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A Note on "Well-Posedness Theory for Hyperbolic Conservation Laws"

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Abstract—In this note, we generalize the recent result on L^1 well-posedness theory for strictly hyperbolic conservation laws to the nonstrictly hyperbolic system of conservation laws whose characteristics are with constant multiplicity.

Keywords—Quasilinear hyperbolic conservation laws, Characteristics with constant multiplicity, Riemann problem, L^1 stability.

1. INTRODUCTION

Consider the following quasilinear system of conservation laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x), \tag{1.2}$$

where $u = (u_1, \dots, u_n)^\top$ is the unknown vector-valued function, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given smooth vector-valued function. When (1.1) is a strictly hyperbolic $n \times n$ system of conservation laws, and each characteristic field is either linearly degenerate or genuinely nonlinear, in [1], Bressan, Crasta and Piccoli prove that there exist a domain $\mathbb{D} \subset L^1$, containing all functions with sufficiently small total variation and a uniformly Lipschitz continuous semigroup $S : \mathbb{D} \times [0,\infty) \to \mathbb{D}$ with the following properties: every trajectory $t \mapsto u(t,\cdot) = S_t u_0$ of the semigroup is a weak, entropy-admissible solution of the initial value problem. For a given domain $\mathbb{D}$, the semigroup S with the

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above properties and local BV structure is unique (cf. [2]). These results yield the uniqueness, continuous dependence, and global stability of weak, entropy-admissible solutions of the Cauchy problem for (1.1) with small initial data. Liu and Yang in [3] and Bressan, Liu and Yang in [4] explicitly define a functional $\Phi = \Phi(u, v)$, equivalent to the L^1 distance between u and v, which is nonincreasing, i.e.,

$$\Phi(u(t), v(t)) - \Phi(u(s), v(s)) \leq O(\varepsilon) \cdot (t - s), \qquad \text{for all } t > s \geq 0,$$

for every pair of ε-approximate solutions u, v with small total variation, generated by the Glimm's scheme or wave front tracking algorithm. The small parameter ε here could be the errors in the wave speeds, the maximum size of rarefaction shock fronts, and the total strength of all nonphysical waves in u and in v, which tends to zero when the grid size of the scheme approaches zero. By the above estimate, they prove that the approximations converge to a unique limit solution, which depends Lipschitz continuously on the initial data, in the L^1 norm. This also implies the existence of the unique standard Riemann semigroup generated by a $n \times n$ system of conservation laws (cf. [1,2]).

In these approaches, one needs to assume that system is strictly hyperbolic, i.e., $\nabla f(u)$ has n distinct real eigenvalues. However, some physical systems are not strictly hyperbolic and possess nonsimple eigenvalues (see [5,6]). Important examples occur in nonlinear elasticity, plasma physics, and magnetohydrodynamics. The study of which includes the system for the general motion of an elastic string (cf. [6,7]), the quasilinear hyperbolic system of conservation laws with rotational degeneracy (cf. [5,8]), the system of magnetohydrodynamics, and the system for reactive flows (cf. [5,9,10]), etc. These examples have nonsimple eigenvalues. Off a closed set $\mathbb{S}$ of codimension one or two (merely two, because these systems have symmetric matrices), these eigenvalues have constant multiplicity. The aim of this note is to generalize the result on L^1 stability theory above for the initial value problem for (1.1) to the nonstrictly hyperbolic system of conservation laws with characteristics with constant multiplicity. Since our main result, Theorem 2.2, holds locally in the phase space, it can be applied to the above systems around every point in the complement (an open set) of $\mathbb{S}$.

2. MAIN RESULT

The following are our main assumptions on system (1.1).

(H1) System (1.1) is hyperbolic, i.e., for any u in the domain Ω under consideration, the matrix $\nabla f(u)$ has n real eigenvalues: $\lambda_i(u)$ $(i = 1, \dots, n)$ and there is a complete set of left (respectively, right) eigenvectors $\{l_1(u), \dots, l_n(u)\}$ (respectively, $\{r_1(u), \dots, r_n(u)\}$). Without loss of generality, we suppose that

$$l_i(u) r_j(u) = \delta_{ij}, \qquad i, j = 1, \dots, n, \tag{2.1}$$

where δ_{ij} stands for the Kronecker's symbol.

(H2) Any eigenvalue of $\nabla f(u)$ has a constant multiplicity. Without loss of generality, we assume that

$$\lambda_1(u) < \cdots < \lambda_m(u) < \lambda_{m+1}(u) \equiv \cdots \equiv \lambda_{m+p}(u) \stackrel{\triangle}{=} \lambda(u) < \lambda_{m+p+1}(u) < \cdots < \lambda_n(u), \tag{2.2}$$

where $1 \leq m \leq n-1$ and $1 \leq p \leq n$.

In particular, when $p = 1$, the system is strictly hyperbolic; while, when $p > 1$, (1.1) is a nonstrictly hyperbolic system with the eigenvalue with constant multiplicity p.

(H3) Any simple eigenvalue $\lambda_i(u)$ is either linearly degenerate (ld) or genuinely nonlinear (gn)

$$\nabla \lambda_i(u) r_i(u) \equiv 0 \ \text{(ld)}, \qquad \nabla \lambda_i(u) r_i(u) \neq 0 \ \text{(gn)}, \tag{2.3}$$

where $r_i(u)$ is the right eigenvector corresponding to $\lambda_i(u)$.

We first consider the Riemann problem for system (1.1) with the following initial data:

$$u = \begin{cases} \hat{u}^+, & x > 0, \\ \hat{u}^-, & x < 0, \end{cases} \tag{2.4}$$

where $\hat{u}^\pm$ are constant vectors. The following theorem is well known.

THEOREM 2.1. *Under Hypotheses H1–H3, if $|\hat{u}^+ - \hat{u}^-|$ is sufficiently small, then Riemann problem (1.1),(2.4) has a unique small amplitude similarity solution. This solution is composed of the $n - p + 2$ constant states, denoted by $\hat{u}^i$ $(i = 0, 1, \dots, m, m + p, \dots, n)$, and $n - p + 1$ small amplitude elementary waves (shocks or centered rarefaction waves corresponding to genuinely nonlinear characteristic fields, contact discontinuities corresponding to linearly degenerate characteristic fields). Moreover, there exist uniquely small parameters ε_i $(i = 1, \dots, n)$ and $\hat{u}^i$ $(i = 0, 1, \dots, m, m + p, \dots, n)$ close to $\hat{u}^-$ such that*

$$\hat{u}^0 = \hat{u}^-, \qquad \hat{u}^n = \hat{u}^+, \tag{2.5}$$

$$\hat{u}^i = g^i\left(\hat{u}^{i-1}, \varepsilon_i\right), \qquad (i = 1, \dots, m, m + p + 1, \dots, n) \tag{2.6}$$

and

$$\hat{u}^{m+p} = g\left(\hat{u}^m, \varepsilon_{m+1}, \dots, \varepsilon_{m+p}\right), \tag{2.7}$$

where g^i, g are smooth functions with respect to ε_i $(i \in \{1, \dots, n\})$ and satisfy

$$g^i\left(u^-, 0\right) = u^-, \ \frac{\partial g^i}{\partial \varepsilon_i}\left(u^-, 0\right) = r_i\left(u^-\right), \qquad (i = 1, \dots, m, m + p + 1, \dots, n), \tag{2.8}$$

$$\frac{\partial g}{\partial \varepsilon_j} \in \operatorname{Ker}\left(\lambda(u) I - \nabla f(u)\right), \qquad (j = m + 1, \dots, m + p), \tag{2.9}$$

$$g\left(u^-, 0 \dots, 0\right) = u^-, \quad \frac{\partial g}{\partial \varepsilon_j}\left(u^-, 0, \dots, 0\right) = r_j\left(u^-\right), \qquad (j = m + 1, \dots, m + p). \tag{2.10}$$

REMARK 2.1. It follows from [11,12] that the eigenvalue $\lambda(u)$ with constant multiplicity p (> 1) must be linearly degenerate, i.e.,

$$\nabla \lambda(u) r_j(u) \equiv 0, \qquad (j = m + 1, \dots m + p),$$

where $r_j(u)$ $(j = m + 1, \dots, m + p)$ are the right eigenvectors corresponding to $\lambda(u)$; the vector-space bundle $\operatorname{Ker}(\lambda(u)I - \nabla f(u))$ spanned by $\{r_{m+1}(u), \dots, r_{m+p}(u)\}$ is complete integrable, hence, for any $u^- \in \Omega$, there exist a neighborhood $\mathcal{N}(u^-) \subset \Omega$ of u^- and $n - p$ independent smooth functions $\varphi_1(u), \dots, \varphi_m(u), \varphi_{m+p+1}(u), \dots, \varphi_n(u)$ such that

$$\nabla \varphi_i(u) r_j(u) \equiv 0, \qquad (i = 1, \dots, m, m + p + 1, \dots, n;\ j = m + 1, \dots, m + p), \quad \forall u \in \mathcal{N}\left(u^-\right),$$

$$\nabla \varphi_i\left(u^-\right) = l_i\left(u^-\right), \qquad (i = 1, \dots, m, m + p + 1, \dots, n),$$

where $l_i(u)$ is the left eigenvector corresponding to $\lambda_i(u)$. Moreover, $\varphi_i(u) = \varphi_i(u^-)$ $(i = 1, \dots, m, m + p + 1, \dots, n)$ determine a p-dimensional connected smooth manifold, $\Sigma(u^-)$, in $\mathcal{N}(u^-)$ such that for any $u \in \Sigma(u^-)$, (u^-, u) forms a contact discontinuity with speed s,

$$s = \lambda(u) = \lambda\left(u^-\right).$$

REMARK 2.2. Theorem 2.1 was first proved by Lax for strictly hyperbolic systems of conservation laws, i.e., $\nabla f(u)$ has n distinct real eigenvalues (cf. [13]). For nonstrictly hyperbolic systems of conservation laws with characteristics with constant multiplicity, Theorem 2.1 can be found in [5], in which this kind of systems is also said to be "strictly hyperbolic" (see [5, pp. 124–126]).

Now we turn to consider the initial value problem for general $n \times n$ system of conservation laws (1.1). When system (1.1) is strictly hyperbolic, Liu and Yang [3] and Bressan *et al.* [4] gave a well-posedness theory for the initial value problem for (1.1). We can generalize the result to the nonstrictly hyperbolic system of conservation laws when characteristics have constant multiplicity.

THEOREM 2.2. WELL-POSEDNESS THEOREM. *Under Hypotheses H1–H3, suppose furthermore that the total variation of the initial data* $u_0(x), v_0(x)$ *of the solutions is sufficiently small, and that* $u_0(x) - v_0(x) \in L^1(\mathbb{R})$. *Then for the exact weak solutions* $u(t,x)$ *and* $v(t,x)$ *of (1.1) constructed by Glimm's scheme or wave front tracking method, there exists a positive constant* C *independent of time* t *such that*

$$\|u(t,x) - v(t,x)\|_{L^1} \le C\|u(s,x) - v(s,x)\|_{L^1},$$

for any s *and* t *with* $0 \le s \le t < \infty$.

PROOF. Theorem 2.2 can be proved in a manner similar to [3,4]. The definition of the nonlinear L^1 functional is the same as in [3,4]. The nonincreasingness of this functional comes from the uniqueness of the solution to the Riemann problem. Notice that the Q_d component in the functional is not defined between the families with the same characteristic speed, and this is not needed. Please refer to [3,4], for the detailed definition of the functional. We omit the details here. ∎

Theorem 2.2 immediately implies the following.

COROLLARY 2.1. UNIQUENESS OF WEAK SOLUTION CONSTRUCTED BY GLIMM'S SCHEME OR WAVE FRONT TRACKING METHOD. *For any given initial data with sufficiently small total variation, the whole sequence of the approximate solutions constructed by Glimm's scheme or wave front tracking method converge to a unique weak solution of (1.1) as the mesh size tends to zero.*

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