}} \circ (\mathcal{U}(q)^T)^{-1}, \tag{3.25}$$

where the linear operator $\mathcal{U}(q) : \mathcal{K}^\perp \rightarrow \mathcal{Q}^\perp$ reads

$$\mathcal{U}(q) = \left(\text{id}_{\vec{\mathcal{G}}} - e^{-\text{ad}_{\vec{\gamma}}} \circ \Gamma' \right) \Big|_{\mathcal{K}^\perp}. \tag{3.26}$$

Since $e^{\text{ad}_{\vec{\gamma}}}$ preserves the scalar product on $\vec{\mathcal{G}}$, and inversion and taking transpose commute, we can rewrite the Hamiltonian H_S (3.23) as

$$H_S = \frac{1}{2} \langle p, p \rangle + \frac{1}{2} \langle \vec{\xi}_{\mathcal{K}^\perp}, (\mathcal{U}(q)^T \mathcal{U}(q))^{-1} (\vec{\xi}_{\mathcal{K}^\perp}) \rangle_{\vec{\lambda}}. \tag{3.27}$$

For $N = 1$, when $\mathcal{Q} = \mathcal{K}$, it was shown in [13] that the extension of $\mathcal{U}(q)^{-1}$ by zero on \mathcal{Q} is a solution of the modified classical dynamical Yang–Baxter equation. There it was important that the automorphism γ' preserves the scalar product on the underlying Lie algebra. This holds for our Γ' if and only if $\lambda_1 = \lambda_2 = \dots = \lambda_N$. It would be interesting to know if $\mathcal{U}(q)$ is still related to a classical dynamical r -matrix for any N with arbitrary parameters $\vec{\lambda}$.

Third, it is proper to mention here that Hochgerner [12] studied reductions of cotangent bundles under the assumption that a single isotropy type occurs for the underlying group action on the configuration space. Our Proposition 3 can be obtained as a special case of his results. Since this would necessitate going into several technicalities, we have preferred to give a simple direct derivation.

Finally, note that the reduced system governed by the Hamiltonian (3.27) is exactly solvable since its solutions can be obtained by the projection method applied to the free flow (3.7), which involves only algebraic operations. There exist general arguments [5,11,24] that also indicate Liouville integrability on the full reduced phase space P_{red} . Indeed, one can easily construct many conserved quantities in involution as follows. For each fixed real parameter $u \neq 0$, define the $\vec{\mathcal{G}}$ -valued function ϕ_u on the unreduced phase space by

$$\phi_u(\vec{g}, \vec{J}, \vec{\xi}) := -\vec{g}^{-1} \vec{J} \vec{g} + \frac{\vec{\xi}}{u}. \tag{3.28}$$

Taking a basis $\{V^a\}$ of $\vec{\mathcal{G}}$, the Poisson brackets of the components $\phi_u^a := \langle \phi_u, V^a \rangle_{\vec{\lambda}}$ satisfy

$$\{\phi_u^a, \phi_v^b\} = f_c^{ab} \left(\frac{u-1}{u-v} \phi_u^c + \frac{v-1}{v-u} \phi_v^c \right) \tag{3.29}$$

for any $u \neq v$, where $[V^a, V^b] = f_c^{ab} V^c$ and summation over the index c is understood. It immediately follows from (3.29) that for any two $\vec{\mathcal{G}}$ -invariant smooth functions h_1 and h_2 on $\vec{\mathcal{G}}$ and for any parameters u and v we have

$$\{h_1 \circ \phi_u, h_2 \circ \phi_v\} = 0. \tag{3.30}$$

This descends to a Poisson commuting family on the reduced phase space. As generators, it is useful to take invariant homogeneous polynomials h_i and extract the coefficients of the powers of u^{-1} from $h_i \circ \phi_u$. Because (3.30) holds for any u and v , the so obtained coefficient functions are also in involution. The u -independent term coming from $h(\vec{X}) = \frac{1}{2} \langle \vec{X}, \vec{X} \rangle_{\vec{\lambda}}$ gives the

main Hamiltonian after reduction, and it is natural to expect that the family just delineated is generically sufficient to ensure its Liouville integrability.

3.2. The examples associated with $\gamma = \text{id}_G$

Our goal now is to provide an explicit formula for the reduced Hamiltonian H_S (3.27) in the case of the identity automorphism γ of an arbitrary connected and simply-connected compact simple Lie group G . During the calculations it proves to be handy to realize the real Lie algebra \mathcal{G} as a compact real form of its complexification $\mathcal{G}^{\mathbb{C}}$. Since the complexification $\mathcal{T}^{\mathbb{C}}$ of \mathcal{T} is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, the pair $(\mathcal{G}^{\mathbb{C}}, \mathcal{T}^{\mathbb{C}})$ uniquely determines a reduced root system Φ . Choose a polarization $\Phi = \Phi_+ \cup \Phi_-$ and let $\Pi \subset \Phi_+$ denote the corresponding set of simple roots, whose number is the rank of G , r . Also, for each root φ we select an appropriate root vector X_φ . On the root vectors we may and shall impose the following two conditions:

- (a) The root vectors are normalized by the conditions

$$\kappa(X_\varphi, X_{-\varphi}) = 1 \quad (\varphi \in \Phi_+), \tag{3.31}$$

where κ denotes a positive multiple of the Killing form of $\mathcal{G}^{\mathbb{C}}$.

- (b) The real Lie algebra \mathcal{G} decomposes as

$$\mathcal{G} = \mathcal{T} \oplus (\oplus_{\varphi \in \Phi_+} \mathbb{R}Y_\varphi) \oplus (\oplus_{\varphi \in \Phi_+} \mathbb{R}Z_\varphi), \tag{3.32}$$

where for each $\varphi \in \Phi_+$ we define

$$Y_\varphi = i \frac{X_\varphi + X_{-\varphi}}{\sqrt{2}} \quad \text{and} \quad Z_\varphi = \frac{X_\varphi - X_{-\varphi}}{\sqrt{2}}. \tag{3.33}$$

Take an arbitrary set of vectors $\{T_j\}_{j=1}^r \subset \mathcal{T}$ satisfying $\kappa(T_j, T_k) = -\delta_{j,k}$ ($1 \leq j, k \leq r$). Clearly the family of vectors T_j ($1 \leq j \leq r$) and Y_φ, Z_φ ($\varphi \in \Phi_+$) gives an orthonormal basis in \mathcal{G} with respect to the Ad-invariant Euclidean scalar product obtained by restricting the complex bilinear form $\langle \cdot, \cdot \rangle = -\kappa$ onto \mathcal{G} . Obviously, this convention agrees with the one used in (2.12) and the definition of $\langle \cdot, \cdot \rangle_{\tilde{\lambda}}$ extends by linearity to the complexification of $\tilde{\mathcal{G}}$.

Recalling the form of the Hamiltonian H_S (3.27), first we wish to examine the action of the linear operator $\mathcal{U}(q)^T \mathcal{U}(q)$ ($q \in \tilde{\mathcal{T}}$, an open Weyl alcove) on a convenient basis of \mathcal{K}^\perp . For this reason, for all $V \in \mathcal{G}^{\mathbb{C}}$ and $1 \leq I \leq N$ we define the vector $\vec{V}^I \in \tilde{\mathcal{G}}^{\mathbb{C}}$ with components $(\vec{V}^I)_J = \delta_J^I V$ ($1 \leq J \leq N$). To clarify our notations, we remark that for \vec{X} (2.5) we have $(\vec{X})_I = X_I$. The family of vectors

$$\vec{T}_j^I \quad (1 \leq j \leq r, 1 \leq I \leq N) \quad \text{and} \quad \vec{X}_\varphi^I \quad (\varphi \in \Phi, 1 \leq I \leq N) \tag{3.34}$$

gives a basis in the complex Lie algebra $\tilde{\mathcal{G}}^{\mathbb{C}}$. Now we record that for each $q \in \mathcal{T}$ the action of the complexification of the linear operator

$$\mathcal{I}(q) := \left(\text{id}_{\tilde{\mathcal{G}}} - e^{-\text{ad}_{\tilde{q}}} \circ \Gamma' \right)^T \circ \left(\text{id}_{\tilde{\mathcal{G}}} - e^{-\text{ad}_{\tilde{q}}} \circ \Gamma' \right) : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}} \tag{3.35}$$

on the basis (3.34) reads

$$\mathcal{I}(q) \vec{T}_j^J = \sum_{I=1}^N \frac{1}{\lambda_I} M_{I,j}(0) \vec{T}_j^I \quad \text{and} \quad \mathcal{I}(q) \vec{X}_\varphi^J = \sum_{I=1}^N \frac{1}{\lambda_I} M_{I,j}(\varphi(q)) \vec{X}_\varphi^I, \tag{3.36}$$

where $M(ix)$ is the $N \times N$ complex matrix with entries

$$M_{I,J}(ix) = (\lambda_I + \lambda_{I+1}) \delta_{I,J} - \lambda_I e^{-ix/\lambda_I} \delta_{I-1,J} - \lambda_{I+1} e^{ix/\lambda_{I+1}} \delta_{I+1,J} \quad (1 \leq I, J \leq N), \tag{3.37}$$

depending on the real parameter $x \in \mathbb{R}$. Here and below, the capital indices are understood modulo N . The factor i occurs since $\varphi(q) \in i\mathbb{R}$ for each $q \in \mathcal{T}$ and $\varphi \in \Phi$. It is also worth mentioning that the function $\mathbb{R} \ni x \mapsto M(ix) \in \mathbb{C}^{N \times N}$ is 2π -periodic with symmetry properties

$$M_{J,I}(ix) = M_{I,J}(-ix) = \overline{M_{I,J}(ix)}. \tag{3.38}$$

On account of the formula (3.27) and the relationship

$$(\mathcal{U}(q)^T \mathcal{U}(q))^{-1} = (\mathcal{I}(q)|_{\mathcal{K}^\perp})^{-1}, \tag{3.39}$$

we need control over the inverse of $M(ix)$. One can verify that $M(ix)$ is invertible if and only if $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Moreover, upon setting

$$b_I = \sum_{k=1}^I \frac{1}{\lambda_k} \quad \text{and} \quad b_{I,J} = b_I - b_J \quad (1 \leq I, J \leq N), \tag{3.40}$$

for all $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ the entries of the inverse matrix $\mathcal{P}(ix) = M(ix)^{-1}$ can be checked to be

$$\mathcal{P}_{I,J}(ix) = e^{-ib_{I,J}x} \left(\frac{1}{4 \sin^2(x/2)} + i \frac{b_{I,J}}{2} \cot(x/2) - \frac{|b_{I,J}|}{2} \right) \quad (1 \leq I, J \leq N). \tag{3.41}$$

This matrix inversion was treated in detail in [20]. We note that $\varphi(q) \in i(\mathbb{R} \setminus 2\pi\mathbb{Z})$ holds for every q from the open Weyl alcove $\check{\mathcal{T}}$ and $\varphi \in \Phi$.

Turning to the study of the degenerate matrix $M(0)$, let us endow the linear space of the complex column vectors \mathbb{C}^N with the inner product

$$\mathbb{C}^N \times \mathbb{C}^N \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}^* \Lambda \mathbf{y} \in \mathbb{C} \tag{3.42}$$

induced by the positive definite diagonal matrix

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N). \tag{3.43}$$

Thinking of the $N \times N$ complex matrices as linear operators acting from the left on the inner product space \mathbb{C}^N , from the formulae (3.37) and (3.41) we can easily deduce the following:

- (i) The operator $\Lambda^{-1}M(0)$ is self-adjoint with respect to the inner product (3.42), the kernel $\mathcal{N} = \ker(\Lambda^{-1}M(0))$ is spanned by the column vector $\mathbf{n} = [1, \dots, 1]^*$, and the orthogonal complement \mathcal{N}^\perp of \mathcal{N} is an invariant subspace.
- (ii) The restriction $(\Lambda^{-1}M(0))|_{\mathcal{N}^\perp}: \mathcal{N}^\perp \rightarrow \mathcal{N}^\perp$ is invertible and for its inverse we have

$$\left((\Lambda^{-1}M(0))|_{\mathcal{N}^\perp} \right)^{-1} = (\pi_\perp \mathcal{P}' \Lambda \pi_\perp)|_{\mathcal{N}^\perp}, \tag{3.44}$$

where the $N \times N$ matrix π_\perp denotes the orthogonal projection onto the subspace \mathcal{N}^\perp , whereas \mathcal{P}' is the $N \times N$ matrix with entries

$$\mathcal{P}'_{I,J} = \frac{(b_{I,J})^2 - |b_{I,J}|}{2} \quad (1 \leq I, J \leq N). \tag{3.45}$$

The operator defined by (3.44) encodes the action of the inverse (3.39) on $\mathcal{K}^\perp \cap \vec{\mathcal{T}}$. It is reassuring to see that the outcome of the elementary algebraic manipulations is consistent with the statements in (2.19) and Lemma 2.

Finally, let us observe that due to the relations (3.39) and (3.36) we now fully control the linear operator $(\mathcal{U}(q)^T \mathcal{U}(q))^{-1}$. This implies the explicit formula of H_S given below.

Proposition 4. *For all $N \geq 1$, at each point $(q, p, \vec{\xi}_{\mathcal{K}^\perp}) \in \check{\mathcal{T}} \times \mathcal{T} \times \vec{\mathcal{O}}_{\text{red}}$, the reduced Hamiltonian H_S (3.27) associated with the trivial automorphism $\gamma = id_G$ takes the form*

$$\begin{aligned}
 H_S = & \frac{1}{2} \langle p, p \rangle + \frac{1}{2} \sum_{j=1}^r \sum_{I, J=1}^N \mathcal{P}'_{I, J} \langle \vec{T}_j^I, \vec{\xi}_{\mathcal{K}^\perp} \rangle_{\vec{\lambda}} \langle \vec{T}_j^J, \vec{\xi}_{\mathcal{K}^\perp} \rangle_{\vec{\lambda}} \\
 & - \frac{1}{2} \sum_{\varphi \in \Phi} \sum_{I, J=1}^N \mathcal{P}_{I, J}(\varphi(q)) \langle \vec{X}_\varphi^I, \vec{\xi}_{\mathcal{K}^\perp} \rangle_{\vec{\lambda}} \langle \vec{X}_{-\varphi}^J, \vec{\xi}_{\mathcal{K}^\perp} \rangle_{\vec{\lambda}}
 \end{aligned} \tag{3.46}$$

with \mathcal{P} and \mathcal{P}' defined in (3.41) and (3.45), respectively.

The above Hamiltonian H_S can be interpreted in terms of ‘particles’ with positions defined by the components of q that interact with each other as well as with the ‘spin variables’ represented by $\vec{\xi}$. It is a system of ‘Sutherland type’ since the interaction exhibits trigonometric dependence on the coordinates, which means that the ‘particles’ move on the circle. For $\mathcal{G} = su(r + 1)$, the explicit formula (3.46) reproduces precisely the integrable spin Sutherland Hamiltonian obtained previously by Blom and Langmann [18,19] and by Polychronakos [20] by means of different methods, and generalizes their system for arbitrary simple Lie algebras.

4. Alternative derivation from twisted current algebra

It is remarkable that certain many-body systems can be obtained by alternative reduction procedures starting from finite-dimensional as well as from infinite-dimensional ‘free systems’. The main ideas in this regard go back to Gorsky and Nekrasov [15,16], who first interpreted the standard trigonometric Sutherland system from an infinite-dimensional standpoint. Their derivation had been exposed in [5,25], too. In effect, it also formed the basis of the derivation of the $\mathcal{G} = su(r + 1)$ special case of H_S (3.46) given by Blom and Langmann [18,19]. We thought it worthwhile to develop the Gorsky–Nekrasov method in our general case. Below we re-derive the same systems as those obtained in Section 3.1, but we will not be fully rigorous as we shall neglect topological details concerning the infinite-dimensional spaces involved.

We start with the infinite-dimensional phase space \tilde{P} consisting of triples of \mathcal{G} -valued functions $(A(x), E(x), \zeta(x))$ on the real line, subject to the following quasi-periodicity condition for $C = A, E, \zeta$:

$$C(x + 1) = \tau'(C(x)) \quad \text{with} \quad \tau := \gamma^{-1}. \tag{4.1}$$

We assume that $\zeta(x)$ is actually a generalized function of the form

$$\zeta(x) = \sum_{k \in \mathbb{Z}} \zeta_k \delta(x - x_k), \tag{4.2}$$

where

$$0 < x_1 < x_2 < \dots < x_N < 1 \tag{4.3}$$

and

$$x_{k+N} = x_k + 1, \quad \zeta_{k+N} = \tau'(\zeta_k), \quad \forall k \in \mathbb{Z}. \tag{4.4}$$

The normalization is such that $\int_0^1 \delta(x - x_k) dx = 1$ for $k = 1, \dots, N$. Moreover, we assume that ζ_k belongs to an adjoint orbit \mathcal{O}'_k of G in \mathcal{G} , identified with \mathcal{G}^* via the scalar product $\langle \cdot, \cdot \rangle$. Regarding the quasi-periodic \mathcal{G} -valued functions A and E , we suppose that they are smooth on each open interval (x_k, x_{k+1}) , and they as well as their derivatives of all orders have finite one-sided limits at every point x_k . For reasons that will become clear shortly, we permit both A , E and all their derivatives to have jumps at the distinguished points x_k .

By using smooth \mathcal{G} -valued quasi-periodic test functions $V(x)$ and $W(x)$, the non-vanishing Poisson brackets on \tilde{P} are specified as

$$\left\{ \int_0^1 \langle A(x), V(x) \rangle dx, \int_0^1 \langle E(x), W(x) \rangle dx \right\} = \int_0^1 \langle V(x), W(x) \rangle dx \tag{4.5}$$

and

$$\{ \langle \zeta_k, V_k \rangle, \langle \zeta_k, W_k \rangle \} = \langle \zeta_k, [V_k, W_k] \rangle, \tag{4.6}$$

where $V_k := V(x_k)$, $W_k := W(x_k)$. The corresponding symplectic form $\tilde{\Omega}$ on \tilde{P} reads

$$\tilde{\Omega} = d\tilde{\theta} + \omega' \quad \text{with} \quad \omega' = \sum_{k=1}^N \omega'_k, \tag{4.7}$$

where ω'_k is the Kirillov–Kostant–Souriau form on the orbit \mathcal{O}'_k and

$$\tilde{\theta} = \int_0^1 \langle E(x), dA(x) \rangle dx. \tag{4.8}$$

We equip the phase space \tilde{P} with the free Hamiltonian

$$\tilde{H} := \frac{1}{2} \int_0^1 \langle E(x), E(x) \rangle dx \tag{4.9}$$

and a Hamiltonian action of a suitable twisted loop group, \tilde{G} . Namely, an element $g \in \tilde{G}$, which is a G -valued function on the line subject to $g(x + 1) = \tau(g(x))$, acts on the triple (A, E, ζ) according to

$$\begin{aligned} A(x) &\mapsto g(x)A(x)g^{-1}(x) - g'(x)g^{-1}(x), & E(x) &\mapsto g(x)E(x)g^{-1}(x), \\ \zeta_k &\mapsto g(x_k)\zeta_k g^{-1}(x_k). \end{aligned} \tag{4.10}$$

We suppose that g and all its derivatives are smooth on each open interval (x_k, x_{k+1}) and have finite one-sided limits similarly to A and E . An important difference is that g is required to be globally continuous on \mathbb{R} (including at the points x_k). These assumptions guarantee that the formulae in (4.10) yield a well-defined action of \tilde{G} on \tilde{P} .

The action (4.10) admits the equivariant momentum map

$$\tilde{\Psi}(x) := [A(x), E(x)] + E'(x) + \zeta(x) \tag{4.11}$$

that takes its values in the distributional dual of the space of \mathcal{G} -valued smooth quasi-periodic functions. In particular, the derivative of $E(x)$ is understood in the distribution sense, i.e., the value $\tilde{\Psi}[V]$ of the functional $\tilde{\Psi}$ on the test function V is

$$\tilde{\Psi}[V] = \int_0^1 (\langle V(x), [A(x), E(x)] \rangle - \langle V'(x), E(x) \rangle) dx + \sum_{k=1}^N \langle V(x_k), \zeta_k \rangle. \tag{4.12}$$

By using this, one can verify that $\tilde{\Psi}$ generates the infinitesimal action according to

$$\begin{aligned} \{A(x), \tilde{\Psi}[V]\} &= [V(x), A(x)] - V'(x), \\ \{E(x), \tilde{\Psi}[V]\} &= [V(x), E(x)], \\ \{\zeta_k, \tilde{\Psi}[V]\} &= [V(x_k), \zeta_k]. \end{aligned} \tag{4.13}$$

The reduction of our concern is defined by imposing the momentum map constraint

$$\tilde{\Psi} = 0 \tag{4.14}$$

and then factorizing by \tilde{G} , as usual. An important fact used in the subsequent analysis is that for every A that appears in \tilde{P} there exists an element $g \in \tilde{G}$ that transforms it into a constant function on \mathbb{R} . The constant can be taken from the closure of the open ‘twisted Weyl alcove’ \check{T}^γ that parametrizes the twisted conjugacy classes in G given by the orbits of the C^γ -action (2.1). For convenience, we present a proof of this fact in [Appendix A](#).

Similarly as we worked in Section 3, we restrict attention to the subset \tilde{P}' consisting of such triples for which $A(x)$ can be gauge transformed to a constant from the interior of the alcove, denoted as

$$\chi \in \check{T}^\gamma. \tag{4.15}$$

Mimicking Section 3.1, we consider the ‘gauge slice’ \tilde{S} of a partial gauge fixing in \tilde{P}' :

$$\tilde{S} := \{ (A, E, \zeta) \mid \tilde{\Psi} = 0, A(x) = \chi \in \check{T}^\gamma \}. \tag{4.16}$$

Clearly the momentum map constraint $\tilde{\Psi} = 0$ requires that the function $E(x)$ takes the form

$$E(x) = e^{-(x-x_k)\text{ad}_x} E_k^+ \quad \text{for } x_k < x < x_{k+1}, \tag{4.17}$$

where the only objects so far undetermined are the constants

$$E_k^+ \in \mathcal{G}. \tag{4.18}$$

On account of the quasi-periodicity, we have $E_{k+N}^+ = \tau'(E_k^+)$ and we also define

$$E_k^- = e^{-(x_k-x_{k-1})\text{ad}_x} E_{k-1}^+, \quad \forall k \in \mathbb{Z}. \tag{4.19}$$

Since the function $E(x)$ has the jump $(E_k^+ - E_k^-)$ at x_k , we find that when applied to the elements of \tilde{S} the constraint $\tilde{\Psi} = 0$ translates into the following equations:

$$\zeta_k + (E_k^+ - E_k^-) = 0, \quad k = 1 \dots, N. \tag{4.20}$$

Now, our key observation is that the correspondence

$$q = \chi, \quad \frac{1}{\lambda_k} = x_k - x_{k-1}, \quad \lambda_k \xi_k = \zeta_k, \quad \lambda_k J_k = E_{k-1}^+, \quad k = 1, \dots, N, \tag{4.21}$$

allows to reformulate (4.20) as the following system of equations:

$$\xi_k + \frac{\lambda_{k+1}}{\lambda_k} J_{k+1} - e^{-\text{ad}_q/\lambda_k} (J_k) = 0, \quad k = 1 \dots, N, \tag{4.22}$$

where $J_{N+1} = \tau'(J_1)$ and $\lambda_{N+1} = \lambda_1$. The crux is that this system of equations is nothing but the componentwise form of the relation (3.13), which represents the momentum map constraint $\Psi = 0$ applied on the gauge slice S in the finite-dimensional symplectic reduction of Section 3.1. Since the residual gauge transformations are in both cases given by the action of the group K (2.28), the relation (4.21) induces a *one-to-one correspondence* between the reduced phase spaces coming from the infinite-dimensional and finite-dimensional reductions:

$$\tilde{P}'_{\text{red}} = \tilde{P}'_{\tilde{\Psi}=0}/\tilde{G} \simeq \tilde{S}/K \simeq S/K \simeq P'_{\Psi=0}/G \simeq P'_{\text{red}}. \tag{4.23}$$

It can be readily checked that *the reduced symplectic structures and also the reduced free Hamiltonians are converted into each other upon this correspondence*. The latter statement follows immediately since on \tilde{S} we have

$$\int_0^1 \langle E(x), E(x) \rangle dx = \sum_{k=1}^N (x_k - x_{k-1}) \langle E_{k-1}^+, E_{k-1}^+ \rangle = \sum_{k=1}^N \lambda_k \langle J_k, J_k \rangle = \langle \vec{J}, \vec{J} \rangle_{\vec{\lambda}}. \tag{4.24}$$

As for the symplectic form, let us first notice that the mapping $\lambda_k \xi_k \leftrightarrow \zeta_k$ converts the (unreduced) Poisson brackets in (3.4) into those in (4.6). In other words, $(\mathcal{O}_k, \omega_k)$ and $(\mathcal{O}'_k, \omega'_k)$ represent the same coadjoint orbit of G in \mathcal{G}^* , identified with \mathcal{G} via the different scalar products $\lambda_k \langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$. Consequently, it is enough to focus on the 1-form $\tilde{\theta}$ (4.8). Regarding this, we have

$$\begin{aligned} \tilde{\theta}|_{\tilde{\mathcal{G}}} &= \int_0^1 \langle E(x), dq \rangle dx = \sum_{k=1}^N (x_k - x_{k-1}) \langle E_{k-1}^+, dq \rangle = \langle \vec{J}, d\vec{q} \rangle_{\vec{\lambda}} \\ &= \langle \vec{J}_{\mathcal{Q}}, d\vec{q} \rangle_{\vec{\lambda}} = \langle p, dq \rangle. \end{aligned} \tag{4.25}$$

The second equality holds since only the \mathcal{T} -part of $E(x)$ contributes, which is constant on each interval (x_{k-1}, x_k) . We have taken into account the correspondence (4.21) together with Eqs. (3.15), (3.17) and (2.13). We conclude that the restriction of Ω on S is mapped into the restriction of $\tilde{\Omega}$ on \tilde{S} , which implies that the respective reduced symplectic structures are converted into each other, as claimed.

The infinite-dimensional phase space that we started with emerges from Yang–Mills theory in $(1 + 1)$ -dimensions with quasi-periodic boundary condition and external non-dynamical charges located at the points x_k . Indeed, $A = A_1$ is the spatial component of the Yang–Mills potential A_μ ($\mu = 0, 1$) in the gauge where A_0 has been set to zero, and E is its canonical conjugate (the ‘color-electric’ field). Because of this interesting physical context [15,18,5,17] it would be desirable to place the above sketched derivation on completely rigorous mathematical ground. We plan to deal with this issue, together with the isomorphism of the full reduced phase spaces $P_{\Psi=0}/G$ and $\tilde{P}_{\tilde{\Psi}=0}/\tilde{G}$, elsewhere.

To finish, note from (4.10) that $A(x)$ transforms in the same way as (at fixed level) the elements of the dual of a centrally extended twisted loop algebra based on \mathcal{G} . In this section we followed the current algebraic derivation of the Sutherland system as presented in the book [25], but (unlike there) we did not assume that $A(x)$ varies in the *smooth* dual. Although such framework may appear advantageous for the symplectic reduction itself, it is problematic since the

flow of the ‘free Hamiltonian’ \tilde{H} (4.9) starting at \tilde{S} at $t = 0$ leaves the space of smooth fields A , because E must have jumps at the points x_k . In fact, our assumptions on A , E and g were designed to ensure that the flow of the Hamiltonian \tilde{H} ,

$$(A(x, t), E(x, t), \zeta_1(t), \dots, \zeta_N(t)) = (A(x, 0) + tE(x, 0), E(x, 0), \zeta_1(0), \dots, \zeta_N(0)), \tag{4.26}$$

stays in the unreduced phase space \tilde{P} on which \tilde{H} is well-defined and enjoys \tilde{G} -symmetry.

5. Conclusion

In this paper we presented two derivations of novel spin Sutherland systems that in special cases were studied earlier [18–20], and described the mechanism whereby the two derivations always yield the same result. The main virtues of the infinite-dimensional derivation are its connection to Yang–Mills theory and that it can be extended for obtaining elliptic generalizations, similarly to the derivation of the standard elliptic Calogero system from the current algebra on the torus [27,5,17]. The finite-dimensional derivation has different advantages. For example, it permits the construction of classical solutions by a purely algebraic projection algorithm. Its main advantage is that, as we outline next, the corresponding quantum-Hamiltonian reduction is also in the range of well-understood group-theoretic methods.

The quantum mechanical analogue of the unreduced classical system (P, Ω, H) is defined by the Hamilton operator

$$\hat{H} = -\frac{1}{2} \Delta_{\vec{G}} \otimes \text{id}_{V_{\vec{v}}}, \tag{5.1}$$

where $\Delta_{\vec{G}}$ is the Laplace–Beltrami operator associated with the Riemannian metric on \vec{G} that corresponds to the invariant scalar product $\langle \cdot, \cdot \rangle_{\vec{G}}$. Now this operator acts on (its usual domain in) the Hilbert space $L^2(\vec{G}, d\mu_{\vec{G}}) \otimes V_{\vec{v}}$. The measure $d\mu_{\vec{G}}$ comes from the metric and the vector space $V_{\vec{v}}$ carries a highest weight representation $\rho_{\vec{v}}$ of the direct product group \vec{G} . This representation is the ‘exterior tensor product’ of representations of the N -factors, which are quantum counterparts of the constituent orbits \mathcal{O}_k of $\vec{\mathcal{O}}$ (3.1), i.e.,

$$V_{\vec{v}} = V_{\nu_1} \boxtimes \dots \boxtimes V_{\nu_N}. \tag{5.2}$$

Any group element $\vec{\eta} \in \vec{G}$ is represented by a unitary operator $U_{\vec{\eta}}$ on the Hilbert space, operating on a $V_{\vec{v}}$ -valued function F according to

$$F \mapsto U_{\vec{\eta}} F = \rho_{\vec{v}}(\vec{\eta}) \circ F \circ C_{\vec{\eta}^{-1}}^{\Gamma}. \tag{5.3}$$

The reduced Hilbert space is provided by the subspace of \vec{G} -singlets,

$$(L^2(\vec{G}, d\mu_{\vec{G}}) \otimes V_{\vec{v}})^{\vec{G}}. \tag{5.4}$$

This is mapped to itself by \hat{H} , and the corresponding restriction defines the reduced quantum Hamiltonian. By arguments similar to those in [14], it is not difficult to show that the reduced quantum Hamiltonian is unitarily equivalent to the operator \hat{H}_{red} given in (5.6) below that acts on (a suitable dense domain inside) the Hilbert space

$$L^2(\check{\mathcal{T}}^{\gamma}, d\mu_{\check{\mathcal{T}}^{\gamma}}) \otimes V_{\vec{v}}^K, \tag{5.5}$$

where $V_{\mathfrak{v}}^K \subset V_{\mathfrak{v}}$ is formed by the vectors fixed by the subgroup K (2.28) and the measure on \mathcal{T}^γ is defined by the Euclidean scalar product $\langle \cdot, \cdot \rangle$. To describe the operator \hat{H}_{red} , consider dual bases T_a and T^b of \mathcal{K}^\perp ($\langle T_a, T^b \rangle_{\vec{\lambda}} = \delta_a^b$) and introduce $\mathcal{I}^{a,b}(q) = \langle T^a, \mathcal{I}(q) T^b \rangle_{\vec{\lambda}}$ using $\mathcal{I}(q)$ in (3.35). Then one obtains

$$\hat{H}_{\text{red}} = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial q_i^2} + \frac{1}{2} \sum_{a,b} \mathcal{I}^{a,b}(q) \rho_{\mathfrak{v}}(T_a) \rho_{\mathfrak{v}}(T_b) + \mathcal{C}_{G,N}^{\gamma, \vec{\lambda}}, \tag{5.6}$$

where the q_i are coordinates on \mathcal{T}^γ with respect to an orthonormal basis and $\mathcal{C}_{G,N}^{\gamma, \vec{\lambda}}$ is a constant. Observe that apart from this constant (which arises from a similarity transformation [14]), the outcome of the quantum Hamiltonian reduction is obtained from the classical Hamiltonian (3.23) via the naive quantization of the kinetic energy and the replacement of $\langle T_a, \vec{\xi} \rangle_{\vec{\lambda}}$ by $\rho_{\mathfrak{v}}(T_a)$ together with restriction to the K -invariant subspace of $V_{\mathfrak{v}}^K$ annihilated by the quantum analogues of the classical constraints enforcing $\vec{\xi}_{\mathcal{K}} = 0$. The constant was calculated¹ in [14] for $N = 1$ and any γ , and now we also calculated it for the systems detailed in Section 3.2, i.e., for $\gamma = \text{id}_G$ and arbitrary N . It turned out that the constant is actually independent of N and of $\vec{\lambda}$ and it reads $\mathcal{C}_{G,N}^{\text{id}, \vec{\lambda}} = -\frac{1}{2} \kappa(\delta, \delta)$ with the ‘Weyl vector’ $\delta = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$.

Finally, let us sketch how representation theory can be used to determine, in principle, the spectrum of the reduced Hamiltonian \hat{H}_{red} . The clue is that the spectrum of $\Delta_{\vec{G}}$ is known from the Peter–Weyl theorem. The theorem says that the representation of $\vec{G} \times \vec{G}$ on $L^2(\vec{G}, d\mu_{\vec{G}})$ that comes from left- and right-multiplications decomposes as

$$L^2(\vec{G}, d\mu_{\vec{G}}) = \bigoplus_{\vec{\Lambda}} (V_{\vec{\Lambda}^*} \boxtimes V_{\vec{\Lambda}}), \tag{5.7}$$

where all ‘components’ Λ_k run over the dominant integral weights of G and $\vec{\Lambda}^*$ is composed from the highest weights Λ_k^* , denoting the highest weight of the contragredient of the representation V_{Λ_k} . The Laplace–Beltrami operator is constant on each subspace $V_{\vec{\Lambda}^*} \boxtimes V_{\vec{\Lambda}}$, taking the value

$$C_2^{\vec{\lambda}}(\vec{\Lambda}) = -\sum_{k=1}^N \lambda_k^{-1} \kappa(\Lambda_k + 2\delta, \Lambda_k). \tag{5.8}$$

It is easy to see from this that the space of singlets in $L^2(\vec{G}, d\mu_{\vec{G}}) \otimes V_{\mathfrak{v}}$ with respect to the representation (5.3) of \vec{G} can be presented as an infinite orthogonal direct sum of the finite-dimensional spaces

$$(V_{\Lambda_2^*} \otimes V_{\Lambda_1} \otimes V_{\nu_1})^G \boxtimes \dots \boxtimes (V_{\Lambda_N^*} \otimes V_{\Lambda_{N-1}} \otimes V_{\nu_{N-1}})^G \boxtimes (V_{\Lambda_1^* \circ \gamma'} \otimes V_{\Lambda_N} \otimes V_{\nu_N})^G. \tag{5.9}$$

The sum is over those vectors $\vec{\Lambda}$ for which the spaces of the G -singlets in all the N 3-fold tensor products in (5.9) are non-trivial. The composition of Λ_1^* with γ' makes sense since γ' acts on the Cartan subalgebra on which the weights are defined. The non-trivial finite dimensional spaces of the form (5.9) are eigensubspaces of the original Hamiltonian \hat{H} , with eigenvalue $-\frac{1}{2} C_2^{\vec{\lambda}}(\vec{\Lambda})$. It is an archetypical Clebsch–Gordan problem to determine these spaces. This infinite collection of finite-dimensional linear algebraic problems is equivalent to the problem of diagonalization for the generalized spin Sutherland Hamiltonian \hat{H}_{red} (5.6).

¹ A notational difference is that in [13,14] the normalized Killing form κ was denoted by $\langle \cdot, \cdot \rangle$.

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Appendix A. Appendix on gauge transformations

The argument presented below is adapted from the description of the coadjoint orbits of the twisted affine Lie algebras [26,21], which generalizes earlier results on the untwisted case.

Take a \mathcal{G} -valued function A on the real line \mathbb{R} that satisfies the assumptions detailed in Section 4. In particular, it is quasi-periodic in the sense that $A(x + 1) = \tau'(A(x))$, where τ' and τ are automorphisms of \mathcal{G} and G induced from an automorphism of the Dynkin diagram. (The trivial automorphism is of course a special case.) Then consider the following differential equation for a G -valued function y_A ,

$$y'_A(x) = y_A(x)A(x), \tag{A.1}$$

where at the distinguished points x_k (see Section 4) this is understood in the sense of one-sided limits. We further impose the initial condition

$$y_A(0) = e \in G \tag{A.2}$$

and also require *continuity of y_A at all points x_k* . It is easily seen that there exists a unique ‘fundamental solution’ y_A that meets these requirements. This boils down to well-known existence and uniqueness statements on each closed interval $[x_k, x_{k+1}]$, and y_A can be constructed by gluing the solutions on each closed interval by continuity. The solution y_A can be written explicitly as an ordered product-integral (‘Wilson line’). It follows that y_A verifies the conditions used in our definition of the group \tilde{G} in Section 4, *except* for the quasi-periodicity condition. Indeed, the quasi-periodicity of A and the uniqueness of y_A imply the relation

$$y_A(x + 1) = y_A(1)\tau(y_A(x)), \quad \forall x \in \mathbb{R}. \tag{A.3}$$

Recall that there exist an element $w \in G$ and a unique $\chi \in \text{cl}(\check{\mathcal{T}}^\gamma)$ for which

$$e^\chi = C_w^\gamma(y_A(1)). \tag{A.4}$$

Here, $\text{cl}(\check{\mathcal{T}}^\gamma)$ is the closure of $\check{\mathcal{T}}^\gamma$ introduced at the beginning of Section 2. The automorphisms γ and τ are inverses of each other (actually equal except possibly for $G = \text{Spin}(8, \mathbb{R})$). Picking w and χ , we now define the G -valued function g_A on \mathbb{R} by

$$g_A(x) = e^{-x\chi}\gamma(w)y_A(x). \tag{A.5}$$

It is trivial to check that g_A satisfies the identity

$$g_A(x)A(x)g_A^{-1}(x) - g'_A(x)g_A^{-1}(x) = \chi, \quad \forall x \in \mathbb{R}, \tag{A.6}$$

and is correctly quasi-periodic

$$g_A(x + 1) = \tau(g_A(x)). \tag{A.7}$$

In conclusion, we have shown that any function A subject to our conditions defining the phase space \tilde{P} can be gauge transformed into a constant χ from an arbitrarily chosen ‘twisted Weyl alcove’ by a gauge transformation defined by an element of the loop group \tilde{G} .

References

- [1] F. Calogero, Solution of the one-dimensional N -body problem with quadratic and/or inversely quadratic pair potentials, *J. Math. Phys.* 12 (1971) 419–436.
- [2] B. Sutherland, Exact results for a quantum many body problem in one dimension, *Phys. Rev. A* 4 (1971) 2019–2021.
- [3] J. Moser, Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. Math.* 16 (1975) 197–220.
- [4] S.N.M. Ruijsenaars, Systems of Calogero–Moser type, in: *Proceedings of the 1994 CRM–Banff Summer School ‘Particles and Fields’*, Springer, 1999, pp. 251–352.
- [5] N. Nekrasov, Infinite-dimensional algebras, many-body systems and gauge theories, in: A.Yu. Morozov, M.A. Olshanetsky (Eds.), *Moscow Seminar in Mathematical Physics*, in: *AMS Transl. Ser. 2*, Amer. Math. Soc., 1999, pp. 263–299.
- [6] A.P. Polychronakos, Physics and mathematics of Calogero particles, *J. Phys. A, Math. Gen.* 39 (2006) 12793–12845, arXiv:hep-th/0607033.
- [7] P. Etingof, *Calogero–Moser Systems and Representation Theory*, European Mathematical Society, 2007.
- [8] M.A. Olshanetsky, A.M. Perelomov, Explicit solution of the Calogero model in the classical case and geodesic flows on symmetric spaces of zero curvature, *Lett. Nuovo Cimento* 16 (1976) 333–339.
- [9] D. Kazhdan, B. Kostant, S. Sternberg, Hamiltonian group actions and dynamical systems of Calogero type, *Commun. Pure Appl. Math.* XXXI (1978) 481–507.
- [10] L.-C. Li, P. Xu, Integrable spin Calogero–Moser systems, *Commun. Math. Phys.* 231 (2002) 257–286, arXiv:math.QA/0105162.
- [11] N. Reshetikhin, Degenerate integrability of spin Calogero–Moser systems and the duality with the spin Ruijsenaars systems, *Lett. Math. Phys.* 63 (2003) 55–71, arXiv:math.QA/0202245.
- [12] S. Hochgerner, Singular cotangent bundle reduction and spin Calogero–Moser systems, *Differ. Geom. Appl.* 26 (2008) 169–192, arXiv:math.SG/0411068.
- [13] L. Fehér, B.G. Puztai, Spin Calogero models obtained from dynamical r -matrices and geodesic motion, *Nucl. Phys. B* 734 (2006) 304–325, arXiv:math-ph/0507062.
- [14] L. Fehér, B.G. Puztai, Twisted spin Sutherland models from quantum Hamiltonian reduction, *J. Phys. A, Math. Theor.* 41 (2008) 194009, arXiv:0711.4015 [math-ph].
- [15] A. Gorsky, N. Nekrasov, Hamiltonian systems of Calogero type and two dimensional Yang–Mills theory, *Nucl. Phys. B* 414 (1994) 213–238, arXiv:hep-th/9304047.
- [16] A. Gorsky, N. Nekrasov, Relativistic Calogero–Moser model as gauged WZW theory, *Nucl. Phys. B* 436 (1995) 582–608, arXiv:hep-th/9401017.
- [17] E. Langmann, Gauge theory approach towards an explicit solution of the (classical) elliptic Calogero–Moser system, *J. Nonlinear Math. Phys.* 12 (Suppl. 1) (2005) 423–439.
- [18] J. Blom, E. Langmann, Novel integrable spin-particle models from gauge theories on a cylinder, *Phys. Lett. B* 429 (1998) 336–342, arXiv:solv-int/9804007.
- [19] J. Blom, E. Langmann, Finding and solving Calogero–Moser type systems using Yang–Mills gauge theories, *Nucl. Phys. B* 563 (1999) 506–532, arXiv:math-ph/9909019.
- [20] A.P. Polychronakos, Generalized Calogero–Sutherland systems from many-matrix models, *Nucl. Phys. B* 546 (1999) 495–502, arXiv:hep-th/9806189.
- [21] S. Mohrdieck, R. Wendt, Integral conjugacy classes of compact Lie groups, *Manuscr. Math.* 114 (2004) 531–547, arXiv:math.QA/0303118.
- [22] V.G. Kac, *Infinite Dimensional Lie Algebras*, third ed., Cambridge, 1990.
- [23] J.-P. Ortega, T.S. Ratiu, *Momentum Maps and Hamiltonian Reduction*, Birkhäuser, 2004.
- [24] N.T. Zung, Torus actions and integrable systems, in: A.V. Bolsinov, A.T. Fomenko, A.A. Oshemkov (Eds.), *Topological Methods in the Theory of Integrable Systems*, Cambridge Scientific Publishers, 2006, pp. 289–328, arXiv:math.DS/0407455.
- [25] B. Khesin, R. Wendt, *The Geometry of Infinite-Dimensional Groups*, Springer, 2009.

- [26] R. Wendt, Weyl's character formula for non-connected Lie groups and orbital theory for twisted affine Lie algebras, *J. Funct. Anal.* 180 (2001) 31–65, arXiv:math/9909059.
- [27] A. Gorsky, N. Nekrasov, Elliptic Calogero–Moser system from two dimensional current algebra, arXiv:hep-th/9401021.