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Rearrangements of Functions on a Locally Compact Abelian Group and Integrability of the Fourier Transform

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We find in this paper the equimeasurable hulls and kernels of some function classes on a locally compact abelian group. These classes consist of all functions for

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1. INTRODUCTION

The purpose of this paper is to study connections between the equimeasurability relation on function classes on a locally compact abelian (LCA) group G and the integrability properties of the Fourier transform

$$F: (L^1 + L^2)(G) \to (C_0 + L^2)(\Gamma),$$

where Γ denotes the dual group of G. We will be dealing with the following two problems which go back to Hardy and Littlewood.

(A) *For a given* $1 < p < \infty$, *characterize those functions* f *on* G *for which at least one function* g *having* $|g|$ *equimeasurable with* $|f|$ *satisfies* $Fg \in L^p(\Gamma)$.

(B) *For a given* $1 < p < \infty$, *characterize those functions* f *on* G *for which all functions* g *having* $|g|$ *equimeasurable with* $|f|$ *satisfy* $Fg \in L^p(\Gamma)$.

These problems are called the equimeasurable hull and equimeasurable kernel problems, respectively.

Problems A and B were originally studied for the circle group T and the group of integers Z by Hardy and Littlewood [HL1, HL2] (see also [Z], Ch. 11). In the case $G = Z$, some modifications should be made in the formulation of the problems (see the definition of classes $A^{p,q}$ and $A_{p,q}$ below). Hardy and Littlewood solved Problem A in the case $1 < p < 2$ and

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Problem B in the case $2 < p < \infty$ for the groups T and Z. The results of Hardy and Littlewood have been extended by Hunt [Hu] who studied the Lorentz space case for the group $G = R^n$. The first result concerning Problems A and B for general LCA groups is due to Hewitt and Ross [HR2]. They solved Problems A and B in the cases considered by Hardy and Littlewood for all infinite discrete abelian groups. Later, Lin [Li] treated the Hardy–Littlewood cases in Problems A and B for a general LCA group and studied an easier problem of rearranging the Fourier transform. He solved this problem for "almost all" LCA groups (see [Li] for the explanation of what "almost all" means in this setting). The typical solution to Problems A and B in the cases mentioned above is given by the Lorentz space $L(p', p)$ where $p^{-1} + (p')^{-1} = 1$.

Problem A for the group T and $p > 2$ was originally considered by the author [G1], [G2], and [G5] who proved that the equimeasurable hull in this case coincides with the space $L^1(T)$. The same result holds for the Fourier coefficients with respect to any orthonormal system on a Lebesgue measure space (see [G2]). Problem B for T and $1 < p < 2$ (and also for the Fourier coefficients with respect to a general orthonormal system) was solved by Cereteli [C1] (see also [G2] where stronger results are given). The solution in Cereteli's case coincides with the set of all constant functions.

The equimeasurable and rearrangements-invariant hulls and kernels of non-invariant function classes have been studied by various authors. Cereteli in [C3, C4] gave a characterization of the hulls and kernels of the Hardy space $Re(H^1)$ and some related spaces. Later, B. Davis [D1] also obtained a description of the rearrangement-invariant hull of $Re(H^1)$. The results of Cereteli and Davis were generalized and extended by Kalton [K1, K2, K3]. The following theorem was proved by the author [G2]: any function from the rearrangement-invariant hull of $Re(H^1)$ can be rearranged in such a way that both the Fourier series and the conjugate series of the resulting rearrangement converge in $L^1(T)$. A description of the equimeasurable hull of the space BMO is due to Bennett, DeVore, and Sharpley [BDVS], [BS] (see also Bonami [B]). B. Davis [D2] obtained a characterization of the equimeasurable hull of function class consisting of all functions for which the ergodic maximal function is integrable (see some related results in [V1], [V2], and [E]).

In this paper, we obtain a complete description of the equimeasurable hulls and kernels in the Lorentz space case for any infinite LCA group (see Theorems 1–4 below). This includes all known results which were mentioned above. We consider not only the original equimeasurability relation of Hardy and Littlewood, but also some other equivalence relations for functions on a LCA group and solve the hull and kernel problems for these relations with very few exceptions where the problems remain open.

2. MAIN RESULTS ON REARRANGEMENT-INVARIANT HULLS AND KERNELS

Throughout the paper, G will denote an infinite locally compact Hausdorff abelian group and Γ will stand for the dual group of G. The Fourier transform $F: L^1(G) \to C_0(\Gamma)$ is defined by

$$Ff(\gamma) = \int_G f(x)(-x, \gamma)\, dx$$

for all $\gamma \in \Gamma$. The restriction of the Fourier transform F to $(L^1 \cap L^2)(G)$ is an isometry with respect to the L^2-norms onto a dense subspace of $L^2(\Gamma)$. Therefore, the Fourier transform may be extended to an isometry $F: L^2(G) \to L^2(\Gamma)$ (Plancherel's Theorem, see [HR1, R]). The inverse operator $F^{-1}: L^2(\Gamma) \to L^2(G)$ can be obtained in the similar way from the inverse Fourier transform $F^{-1}: L^1(\Gamma) \to C_0(G)$ defined by

$$F^{-1}g(x) = \int_\Gamma g(\gamma)(x, \gamma)\, d\gamma$$

for all $x \in G$.

We will use an appropriate normalization of the Haar measures of G and Γ for which the Fourier inversion formula holds (see [R], 1.1.3 and 1.5.3). It will always be assumed with no loss of generality that for a compact group G the Haar measure m_G satisfies $m_G(G) = 1$, while for a discrete group G, m_G will always be the counting measure.

It is well known that the Haar measure of G has the following property. Any two measurable sets E_1 and E_2 of equal finite measure have the same metric structure *mod* 0. This means that there exist measurable sets $\tilde{E}_1 \subset E_1$, $\tilde{E}_2 \subset E_2$ and a measurable one-to-one transformation $\omega: \tilde{E}_1 \to \tilde{E}_2$ such that $m(\tilde{E}_1) = m(E_1)$, $m(\tilde{E}_2) = m(E_2)$, and both ω and ω^{-1} are measure preserving transformations.

We consider the Fourier transform and the inverse Fourier transform as the linear operators $F: (L^1 + L^2)(G) \to (C_0 + L^2)(\Gamma)$, and $F^{-1}: (L^1 + L^2)(\Gamma) \to (C_0 + L^2)(G)$.

The Lorentz space $L(p, q)(G)$ where $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ is defined by the following

$$f \in L(p, q)(G) \Leftrightarrow \|f\|^*_{p,q} = \left\{ \int_0^\infty f^*(t)^q \, t^{(q/p)-1}\, dt \right\}^{1/q} < \infty$$

for $1 < p < \infty$, $1 \leqslant q < \infty$. For $q = \infty$, we have

$$\|f\|^*_{p,\infty} = \sup\{ t^{1/p} f^*(t) \}.$$

In the definition above, the function f is complex and f^* denotes the monotonic rearrangement of $|f|$ on $(0, \infty)$. The function f^* is the inverse function to the distribution function

$$D(y, f) = m(x \in G\colon |f(x)| \geqslant y), \qquad y \geqslant 0.$$

It is known that the functional $\|\cdot\|^*_{p,q}$ is a quasi-norm on $L(p, q)(G)$ and there exists a norm $\|\cdot\|_{p,q}$ on $L(p, q)(G)$ equivalent to the quasi-norm $\|\cdot\|^*_{p,q}$ (see more facts concerning the Lorentz spaces in [Hu] and [SW]).

Suppose p and q are given as above. In this paper, we will primarily be concerned with the following function classes on G

$$A^{p,q}(G) = \{f \in (L^1 + L^2)(G)\colon Ff \in L(p, q)(\Gamma)\}$$

and

$$A_{p,q}(G) = \{f \in (L^\infty + L^2)(G)\colon f = F^{-1}g \quad \text{for some } g \in L(p, q) \cap (L^1 + L^2)(\Gamma)\}.$$

The main reason why we introduce two different classes $A^{p,q}(G)$ and $A_{p,q}(G)$ is that for some groups and some values of p and q only one of these classes really captures the $L(p, q)$-integrability properties of the Fourier transform while the other one trivially coincides with $L^2(G)$. For example, if the group G is discrete, then $A^{p,q}(G) = L^2(G)$ for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, and if the group G is compact, then $A_{p,q}(G) = L^2(G)$ for $2 < p < \infty$, $1 \leqslant q \leqslant \infty$.

Next we introduce some equivalence relations on the space $(L^1 + L^\infty)(G)$ and analyse how the classes $A^{p,q}(G)$ and $A_{p,q}(G)$ behave with respect to these relations.

Definition 1. Let f be a complex function in $(L^1 + L^\infty)(G)$. Then the distribution of f is the following measure on the Borel σ-algebra of the complex plane C:

$$d(E, f) = m(x \in G\colon f(x) \in E),$$

where $E \in B$ and m is the Haar measure of G.

It is not difficult to see that $d(E, f)$ can be uniquely determined from the following joint distribution functions

$$D^{++}(y_1, y_2, f) = m(x \in G\colon (Re\, f)^+(x) \geqslant y_1, (Im\, f)^+(x) \geqslant y_2),$$

$$D^{+-}(y_1, y_2, f) = m(x \in G\colon (Re\, f)^+(x) \geqslant y_1, (Im\, f)^-(x) \geqslant y_2),$$

$$D^{-+}(y_1, y_2, f) = m(x \in G\colon (Re\, f)^-(x) \geqslant y_1, (Im\, f)^+(x) \geqslant y_2),$$

and

$$D^{--}(y_1, y_2, f) = m(x \in G: (Re\, f)^-(x) \geqslant y_1, (Im\, f)^-(x) \geqslant y_2)$$

for $y_1 \geqslant 0$ and $y_2 \geqslant 0$. Here $Re\, f$ and $Im\, f$ denote the real and imaginary parts of the function f. Also, for a real function f, we use the notation $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$. For such a function, we need only two distribution functions $D^+(y, f) = m(x \in G: f^+(x) \geqslant y)$ and $D^-(y, f) = m(x \in G: f^-(x) \geqslant y)$ for $y \geqslant 0$.

Definition 2. Two functions f and g from the space $(L^1 + L^\infty)(G)$ are called equimeasurable if $D^{++}(y_1, y_2, f) = D^{++}(y_1, y_2, g)$, $D^{+-}(y_1, y_2, f) = D^{+-}(y_1, y_2, g)$, $D^{-+}(y_1, y_2, f) = D^{-+}(y_1, y_2, g)$, and $D^{--}(y_1, y_2, f) = D^{--}(y_1, y_2, g)$ for all $y_1 \geqslant 0$ and $y_2 \geqslant 0$.

If the functions f and g are real, the equimeasurability of f and g reduces to $D^+(y, f) = D^+(y, g)$ and $D^-(y, f) = D^-(y, g)$ for all $y \geqslant 0$. For complex functions f and g, the equimeasurability of $|f|$ and $|g|$ is equivalent to $D(y, f) = D(y, g)$, $y \geqslant 0$.

Let us define the following functions for any $f \in (L^1 + L^\infty)(G)$,

$$f_1(x) = \begin{cases} f(x) & \text{if } Re\, f(x) > 0, \quad Im\, f(x) \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & \text{if } Re\, f(x) \leqslant 0, \quad Im\, f(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_3(x) = \begin{cases} f(x) & \text{if } Re\, f(x) < 0, \quad Im\, f(x) \leqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_4(x) = \begin{cases} f(x) & \text{if } Re\, f(x) \geqslant 0, \quad Im\, f(x) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

At this point we introduce the following equivalence relations on $(L^1 + L^\infty)(G)$:

$$f R_1 g \Leftrightarrow |f| \text{ and } |g| \text{ are equimeasurable}$$

$$f R_2 g \Leftrightarrow |f_k| \text{ and } |g_k| \text{ are equimeasurable for all } 1 \leqslant k \leqslant 4$$

$$f R_3 g \Leftrightarrow f \text{ and } g \text{ are equimeasurable}$$

and

$$fR_4g \Leftrightarrow (Re\ f)\ R_3(Re\ g) \text{ and } (Im\ f)\ R_3(Im\ g)$$

Definition 3. We say that a function $f \in (L^1 + L^\infty)(G)$ is a rearrangement of a function $g \in (L^1 + L^\infty)(G)$ iff there exists a Haar measure preserving invertible *mod* 0 transformation $\omega: G \to G$ such that $f = g \circ \omega$.

Let us define one more equivalence relation on $(L^1 + L^\infty)(G)$:

$$fR_5g \Leftrightarrow f \text{ is a rearrangement of } g.$$

It can be shown that $fR_5g \Rightarrow fR_3g \Rightarrow fR_2g \Rightarrow fR_1g$ and $fR_3g \Rightarrow fR_4g$. On the other hand, fR_4g does not in general imply fR_1g. Indeed, we may take $G = R^1$, $f = \chi_{[0,1]} + i\chi_{[0,1]}$, and $g = \chi_{[0,1]} + i\chi_{[1,2]}$. It is clear that fR_4g. However, f and g are not R_1-equivalent because $|f| = \sqrt{2}\,\chi_{[0,1]}$ and $|g| = \chi_{[0,2]}$. The equivalence relation R_2 is in some sense intermediate between R_1 and R_3. We use it when it is difficult to work with the more complicated relations R_3 and R_5.

A subset B of $(L^1 + L^\infty)(G)$ is called R_i-invariant if $(f \in B)\ \&\ (gR_if) \Rightarrow g \in B$.

Definition 4. For $1 \leqslant i \leqslant 5$, the R_i-invariant hull $(\bar{A})_i$ of a set $A \subset (L^1 + L^\infty)(G)$ is the intersection of all R_i-invariant sets containing A.

Definition 5. For $1 \leqslant i \leqslant 5$, the R_i-invariant kernel $(\underline{A})_i$ of a set $A \subset (L^1 + L^\infty)(G)$ is the union of all R_i-invariant sets contained in A.

The hull of A coincides with the smallest invariant set containing A, while the kernel of A is the largest invariant set contained in A. The problem of describing the invariant hulls and kernels of sets with respect to a given equivalence relation on a larger set was posed and originally studied by Cereteli [C2, C4, C5].

The objective of this paper is to characterize the R_i-invariant hulls and kernels of the classes $A^{p,q}$ and $A_{p,q}$. First we rewrite some of the known results mentioned in the introduction, using the language of invariant hulls and kernels.

In our notation, the results of Hardy and Littlewood can be formulated as follows. For $1 < p < 2$,

$$(\bar{A}^{p,p})_1\,(T) = L(p', p)(T) \tag{1}$$

where $1/p + 1/p' = 1$.

For $2 < p < \infty$,

$$(\underline{A}^{p,p})_1\,(T) = L(p', p)(T). \tag{2}$$

For $1 < p < 2$,

$$(\overline{A}_{p,\,p})_1\,(Z) = L(p',\, p)(Z). \tag{3}$$

For $2 < p < \infty$

$$(\underline{A}_{p,\,p})_1\,(Z) = L(p',\, p)(Z). \tag{4}$$

The author's results from [G1] and [G2] are the following:

$$(\overline{A}^{p,\,p})_i\,(T) = L^1(T) \tag{5}$$

for the real spaces $A^{p,\,p}$, $1 \leqslant i \leqslant 5$, and $2 < p < \infty$, and also

$$\left(\overline{\bigcap_{p>2} A^{p,\,p}}\right)_i (T) = L^1(T),$$

while Cereteli's result from [C1] can be formulated as

$$(\underline{A}^{p,\,p})_i\,(T) = \{f\colon f = const\} \tag{6}$$

for the real spaces $A^{p,\,p}$, $1 < p < 2$ and $1 \leqslant i \leqslant 5$. Let us also mention the following result

$$(\underline{A}_{1,\,1})_i\,(G) = L^2(G) \tag{7}$$

for any infinite discrete abelian group G and $1 \leqslant i \leqslant 5$. This result with an additional restriction on the group G has been obtained in [H]. The restriction has been removed in [HR1], V. 2, p. 437.

It is clear that $(\overline{A}^{2,\,2})_i\,(G) = (\underline{A}^{2,\,2})_i\,(G) = (\overline{A}_{2,\,2})_i\,(G) = (\underline{A}_{2,\,2})_i\,(G) = L^2(G)$ for all groups G and $1 \leqslant i \leqslant 5$. Moreover, it follows from the definitions that for a discrete abelian group G we have

$$(\overline{A}^{p,\,q})_i\,(G) = L^2(G), \tag{8}$$

$$(\underline{A}^{p,\,q})_i\,(G) = L^2(G) \tag{9}$$

for all $1 \leqslant i \leqslant 5$, $1 < p < 2$, and $1 \leqslant q \leqslant \infty$. Similarly, for a compact abelian group G, we have

$$(\overline{A}_{p,\,q})_i\,(G) = L^2(G), \tag{10}$$

$$(\underline{A}_{p,\,q})_i\,(G) = L^2(G) \tag{11}$$

for all $1 \leqslant i \leqslant 5$, $2 < p < \infty$, and $1 \leqslant q \leqslant \infty$.

The next theorems provide a description of the rearrangement-invariant and equimeasurable hulls and kernels of the sets $A^{p,\,q}(G)$ and $A_{p,\,q}(G)$.

THEOREM 1. (i) *Let G be a LCA group. Then*

$$(\bar{A}^{p,q})_i(G) = L(p', q)(G) \cap (L^1 + L^2)(G) \tag{12}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $i = 1, 2, 4$.

(ii) *Let G be a LCA group. Then*

$$(\bar{A}^{p,q})_i(G) = (L^1 + L^2)(G) \tag{13}$$

for $2 < p < \infty$, $1 \leqslant q \leqslant \infty$, *and* $1 \leqslant i \leqslant 5$.

Part (i) of Theorem 1 holds trivially in the discrete case for all $1 \leqslant i \leqslant 5$. This has already been mentioned above (see (8)). I do not know whether (12) in Theorem 1 holds for $i = 3, 5$ in the non-discrete case. For example, I do not know whether $(\bar{A}^{p,p})_5(T) = L(p', p)$ for $1 < p < 2$. Theorem 1 contains (1) and (5) as special cases.

THEOREM 2. (i) *Let G be a LCA group. Then*

$$(\bar{A}_{p,q})_i(G) = L(p', q)(G) \tag{14}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $i = 1, 2, 4$.

(ii) *Let G be a LCA group. Then*

$$(\bar{A}_{p,q})_i(G) = L^2(G) \tag{15}$$

for $2 < p < \infty$, $1 \leqslant q \leqslant \infty$, *and* $1 \leqslant i \leqslant 5$.

Part (ii) of Theorem 2 for a compact group G has already been mentioned above (see (10)). I do not know whether (14) in Theorem 2 holds for $i = 3, 5$. Part (i) of Theorem 2 contains (3) and some results of Hewitt and Ross [HR2] as special cases.

THEOREM 3. (i) *Let G be a non-compact non-discrete LCA group. Then*

$$(\underline{A}^{p,q})_i(G) = \{0\} \tag{16}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $1 \leqslant i \leqslant 5$.

(ii) *Let G be a compact abelian group. Then*

$$(\underline{A}^{p,q})_i(G) = \{f: f = const\} \tag{17}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $1 \leqslant i \leqslant 5$.

(iii) *Let G be an infinite discrete LCA group. Then*

$$(\underline{A}^{p,q})_i (G) = L^2(G) \tag{18}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $1 \leqslant i \leqslant 5$.

(iv) *Let G be an infinite LCA group. Then*

$$(\underline{A}^{p,q})_i (G) = L(p', q)(G) \tag{19}$$

for $2 < p < \infty$, $1 \leqslant q \leqslant \infty$, *and* $i = 1, 2, 4$.

Theorem 3 contains (2), (6), and also (9). I do not know if Part (iv) of Theorem 3 holds for $i = 3, 5$. However, if we restrict ourselves to the real class $A_r^{p,q}(G)$ consisting of all real functions from the class $A^{p,q}(G)$ then formula (19) holds with $i = 3, 5$. (See Remark 3 below).

THEOREM 4. (i) *Let G be a non-compact non-discrete LCA group. Then*

$$(\underline{A}_{p,q})_i (G) = \{0\} \tag{20}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $1 \leqslant i \leqslant 5$.

(ii) *Let G be a compact abelian group. Then*

$$(\underline{A}_{p,q})_i (G) = \{f : f = const\} \tag{21}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $1 \leqslant i \leqslant 5$.

(iii) *Let G be a discrete abelian group. Then*

$$(\underline{A}_{p,q})_i (G) = L^2(G) \tag{22}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $1 \leqslant i \leqslant 5$.

(iv) *Let G be an infinite LCA group. Then*

$$(\underline{A}_{p,q})_i (G) = L(p', q)(G) \cap L^2(G) \tag{23}$$

for $2 < p < \infty$, $1 \leqslant q \leqslant \infty$, *and* $i = 1, 2, 4$.

Theorem 4 contains (4), (9), and some results of Hewitt and Ross [HR2] as special cases. I do not know if Part (iv) of Theorem 4 holds for $i = 3, 5$.

The results of Hunt [Hu] are special cases of part (i) of Theorem 2 and part (iv) of Theorem 3. Note that Theorems 1–4 contain stronger results than those of Lin [Li].

Let us introduce the following notation: For $f \in (\bar{A}^{p,q})_i(G)$,

$$I_i^{p,q}(f) = \inf_{g:\, gR_i f} \|Fg\|_{L(p,q)(\Gamma)}.$$

For $f \in (\bar{A}_{p,q})_i\,(G)$,

$$I_{p,q,i}(f) = \inf_{g:\,(gR_i f)\,\&\,(F^{-1}h=g)} \|h\|_{L(p,q)(\Gamma)}.$$

For $f \in (\underline{A}^{p,q})_i\,(G)$,

$$S_i^{p,q}(f) = \sup_{g:\, gR_i f} \|Fg\|_{L(p,q)(\Gamma)}.$$

For $f \in (\underline{A}_{p,q})_i\,(G)$,

$$S_{p,q,i}(f) = \sup_{g:\,(gR_i f)\,\&\,(F^{-1}h=g)} \|h\|_{L(p,q)(\Gamma)}.$$

These quantities arise in the norm estimates in Theorem 1–4 (see Remark 4 in the end of the paper.)

3. PRELIMINARIES

In the sequel, we will need some known results concerning Lorentz spaces.

THEOREM 5 (Multiplication theorem). *Suppose numbers p, p_0, p_1, q, q_0, and q_1 are given satisfying $1 < p,\ p_0,\ p_1 < \infty$, $1 \leqslant q,\ q_0,\ q_1 \leqslant \infty$, $1/p = 1/p_0 + 1/p_1$, and $1/q = 1/q_0 + 1/q_1$. Then there exists a constant $c > 0$ such that*

$$\|fg\|_{p,q} \leqslant c\,\|f\|_{p_0,q_0}\,\|g\|_{p_1,q_1}$$

for any functions $f \in L(p_0, q_0)$ and $g \in L(p_1, q_1)$. The constant c depends only on p, p_0, p_1, q, q_0, and q_1.

THEOREM 6 (Convolution theorem). *Suppose numbers p, p_0, p_1, q, q_0, and q_1 are given satisfying $1 < p,\ p_0,\ p_1 < \infty$, $1 \leqslant q,\ q_0,\ q_1 \leqslant \infty$, $1/p = 1/p_0 + 1/p_1 - 1$, and $1/q = 1/q_0 + 1/q_1$. Then there exists a constant $c > 0$ such that*

$$\|f * g\|_{p,q} \leqslant c\,\|f\|_{p_0,q_0}\,\|g\|_{p_1,q_1}$$

for any functions $f \in L(p_0, q_0)$ and $g \in L(p_1, q_1)$. The constant c depends only on p, p_0, p_1, q, q_0, and q_1.

Theorems 5 and 6 can be found in [Hu].

The next theorem concerns the behavior of the Fourier transform in the Lorentz spaces.

THEOREM 7. *Let G be a LCA group. Then*

$$\|Ff\|_{L(p',q)(\Gamma)} \leqslant c\,\|f\|_{L(p,q)(G)}$$

for $1 < p < 2$, $1 \leqslant q \leqslant \infty$, *and* $1/p = 1/p' = 1$.

Theorem 7 follows from the Plancherel Theorem, the boundedness of the Fourier transform from $L^1(G)$ to $L^\infty(\Gamma)$, and the interpolation theorem for the Lorentz spaces (see [St, SW]).

Let G be a LCA group and f be a non-negative measurable function such that its level sets $E(y, f) = \{x \in G : f(x) \geqslant y\}$, $y > 0$ satisfy $mE(y, f) < \infty$, $y > 0$. The family $E(y, f)$ is monotonically decreasing and left-continuous. (This means $\bigcap_{z<y} E(z, f) = E(y, f)$.)

DEFINITION 6. Let G be a non-discrete LCA group. Suppose a number r is given such that $0 < r \leqslant mG$. An r-pyramid on G is a family $\{E_t\}$, $0 \leqslant t \leqslant r$ of measurable subsets of G such that

1. $E_{t_1} \subset E_{t_2}$ for $t_1 \leqslant t_2 \leqslant r$.
2. $mE_t = t$ for $0 \leqslant t \leqslant r$.

Note that our definition of a pyramid is similar to the definition of resolutions of elements of a Boolean algebra (see [Lu]) or μ-resolutions of measurable sets (see [CR]).

DEFINITION 7. Let G be any discrete abelian group. Suppose an integer s is given such that $0 < s \leqslant \infty$. An s-pyramid on G is a family $\{E_n\}$, $0 < n \leqslant s$ of subsets of G such that

1. $E_{n_1} \subset E_{n_2}$ for $0 < n_1 \leqslant n_2 \leqslant s$.
2. $mE_n = n$ for $0 < n \leqslant s$.

DEFINITION 8. Suppose an r-pyramid P is given. We call the set E_0 the top of the pyramid P. The set E_r is called the base of P. In this discrete case, the set E_1 is called the top of an s-pyramid P and the set E_s is called its base. We denote the top of P by $T(P)$ and the base of P by $B(P)$.

If $x \in T(P)$, then we may consider a new pyramid $P[x] = \{E_t[x]\} = \{E_t - x\}$. The new pyramid satisfies $0 \in T(P[x])$ where 0 denotes the identity in G. In the discrete case, the top $T(P)$ consists of one element a and $P[a] = \{E_n[a]\} = \{E_n - a\}$.

Building pyramids on groups is important for our purposes because they equip us with a general notion of monotonicity. More exactly, if an r-pyramid $P=\{E_t\}$ is given and f is a non-negative function such that its level sets have finite measure and $m(f>0)=r$, then we can rearrange f along P in the following way. Our goal is to construct a function $g\geqslant 0$ equimeasurable with f and such that $E(y, g)\in P$ for all $y>0$. It is easy to see that

$$f(x)=\int_0^\infty \chi_{E(y,\,f)}(x)\,dy.$$

The corresponding rearrangement is given by

$$g(x)=\int_0^\infty \chi_{A(y)}(x)\,dy$$

where $A(y)=E_{D(y,\,f)}$. We call g the monotonic rearrangement of f along the pyramid P. The integral formulas for both f and g should be understood in the pointwise sense. In order to use the interpolation theory, we need representations involving vector valued functions.

Assuming $f\in L(p,q)(G)$ with $1<p<\infty$, $1\leqslant q<\infty$, we see that the mapping $y\to\chi_{A(y)}$ is Pettis integrable in $L(p,q)(G)$ and

$$g=(P)-\int_0^\infty \chi_{A(y)}\,dy.$$

Here $(P)-\int$ denotes the Pettis integral (see all the necessary definitions in [DU]). If in addition $f\in(L^1+L^2)(G)$, then

$$Fg=(P)-\int_0^\infty F\chi_{A(y)}\,dy. \tag{24}$$

Formula (24) justifies the idea of constructing special pyramids $P=\{E_t\}$ $0<t\leqslant r$ satisfying an additional condition

$$\|F\chi_{E_t}\|_{L^{p'}(\Gamma)}\leqslant c(mE_t)^{1/p}=ct^{1/p} \tag{25}$$

for any $0<t\leqslant r$, $2\leqslant p<\infty$, and $1\leqslant q\leqslant\infty$. This will be done in Section 4. By Theorem 7, inequality (25) holds for $1<p<2$.

In the sequel, we will often need to separate non-intersecting pyramids. Two pyramids $P_1=\{E_t^1\}$ with $0<t\leqslant r_1$ and $P_2=\{E_t^2\}$ with $0<t\leqslant r_2$ are called non-intersecting if $m(E_{r_1}^1\cap E_{r_2}^2)=0$. The pyramids P_1 and P_2 can be separated if the characteristic functions of their bases are Fourier multipliers in $L(p,q)(\Gamma)$ for all $1<p<2$ and $1\leqslant q\leqslant\infty$, DEFINITION 9. A function $h \in L^\infty(G)$ is called a Fourier multiplier in $L(p, q)(\Gamma)$ provided for any function f such that $f = Fg$ with $g \in L(p, q)(\Gamma)$, one has $hf = Fe$ with $e \in L(p, q)(\Gamma)$. Moreover, the estimate $\|e\|_{p,q} \leqslant c\,\|g\|_{p,q}$ should hold with some positive constant c independent of g.

The set of multipliers in $L(p, q)(\Gamma)$ will be denoted $M(L(p, q)(\Gamma))$. The norm of the operator $g \to e$ is called the multiplier norm of h (see more information on Fourier multipliers on groups in [L, HR1]).

In order to use (24) and (25) for our purposes, we will need some definitions and results from interpolation theory. The reader is referred to [BL] for basic facts on the interpolation of linear operators. The symbol $\bar{A} = (A_1, A_2)$ denotes a Banach couple, $\Sigma(\bar{A})$ denotes the space $A_1 + A_2$, while $\Delta(\bar{A})$ is the space $A_1 \cap A_2$. The real interpolation spaces generated by the K-method are denoted by $\bar{A}_{\theta,p}$. In the sequel, we use an estimate for the Pettis integral in the interpolation spaces $\bar{A}_{\theta,p}$ for a vector-valued function $\lambda\colon (0, \infty) \to \Delta(\bar{A})$ obtained in [G3] and [G4]. We call the mapping λ totally scalarly measurable in $\Sigma(\bar{A})$ if the family of functionals $\gamma \in \Sigma(\bar{A})^*$ for which the functions $\lambda_\gamma(x) = \gamma(\lambda(x))$, $x > 0$ are Lebesgue measurable, separates points on $\Sigma(\bar{A})$.

Assume the mapping λ satisfies $\|\lambda(t)\|_{A_1} \leqslant \omega_1(t)$, $\|\lambda(t)\|_{A_2} \leqslant \omega_2(t)$, $t > 0$.

DEFINITION 10. We say that the couple (ω_1, ω_2) is admissible if there exist numbers M, τ and a function γ defined on $(0, M)$ such that $0 < M \leqslant \infty$, $0 < \tau < 1$, the function γ is positive and non-decreasing on $(0, M)$, and the following properties hold: $\omega_2(t) = \omega_1(t)^\tau\,\gamma(t)$ for $0 < t < M$ and $\omega_1(t) = \omega_2(t) = 0$ for $t > M$

For every Lebesgue measurable set $E \subset (0, \infty)$, $1 \leqslant p \leqslant \infty$ and $0 < \theta < 1$, denote

$$I_E(\theta, p, \omega_1, \omega_2) = \begin{cases} \left\{\int_E [\omega_1(t)^{1-\theta}\,\omega_2(t)^\theta]^p\, t^{p-1}\, dt\right\}^{1/p} & \text{if } \ 0 < \theta < 1, \quad 1 \leqslant p < \infty \\ \sup_{t \in E} \{\omega_1(t)^{1-\theta}\,\omega_2(t)^\theta\, t\} & \text{if } \ 0 < \theta < 1, \quad p = \infty \end{cases}$$

DEFINITION 11. We call a Banach couple $\bar{A} = (A_1, A_2)$ admissible if at least one of the following conditions holds: (i) One of the spaces A_1, A_2 is reflexive. (ii) One of the spaces A_1, A_2 is separable. (iii) One of the spaces $\bar{A}_{\theta,p}$ is reflexive. (iv) One of the spaces $\bar{A}_{\theta,p}$ is separable.

The following theorem gives an estimate for the Pettis integral in the interpolation spaces provided estimates for the function are known.

THEOREM 8 (See [G3], [G4]). *Let $\bar{A}$ be an admissible Banach couple and $\lambda: (0, \infty) \to \Delta(\bar{A})$ be a totally scalarly measurable mapping in $\Sigma(\bar{A})$. Assume the above mentioned estimates hold for λ where the pair (ω_1, ω_2) is admissible. Assume also that $I_{R^+}(\theta, p, \omega_1, \omega_2) < \infty$ for some $0 < \theta < 1$ and $1 < p < \infty$. Then the mapping λ is Pettis integrable in $\bar{A}_{\theta, p}$ and the estimate*

$$\left\| \bar{A}_{\theta, p} - \int_E \lambda(t)\, dt \right\|_{\bar{A}_{\theta, p}} \leqslant c I_E(\theta, p, \omega_1, \omega_2)$$

holds for any Lebesgue measurable set E. The constant c depends only on θ, p, τ, and γ where τ and γ are the constants arising in the admissibility condition for (ω_1, ω_2).

The next two theorems from [G3], [G4] explain what happens in the extreme cases $p = 1$ and $p = \infty$.

THEOREM 9. *Suppose all conditions of Theorem 8 hold, except maybe the admissibility condition for (ω_1, ω_2). Assume also that $I_{R^+}(\theta, 1, \omega_1, \omega_2) < \infty$ for some $0 < \theta < 1$. Then λ is Pettis integrable in $\Sigma(\bar{A})$, $\Sigma(\bar{A}) - \int_E \lambda(t)\, dt \in \bar{A}_{\theta, 1}$, and*

$$\left\| \Sigma(\bar{A}) - \int_E \lambda(t)\, dt \right\|_{\bar{A}_{\theta, 1}} \leqslant c I_E(\theta, 1, \omega_1, \omega_2)$$

for any Lebesgue measurable set $E \subset (0, \infty)$. The constant c depends only on θ.

THEOREM 10. *Suppose all conditions of Theorem 8 hold and $I_{R^+}(\theta, \infty, \omega_1, \omega_2) < \infty$ for some $0 < \theta < 1$. Then λ is Pettis integrable in $\Sigma(\bar{A})$, $\Sigma(\bar{A}) - \int_E \lambda(t)\, dt \in \bar{A}_{\theta, \infty}$, and*

$$\left\| \Sigma(\bar{A}) - \int_E \lambda(t)\, dt \right\|_{\bar{A}_{\theta, \infty}} \leqslant c I_E(\theta, \infty, \omega_1, \omega_2)$$

for any Lebesgue measurable set $E \subset (0, \infty)$. The constant c depends only on θ, τ, and γ.

The next corollary is useful when one needs to interpolate in the Lorentz spaces knowing the boundedness of a linear operator only on some family of characteristic functions.

Suppose $\bar{A}$ is an admissible Banach couple continuously embedded into a Hausdorff locally convex linear topological space X. Let (S, Σ, μ) be a measure space. Assume a linear operator T to be bounded from $L(p_1, 1) + L(p_2, 1)$ into X where p_1 and p_2 are given and $1 \leqslant p_1 < p_2 < \infty$. Under these conditions, the following assertion is true.

Corollary 1 (See [G3], [G4]). *Suppose a family D of measurable sets is given such that*

$$\|T\chi_E\|_{A_1} \leqslant c_1(\mu E)^{1/p_1},$$

and

$$\|T\chi_E\|_{A_2} \leqslant c_2(\mu E)^{1/p_2}$$

for all $E \in D$ *and some* $1 \leqslant p_1 < p_2 < \infty$. *Then*

$$\|Tf\|_{\bar{A}_{\theta,p}} \leqslant c\,\|f\|_{L(q,p)}$$

for any function $f \geqslant 0$ *for which* $E(t, f) \in D$ *for all* $t > 0$. *Here* $1/q = 1/p_1 + 1/p_2$, $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$, *and the constant c depends only on* θ, p_1, p_2, c_1, *and* c_2.

We next state some structure theorems for LCA groups.

Theorem 11 (See [EHR]). *Any infinite abelian group contains a subgroup isomorphic to one of the following groups*: Z, $Z(r^\infty)$, $P^*_{n\in N}Z(r_n)$, *or* $Z(r)^{\aleph_0^*}$.

In Theorem 11, Z is the group of integers, $Z(r^\infty)$ is the dual group of the group $\Delta(r)$ of r-adic integers, $Z(p)$ is the cyclic group of order p, P^* denotes the weak direct product of groups, and $Z(r)^{\aleph_0^*}$ is the weak direct product of a countable set of copies of $Z(r)$. The definitions of these groups can be found in [HR1].

Theorem 12 (See [HR1], [R]). *Every LCA group has an open subgroup* G_1 *which is the direct sum of a compact group H and a Euclidean space* R^n *with* $n \geqslant 0$.

Let G be a LCA group and H be a closed subgroup. Denote by G/H the quotient group of G modulo H. Let m_G, m_H, and $m_{G/H}$ denote the Haar measures of the indicated groups. Then for any $f \in L^1(G)$ the integral $\int_H f(x+y)\,dm_H(y)$ exists for almost all $x \in G$ and depends only on the equivalence class ξ in G/H containing x. Moreover, the following formula holds

$$\int_G f\,dm_G = \int_{G/H} dm_{G/H}(\xi) \int_H f(x+y)\,dm_H(y) \tag{26}$$

(see [R], p. 54).

Let Γ denote the dual group of G and H^0 denote the annihilator of H. Then the dual group of H is Γ/H^0. Denote by τ the canonical homomorphism $\tau\colon \Gamma \to \Gamma/H^0$. The following theorem was proved by de Leeuw [dL] and Saeki [S] (see also [L]).

THEOREM 13. *If* $h \in L^\infty(\Gamma/H^0)$, $\hat{h} = h \circ \tau$, *and* $1 \leqslant p \leqslant 2$, *then*

$$h \in M(L^p(H)) \Leftrightarrow \hat{h} \in M(L^p(G)).$$

It follows from formula (26) that if G is a discrete abelian group, H is an infinite subgroup and H' is a subset of H, then

$$\chi_{H'} \in M(L^p(\Gamma/H^0)) \Rightarrow \chi_{H'} \in M(L^p(\Gamma)). \tag{27}$$

Suppose G is a compact abelian group and Γ is its dual group.

DEFINITION 12. A set $Q \subset \Gamma$ is called a $\Lambda(p)$-set, $1 < p < \infty$, if there is a constant c such that

$$\|f\|_{L^p(G)} \leqslant c\, \|f\|_{L^1(G)}$$

for all $f \in L^1(G)$ with *supp*$(Ff) \subset Q$.

We have chosen this definition of the $\Lambda(p)$-sets because it is most appropriate for our purposes (see [R], [HR1], and [LR] for more details concerning $\Lambda(p)$-sets.) In the sequel, we will need the following property of $\Lambda(p)$-sets: *for* $p > 2$, *Definition* 12 *is equivalent to the existence of a constant* c *such that*

$$\|f\|_{L^p(G)} \leqslant c\, \|f\|_{L^2(G)} \tag{28}$$

for all $f \in L^2(G)$ *with supp*$(Ff) \subset Q$ (see [LR], p. 54).

We need the following theorem (see [R, HR1] where a stronger result concerning Sidon sets is given).

THEOREM 14. *Every infinite set* $Q \subset \Gamma$ *contains an infinite subset* $\tilde{Q}$ *which is a* $\Lambda(p)$-*set for all* $1 < p < \infty$.

4. PYRAMIDS ON LOCALLY COMPACT GROUPS AND PROOFS OF MAIN RESULTS IN THE HARDY–LITTLEWOOD CASES

Proofs given in this section are developed by means of several lemmas. In them, we construct special non-intersecting pyramids on a LCA group G. There is nothing magic in the number of pyramids (four) in

Lemmas 1–3. We need this number of pyramids in order to be able to deal with the equivalence relation R_2.

LEMMA 1. *Let G be a discrete abelian group. Then there exist four non-intersecting ∞-pyramids $P_i = \{E^i_n\}$ on G and positive constants c, c', and γ independent of n such that the following are satisfied.*

1. *For $1 \leqslant n < \infty$, $1 \leqslant i \leqslant 4$, and $2 \leqslant p < \infty$,*

$$\|F\chi_{E^i_n}\|_{L^{p'}(\Gamma)} \leqslant cn^{1/p}$$

2. *For $1 < p < 2$ and $1 \leqslant i \leqslant 4$,*

$$\chi_{B(P_i)} \in M(L^p(\Gamma)).$$

3. *For all $x \in E^i_n[m_i]$, $1 \leqslant n < \infty$, and $1 \leqslant i \leqslant 4$,*

$$m\{x - E^i_n[m_i]\} \cap \{E^i_{cn}[m_i] \cup (-E^i_{cn}[m_i])\} \geqslant \gamma n.$$

LEMMA 2. *Let G be a compact abelian group. Suppose numbers $r_i > 0$ with $1 \leqslant i \leqslant 4$ are given such that $r_1 + r_2 + r_3 + r_4 = 1$. Then there exist four non-intersecting r_i-pyramids $P_i = \{E^i_t\}$ on G and a positive constant c independent of t such that*

1. *For $0 < t < r_i$, $1 \leqslant i \leqslant 4$, and $2 \leqslant p < \infty$,*

$$\|F\chi_{E^i_t}\|_{L^{p'}(\Gamma)} \leqslant ct^{1/p}.$$

Moreover, for some numbers r_i as above, there exist positive constants c' and γ such that the following two conditions hold together with condition 1:

2. *For all $1 < p < 2$ and $1 \leqslant i \leqslant 4$,*

$$\chi_{B(P_i)} \in M(L^p(\Gamma)).$$

3. *There exists $x_i \in T[P_i]$ such that*

$$m\{x - E^i_t[x_i]\} \cap \{E^i_{ct}[x_i] \cup (-E^i_{ct}[x_i])\} \geqslant \gamma t$$

for all $x \in E^i_t[x_i]$, $0 \leqslant t \leqslant r_i/c'$, and $1 \leqslant i \leqslant 4$.

LEMMA 3. *Let G be a locally compact non-compact non-discrete abelian group. Then there exist four non-intersecting ∞-pyramids $P_i = \{E^i_t\}$ on G and positive constants c, c', and γ independent of t such that the following are satisfied*:

1. *For $0 < t < \infty$, $1 \leqslant i \leqslant 4$, and $2 \leqslant p < \infty$,*

$$\|F\chi_{E^i_t}\|_{L^{p'}(\Gamma)} \leqslant ct^{1/p},$$

2. *For* $1<p<2$ *and* $1\leqslant i\leqslant 4$,

$$\chi_{B(P_i)}\in M((L^p(\Gamma)).$$

3. *There exists* $x_i\in T(P_i)$ *such that*

$$m\{x-E_t^i[x_i]\}\cap\{E_{ct}^i[x_i]\cup(-E_{ct}^i[x_i])\}\geqslant\gamma t$$

for all $x\in E_t^i[x_i]$, $0\leqslant t\leqslant\infty$, *and* $1\leqslant i\leqslant 4$.

We will use part one of Lemmas 1–3 when we need to construct rearrangements with well behaved level sets. Part 2 allows us to separate the positive and negative parts of the real and complex parts of a function, while part 3 will be helpful in dealing with convolutions.

Remark 1. For some groups G, we will get Lemma 1 with the following condition which is stronger than that in part 3: There exists an integral constant $c>0$ independent of n such that

$$x-E_n^i[m_i]\subset E_{cn}^i[m_i]\cup(-E_{cn}^i[m_i])$$

for all $x\in E_n^i[m_i]$, $1\leqslant n\leqslant n_i/c$, and $1\leqslant i\leqslant 4$. Similarly, for some non-discrete groups G, we get the following strengthening of part 3 of Lemmas 2–3: There exist $x_i\in T(P_i)$ and a constant $c>0$ independent of n such that

$$x-E_t^i[x_i]\subset\{E_{ct}^i[x_i]\cup(-E_{ct}^i[x_i])\}$$

for all $x\in E_t^i[x_i]$, $0\leqslant t<r_i/c$, and $1\leqslant i\leqslant 4$.

Proof of Lemma 1. It is sufficient to prove Lemma 1 for the special groups listed in Theorem 11. This follows from (26), (27), and Theorem 11.

$G=Z$. We construct pyramids $P_i=\{E_n^i\}$, $1\leqslant i\leqslant 4$ on Z in the following way. Their bases coincide with the set of even positive integers, the set of odd positive integers, the set of even negative integers, and the set of odd negative integers, respectively. As to the sets E_n^i, they are simply the sets of first n elements of the corresponding bases. The interval-like structure of the sets E_n^1 allows us to prove part 1 of Lemma 1. The argument here is based on the following inequality which is not difficult to check:

$$\left\{\int_0^{2\pi}\left|\sum_{k=0}^{n-1}e^{2ikx}\right|^{p'}dx\right\}^{1/p'}\leqslant cn^{1/p}.$$

Part 2 of Lemma 1 follows from the elementary properties of the Fourier multipliers and from the fact that the characteristic functions of finite subsets of Z and the characteristic function of the set $\{n\in Z: n\geqslant 0\}$ belong to

$M(L^p(T))$ (the previous assertion follows from the boundedness of the conjugate function in the space $L^p(T)$ for $1<p<\infty$). Finally, one easily checks that the property in Remark 1 holds with $c=2$.

$G=P^*_{n\in N}Z(p_n)$, $p_n\geqslant 2$. The dual group G in this case is $\Gamma = P_{n\in N}Z(p_n)$. The pyramid P_1 on G is constructed as follows. The set E^1_1 contains just one element $\{0, 0, \ldots\}$. The set E^1_2 is equal to $E^1_1\cup\{0, 0, 1, 0, \ldots\}$. Then we add $\{0, 0, 2, 0, \ldots\}$ to the set E^1_2 and get E^1_3. After that we add the following elements one by one: $\{0, 0, 3, 0, \ldots\}$, ... $\{0, 0, p_3-1, 0, \ldots\}$, $\{0, 0, 0, 1, 0, \ldots\}$, $\{0, 0, 1, 1, 0, \ldots\}$, ... $\{0, 0, p_3-1, 1, 0, \ldots\}$, ...$\{0, 0, 0, 2, 0, \ldots\}$, ... $\{0, 0, p_3-1, 2, 0, \ldots\}$, ... $\{0, 0, p_3-1, p_4-1, 0, \ldots\}$, etc. As to the pyramids P_2, P_3, and P_4, they are taken to be some shifted copies of P_1. The only difference between them and P_1 is in the values of their first two components. The pyramid P_2 has $n_1=0$, $n_2=1$, while P_3 has $n_1=1$, $n_2=0$ and P_4 has $n_1=1$, $n_2=1$.

We prove Lemma 1 for P_1. It is clear that this will imply the same for P_2, P_3, and P_4. Part 1 of Lemma 1 for P_1 can be checked as follows. For any cyclic group $Z(r)$ and any subset E of $Z(r)$ satisfying $E=\{0, 1, 2, ..., j-1\}$ with $1\leqslant j\leqslant r$, one gets

$$\begin{aligned} I^{p'} &= \frac{1}{r}\sum_{m=0}^{r-1}|F\chi_E(m)|^{p'} \\ &= \frac{1}{r}\sum_{m=0}^{r-1}\left|\sum_{l=0}^{j-1} exp\left\{\frac{2\pi iml}{r}\right\}\right|^{p'} \leqslant \frac{1}{r}j^{p'} + \frac{1}{r}\sum_{m=1}^{r-1}\frac{|\sin(\pi mj/r)|^{p'}}{|\sin(\pi m/r)|^{p'}} \\ &\leqslant \frac{1}{r}j^{p'} + \frac{2}{r}\sum_{m=1}^{r/2}\frac{|\sin(\pi mj/r)|^{p'}}{|\sin(\pi m/r)|^{p'}} \leqslant \frac{N}{r}j^{p'} + \frac{2}{r}\sum_{m=N}^{r/2}\frac{1}{|\sin(\pi m/r)|^{p'}} \end{aligned}$$

with some integer N which will be chosen later. It follows that

$$I\leqslant\left\{\frac{N}{r}j^{p'} + \frac{c}{r}\sum_{m=N}^{r/2}\frac{r^{p'}}{m^{p'}}\right\}^{1/p'} \leqslant \left\{\frac{N}{r}j^{p'} + c\left(\frac{r}{N}\right)^{p'-1}\right\}^{1/p'}$$

with $c>0$ depending only on p. Now we choose $N=[c^{1/p'}r/j]$ and get

$$I\leqslant \tilde{c}j^{1/p}. \tag{29}$$

Also, for the set $E=\{0\}\times\{0\}\times Z(p_3)\times\cdots\times Z(p_k)\times\{0\}\times\{0\}\times\cdots$ with $k\geqslant 3$ we have

$$F\chi_E(q) = F\chi_E((q_1, q_2, \ldots)) = \begin{cases} p_3\cdots p_k & \text{if } \quad q_3=\cdots=q_k=0 \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

Now part 1 of Lemma 1 follows from (29) and (30).

The base $B(P_1)$ of the pyramid P_1 is given by $B(P_1)=\{0\}\times\{0\}\times P^*_{n\geqslant 3}\, Z(p_n)$. Since the function $\chi_{B(P_1)}$ is equal to the Fourier transform of the function $\chi_{Z(p_1)}\times\chi_{Z(p_2)}\times\prod_{n\geqslant 3} p_n\delta_n$ on $P_{n\in N}Z(p_n)$ where

$$\delta_n(q_n)=\begin{cases}1 & \text{if } \; q_n=0\\ 0 & \text{if } \; 1\leqslant q_n\leqslant p_n-1\end{cases}$$

we obtain

$$\chi_{B(P_1)}\in M(L^p(P_{n\in N}Z(p_n))).$$

This establishes part 2 of Lemma 1.

Finally, the property in Remark 1 with $c=1$ follows easily from the definitions.

$G=Z(r^\infty)$. For a prime number r, the elements of the group $Z(r^\infty)$ have the following form:

$$t=e^{2\pi i(l/r^n)} \tag{31}$$

with $0\leqslant l<r^n$ and $n\geqslant 1$. We say that the order of t is equal to m if t is representable in the form (31) with $n=m$ but not with any $n<m$.

The first pyramid P_1 on G is constructed as follows. We start with $E_1=\{1\}$ and get the next several sets E_n by adding successively the points of order one until they are exhausted. Then we continue in the same manner with the points of order 5; after that we proceed with the points of order 9 etc. The pyramid P_2 is built similarly using the points of order 2, 6, 10, etc. The pyramid P_3 contains the points of order 3, 7, 11, etc., and finally, the pyramid P_4 contains the points of order 4, 8, 12, etc.

Consider the following set

$$E_{k,n}=\{e^{2\pi i(m/r^k)}: 0\leqslant m\leqslant n<r^k\}.$$

Using (29), we get

$$\begin{aligned}&\left\{\int_{\varDelta_r}|F\chi_{E_{k,n}}(\gamma)|^{p'}\,d\gamma\right\}^{1/p'}\\&\quad=\left\{\frac{1}{r^k}\sum_{x_0,\,\ldots,\,x_{k-1}=0}^{r-1}\left|\sum_{m=0}^{n}e^{2\pi i(m/r^k)(x_0+x_1r+\cdots+x_{k-1}r^{k-1})}\right|^{p'}\right\}^{1/p'}\\&\quad=\left\{\frac{1}{r^k}\sum_{s=0}^{r^k-1}\left|\sum_{s=0}^{n}e^{(2\pi ims)/(r^k)}\right|^{p'}\right\}^{1/p'}\leqslant c_pn^{1/p}.\end{aligned} \tag{32}$$

Note that we have $\Gamma=\varDelta_r$.

Each set E_n^1 belonging to the pyramid P_1 can be represented as $E_n^1 = \bar{E}_n \cup \tilde{E}_n$ where

$$\bar{E}_n = \bigcup_{k=0}^{s(n)} H_{1+4k} \tag{33}$$

and the set $\tilde{E}_n$ consists of several initial elements of order $\tau = 1 + 4(s(n)+1)$ of the group $Z(r^\infty)$. In (33), the symbol H_j denotes the complete set of elements of order j.

Using (32) and the formula $H_l = E_{l, r^l - 1} \backslash E_{l-1, r^{l-1}-1}$, we get

$$\|F\chi_{H_l}\|_{L^{p'}(\Delta_r)} \leqslant cr^{l/p} \leqslant c(mH_l)^{1/p}. \tag{34}$$

It follows from (33) and (34) that

$$\begin{aligned} \|F\chi_{\bar{E}_n}\|_{L^{p'}(\Delta_r)} &\leqslant \sum_{k=0}^{s(n)} \|Ff\chi_{H_{1+4k}}\|_{L^{p'}(\Delta_r)} \leqslant c \sum_{k=0}^{s(n)} r^{4k/p} \\ &\leqslant cr^{(4(s(n)+1))/p} \leqslant c_{p,r}(m\bar{E}_n)^{1/p}. \end{aligned} \tag{35}$$

Let us denote by j the number of elements in the set $\tilde{E}_n$. Then we have $\tilde{E}_n = E_{\tau,d} \backslash E_{\tau-1,e}$, where the set $E_{\tau,d}$ consists of all elements of $Z(r^\infty)$ of order $\leqslant \tau$ preceding the last element of $\tilde{E}_n$ in the order inherited from T and the set $E_{\tau-1,e}$ has similar structure with $\tau - 1$ instead of τ. Denote by $d(k)$ the number of elements of $Z(r^\infty)$ of order k with $k \leqslant \tau$, preceding the last element of $\tilde{E}_n$ in the order of T. Then $j = k(\tau) \geqslant rk(\tau-1) \geqslant \ldots \geqslant r^{\tau-1}k(1)$. It follows that

$$mE_{\tau,d} \leqslant cm\tilde{E}_n. \tag{36}$$

Therefore, by (32) and (36),

$$\begin{aligned} \|F\chi_{\tilde{E}_n}\|_{L^{p'}(\Delta_r)} &\leqslant \|F\chi_{E_{r,d}}\|_{L^{p'}(\Delta_r)} + \|F\chi_{E_{r-1,e}}\|_{L^{p'}(\Delta_r)} \\ &\leqslant c(mE_{\tau,d} + mE_{\tau-1,e})^{1/p} \leqslant c(mE_{\tau,d})^{1/p} \leqslant c(m\tilde{E}_n)^{1/p}. \end{aligned}$$

Finally, using (35) and the previous inequality, we get part 1 of Lemma 1 for the group $Z(r^\infty)$.

In order to prove part 2 of Lemma 1 in the case $G = Z(r^\infty)$, we need some results of Taibleson [T] concerning the Fourier multipliers for $L^p(\Omega_r)$ where Ω_r is the group of r-adic numbers. The group Δ_r is a compact subgroup of Ω_r. Denote by H_0 its annihilator in Ω_r (the group Ω_r is self-dual) and consider the standard homomorphism $\pi\colon \Omega_r \to \Omega_r/H_0 = Z(r^\infty)$. Taibleson proved that any bounded radial function h on Ω_r belongs to $M(L^p(\Omega_r))$ for $1<p<2$ (see the definition of radial functions in Taibleson's book). It follows from Theorem 13 that if g is a bounded function on $Z(r^\infty)$ for which $g\circ\pi$ is radial on Ω_r, then $g\in M(L^p(Z(r^\infty)))$. The base of the pyramids P_1 constructed above has the following form:

$$B[P_1]=\bigcup_{k=0}^{\infty} H_{1+4k}.$$

If $g=\chi_{B[P_1]}$, then the function $g\circ\pi$ is radial and hence $\chi_{B[P_1]}\in M(L^p(Z(r^\infty)))$. Similar reasoning applies to P_i with $2\leqslant i\leqslant 4$. This proves part 2 of Lemma 1.

We now prove part 3 for $Z(r^\infty)$. Let E_n^1 be any set belonging to the pyramid P_1. Then E_n^1 can be represented $E_n^1=\bar{E}_n\cup\tilde{E}_n$ as above. We need the following fact: if $j<k$, then

$$H_j+H_k\subset H_k. \tag{37}$$

However, the sum of two elements of the same order may have a smaller order. This is the reason why we get property 3 in Lemma 1 instead of the stronger property in Remark 1.

We have

$$\begin{aligned} m\left(\bigcup_{k=0}^{s(n)} H_{1+4k}\right) &= \sum_{k=0}^{s(n)} (r^k-r^{k-1}) \\ &\geqslant c\sum_{k=0}^{s(n)+1} (r^k-r^{k-1}) = cm\left(\bigcup_{k=0}^{s(n)+1} H_{1+4k}\right). \end{aligned} \tag{38}$$

Using inclusion (37) and the fact that $m_1=0$ for the pyramid P_1, we see that if $x\in E_n^1$ has order $1+4(s(n)+1)$, then $x-H_{1+4k}\subset H_{1+4(s(n)+1)}$ for $0\leqslant k\leqslant s(n)$. Let c be the constant in (38). Then, using (38), we obtain

$$\begin{aligned} m\{(x-E_n^1)\cap E_{cn}^1\} &\geqslant m\{(x-H_{1+4s(n)})\cap H_{1+4(s(n)+1)}\}=mH_{1+4s(n)} \\ &= r^{1+4s(n)}-r^{4s(n)}\geqslant \gamma\sum_{k=0}^{s(n)+1}(r^k-r^{k-1})\geqslant \gamma m E_n^1. \end{aligned}$$

Similar reasoning applies when the order of $x\in E_n^1$ is less than $1+4(s(n)+1)$. This shows that part 3 of Lemma 1 holds for the group $G=Z(r^\infty)$.

Since all the special groups have already been considered, the proof of Lemma 1 is now complete.

Proof of Lemma 2. By Theorem 11, any compact group G has a closed subgroup H such that the quotient group G/H is isomorphic to one of the following groups: T, $P_{n\geqslant 1}Z(p_n)$, or Δ_r. These groups are the circle group, the infinite product of cyclic groups, and the group of r-adic integers, respectively. The canonical homomorphism $\tau: G \to G/H$ is Haar measure preserving. Hence we can construct a pyramid on a group from one on a quotient group by taking the inverse images with respect to the homomorphism τ. Using formula (26) and the continuity of τ, we show that parts 1 and 2 in Lemma 2 are conserved under τ. It is easy to see that part 3 is also conserved. Hence, it is sufficient to prove Lemma 2 for the special groups mentioned above.

$G = T$. Suppose the numbers r_1, r_2, r_3, and r_4 are given such that $r_1 + r_2 + r_3 + r_4 = 1$. It is clear that we may consider the interval $[0, 1]$ instead of T and build the pyramids P_i on $[0, 1]$. They will have four closed essentially non-intersecting intervals I_i satisfying $mI_i = r_i$ as their bases. The set E^i_t belonging to the pyramid P_i coincides with the closed interval of measure t having the same center as the interval $B(P_1)$. Part 1 of Lemma 2 for characteristic functions of intervals follows from straightforward calculations. As the characteristic function of any interval is in $M(L^p(Z))$ (this follows from the results of Hirschman [Hi] on the multipliers having bounded variation), we have an even stronger property than that in part 2 of Lemma 2. Finally, it is easy to see that the property in Remark 2 holds with $c = 2$. Therefore, Lemma 2 is true for the group T.

$G = P_{n\geqslant 1}Z(p_n)$. The Haar measures of the groups $P_{n\geqslant 1}Z(p_n)$ and T are isomorphic. The standard measure preserving invertible *mod* 0 transformation $\omega: P_{n\geqslant 1}Z(p_n) \to T$ is obtained as follows. First we subdivide the circle T into p_0 closed essentially non intersecting subarcs of equal length (we call this partition π_0). Then we subdivide each of these subarcs into p_2 closed subarcs of equal length (partition π_1) etc. After that we define $\omega(p) = \omega(\{q_n\})$ with $q \in P_{n\geqslant 1}Z(p_n)$ to be the unique point belonging to the intersection of the arcs corresponding to the integers $\{q_n\}$ in the partitions above. The explicit formula for ω is

$$\omega(\{q_n\}) = \sum_{n=0}^{\infty} \frac{q_n}{p_0 \cdots p_n}.$$

The next step is to build a 1-pyramid P_G on G as follows. The pyramid P_G will be the inverse image with respect to ω of some 1-pyramid $\tilde{P} = \{\tilde{E}_t\}$ on the circle group T (we identify the subsets of T and $[0, 1]$ as usual). Set

$$S = \sum_{i=0}^{\infty} \frac{1}{p_0 \cdots p_i}$$

and define the sets $\tilde{E}_t$ as follows: $\tilde{E}_t = [0, t/2] \cup [S - t/2, S]$ for $0 < t < S$ and $\tilde{E}_t = [0, t]$ for $S < t \leqslant 1$. Next we define the pyramid P_G by $P_G = \{\omega^{-1}(\tilde{E}_t)\}$.

Denote $\Delta_1 = [0, t/2]$, $\Delta_2 = [S - t/2, S]$ with $0 < t < S$. Assume $\omega(x) \in \Delta_1$ and $\omega(x') \in \Delta_2$. Then

$$0 \leqslant \sum \frac{x_i}{p_0 \cdots p_i} \leqslant \frac{t}{2},$$

and

$$S - \frac{t}{2} \leqslant \sum \frac{x'_i}{p_0 \cdots p_i} \leqslant S.$$

It follows that

$$1 \leqslant \sum \frac{p_i - x'_i}{p_0 \cdots p_i} \leqslant 1 + \frac{t}{2}$$

and

$$1 \leqslant \omega(x - x') = \sum \frac{x_i + p_i - x'_i}{p_0 \cdots p_i} \leqslant 1 + t.$$

This implies part 3 of Lemma 2 for the pyramid P_G in the case $x \in \omega^{-1}\Delta_1$. If $\omega(x) \in \Delta_2$ and $\omega(x') \in \Delta_1$, then the same reasoning allows us to estimate $\omega(x' - x)$ and get part 3 of Lemma 2 for $x \in \omega^{-1}\Delta_2$. We thus conclude that Part 3 of Lemma 3 holds for P_G.

Our next goal will be checking the inequality in part 1 of Lemma 2 for the pyramid P_G. Since the sets constituting the pyramid $\tilde{P}$ consist of at most 2 closed arcs in T, it is sufficient to prove that

$$\left\{\sum_{\gamma} |c_\gamma(\chi_I)|^{p'}\right\}^{1/p'} \leqslant c(m(I))^{1/p} \tag{39}$$

where $2 < p < \infty$ and the constant $c > 0$ is independent of I. In (39), I is any subinterval of $[0, 1]$, χ_I is its characteristic function, $\gamma \in P^*_{n \geqslant 0} Z(p_n)$, and c_γ are the Fourier coefficients with respect to the orthogonal system $\Phi = \{\phi_\gamma\} = \{\gamma \circ \omega^{-1}\}$. It is clear that part 1 of Lemma 2 for the pyramid P_G follows from (39).

In order to prove (39), we need the following condition to be true: *if*

$$\gamma = \{m_0, m_1, ..., m_n, 0, 0, \ldots\} \tag{40}$$

for some n, $m_n > 0$, *and* $0 \leqslant m_k < p_k$ *with* $0 \leqslant k \leqslant n$, *and if the interval* I *belongs to* π_j *with* $j < n$, *then*

$$\int_I \phi_\gamma \, dt = 0. \tag{41}$$

Formula (41) can be proven in the following way. It is sufficient to consider the case $n = j + 1$. Since

$$\gamma(x) = e^{2\pi i((m_0 x_0/p_0) + \cdots + (m_n x_n/p_n))}$$

for $x = (x_k) \in P_{k \geqslant 0} Z(p_k)$, we have

$$\int_I \phi_\gamma(t) \, dt = c \, \frac{1}{p_0 p_1 \cdots p_n} \sum_{k=0}^{p_n - 1} e^{2\pi i m_n k/p_n} = 0$$

This proves (41).

We have

$$c_\gamma(\chi_I) = \int_I \phi_\gamma(t) \, dt$$

where I is any subinterval of $[0, 1]$ and γ is given by (40). Denote $s = \max\{n\colon (p_0 \cdots p_n)^{-1} \geqslant m(I)\}$. We have

$$\frac{1}{p_0 \cdots p_s} \geqslant m(I) > \frac{1}{p_0 \cdots p_{s+1}}. \tag{42}$$

It follows that there exists an integer l such that $1 \leqslant l < p_{s+1}$ and

$$\frac{1}{p_0 \cdots p_s l} \geqslant m(I) > \frac{1}{p_0 \cdots p_s (l+1)}. \tag{43}$$

We obtain

$$\sum_{\gamma \in \Gamma} |c_\gamma(\chi_I)|^{p'} = \sum_{\gamma \in \Gamma_1} + \sum_{\gamma \in \Gamma_2} + \sum_{\gamma \in \Gamma_3} = I_1 + I_2 + I_3 \tag{44}$$

where $\Gamma_1 = \Gamma_{11} \cup \Gamma_{12} \cup \Gamma_{13}$ with

$$\Gamma_{11} = \{\gamma \in \Gamma : \gamma = (m_0, ..., m_s, 0, 0, ...), 0 \leqslant m_i \leqslant p_i - 1, 0 \leqslant i \leqslant s\};$$

$$\Gamma_{12} = \{\gamma \in \Gamma : \gamma = (m_0, ..., m_s, m_{s+1}, 0, 0, ...),$$
$$0 \leqslant m_i \leqslant p_i - 1, 0 \leqslant i \leqslant s, 0 \leqslant m_{s+1} \leqslant l\};$$

$$\Gamma_{13} = \{\gamma \in \Gamma : \gamma = (m_0, ..., m_s, m_{s+1}, 0, 0, ...),$$
$$0 \leqslant m_i \leqslant p_i - 1, 0 \leqslant i \leqslant s, p_{s+1} - l \leqslant m_{s+1} \leqslant p_{s+1} - 1\};$$

$$\Gamma_2 = \{\gamma \in \Gamma : \gamma = (m_0, ..., m_s, m_{s+1}, 0, 0, ...),$$
$$0 \leqslant m_i \leqslant p_i - 1, 0 \leqslant i \leqslant s, l+1 \leqslant m_{s+1} \leqslant p_{s+1} - l + 1\};$$

$$\Gamma_3 = \{\gamma \in \Gamma : \gamma = (m_0, ..., m_k, 0, 0, ...),$$
$$0 \leqslant m_i \leqslant p_i - 1, 0 \leqslant i \leqslant k, k \geqslant s+2, m_k > 0\}.$$

From the inequality $|c_\gamma(\chi_I)| \leqslant m(I)$ and from (43), we obtain

$$I_1^{1/p'} \leqslant c(p_0 \cdots p_s l m(I)^{p'})^{1/p'}. \tag{45}$$

Using (41), we get

$$c_\gamma(\chi_I) = \int_{\Delta_1} \overline{\phi_\gamma}\, dt + \int_{\Delta_2} \overline{\phi_\gamma}\, dt \tag{46}$$

for $\gamma = (m_0, ..., m_n, 0, 0, ...)$ and $1 \leqslant m_n \leqslant p_n - 1$, where Δ_1 and Δ_2 are some subintervals of I such that $m(\Delta_1), m(\Delta_2) \leqslant (p_0 \cdots p_{n-1})^{-1}$.

The intervals Δ_1 and Δ_2 consist of several intervals belonging to the partition π_n and of the intervals Δ'_1 and Δ'_2 such that $m(\Delta'_1)$, $m(\Delta'_2) \leqslant (p_0 \cdots p_n)^{-1}$. It follows from the previous inequality and (46) that

$$|c_\gamma(\chi_I)| \leqslant \frac{c}{p_0 \cdots p_n} + \frac{1}{p_0 \cdots p_n} \left| \sum_{j=s_1}^{p_n - 1} e^{2\pi i j m_n / p_n} \right| + \frac{1}{p_0 \cdots p_n} \left| \sum_{j=0}^{s_2} e^{2\pi i j m_n / p_n} \right|$$
$$\leqslant \frac{c}{p_0 \cdots p_n} + \frac{c}{p_0 \cdots p_n} \, \frac{1}{\sin(\pi m_n / p_n)}.$$

Now using the inequality $\sin x \geqslant (2/\pi) x$, $0 \leqslant x \leqslant \pi/2$, we get

$$|c_\gamma(\chi_I)| \leqslant \frac{c}{p_0 \cdots p_{n-1}} \min\left(\frac{1}{m_n}, \frac{1}{p_n - m_n}\right). \tag{47}$$

From (47), we obtain

$$I_3 \leqslant \sum_{k=s+2}^{\infty} p_0 \cdots p_{k-1} \sum_{m_k=1}^{p_k-1} \frac{1}{(p_0 \cdots p_{k-1})^{p'}} \min\left(\frac{1}{m_k}, \frac{1}{p_k - m_k}\right)^{p'}$$
$$\leqslant c \sum_{k=s+2}^{\infty} \frac{1}{(p_0 \cdots p_{k-1})^{p'-1}}.$$

As

$$\frac{1}{(p_0 \cdots p_{k-1})^{p'-1}} \leqslant c \left(\frac{1}{(p_0 \cdots p_{k-1})^{p'-1}} - \frac{1}{(p_0 \cdots p_k)^{p'-1}}\right),$$

we get

$$I_3^{1/p'} \leqslant c(p_0 \cdots p_{s+1})^{-1/p}.$$

Now (42) gives

$$I_3^{1/p'} \leqslant cm(I)^{1/p}. \tag{48}$$

Finally, (47) and (42) imply

$$I_2^{1/p'} \leqslant c \left\{ p_0 \cdots p_s \sum_{m_{s+1}=l+1}^{p_{s+1}-l+1} \frac{1}{(p_0 \cdots p_s)^{p'}} \min\left(\frac{1}{m_{s+1}}, \frac{1}{p_{s+1} - m_{s+1}}\right)^{p'} \right\}^{1/p'}$$
$$\leqslant c \left\{ \frac{1}{(p_0 \cdots p_s)^{p'-1}} \sum_{m_{s+1}=l+1}^{\infty} \frac{1}{(m_{s+1})^{p'}} \right\}^{1/p'}$$
$$\leqslant c \frac{1}{[p_0 \cdots p_s(l+1)]^{1/p}} \leqslant cm(I)^{1/p}. \tag{49}$$

Consequently, inequality (39) follows from (44), (45), (48), and (49). This proves the inequality in part 1 of Lemma 2 for the pyramid P_G. Moreover, inequality (39) implies part 1 of Lemma 2. Indeed, we can construct four non-intersecting r_i-pyramids $\widetilde{P}_i$ on $[0, 1]$ consisting of intervals and take their inverse images with respect to ω to be the pyramids P_i.

The next step in the proof of Lemma 2 in the case $G = P_{n \geqslant 0} Z(p_n)$ is the following. We will find numbers r'_i with $r'_1 + r'_2 + r'_3 + r'_4 = 1$ and build r'_i-pyramids P_i satisfying conditions 1, 2, and 3 in Lemma 2.

Set

$$r'_1 = r'_2 = r'_3 = \frac{1}{p_0 p_1}, \qquad r'_4 = \frac{p_0 p_1 - 3}{p_0 p_1},$$

and consider a 1-pyramid $P_H = \{W_t\}$ constructed exactly as above with the group

$$H = P_{n \geqslant 2} Z(p_n) \tag{50}$$

in place of the group G. Define four r_i'-pyramids P_i on G as follows. For $1 \leqslant i \leqslant 3$, we set $P_i = \{E_t^i\}$ where $E_t^i = \overline{x}_i \times W_{t(r_i')-1}$ and $x_1 = (0, 1)$, $x_2 = (1, 0)$, $x_3 = (1, 1)$. In addition, $P_4 = \{E_t^4\}$ with $E_t^4 = A \times W_{t(r_4')-1}$ and $A = Z(p_0) \times Z(p_1) \setminus \{x_1, x_2, x_3\}$.

It is clear that the pyramids P_1, P_2 and P_3 are obtained by shifting the pyramid

$$P = \{E_t\} = \{(0, 0) \times W_{t(r_1')-1}\}, \tag{51}$$

while the pyramid P_4 can be represented as a disjoint finite union of such shifts. Using this, we deduce the validity of parts 1 and 2 of Lemma 2 for the pyramids P_i from that for the pyramid P_H. It is clear that part 3 of Lemma 2 for the pyramid P_H also holds. This immediately implies part 3 of Lemma 2 for P and for P_1, P_2 and P_3 as well because part 3 of Lemma 2 is preserved under shifts. For the pyramid P_4, we use the fact that it contains the pyramid P as a subpyramid and easily get part 3 of Lemma 2 for it.

This completes the proof of Lemma 2 for the group $G = P_{n \geqslant 0} Z(p_n)$.

$G = \Delta_r$. Since the group Δ_r is indistinguishable from the group $P_{n \geqslant 0} Z(p_n)$ with $p_n = r$ both as a topological and metric space, the proof of Lemma 2 for Δ_r follows the same lines as in the previous case. First we build a 1-pyramid P_G on Δ_r exactly as above with the only difference being that now we set

$$S = \sum_{n \geqslant 1} \frac{p_i - 1}{p_0 \cdots p_i}.$$

The reason we have this difference with the previous case is the following. If $x = (x_n) \in \Delta_r$, then $-x = (p_0 - x_0, p_1 - 1 - x_1, ..., p_n - 1 - x_n, \ldots)$. Now we can prove part 3 of Lemma 2 for P_G exactly as before. The proof of part 1 is also similar. All we need here is to get (39) with $\Gamma = Z(r^\infty)$. If $\gamma \in Z(r^\infty)$, then

$$\gamma(x) = exp \left\{ \frac{2\pi i}{r^{n+1}} (m_0 + m_1 r + \cdots + m_n r^n)(x_0 + x_1 r + \cdots + x_n r^n) \right\} \tag{52}$$

where $x = (x_k) \in \Delta_r$, n is some integer, and $0 \leqslant m_k \leqslant r$ for $0 \leqslant k \leqslant n$. Now we define an element of the group $P^*_{n \geqslant 0} Z(r)$ by formula (40). If we prove (41) for the functions $\phi_\gamma = \gamma \circ \omega^{-1}$, then the proof of part 1 for P_G can be completed exactly as in the previous case. If the interval I belongs to the partition π_{n-1}, then for $x \in \omega^{-1}(I)$ we have

$$\gamma(x) = c\ exp\left\{\frac{2\pi imx_n r^n}{r^{n+1}}\right\} = c\ exp\left\{\frac{2\pi imx_n}{r}\right\} = c\ exp\left\{\frac{2\pi im_0 x_n}{r}\right\}$$

where c is a complex constant such that $|c| = 1$ and $m = m_0 + \cdots + r^n m_n$. As $m_0 \neq 0$ and $m_n \neq 0$, we obtain

$$\sum_{x_n=0}^{r-1} \exp\left\{\frac{2\pi im_0 x_n}{r}\right\} = 0.$$

This proves (41) and establishes part 1 of Lemma 2 for the pyramid P_G.

Next we construct the pyramids P_i and the r^{-2}-pyramid P on Δ_r exactly as in the case of the group $P_{n \geqslant 0} Z(r)$ and obtain part 3 of Lemma 2 exactly in the same way as before.

We will prove part 2 of Lemma 2 by showing that

$$\chi_{(0,0) \times H} \in M(L^p(Z(r^\infty))), \tag{53}$$

where $H = P_{n \geqslant 2} Z(r)$.

In order to prove (53), let us suppose $\gamma \in Z(r^\infty)$ is given by (52) and $m_n \neq 0$. Then we have

$$I = F\chi_{(0,0) \times H}(\gamma) = \frac{1}{r^{n+1}} \sum_{x = (x_0, \ldots, x_n, 0 \cdots)} \chi_{(0,0) \times H}(x)\, \bar{\gamma}(x).$$

If $n > 1$, we get

$$I = \frac{1}{r^{n+1}} \sum_{x_{s+1}, \ldots, x_n = 0}^{r-1} \exp \frac{-2\pi im(x_{s+1} + \cdots + x_n r^{n-s-1})}{r^{n-s}} = 0$$

where $m = m_0 + rm_1 + \cdots + r^n m_n$. Moreover, if $n \geqslant 1$, then $I = r^{-2}$. It follows that

$$\|F\chi_{x_0 \times H}\|_{L^1(Z(r^\infty))} \leqslant 1.$$

This proves (53).

Next we prove part 1 of Lemma 2. It is sufficient to prove it for the pyramid P defined in (51). We have

$$J = F\chi_{E_t}(\gamma) = \int_{\Delta_r} \chi_{E_t}(x) \exp\left\{\frac{-2\pi im}{r^{n+1}}(x_0 + \cdots + x_n r^n)\right\} dm_G.$$

If $n \leqslant 1$, then $J=0$. If $n>1$, then

$$J=\frac{1}{r^2}\sum_{x_0,\,x_1=0}^{r-1} \exp\left\{\frac{2\pi i m}{r^{n+1}}(x_0+x_1 r)\right\}$$
$$\times \int_H \chi_{W_{tr^2}}(x_2,\ldots) \exp\left\{\frac{2\pi i m}{r^{n-1}}(x_2+\cdots+x_n r^{n-2})\right\} dm_H. \qquad (54)$$

Let us denote $x_0'=x_2$, $x_1'=x_3, \ldots,$ $\tilde{m}=m_0+\cdots+m_{n-2}r^{n-2}$, $\bar{m}=m_{n-1}r^{n-1}+m_n r^n$, $\tilde{x}=x_0+x_1 r$, and $\overline{x'}=x_2+\cdots+x_n r^{n-2}$. It follows from (54) that for $\tilde{m}\neq 0$,

$$J=\frac{1}{r^2}\sum_{x_0,\,x_1}^{r-1} \exp\left\{\frac{-2\pi i(\tilde{m}+\bar{m})\,\tilde{x}}{r^{n+1}}\right\} \int_H \chi_{W_{tr^2}}(x') \exp\frac{-2\pi i \tilde{m}\overline{x'}}{r^{n-1}}\, dm_H(x')$$
$$=\frac{1}{r^2}\sum_{x_0,\,x_1}^{r-1} \exp\left\{\frac{-2\pi i(\tilde{m}+\bar{m})\,\tilde{x}}{r^{n+1}}\right\} F\chi_{W_{tr^2}}(\tilde{m}).$$

Therefore, $|F\chi_{E_t}(\gamma)| \leqslant |F\chi_{W_{tr^2}}(\tilde{m})|$ and part 1 of Lemma 2 follows from the validity of part 1 of Lemma 2 for the pyramid P_H. This finishes the proof of Lemma 2 for the group Δ_r.

Thus Lemma 2 holds for all special groups and therefore the proof of Lemma 2 is now completed.

Proof of Lemma 3. By Theorem 12, there exists an open subgroup U of the group G topologically isomorphic to $R^a \times K$ for some compact group K and a non-negative integer a. First we consider the case $a \geqslant 1$. It is clear that the group G/U is discrete. Denote by H the annihilator of U. Using (26), we get for any $g \in (L^1+L^2)(U)$,

$$\int_\Gamma |Fg(\gamma)|^{p'}\, d\gamma = \int_{\Gamma/H} dm_{\Gamma/H}(\xi) \int_H |Fg(\gamma+\gamma')|^{p'}\, dm_H(\gamma')$$
$$=\int_{\Gamma/H} \left|\int_U g(y)\,\bar{\xi}(y)\, dm_U(y)\right|^{p'} \int_H dm_H(\gamma')$$
$$=\int_{\Gamma/H} |Fg(\xi)|^{p'}\, dm_{\Gamma/H}(\xi). \qquad (55)$$

Therefore, it is sufficient to build for ∞-pyramids on the group $R^a \times K$ satisfying the conditions of Lemma 3. Then we easily prove Lemma 3 for the group G. Indeed, part 1 of Lemma 3 for G follows from (55). Part 2 can be obtained from (27), while part 3 is straightforward.

Part 2 of Lemma 3 for the group $R^a \times K$ in the case $a \geqslant 2$ is slightly easier than that for $a = 1$; hence we consider it first. We give the proof for $a = 2$. The case $a > 2$ is similar.

Consider the family of squares $\{C_t = [0, \sqrt{t}]^2 : t > 0\}$ belonging to the first quadrant $Q_1 = \{x = (x_1, x_2) \in R^2 : x_1 \geqslant 0, x_2 \geqslant 0\}$ of R^2. The pyramid P_1 in $R^2 \times K$ is defined by $P_1 = \{C_t \times K\}$. Part 1 of Lemma 3 is easy to check for this pyramid. In order to prove part 2, we use the fact that the characteristic function of the first quadrant is a Fourier multiplier in $L^p(R^2)$. This follows from the boundedness of the Hilbert transform in $L^p(R)$ for $p > 1$. Part 3 is easy to check. Three more pyramids can be obtained by using the same construction for the second, third, and fourth quadrants, respectively. This completes the proof for $a \geqslant 2$.

Next we consider the case $a = 1$. Let us define the following set

$$W = \left(\bigcup_{k=0}^{\infty} \left[k - \frac{1}{8}, k + \frac{1}{8} \right] \right) \times K. \tag{56}$$

This set will be the base of the pyramid P_1. The pyramid itself is given by $E_t^1 = [-\frac{1}{8}, -\frac{1}{8} + t] \times K$ for $0 < t \leqslant \frac{1}{4}$; $E_t^1 = ([-\frac{1}{8}, \frac{1}{8}] \cup [\frac{7}{8}, \frac{7}{8} + t - \frac{1}{4}]) \times K$ for $\frac{1}{4} < t \leqslant \frac{2}{4}$, etc.

Part 3 of lemma 3 for the pyramid P_1 follows from the following observation. Suppose $m/4 < t \leqslant (m+1)/4$. Then

$$E_t^1 = \bigcup_{k=0}^{m-1} \left[k - \frac{1}{8}, k + \frac{1}{8} \right] \cup \left[m - \frac{1}{8}, m - \frac{1}{8} + t - \frac{m}{4} \right] = \bigcup_{k=0}^{m-1} I_k \cup I'_m.$$

If $x \in I_k$ with $0 \leqslant k \leqslant m-1$ and j is between 0 and $m-1$, then there exists a subset $\tilde{I}_j$ of I_j such that $m(\tilde{I}_j) = \frac{1}{8}$ and $x - \tilde{I}_j \subset E_t^1 \cup (-E_t^1)$. Similar reasoning applies in the case $x \in I'_m$. This proves part 3 of Lemma 3 for the pyramid P_1.

We next define the set

$$V = \bigcup_{k=-\infty}^{\infty} \left[k - \frac{1}{8}, k + \frac{1}{8} \right],$$

and note that the following corollary from Hirschman's theorem [Hi] on the Fourier multipliers of bounded variation holds: $\chi_{[-1/8, 1/8]} \in M(L^p(Z))$. By this corollary and Theorem 13,

$$\chi_V \in M(L^p(R)). \tag{57}$$

Now the boundedness of the Hilbert transform in $L^p(R)$ with $p > 1$, (56), and (57) give $\chi_W \in M(L^p(R))$. This implies part 2 of Lemma 3 for the pyramid P_1.

In order to establish part 1, we make the following observation. The inequality in part 1 of Lemma 3 holds for the family of all subintervals of R. Hence, part 1 will be established if we prove the same inequality for the family of sets

$$L_m = \bigcup_{k=0}^{m} \left[k - \frac{1}{8}, k + \frac{1}{8} \right], \qquad m \geqslant 0.$$

The proof of the above-mentioned inequality is left as an exercise for the reader.

We construct three more pyramids P_2, P_3, and P_4 exactly in the same way by shifting P_1 by $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$, respectively. Since all the required properties are unaffected by shifts, Lemma 3 holds for $a = 1$.

Next we consider the case when G contains an open compact subgroup K. Let us denote $H = G/K$. The group H is discrete and hence there are four ∞-pyramids $Q_i = \{\tilde{Q}^i_t\}$ with $1 \leqslant i \leqslant 4$ on H satisfying the conditions of Lemma 1. We may also construct a 1-pyramid $W = \{\tilde{W}_t\}$ on K satisfying the conditions of Lemma 2. These auxiliary pyramids will help us to construct the ∞-pyramids $P_i = \{E^i_t\}$ on G. For example, the pyramid P_1 is obtained as follows. It is easy to see that the pyramid Q_1 provides a numeration for its base $B(Q_1)$ in H. Consider the coset $x_1 + K$ of K corresponding to the first element of $B(Q_1)$ in this numeration and define $E^1_t = x_1 + \tilde{W}_t$ for $0 < t \leqslant 1$. Next we consider the coset $x_2 + K$ of K corresponding to the second element of $B(Q_1)$ and set $E^1_t = E^1_1 \cup (x_2 + \tilde{W}_t)$ for $1 < t \leqslant 2$. Now it is clear how to continue this construction by induction.

We have

$$B(P_1) = \bigcup_{k=0}^{\infty} (x_k + K).$$

Since $\chi_{B(Q_1)} \in M(L^p(H))$, we get $\chi_{B(P_1)} \in M(L^p(G))$, by Theorem 13. This proves part 2 of Lemma 3 for the pyramid P_1.

Denote by K° the annihilator of K. As H is discrete, K° is compact. For $0 < t \leqslant 1$, we have by (26) and part 1 of Lemma 2 that

$$\begin{aligned} \left\{ \int_\Gamma |F\chi_{E^1_t}(\gamma)\, dm_\Gamma \right\}^{1/p'} &= \left\{ \int_{\Gamma/K^\circ} dm_{\Gamma/K^\circ}(\xi) \int_{K^\circ} |F\chi_{E^1_t}(\gamma + \gamma')|^{p'}\, dm_{K^\circ}(\gamma') \right\}^{1/p'} \\ &= \left\{ \int_{\Gamma/K^\circ} |F\chi_{E^1_t}(\xi)|^{p'}\, dm_{\Gamma/K^\circ}(\xi) \right\}^{1/p'} \\ &\leqslant c_p (mE^1_t)^{1/p}. \end{aligned} \tag{58}$$

Now consider a family of sets given by

$$A_j = \bigcup_{k=0}^{j} (x_k + K), \qquad j \geqslant 0. \tag{59}$$

By (26) and part 1 of Lemma 1, we have

$$\begin{aligned}\left\{\int_{\Gamma} |F\chi_{A_j}(\gamma)|^{p'}\, dm_{\Gamma}\right\}^{1/p'} &= \left\{\int_{\Gamma/K^{\circ}} dm_{\Gamma/K^{\circ}}(\xi) \int_{K^{\circ}} |F\chi_{A_j}(\gamma+\gamma')|^{p'}\, dm_{K^{\circ}}(\gamma')\right\}^{1/p'} \\ &= \left\{\int_{K^{\circ}} |F\chi_{A_j}(\gamma')|^{p'}\, dm_{K^{\circ}}(\gamma')\right\}^{1/p'} \\ &\leqslant c_p j^{1/p}, \qquad j \geqslant 1. \end{aligned} \tag{60}$$

Now part 1 of Lemma 3 follows from P_1 because any set E_t^1 consists of a set of type (59) and a shift of a set E_y^1 for some $0 < y < 1$.

Finally, part 3 of Lemma 3 for P_1 can be obtained by combining the following facts: part 3 of Lemma 3 for $0 < t < c^{-1}$, the inclusion $E_t^1 - E_t^1 \subset K$ for $c^{-1} \leqslant t \leqslant 1$, and part 3 of Lemma 1 for the pyramid Q_1.

The pyramids P_i with $2 \leqslant i \leqslant 4$ are constructed exactly in the same way as the pyramid P_1 but from Q_i instead of Q_1. It is clear that the same reasoning applies for P_i with $2 \leqslant i \leqslant 4$.

This finishes the proof of Lemma 3.

Proof of (12), (14), (19), *and* (23). Using Theorem 7 and the Fourier Inversion Theorem, we get the following inclusions for any LCA group G:

$$(\bar{A}^{p,q})_i(G) \subset L(p',q)(G) \cap (L^1 + L^2)(G), \qquad 1 < p < 2,\ 1 \leqslant q \leqslant \infty,\ 1 \leqslant i \leqslant 5, \tag{61}$$

$$(\bar{A}_{p,q})_i(G) \subset L(p',q)(G), \qquad 1 < p < 2,\ 1 \leqslant q \leqslant \infty,\ 1 \leqslant i \leqslant 5, \tag{62}$$

$$L(p',q)(G) \subset (\underline{A}^{p,q})_i(G), \qquad 2 < p < \infty,\ 1 \leqslant q \leqslant \infty,\ 1 \leqslant i \leqslant 5, \tag{63}$$

and

$$L(p',q)(G) \cap L^2(G) \subset (\underline{A}_{p,q})_i(G),\ 2 < p < \infty,\ 1 \leqslant q \leqslant \infty,\ 1 \leqslant i \leqslant 5. \tag{64}$$

The reversed inclusions for the equivalence relations under consideration (R_i with $i=1, 2, 4$) can be obtained in the following way. In order to prove the reversed inclusions for (61) and (62), we first construct pyramids satisfying property 1 in Lemmas 1–3, then rearrange the functions from the corresponding Lorentz spaces along the constructed pyramids, and finally apply Corollary 1.

Let us illustrate this by proving

$$L(p', q)(G) \subset (\bar{A}_{p,q})_2(G) \tag{65}$$

for a compact LCA group G, $1 < p < 2$, and $1 \leqslant q \leqslant \infty$.

Assume

$$f \in L(p', q)(G) \tag{66}$$

and consider the functions f_j in the definition of the equivalence relation R_2. Denote $s_j = m(x \in G: |f_j(x)| > 0)$ for $1 \leqslant j \leqslant 4$ and choose any non-negative numbers r_j such that $r_j \geqslant s_j$ and $r_1 + r_2 + r_3 + r_4 = 1$. By Lemma 2, there exist four r_j-pyramids P_j satisfying part 1 of Lemma 2. Let us rearrange the function $|f_j|$ along the pyramid R_j and denote the new function by f_j^*. Then the function

$$g = f_1^* - f_3^* + if_2^* - if_4^* \tag{67}$$

satisfies $gR_2 f$. Our next goal is to prove

$$g \in A_{p,q}(G). \tag{68}$$

Let us first prove

$$f_1^* \in A_{p,q}(G). \tag{69}$$

Denote $h_n(x) = \min\{n, f_1^*(x)\}$ for $n \geqslant 1$. Using the Fourier Inversion Theorem we get $h_n = F^{-1} l_n$ where $l_n = F h_n$. Now part 1 of Lemma 2 and Corollary 1 give

$$\|l_n - l_m\|_{L(p,q)(\Gamma)} \leqslant c \, \|h_n - h_m\|_{L(p',q)(G)}.$$

for $n > m$. Since (66) holds, the right side of the previous inequality tends to 0 as $n, m \to \infty$. Hence l_n tends to some function e belonging to the space $L(p, q)(\Gamma)$. It is clear that $f_1^* = F^{-1} e$ which proves (69). The same reasoning applies in the cases $j = 2, 3, 4$. This gives (68). Therefore, inclusion (65) is valid. Now (62) and (65) imply (14) in the case considered above. The remaining cases are similar. This establishes (12) and (14).

On the other hand, the proofs of the reversed inclusions for (63) and (64) are more complicated. They need the full-scale Lemmas 1–3. We will again consider the case of a compact group G, $2 < p < \infty$, $1 \leqslant q \leqslant \infty$, and $i = 2$ and prove the inclusion

$$(\underline{A}^{p,q})_2 (G) \subset L(p', q)(G). \tag{70}$$

Suppose $f \in (\underline{A}^{p,q})_2 (G)$. Then for every function $\tilde{f}$ such that $\tilde{f} R_2 f$ we have

$$\|F\tilde{f}\|_{L(p,q)(\Gamma)} < \infty. \tag{71}$$

By Lemma 2, there exist four non-intersecting r_j-pyramids P_j satisfying parts 1–3 of Lemma 2. Here r_j are some numbers independent of f and such that $r_1 + r_2 + r_3 + r_4 = 1$. Set $s_j = m(x \in G: |f_j(x)| > 0)$ where f_j denotes functions arising in the definition of the equivalence relation R_2. A natural idea here would be to use the rearrangements of functions $|f_j|$ along the pyramids P_j. However, this is possible only if $s_j \leqslant r_j$. Therefore, some more technical work is needed.

We consider the special case when $s_1 > r_1$, $s_2 \leqslant r_2$, $s_3 \leqslant r_3$, $s_4 \leqslant r_4$ and give the proof in this case. The remaining cases can be dealt with in the same way.

First we rearrange the functions $|f_j|$, $2 \leqslant j \leqslant 4$ along the corresponding pyramids P_j and get the monotonic rearrangements f_j^*, $2 \leqslant j \leqslant 4$. As we have already mentioned, we cannot rearrange $|f_1|$ along P_1 because $s_1 > r_1$. However, using the functions

$$g_{1,\alpha} = |f_1| \chi_{\{x: |f_1(x)| \geqslant \alpha\}}, \qquad h_{1,\alpha} = |f_1| \chi_{\{x: |f_1(x)| < \alpha\}}, \qquad \alpha > 0,$$

we can represent the function $|f_1|$ as follows

$$|f_1| = g_1 + h_1. \tag{72}$$

Here we have

$$\|h_1\|_{L(p',q)(G)} \leqslant c \, \|g_1\|_{L(p',q)(G)} \tag{73}$$

and

$$r_1 = m(x \in G: g_1(x) > 0). \tag{74}$$

Now we can rearrange the function g_1 along the pyramid P_1 and get the P_1-monotonic rearrangement g_1^*. As for the function h_1, we first consider a function h equimeasurable with h_1 and supported in the set

$$A_2 \cup A_3 \cup A_4 = (B(P_2) \backslash supp(f_2^*)) \cup (B(P_3) \backslash supp(f_3^*)) \cup (B(P_4) \backslash supp(f_4^*)).$$

Then we denote $l_j = h\chi_{A_j}$, $2 \leqslant j \leqslant 4$. It is clear that the function

$$f^* = g_1^* + l_2 + l_3 + l_4 - f_3^* + if_2^* - if_4^* \tag{75}$$

satisfies

$$fR_2 f^*. \tag{76}$$

By the multiplier property for the bases of the pyramids P_j (see part 2 of Lemma 2), we get

$$\|Fg_1^*\|_{L(p,q)(\Gamma)} \leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)}, \tag{77}$$

$$\|F(if_2^* + l_2)\|_{L(p,q)(\Gamma)} \leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)}, \tag{78}$$

$$\|F(-f_3^* + l_3)\|_{L(p,q)(\Gamma)} \leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)}, \tag{79}$$

and

$$\|F(-if_4^* + l_4)\|_{L(p,q)(\Gamma)} \leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)}. \tag{80}$$

Now it follows from (73), (78)–(80), and Theorem 7 that for $2 \leqslant j \leqslant 4$ we have

$$\begin{aligned} \|Ff_j^*\|_{L(p,q)(\Gamma)} &\leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)} + \|Fl_j\|_{L(p,q)(\Gamma)} \\ &\leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)} + c\, \|l_j\|_{L(p',q)(G)} \\ &\leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)} + c\, \|h\|_{L(p',q)(G)} \\ &\leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)} + c\, \|g_1^*\|_{L(p',q)(G)}. \end{aligned} \tag{81}$$

Our next goal is to prove that

$$\|g_1^*\|_{L(p',q)(G)} \leqslant c\, \|Fg_1^*\|_{L(p,q)(\Gamma)} \tag{82}$$

and

$$\|f_j^*\|_{L(p',q)(G)} \leqslant c\, \|Ff_j^*\|_{L(p,q)(\Gamma)} \tag{83}$$

for $2 \leqslant j \leqslant 4$. Then (71), (72), (75), (76), (77), (80), (82), and (83) give

$$\begin{aligned} \|f\|_{L(p',q)(G)} &\leqslant c\, \|g_1^*\|_{L(p',q)(G)} + \sum_{2 \leqslant j \leqslant 4} \|f_j^*\|_{L(p',q)(G)} \\ &\leqslant c\, \|Fg_1^*\|_{L(p,q)(\Gamma)} + c \sum_{2 \leqslant j \leqslant 4} \|Ff_j^*\|_{L(p,q)(\Gamma)} \\ &\leqslant c\, \|Ff^*\|_{L(p,q)(\Gamma)} < \infty. \end{aligned} \tag{84}$$

Therefore $f \in L(p', q)(G)$ and

$$\|f\|_{L(p',q)(G)} \leqslant \sup_{fR_2 g} \|Fg\|_{L(p,q)(\Gamma)}. \tag{85}$$

This proves inclusion (70).

Inequalities (83) and (84) can be obtained from the following lemma.

LEMMA 4. *Let G be a compact group and $P = \{E_t\}$ be some r-pyramid on G with $0 < r < 1$. Assume the pyramid P satisfies conditions* 1 *and* 3 *in Lemma* 2. *Then for every $2 < p < \infty$, $1 \leqslant q \leqslant \infty$, and every P-monotonic non-negative function $g \in (L^1 + L^2)(G)$ one has*

$$\|Fg\|_{L(p,q)(\Gamma)} < \infty \Rightarrow \|g\|_{L(p',q)(G)} < \infty$$

and

$$\|g\|_{L(p',q)(G)} \leqslant c\, \|Fg\|_{L(p,q)(\Gamma)}$$

with $c > 0$ independent of g.

Remark 2. Similar assertion holds with minor adjustments for the discrete groups and locally compact non-compact non-discrete groups.

Proof of Lemma 4. With no loss of generality we assume $0 \in \bigcap_{t>0} E_t$. Suppose g is a non-negative P-monotonic function on G satisfying

$$\|Fg\|_{L(p,q)(\Gamma)} < \infty. \tag{86}$$

With no loss of generality, we may restrict ourselves to the case

$$m(g = t) = 0 \tag{87}$$

for every $t > 0$ (use Theorem 7 to prove this). Next fix a number s with $1 < s < 2$ such that it is so close to 2 that the number α defined by

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{\alpha'} \tag{88}$$

satisfies $\alpha > 1$.

Let τ be a positive function on G such that the function $\tau\chi_{B(P)}$ is P-monotonic, $\tau(x) = 1$ for $x \in G \backslash B(P)$, and moreover the monotonic rearrangement of $\tau\chi_{B(P)}$ on $(0, r)$ is given by

$$\tau^*(t) = t^{-1/\alpha}. \tag{89}$$

Define a function $\tilde{\tau}$ by $\tilde{\tau}(x)=\tau(-x)$. Then it is clear that

$$\|\tau\|_{L(\alpha,\infty)(G)}=\|\tilde{\tau}\|_{L(\alpha,\infty)(G)}=c.$$

By Corollary 1, we have

$$\|F\tau\|_{L(\alpha',\infty)(\Gamma)}=\|F\tilde{\tau}\|_{L(\alpha',\infty)(\Gamma)}\leqslant c. \tag{90}$$

Using (88), (90), and Theorem 5, we get

$$\|F(g*(\tau+\tilde{\tau}))\|_{L(s,q)(\Gamma)}\leqslant c\ \|Fg\|_{L(p,q)(\Gamma)}. \tag{91}$$

Therefore by Theorem 7 applied to the inverse Fourier transform, we obtain from (91),

$$\|g*(\tau+\tilde{\tau})\|_{L(s',q)(G)}\leqslant c\ \|Fg\|_{L(p,q)(\Gamma)}. \tag{92}$$

For $x\in G$, denote $A_x=\{g\geqslant g(x)\}$. Then we have the following estimate

$$\begin{aligned}(g*(\tau+\tilde{\tau}))(x)&=\int_G[\tau(x-y)+\tilde{\tau}(x-y)]\ g(y)\ dm(y)\\ &\geqslant g(x)\int_{A_x}[\tau(x-y)+\tilde{\tau}(x-y)]\ dm(y)\\ &\geqslant g(x)\int_{x-A_x}[\tau(z)+\tilde{\tau}(z)]\ dm(z).\end{aligned} \tag{93}$$

The set A_x belongs to the pyramid P. Therefore, $A_x=E_t$ where $t=mA_x$. Let c be the constant in part 3 of Lemma 2. Then if the number t above satisfies $t<c^{-1}$, we have from (93) and part 3 of Lemma 2 that

$$(g*(\tau+\tilde{\tau}))(x)\geqslant g(x)\int_{(x-A_x)\cap(E_{ct}\cup(-E_{ct}))}[\tau(z)+\tilde{\tau}(z)]\ dm(z). \tag{94}$$

From (89), we get $\tau(z)\geqslant(ct)^{-1/\alpha}$ for $z\in E_{ct}$ and $\tilde{\tau}(z)\geqslant(ct)^{-1/\alpha}$ for $z\in(-E_{ct})$. Now (94) and part 3 of Lemma 2 give

$$(g*(\tau+\tilde{\tau}))(x)\geqslant\tilde{c}g(x)\ t^{1/\alpha'} \tag{95}$$

for $t=mA_x\leqslant c^{-1}$.

On the other hand, as $\tau(z)\geqslant c$ and $\tilde{\tau}(z)\geqslant c$ for $z\in G$, we get from (93)

$$(g*(\tau+\tilde{\tau}))(x)\geqslant\tilde{c}g(x)\ t^{1/\alpha'} \tag{96}$$

for $t > c^{-1}$. Finally, (95) in combination with (96) imply

$$(g * (\tau + \tilde{\tau}))(x) \geqslant \tilde{c} g(x)\, m(g \geqslant g(x))^{1/\alpha'} \tag{97}$$

for all $x \in G$.

Denote

$$H(x) = g(x)\, m(g \geqslant g(x))^{1/\alpha'}, \qquad J(x) = m(g \geqslant g(x))^{-(1/\alpha')}.$$

Then (92) and (97) give

$$\|H\|_{L(s', q)(G)} \leqslant c\, \|Fg\|_{L(p, q)(\Gamma)}. \tag{98}$$

Since $g(x) = H(x)\, J(x)$, we obtain from (88), (98), and Theorem 5 that

$$\|g\|_{L(p', q)(G)} \leqslant c\, \|J\|_{L(\alpha', \infty)(G)}\, \|Fg\|_{L(p, q)(\Gamma)}. \tag{99}$$

Our next goal is to prove that

$$\|J\|_{L(\alpha', \infty)(G)} \leqslant 1. \tag{100}$$

Using (87), we obtain for every $s > 0$ and every $x \in G$ that $m(g \geqslant g(x)) \leqslant s \Rightarrow g(x) \geqslant g^*(s)$ where g^* denotes the monotonic rearrangement of g on $[0, 1]$. Now using (87), we get

$$m\{x\colon m(g \geqslant g(x)) \leqslant s\} \leqslant m\{x\colon g(x) \geqslant g^*(x)\} = m\{t\colon g^*(t) \geqslant g^*(s)\} = s.$$

Therefore, $D(s; J) = m\{x\colon m(g \geqslant g(x)) \leqslant s^{-\alpha'}\} \leqslant s^{-\alpha'}$ and $J^*(t) \leqslant t^{-1/\alpha'}$. This gives (100). Now Lemma 4 follows from (99).

As we have already mentioned above, Lemma 4 proves (82) and (83). Hence, inclusion (70) holds. This in combination with (69) proves (19) for all compact groups G and $i = 2$. The case of a non-compact group and $i = 2$ is easier than that of a compact group. When we finish constructing four ∞-pyramids P_j in Lemmas 1 or 3, we consider the function g given by (67). Then we separate the functions f_j using the multiplier property for the bases of the pyramids given in part 2 of Lemmas 1 and 3. In the end, we use lemmas similar to Lemma 4.

Now we consider the case $i = 4$. Let us illustrate how to prove inclusion (70) for $i = 4$. Suppose $f \in (\underline{A}^{p, q})_4\,(G)$ and (71) holds for every $\tilde{f}$ with $\tilde{f} R_4 f$. As $Re(\tilde{f}) = 1/2(\tilde{f} + \bar{\tilde{f}})$, we have

$$\|F(Re(\tilde{f}))\|_{L(p, q)(\Gamma)} \leqslant c\, \|F\tilde{f}\|_{L(p, q)(\Gamma)}.$$

The class $\{Re(\tilde{f}): fR_4\tilde{f}\}$ coincides with the class $\{h: hR_2 Re(\tilde{f})\}$. Hence, we may apply the reasoning which was used in the case $i=2$ and conclude that

$$\|Re(f)\|_{L(p',q)(G)} \leqslant c \sup_{\{\tilde{f}: \tilde{f}R_4 f\}} \|F\tilde{f}\|_{L(p,q)(\Gamma)}.$$

Similar reasoning applies to $Im(f)$. This gives (70) for $i=4$ in the case of a compact group G. The remaining cases are similar.

Remark 3. Suppose $f, g \in (L^1+L^2)(G)$ are real functions such that fR_4g. Then for every $1<s<\infty$, $1 \leqslant q \leqslant \infty$, and $\varepsilon>0$ there exists a function h satisfying $hR_5 f$ and

$$\|h-g\|_{L(s,q)(G)} \leqslant \varepsilon.$$

This can be shown using approximation by simple functions and the metric equivalence of measurable sets of equal finite measure. After that we use the validity of formula (19), Theorem 7, and the previous fact with $s=p'$ in order to establish the case $i=3, 5$ for formula (19) for the real classes $A_r^{p,q}(G)$.

5. REARRANGEMENTS WITH UNIFORMLY SMALL FOURIER TRANSFOR, $\Lambda(p)$-SETS, AND REMAINING PROOFS

It has already been mentioned above that formula (15) for a compact group G and formula (18) for a discrete group G follow from definitions and Plancherel's theorem.

Proof of formulas (13) *and* (15) *for a discrete group G.* We prove formula (13) for a discrete group G and $i=5$. The remaining cases are similar. We use some ideas from the proof of Helgason's result in [H] and [HR1]. The inclusion $(\bar{A}^{p,q})_5(G) \subset (L^1+L^2)(G) = L^2(G)$ follows from definitions. In order to prove the reversed inclusion, we assume $f \in L^2(G)$ and denote $S=\{x \in G: f(x) \neq 0\}$. This set is at most countable. By Theorem 14, there exists a countable set $\Lambda \subset G$ which is a $\Lambda(t)$-set for all $1<t<\infty$. Let us first consider the case when both sets $S \backslash \Lambda$ and $\Lambda \backslash S$ are infinite. Then we may construct a permutation $\pi: G \to G$ such that

$$S \subset \pi(\Lambda). \tag{101}$$

The permutation π in (101) coincides with the identity mapping on the set $[G \backslash (S \cup \Lambda)] \cup (S \cap \Lambda)$ and is a one-to-one mapping between Λ and S. It is clear that $supp(f\circ\pi)\subset\Lambda$ and hence, by (28) and the interpolation theorem for the Lorentz spaces, we get

$$\|F^{-1}(f\circ\pi)\|_{L(p,q)(\Gamma)}\leqslant c_{p,q}\,\|F^{-1}(f\circ\pi)\|_{L^2(\Gamma)}=c_{p,q}\,\|f\|_{L^2(G)}.$$

Taking the conjugates, we complete the proof in the case under consideration.

In the case when the set $S\backslash\Lambda$ is finite, we may take the identity mapping on G as the permutation π.

The case when the set $\Lambda\backslash S$ is finite and the set $S\backslash\Lambda$ is infinite is slightly more complicated. In this case, the set $\Lambda\cap S$ is infinite. First we find an infinite set $Q\subset\Lambda\cap S$ such that

$$\|f\chi_Q\|_{L(p',q)(G)}\leqslant\|f\|_{L^2(G)}. \tag{102}$$

Then we construct a permutation $\pi\colon G\to G$ such that π is the identity mapping on the set $G\backslash(Q\cup(S\backslash\Lambda))$ and is a one-to-one mapping of Q onto $S\backslash\Lambda$. Using the properties of $\Lambda(t)$-sets, inequality (102), Theorem 7, and the inclusion $S\backslash Q\subset\pi(\Lambda\cap S)$, we get

$$\begin{aligned}&\|F^{-1}(f\circ\pi)\|_{L(p,q)(\Gamma)}\\&\quad\leqslant\|F^{-1}[(f\circ\pi)\,\chi_{\pi^{-1}(Q)}]\|_{L(p,q)(\Gamma)}+\|F^{-1}[(f\circ\pi)\,\chi_{S\backslash\pi^{-1}(Q)}]\|_{L(p,q)(\Gamma)}\\&\quad\leqslant c\,\|f\|_{L^2(G)}.\end{aligned}$$

Taking conjugates, we get the same result with F instead of F^{-1}.

We have shown that

$$\inf_{\pi}\|F(f\circ\pi)\|_{L(p,q)(\Gamma)}\leqslant c\,\|f\|_{L^2(G)}.$$

This completes the proof of (13) in the discrete case.

Next we sketch the proof of Helgason's result (7) and also give a corresponding norm estimate. Suppose f is such that $f\circ\pi=F^{-1}g_\pi$ for every permutation $\pi\colon G\to G$ and some $g_\pi\in L^1(\Gamma)$. Let S and Λ be the sets from the previous proof and consider the three cases involving cardinality of the sets $S\backslash\Lambda$ and $\Lambda\backslash S$ as above. Then reasoning as in the previous proof we get Helgason's theorem with the following norm estimate

$$\|f\|_{L^2(G)}\leqslant c\sup_{gR_if}\|Fg\|_{L^1(\Gamma)}. \tag{103}$$

For example, if the sets $S\backslash\Lambda$ and $\Lambda\backslash S$ are both infinite, we construct a permutation π as above and get

$$\|f\|_{L^2(G)}\leqslant c\,\|g_\pi\|_{L^2(\Gamma)}.$$

This shows that (103) holds. In the third case of the previous proof, we use the inequality

$$\|f\chi_Q\|_{L^2(G)} \leqslant \sup_{\pi} \|g_\pi\|_{L^1(\Gamma)} \tag{104}$$

and the requirement for the left side of (104) to be finite instead of (102) in the construction of the auxiliary set Q.

Using Helgason's theorem, we easily get formula (22) in Theorem 4.

Now we formulate two lemmas which have independent interest. The first of them is concerned with uniformly small Fourier transforms of rearrangements.

LEMMA 5. *Let G be a non-discrete LCA group. Suppose $h \in L^1(G)$, $g \in L^\infty(G)$, and $\int_G h\, dm_G = 0$. Then for every $\varepsilon > 0$ there exists a Haar measure preserving invertible mod 0 transformation ω_ε: supp$(h) \to$ supp(h) such that*

$$\|F(g(h \circ \omega_\varepsilon))\|_{L^\infty(\Gamma)} \leqslant \varepsilon. \tag{105}$$

LEMMA 6. *Let $2 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Assume also that all conditions in Lemma 5 hold. Then the conclusion of Lemma 5 holds with the estimate*

$$\|F(g(h \circ \omega_\varepsilon))\|_{L(p,q)(\Gamma)} \leqslant \varepsilon. \tag{106}$$

instead of (105).

We get the following corollaries from (105) and (106):

COROLLARY 2. *Assume G is a compact abelian group. Then for every $f \in L^1(G)$ we have*

$$\inf_{hR_5 f} \|Fh\|_{L^\infty(\Gamma)} = \left|\int_G f\, dm_G\right|. \tag{107}$$

COROLLARY 3. *Assume G is a compact abelian group. Then for every $f \in L^1(G)$, $2 < p < \infty$, and $1 \leqslant q \leqslant \infty$, we have*

$$\inf_{hR_5 f} \|Fh\|_{L(p,q)(\Gamma)} = \left|\int_G f\, dm_G\right|. \tag{108}$$

We first show how to derive Lemma 6 from Lemma 5. Consider a representation $h = \sum_{j=1}^{\infty} h_j$ such that $h_j \in L^2(G)$, $\int_G h_j\, dm_G = 0$ for every $j \geqslant 1$, and the supports of h_j are pairwise disjoint. It is not difficult to show that such a representation exists. Next take h_j instead of the function h in Lemma 5 and apply this lemma with $\varepsilon = \varepsilon_j$. We get

$$\|F(g(h_j \circ \omega_j))\|_{L^\infty(\Gamma)} \leqslant \varepsilon_j$$

for some ω_j: $supp(h_j) \to supp(h_j)$. Now define a transformation ω: $G \to G$ as follows. It coincides with the identity transformation on $G \backslash (supp(h))$ and with the transformation ω_j on $supp(h_j)$. Then it is clear that

$$\begin{aligned}
\|F(g(h \circ \omega))\|_{L(p,q)(\Gamma)} &\leqslant \sum_{j \geqslant 1} \|F(g(h_j \circ \omega_j))\|_{L(p,q)(\Gamma)} \\
&\leqslant c \sum_{j \geqslant 1} \|F(g(h_j \circ \omega_j))\|_{L^2(\Gamma)}^{2/p} \|F(g(h_j \circ \omega_j))\|_{L^\infty(\Gamma)}^{1-(2/p)} \\
&\leqslant c \sum_{j \geqslant 1} \|gh_j\|_{L^2(G)}^{2/p} \varepsilon_j^{1-(2/p)}. \qquad (109)
\end{aligned}$$

Since (109) holds with any numbers ε_j, we get Lemma 6 from (109).

Corollary 2 follows from Lemma 5 if we consider a function $l = f - \int_G f \, dm_G$ and apply Lemma 5 to $h = l$ and $g = 1$. We also need an easy inequality

$$\|Fh\|_{L(p,q)(\Gamma)} \geqslant \left| \int_G h \, dm_G \right|$$

to finish the proof. Corollary 3 follows from Lemma 6 in a similar way.

Proof of Lemma 5. Using regularity of the Haar measure and approximating the function h in $L^1(G)$ by simple functions, we may reduce the lemma to the case when h is a simple compactly supported function

$$h(x) = \sum_{m=1}^{M} c_m \chi_{A_m}(x). \qquad (110)$$

Here c_m are some non-zero complex numbers, the sets A_m have compact closure and do not intersect, and

$$\int h \, dm_G = 0. \qquad (111)$$

We will assume in the rest of the proof that h is given by (110) and prove Lemma 5 for such a function.

Denote $A=\bigcup_{m=1}^{M} A_m$. Since the Haar measure of G is non-atomic, we may construct a sequence $\pi_j=\{E_1^j, ..., E_{J(j)}^j\}$ of measurable partitions of A such that π_{j+1} is a refinement of π_j and

$$\lim_{j\to\infty} T_j = \lim_{j\to\infty} \max_{1\leqslant l\leqslant J(j)} \{m_G(E_l^j)\} = 0. \tag{112}$$

For each $1\leqslant j<\infty$ and $1\leqslant l\leqslant J(j)$, fix a function

$$\alpha_{j,l} = \sum_{m=1}^{M} c_m \chi_{E_{l,m}^j} \tag{113}$$

where $\{E_{l,1}^j, ..., E_{l,M}^j\}$ is a fixed measurable partition of the set E_l^j such that

$$m_G(E_{l,m}^j) = \frac{m_G(A_m)\, m_G(E_l^j)}{m_G(A)}.$$

It follows from (111) that

$$\int \alpha_{j,l}\, dm_G = 0. \tag{114}$$

Consider a function class U on A defined in the following way. A function u is in U iff there exists an integer s and a finite sequence of integers n_l with $1\leqslant l\leqslant J(n)$ such that $s\leqslant n_1<\cdots<n_{J(s)}$ and

$$u=\sum_{k=1}^{J(s)} \alpha_k. \tag{115}$$

In (115), the function α_k coincides with the sum of functions of form (113) corresponding to all elements of the partition π_{n_k} which are subsets of the set E_k^s. It is clear that if $u\in U$, then $hR_5 u$.

For every L^1-function τ on A, denote by $Pr_j(\tau)$ the projection

$$Pr_j(\tau) = \sum_{l=1}^{J(j)} \frac{1}{m_G(E_l^j)} \left(\int_{E_l^j} \tau\, dm_G\right) \chi_{E_l^j}$$

of τ into the space of Λ-measurable L^1-functions on A where Λ denotes the σ-algebra generated by the partitions π_j with $1\leqslant j<\infty$. It is known that

$$\lim_{j\to\infty} \|\Sigma(\tau) - Pr_j(\tau)\|_{L^1(A)} = 0 \tag{116}$$

where Σ is the conditional expectation with respect to Λ and the convergence in formula (116) is uniform with respect to any compact subset of $L^1(A)$ (see [DU]).

Since the class U is countable, the Fourier transforms Fu of all functions $u \in U$ are supported in a σ-compact set $B \subset \Gamma$. Fix a representation $B = \bigcup_{k=1}^{\infty} B_k$ where B_k is a non-decreasing sequence of compact subsets of Γ such that for every $\varepsilon > 0$ and $u \in U$ there is a number s such that $|Fu(\gamma)| \leqslant \varepsilon$ for all $\gamma \in B \backslash B_s$.

The following formula follows from the definition of the conditional expectation Σ:

$$\int_W \Sigma(\bar{\gamma} g)\, dm_G = \int_W \bar{\gamma} g\, dm_G \tag{117}$$

for every $\gamma \in \Gamma$ and $W \in \Lambda$. Since the compactness of B_k in Γ implies that of the set $\{g\bar{\gamma}: \gamma \in B_k\}$ in the space $L^1(\Lambda)$ (this follows from 1.2.6 in [R]), we may apply (116) and (117) and get

$$\lim_{j \to \infty} \sup_{\gamma \in B_k} \sup_{u \in U} \left| F(gu)(\gamma) - \int [Pr_j(\bar{\gamma} g)]\, u\, dm_G \right| = 0 \tag{118}$$

for every $u \in U$ and $k \geqslant 1$.

Let $\varepsilon > 0$. It follows from (118) that there exists an increasing sequence of integers $\{j_k\}$ such that

$$\sup_{\gamma \in B_k} \sup_{u \in U} \left| F(gu)(\gamma) - \int [Pr_{j_k}(\bar{\gamma} g)]\, u\, dm_G \right| \leqslant \frac{\varepsilon}{2}.$$

We conclude from this inequality that it is sufficient to prove that for every $\delta > 0$

$$\sup_{\gamma \in B} |\tilde{F}(u)(\gamma)| \leqslant \delta \tag{119}$$

where

$$\tilde{F}(u)(\gamma) = \begin{cases} \int [Pr_{j_1}(\bar{\gamma} g)]\, u\, dm_G & \text{if} \quad \gamma \in B_1 \\ \int [Pr_{j_k}(\bar{\gamma} g)]\, u\, dm_G & \text{if} \quad \gamma \in B_k \backslash B_{k-1}, \quad k \geqslant 2 \end{cases}$$

Let $\delta > 0$. Our goal is to find integers $s, n_1, ..., n_{J(s)}$ such that the corresponding function u_δ satisfies

$$\|\tilde{F}(u_\delta)\|_{L^\infty(B)} \leqslant \delta. \tag{120}$$

We choose the integer s so that

$$T_s \leqslant \frac{\delta}{2\, \|h\|_{L^\infty(G)}\, \|g\|_{L^\infty(G)}} \tag{121}$$

where h is defined in (110). This can be done by (112). Then we define the numbers $\{n_k\}$ by induction in the following way. We set $n_1 = s$. This determines the function α_1 in (115). Next we choose $n_2 > n_1$ such that $|\tilde{F}(\alpha_1)(\gamma)| \leqslant \delta/2$ for all $\gamma \in B \backslash B_{n_2}$. In general, we choose $n_{l+1} > n_l$ so that

$$|\tilde{F}(\alpha_1 + \cdots + \alpha_l)(\gamma)| \leqslant \frac{\delta}{2} \tag{122}$$

for all $\gamma \in B \backslash B_{n_{l+1}}$ and $1 \leqslant l \leqslant J(s) - 1$.

Now we define a function $u \in U$ as in (115) and check that it satisfies (120). Let $\gamma \in B_{n_{l+1}} \backslash B_{n_l}$ for some $1 \leqslant l \leqslant J(s) - 1$. In order to prove (120), we use the representation

$$u = u_1 + u_2 + u_3 = \sum_{k=1}^{l-1} \alpha_k + \alpha_l + \sum_{k=l+1}^{J(s)} \alpha_k.$$

Then we use (122) to estimate $\|\tilde{F}(u_1)\|_{L^\infty(B)}$ and (121) to estimate $\|\tilde{F}(u_2)\|_{L^\infty(B)}$. Finally, it follows from the definitions that $\tilde{F}(u_3)(\gamma) = 0$ for all $\gamma \in B_{n_{l+1}} \backslash B_{n_l}$. This allows is to prove (119) and this finish the proof of Lemma 5.

Proof of formula (13) *for a compact group G.* Formula (108) implies (13) with $i = 5$ for a compact group G. The cases $1 \leqslant i \leqslant 4$ follow easily.

Proof of formulas (16), (17), (20), *and* (21). Let us prove formula (16) with $i = 5$. The proof of the rest of the formulas listed above is similar.

Suppose G is a non-compact non-discrete LCA group and a function $f \in (L^1 + L^2)(G)$ is given such that

$$\|Fh\|_{L(p,\,q)(\Gamma)} < \infty \tag{123}$$

for some $1 < p < 2$, $1 \leqslant q \leqslant \infty$, and all h with $hR_5 f$. Our goal is to prove that this implies $f = 0$.

Assume $f \neq 0$. Then there exists a compact set $K \subset G$ of positive measure such that f is not an identically constant function on K. It follows that there exist two non-intersecting sets $K_1, K_2 \subset K$ for which $m_G(K_1) = m_G(K_2) = \delta$ and $\int_{K_1} f\, dm_G \neq \int_{K_2} f\, dm_G$. Now we can find a small positive number ρ such that

$$\int_{S_1} f\, dm_G \neq \int_{S_2} f\, dm_G \tag{124}$$

for any $S_j \subset K_j$ with $m_G(K_j \backslash S_j) < \rho$ and $j = 1, 2$.

Consider a sequence of functions $\tau_n(x)=\chi_{E_n}(x)-\alpha_n\chi_{H_n}(x)$ such that the sets E_n and H_n do not intersect, $E_n\cup H_n=K$, and

$$\sum m_G(H_n)\leqslant\rho/2. \tag{125}$$

The constants α_n above were chosen in such a way that $\int_G\tau_n\,dm_G=0$. Applying Lemma 6 to $h=\tau_n$, $g=\chi_{K_1}$, and to a sequence ε_n tending to 0, we see that there exists $\omega_n\colon K\to K$ such that

$$\|F(\chi_{K_1}(\tau_n\circ\omega_n))\|_{L(p,\,q)(\Gamma)}\leqslant\varepsilon_n \tag{126}$$

for every $n\geqslant 1$.

We have $\tilde{\tau}_n=\tau\circ\omega_n=\chi_{M_n}-\alpha_n\chi_{P_n}$ where $M_n=\omega_n^{-1}(E_n)$ and $P_n=\omega_n^{-1}(H_n)$. Denote $A=\bigcup P_n$. Then (125) gives

$$m_G(A)\leqslant\rho/2. \tag{127}$$

Consider the sets $K_3=K_1\backslash A$ and $K_4=K_2\backslash A$. Using (127) and changing one of the sets a little bit, we construct sets $S_1\subset K_1$ and $S_2\subset K_2$ such that (124) holds and moreover $m_G(S_1)=m_G(S_2)$.

Next we define a Haar measure preserving invertible transformation $\omega\colon G\to G$ as follows. It coincides with the identity transformation on the set $G\backslash(S_1\cup S_2)$ and has the property $\omega(S_1)=S_2$, $\omega(S_2)=S_1$.

It is known that the following generalization of the Parseval identity holds (see [HR1], p. 249). For every $h_1\in L^p(G)$, $H_2\in L^p(\Gamma)$ with $1\leqslant p\leqslant 2$,

$$\int_G h_1\overline{(F^{-1}h_2)}\,dm_G=\int_\Gamma (Fh_1)\,\overline{h_2}\,dm_\Gamma. \tag{128}$$

Applying (128) with $h_1=\chi_{K_1}\tilde{\tau}_n$ and $h_2=Ff-F(f\circ\omega)$, we get

$$\int_G\overline{(f-f\circ\omega)}\,\chi_{K_1}\tilde{\tau}_n\,dm_G=\int_\Gamma F(\chi_{K_1}\tilde{\tau}_n)\overline{(Ff-F(f\circ\omega))}\,dm_\Gamma. \tag{129}$$

Denote the left side of (129) by I. We have $\tilde{\tau}_n(x)=1$ for $x\in S_1$. Therefore, using (124) and the definition of ω, we get

$$I=\int_{S_1}\overline{f-f\circ\omega}\,dm_G=\overline{\int_{S_1}f\,dm_G}-\overline{\int_{S_2}f\,dm_G}\neq 0. \tag{130}$$

On the other hand, (129), (123), (126) and Hölder's inequality for the Lorentz spaces (see [Hu]) give

$$|I|\leqslant\|F(\chi_{K_1}\tilde{\tau})\|_{L(p',\,q')(\Gamma)}\,\|Ff-F(f\circ\omega)\|_{L(p,\,q)(\Gamma)}\leqslant M\varepsilon_n$$

for all $n \geqslant 1$. This inequality contradicts (130). Therefore, $f = 0$. In the case when the group G is compact, the conclusion is $f = const$. This completes the proof of the formulas under consideration.

Proof of formula (15) *for a non-compact, non-discrete group G.* Let us consider the case $i = 5$. The remaining cases follow from this one. It follows from definitions, that

$$(\bar{A}_{p,q})_5 (G) \subset L^2(G). \tag{131}$$

In order to prove the reversed inclusion, we use the structure theorem (Theorem 12). Assume G contains an open subgroup S topologically isomorphic with $R^n \oplus H$ where H is a compact group and $n > 0$. Let $f \in L^2(G)$ and consider a representation

$$supp(f) = \bigcup a_j \tag{132}$$

where the sets A_j are pairwise disjoint and $0 < m_G(A_j) < \infty$. Then we can choose nonintersecting sets $B_j \subset S$ such that $m_G(B_j) = m_G(A_j)$ for all $j \geqslant 1$. If the sets $supp(f)$ and $B = \bigcup B_j$ do not intersect, we can define a transformation $\omega: G \to G$ in the following way. It coincides with the identity transformation on the set $G \backslash (supp(f) \cup B)$ and with some measure preserving invertible transformation: $\omega_j: B_j \to supp(f)$ on B_j. It follows that $supp(f \circ \omega) \subset S$.

Suppose the sets $supp(f)$ and B intersect. Then we use a similar but more complicated reasoning involving auxiliary functions with small $L(p', q)(G)$-norm and Theorem 7 (see the beginning of Section 5) and use formula (26) to show that we may restrict ourselves to the following two cases. One of them is $G = R^n \oplus H$ with a compact group H and $n \geqslant 1$ while the other one is the case when G contains a compact open subgroup K.

Assume first that $G = R^n \oplus H$ with $n \geqslant 1$. The group Z^n is topologically isomorphic to a discrete subgroup K of R^n. By Theorem 14, there exists an infinite set $Q \subset K$ which is a $\Lambda(t)$-set for all $1 < t < \infty$. Consider the cubes $C_l = \{ y = (y_1, ..., y_n): l_j \leqslant y_j \leqslant l_j + 1,\ \ 1 \leqslant j \leqslant n \}$ in R^n such that $l = (l_1, ..., l_n) \in Q$ where Q is as above and denote

$$B = \bigcup_{l \in Q} (C_l \times H). \tag{133}$$

Let $f \in L^2(G)$. We may restrict ourselves to the case $supp(f) \subset B$. Indeed, if the set $supp(f)$ does not intersect the set B defined in (133), then using representation (132) and choosing the sets B_j in B for which $m_G(B_j) = m_G(A_j)$, we may construct the transformation $\omega: G \to G$ as above and see that $supp(f\circ\omega)\subset B$. If the sets $supp(f)$ and B intersect, we may use auxiliary functions with small $L(p', q)(G)$-norm and Theorem 7 as above to conclude that with no loss of generality we may assume $supp(f)\subset B$ where B is given by (133). Now we only need to prove that every such function satisfies

$$\|Ff\|_{L(p,q)(\Gamma)}\leqslant c\,\|f\|_{L^2(G)}. \tag{134}$$

In order to this, assume $supp(f)\subset B$ holds. Denote $A_l=C_l\times H$ where $l\in Q$ and set $g=\sum_{l\in Q}\alpha_l\chi_{A_l}$ with

$$\alpha_l=\int_{A_l} f\,dm_G.$$

Then we have

$$\int_{A_l}(f-g)\,dm_G=0$$

for every $l\in Q$. Fix ε_l such that $\sum\varepsilon_l\leqslant\|f\|_{L^2(G)}$. By Lemma 5, we may find measure preserving invertible transformations ω_l: $A_l\to A_l$ such that

$$\|F[(f\circ\omega_l-\alpha_l)\,\chi_{A_l}]\|_{L(p,q)(\Gamma)}\leqslant\varepsilon_l. \tag{135}$$

Now we define a transformation ω: $G\to G$ in the following way. It coincides with the identity transformation outside B and with ω_l on A_l. Now (135) gives

$$\|F(f\circ\omega-g\circ\omega)\|_{L(p,q)(\Gamma)}\leqslant\|f\|_{L^2(G)}. \tag{136}$$

Denote the dual group of H by H^*. Then we have for $\xi\in R^n$ and $\delta\in H^*$,

$$\begin{aligned}F(g\circ\omega)(\xi,\delta)=Fg(\xi,\delta)&=\sum_{l\in Q}\alpha_l F\chi_{A_l}(\chi,\delta)\\&=\sum_{l\in Q}\alpha_l e^{2\pi i l\cdot\xi}F\chi_{A_0}(\xi,\delta)\\&=\sum_{l\in Q}\alpha_l e^{2\pi i l\cdot\xi}\int_{C_0}e^{-2\pi i\cdot x}\,dx\int_H\bar{\delta}(y)\,dm_H(y).\end{aligned}$$

Therefore,

$$|F(g\circ\omega)(\xi,\delta)|=\frac{|\sin\pi\xi_1|\cdots|\sin\pi\xi_n|}{\pi^n\,|\xi_1\cdots\xi_n|}\left|\sum_{l\in Q}\alpha_l e^{2\pi i l\cdot\xi}\right|$$

for $\delta = 0$ and $F(g \circ \omega)(\xi, \delta) = 0$ otherwise. Thus we have for every $t > 1$,

$$\begin{aligned}\|F(g\omega)\|_{L^t(\Gamma)} &= \left\{\int_{R^n} |F(g \circ \omega)(\xi, 0)|^t \, d\xi\right\}^{1/t} \\ &= \left\{\int_{C_0} \sum_{k \in Z^n} |F(g \circ \omega)(\xi + k, 0)|^t \, d\xi\right\}^{1/t} \\ &\leqslant \left\{c_t \int_{C_0} \left|\sum_{l \in Q} \alpha_l e^{2\pi i l \cdot \xi}\right|^t d\xi\right\}^{1/t}.\end{aligned}$$

Using the definition of $\Lambda(t)$-sets and the interpolation theorem for the Lorentz spaces, we obtain

$$\|F(g \circ \omega)\|_{L(p,q)(\Gamma)} \leqslant c \left\{\sum_{l \in Q} |\alpha_l|^2\right\}^{1/2} \leqslant c \, \|f\|_{L^2(G)}.$$

Now we see from (136) and the previous inequality that the proof in this case is completed.

Next we consider the case when the group G has a compact open subgroup H. Then the quotient group G/H is discrete and thus contains a countable set Q that is a $\Lambda(t)$-set for all $t > 1$ by Theorem 14. Denote by π the standard homomorphism of H onto G and set

$$B = \bigcup_{\xi_j \in Q} \pi^{-1}(\xi_j).$$

The set B is simply the union of all cosets belonging to Q. We may restrict ourselves to the case $f \in L^2(G)$ with $supp(f) \subset B$ as in the previous proof. For such a function f, we set $g = \sum_{\xi_j \in Q} \alpha_j \chi_{\pi^{-1}(\xi_j)}$ with

$$\alpha_j = \int_{\pi^{-1}(\xi_j)} f \, dm_G.$$

Then we define a transformation ω exactly as above and get (136). After that we use formula (26) and obtain

$$\begin{aligned}F(g \circ \omega)(\gamma) &= \int_{G/H} dm_{G/H}(\xi) \, \bar{\gamma}(\xi) \int_H g(x + \xi) \, \bar{\gamma}(x) \, dm_H(x) \\ &= \int_{G/H} dm_{G/H}(\xi) \, \bar{\gamma}(\xi) \sum_{\xi_j \in Q} \alpha_j \int_H \chi_{\xi_j}(x + \xi) \, \bar{\gamma}(x) \, dm_H(x).\end{aligned}$$

It follows that if γ belongs to the annihilator H° of H, then

$$F(g\circ\omega)(\gamma)=\sum_{\xi_j\in Q}\alpha_j\bar{\gamma}(\xi_j).$$

Also, we have $F(g\circ\omega)(\gamma)=0$ for γ outside H°. Finally we get, using the properties of the $\Lambda(t)$-sets,

$$\|F(g\circ\omega)\|_{L(p,\,q)(\Gamma)}=\|F(g\circ\omega)\|_{L(p,\,q)(H^\circ)} \leqslant c\left\{\sum_{\xi_j\in Q}|\alpha_j|^2\right\}^{1/2}\leqslant c\,\|f\|_{L^2(G)}.$$

Now the proof can be completed as in the previous case.

Finally, formula (13) for a non-compact non-discrete group G can be obtained by combining the previous proof and the proof of formula (13) for a compact group.

This ends the proof of Theorems 1–4.

Remark 4. This remark concerns norm estimates related to Theorems 1–4. We have already mentioned some of them (see (85), (103), and (107)). Let us formulate some more. We have the following estimate for all functions $f\in(L^1+L^2)(G)$:

$$c_1\,\|f\|_{L(p',\,q)(G)}\leqslant I_i^{p,\,q}(f)\leqslant c_2\,\|f\|_{L(p',\,q)(G)}$$

for $1<p<2$, $1\leqslant q\leqslant\infty$, and $i=1, 2, 4$. This inequality corresponds to part (i) of Theorem 1. Part (ii) of Theorem 2 has the following estimate

$$c_1\,\|f\|_{L(p',\,q)(G)}\leqslant I_{p,\,q,\,i}(f)\leqslant c_2\,\|f\|_{L(p',\,q)(G)}$$

for $1<p<2$, $1\leqslant q\leqslant\infty$, and $i=1, 2, 4$. In the case of kernels, we have for any $f\in(L^1+L^2)(G)$,

$$c_1\,\|f\|_{L^2(G)}\leqslant S_i^{p,\,q}(f)\leqslant c_2\,\|f\|_{L^2(G)}$$

for $1<p<2$, $1\leqslant q\leqslant\infty$, and $1\leqslant i\leqslant 5$. This corresponds to part (iii) of Theorem 3. More precisely, if $f\in(\underline{A}^{p,\,q})_i(G)$, then $S_i^{p,\,q}(f)$ is finite and the first inequality in the previous formula holds. A similar formula holds for part (iii) of Theorem 4.

Let us recall that the only case when we can give a precise formula and not only an estimate is the case of a compact abelian group G, $2 < p < \infty$, and $1 \leqslant i \leqslant 5$. Then we have

$$I_i^{p,q}(f) = \left| \int_G f \, dm_G \right|$$

(see Corollary 2).

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