Very Cuspidal Representations of $p$-Adic Symplectic Groups

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INTRODUCTION

Let $n$ be an integer greater than or equal to 2 and $F$ a non-archimedean local field of residue characteristic $p$. Howe [13] constructed supercuspidal representations of $GL_n(F)$ in the tamely ramified case where $n$ is prime to $p$. Carayol [8] introduced the notion of very cuspidal representations of open compact subgroups modulo the center of $GL_n(F)$ for arbitrary $n$ and proved that, by induction, such representations give supercuspidal representations of $GL_n(F)$. Following the method of Howe [13], Morris [17, 18] gave classes of supercuspidal representations, associated to certain tamely ramified anisotropic maximal $F$-tori, of certain classical groups containing a symplectic group $Sp_{2n}(F)$, where $p > 2$ is assumed. Bushnell and Kutzko [7] determined the admissible dual of $GL_n(F)$. In [7], the result of [8] plays an important role. In this paper, we generalize [8] for $GL_n(F)$ to get the analogues for $Sp_{2n}(F)$: constructing classes of supercuspidal representations of $Sp_{2n}(F)$ from very cuspidal representations, associated to certain arbitrarily ramified anisotropic maximal $F$-tori, and computing their formal degrees. These supercuspidal representations contain similar tamely ramified supercuspidal representations of $Sp_{2n}(F)$ which are of level 1 and associated to cuspidal data of rank 1 in [16, 17]. Further, they give examples of anisotropic unrefined minimal $K$-types, due to Moy and Prasad [20, 21] for $Sp_{2n}(F)$.

The contents of this paper are as follows: In Section 1, we define the symplectic group $G = Sp_{2n}(F)$ from a nondegenerate alternating form on a vector space $X$ over $F$, and by Bruhat–Tits theory [3–5], we construct a flag $\mathcal{L}_0 = \{L_i\}_{i \in \mathbb{Z}}$ of $\mathfrak{sl}$-lattices in $X$ associated with a proper subset $\theta$ of a
basis $\tilde{\Pi}$ of the affine root system of $G$, where $\mathfrak{o}$ denotes the maximal order of $F$. From this $\mathfrak{Z}_g$, we get a hereditary $\mathfrak{o}$-order $\mathfrak{A}_g$ of $M_{2n}(F)$, the ring of $2n \times 2n$ matrices with entries in $F$, and its Jacobson radical $\mathfrak{P}_g$, in an obvious manner (cf. [23]). The parahoric subgroup $P = P(\theta)$ of $G$ and its filtration $\{P_i\}_{i \geq 1}$ are defined by $P = G \cap \mathfrak{A}_g$ and $P_i = (1 + \mathfrak{P}_g^i) \cap G$ for $i \geq 1$ (cf. [8, 6, 7, 17, 18]). In Section 2, we determine a parahoric subgroup $P = P(\theta)$ of $G$ and an integer $M$ such that $\mathfrak{P}_g^M \cap \mathfrak{P}$ contains an elliptic regular semisimple element $\alpha$ of a simple block form (cf. 2.6 and 2.7), where $\mathfrak{P}$ denotes the Lie algebra of $G$. For such $P = P(\theta)$ and $M$, we define a very cuspidal element in $\mathfrak{p}$ which is an analogue of an $(s)$-cuspidal element in [8, 3.2] and also a principal element in [17, 2.23]. In Section 3, we define a very cuspidal representation $\rho$ of $P = P(\theta)$ in terms of very cuspidal elements in $\mathfrak{p}$. We prove that the representation $\pi$ of $G$ induced from $\rho$ is irreducible and supercuspidal. In Section 4, we compute the degree $\deg(\rho)$. Further, we see that $\deg(\rho)$ equals the formal degree $d(\pi)$ of $\pi$, with the volume of $P$ one, and hence get the value of $d(\pi)$.

Notations. Let $\mathbb{Z}$ and $\mathbb{R}$ be the ring of integers and the field of real numbers, respectively. Let $F$ be a non-archimedean local field of residue characteristic $p > 2$. Let $\mathfrak{o}$ and $\mathfrak{p}$ be the maximal order of $F$ and the maximal ideal of $\mathfrak{p}$, respectively, and $\omega$ a prime element of $\mathfrak{p}$. Denote the residue class field of $F$ by $\overline{F} = \mathfrak{o}/\mathfrak{p}$, and the order of $\overline{F}$ by $q$. Let $R$ be a ring. For integers $n, m \geq 1$, denote the ring of $n \times n$ matrices (resp. $n \times m$ matrices) with entries in $R$ by $M_n(R)$ (resp. $M_{n,m}(R)$). Denote the multiplicative group of invertible elements in $R$ by $R^\times$. Finally, $|S|$ stands for the number of elements of a finite set $S$.

1. PARAHORIC SUBGROUPS

1.1. Let $n$ be an integer greater than or equal to 2 and $X$ a vector space of dimension $2n$ over $F$. Let $e_{-1}, \ldots, e_{-n}, e_1, \ldots, e_1$ be a basis of $X$ over $F$ and put $I = \{\pm 1, \ldots, \pm n\}$. By this basis, we identify $X$ with $F^{2n}$, the $2n$ copies of $F$. Define an alternating form $f$ on $X = F^{2n}$ by

$$f(x, y) = \sum_{i=1}^{n} (x_{-i}y_i - x_iy_{-i})$$

for $x = (x_{-1}, \ldots, x_{-n}, x_1, \ldots, x_1)$, $y = (y_{-1}, \ldots, y_{-n}, y_n, \ldots, y_1) \in X = F^{2n}$.

Let $G = \text{Sp}_{2n}$ be the symplectic group defined by this form $f$. This is an algebraic group defined over $F$. Let $\mathfrak{g}$ be the Lie algebra of $G$. Denote the group of $F$-rational points in $G$ by $G = G(F)$, and the algebra of $F$-rational points in $\mathfrak{g}$ by $\mathfrak{g} = \mathfrak{g}(F)$. By $X = F^{2n}$, we may identify $M_{2n}(F)$ with
End_F(X), the ring of \( F \)-endomorphisms on \( X \). Thus \( GL_2n(F) = M_{2n}(F)^s = End_F(X)^s \) and

\[
G = Sp_{2n}(F) = \{ g \in GL_{2n}(F) \mid f(gv, gw) = f(v, w) \text{ for } v, w \in X \}.
\]

For \( i, j \in I \), define \( E_{ij} \in M_{2n}(F) \) by \( E_{ij}e_k = \delta_{jk}e_i \), where \( \delta_{jk} \) is the Kronecker delta. Put

\[
J_{2n} = \sum_{i=1}^{n} (E_{-i, i} - E_{i, -i}).
\]

Then the mapping \( x \mapsto J_{2n}^{-1}xJ_{2n} \) is an involution on \( M_{2n}(F) \), where \( 'x \) denotes the transpose of \( x \). Denote this involution by \( \sigma \). Then it is well known that

\[
G = \{ g \in GL_{2n}(F) \mid \sigma g \cdot g = 1_{2n} \}
\]

and

\[
\mathfrak{g} = \{ x \in M_{2n}(F) \mid \sigma x + x = 0_{2n} \},
\]

where \( 1_{2n} \) and \( 0_{2n} \) are the identity and the zero matrices of size \( 2n \times 2n \), respectively.

1.2. Let \( D \) be the maximal \( F \)-split torus of diagonal elements in \( G \) and \( \mathfrak{a} \) the Lie algebra of \( D \). Let \( D = D(F) \) and \( \mathfrak{a} = \mathfrak{a}(F) \) be as in 1.1. Then

\[
D = \{ \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mid t_1, \ldots, t_n \in F^\times \}.
\]

Set

\[
d(t_1, \ldots, t_n) = \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}).
\]

For \( 1 \leq i \leq n \), set \( H_i = E_{ii} - E_{-i, i} \). Then \( \alpha = \sum_{i=1}^{n} FH_i \). Define the mappings \( a_i : \mathfrak{a} \to F, i = 1, \ldots, n, \) by

\[
a_i \left( \sum_{j=1}^{n} x_j H_j \right) = x_i
\]

for \( \sum_{j=1}^{n} x_j H_j \in \mathfrak{a} \). Set

\[
a_{-i} = -a_i, \quad a_{ij} = a_i + a_j \quad \text{for } i, j \in I, i \neq \pm j.
\]

Then \( \Sigma = \{ a_{ij} \mid i, j \in I, i \neq \pm j \} \cup \{ 2a_i \mid i \in I \} \) is the root system of \( G \) relative to \( D \). For each \( a \in \Sigma \), define \( E_a \in \mathfrak{g} \) by

\[
[H, E_a] = HE_a - E_a H = a(H)E_a \quad \text{for } H \in \mathfrak{a}.
\]
The one-parameter subgroups $x_a : G_a \to G$, $a \in \Sigma$, are given by
\[ x_a(t) = 1_{2n} + te_a \]
for $t \in G_a(F) = F$, where $G_a$ is the additive algebraic group defined over $F$.

1.3. We define subgroups of $G$ as follows:
\[ U_a = x_a(F) \quad \text{for } a \in \Sigma, \]
\[ U_{a,k} = x_a(\mathcal{A}^k) \quad \text{for } a \in \Sigma, k \in \mathbb{Z}, \]
\[ U_{0,0} = \{d(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in \mathcal{A}^n\}, \]
and
\[ U_{0,k} = \{d(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in 1 + \mathcal{A}^k\} \quad \text{for } k \geq 1. \]
Then $U_{0,0}$ and $U_{0,k}$ are subgroups of $D$.

1.4. Set $\Phi = (\Sigma \cup \{0\}) \times \mathbb{Z}$. For $\beta = (a, k) \in \Phi$, put
\[ U_\beta = U_{a,k}. \]
Take
\[ \Pi = \{a_{1,-2}, a_{2,-3}, \ldots, a_{n-1,-n}, 2a_n\} \]
as a basis of the root system $\Sigma$. Set
\[ \xi_0 = (-2a_1, 1), \quad \xi_i = (a_{i,-i-1}, 0), 1 \leq i \leq n-1, \quad \xi_n = (2a_n, 0). \]
Then it follows from [14, Sect. 1] that $\Phi$ is a root system and
\[ \widetilde{\Pi} = \{\xi_0, \xi_1, \ldots, \xi_n\} \]
is a basis of $\Phi$. Each root $\beta \in \Phi$ is uniquely written in the form $\beta = \sum_{i=0}^n b_i \xi_i$, $b_i \in \mathbb{Z}$. By the total ordering in $\Phi$ given by the basis $\widetilde{\Pi}$, $\beta = (a, k)$ is positive, say $\beta > 0$, if and only if “$k > 0$” or “$k = 0$ and $a > 0$” (cf. [12, 2.1]). Set $\Phi_+ = \{\beta \in \Phi \mid \beta > 0\}$.

1.5. For a proper subset $\theta \subseteq \widetilde{\Pi}$, let $P(\theta)$ be the subgroup of $G$ generated by $U_\beta$ with $\beta \in \Phi_+$ or $-\beta \in \theta$. Write
\[ P(\theta) = \langle U_\beta \mid \beta \in \Phi_+ \cup -\theta \rangle. \]
This is called a (standard) parahoric subgroup of $G$. 
1.6. Denote the building of $G$ over $F$ by $\mathcal{B}(G,F)$ and the apartment of $\mathcal{B}(G,F)$ associated with $D$ by $\mathbf{A}$ (see [3, 7.4] or [26, 2.1]). By the definition, $G$ acts on $\mathcal{B}(G,F)$ and $\mathcal{B}(G,F) = \bigcup_{g \in G} \mathbf{A}$. For each point $p \in \mathbf{A}$, we denote the group of all the elements of $G$ fixing $p$ by $\tilde{P}_p$ (see [3, 7.1.8] for its original definition). We shall see $P(\theta) = \tilde{P}_p$ for some point $p \in \mathbf{A}$.

1.7. Let $\mathbb{G}$ be the multiplicative algebraic group defined over $F$. We denote the one-parameter subgroup of $D$ by $X_*(D) = \text{Hom}(\mathbb{G},D)$ and the character group of $D$ by $X^*(D) = \text{Hom}(D,\mathbb{G})$. For each homomorphism $\iota: D \to F^\times$, there is a unique homomorphism $\iota_1: \mathbb{G} \to D$ such that the differential $d \alpha = \alpha_i$. In fact, $\alpha_i(d(t_1, \ldots, t_n)) = t_i$, $1 \leq i \leq n$.

We may identify $\alpha_i$ with $\alpha_j$, $1 \leq i \leq n$, and put $X_*(D) = X^*(D)$. Set

$$V = X_*(D) \otimes_\mathbb{Z} \mathbb{R}.$$ 

We extend the canonical pairing $\langle \ , \rangle$ on $X^*(D) \times X_*(D) = X_*(D) \times X_*(D)$ to $X_*(D) \times V$ by linearity. Denote also this extension by $\langle \ , \rangle$. The set $\mathbf{A}$ can be identified with $V$, and a point $p$ in $\mathbf{A} = V$ corresponds uniquely to $(p_i) \in \mathbb{R}^n$ via

$$p_i = \langle a_i, p \rangle, \quad 1 \leq i \leq n$$

(cf. [5, p. 181]). Thus we may set $p = (p_i)$ and $\mathbf{A} = V = \mathbb{R}^n$.

Each root $\beta = (a,k)$ in $\Phi$ can be regarded as an affine linear function $p \mapsto \langle a, p \rangle + k$ on $\mathbf{A} = V$ (cf. [26, 1.1]). Set

$$\langle \beta, p \rangle = \langle a, p \rangle + k.$$ 

1.8. Let $\mathcal{C}$ be the fundamental chamber in $\mathbf{A} = V = \mathbb{R}^n$ defined by $\langle \xi_i, v \rangle > 0$ for all $\xi_i \in \tilde{\Pi}$ (see 1.4). Then

$$\mathcal{C} = \{ v = (v_i) \in V = \mathbf{A} = \mathbb{R}^n | 0 < v_n < v_{n-1} < \cdots < v_1 < 1/2 \}$$

(cf. [26, 1.15]). For $\theta \subset \tilde{\Pi}$, there is a point $p$ in the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ in $\mathbf{A} = V = \mathbb{R}^n$ which satisfies

$$\langle \beta, p \rangle = 0 \quad \text{for only all } \beta \in \theta.$$ 

Such a point $p$ is not always unique. However, the subset $\mathcal{F}$ of all such points in $\overline{\mathcal{C}}$ is the closure of a facet of $V = \mathbf{A} = \mathbb{R}^n$. Thus, by [1, Chap. IV,
Sect. 2, $\mathcal{F}$ is uniquely determined by $\theta \subseteq \tilde{\Pi}$.

1.9. Proposition. For $\theta \subseteq \tilde{\Pi}$, let $p \in V = \mathcal{A} = \mathbb{R}^n$ be a point satisfying $\langle \beta, p \rangle = 0$ for only all $\beta \in \theta$. Then

$$P(\theta) = \hat{P}_p$$

and it is its own normalizer in $G$.

Proof. This proposition is well known (cf. [14, Prop. 2.29] and [3, Remarque 7.1.10]).

1.10. A subset $\theta \subseteq \tilde{\Pi}$ satisfies one among the conditions: (1) $\xi_0, \xi_n \notin \theta$, (2) $\xi_0 \notin \theta$ and $\xi_n \in \theta$, (3) $\xi_0 \in \theta$ and $\xi_n \notin \theta$, and (4) $\xi_0, \xi_n \in \theta$.

In fact, the subsets $\theta \subseteq \tilde{\Pi}$ are given as follows: There is a partition $n = n_1 + \cdots + n_t$, $t \geq 1$ and $n_i \geq 1$, and according to (1), (2), (3), and (4),

(a) $\theta = \tilde{\Pi} - \{\xi_0, \xi_n, \xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n_1+n_2-1}, \xi_n\}$ with $t \geq 1$,

(b) $\theta = \tilde{\Pi} - \{\xi_0, \xi_n, \xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n_1+n_2-1}\}$ with $t \geq 1$,

(c) $\theta = \tilde{\Pi} - \{\xi_0, \xi_{n_1+n_2}, \xi_{n_1+n_2+1}, \ldots, \xi_{n+n_1-1}, \xi_n\}$ with $t \geq 1$, and

(d) $\theta = \tilde{\Pi} - \{\xi_0, \xi_{n_1+n_2}, \xi_{n_1+n_2+1}, \ldots, \xi_{n+n_1-1}\}$ with $t \geq 2$.

1.11. Lemma [5, Lemma 3.10]. A point $p = (p_i) \in V = \mathcal{A} = \mathbb{R}^n$ corresponds to the flag of $\mathcal{O}$-lattices $(L(c))_{c \in \mathbb{R}}$ in $X = F^{2n}$ defined by

$$L(c) = \sum_{i \in I} a^{n_i(c)} e_i, \quad c \in \mathbb{R},$$

where $n_i(c) \in \mathbb{Z}$ with $p_i + c \leq n_i(c) < p_i + c + 1$, $i \in I$ (note that $p_{i-1} = -p_i$ for $i \geq 0$, because $a_{i-1} = -a_i$).

Proof. This follows at once from [5, Lemma 3.10], because $c$ in the assertion of that lemma is equal to one in our split case.

1.12. For the flag of $\mathcal{O}$-lattices $(L(c))_{c \in \mathbb{R}}$ in Lemma 1.11, it can be suitably indexed by $\mathbb{Z}$ as $(L_i)_{i \in \mathbb{Z}}$ and there is a unique integer $s \geq 1$ such that

$$\tilde{\omega}L_i = L_{i+s}, \quad i \in \mathbb{Z}$$

(cf. [4, 1.8]). We call this integer $s$ the period of $(L_i)_{i \in \mathbb{Z}}$, and call a subsequence of the form

$$L_0 \supseteq L_1 \supseteq \cdots \supseteq L_{s-1} \supseteq \tilde{\omega}L_0$$

a slice of $(L_i)_{i \in \mathbb{Z}}$ of origin $L_0$ (see [4, 1.7]). Put

$$\lambda_i = \dim_F(L_{i-1}/L_i), \quad 1 \leq i \leq s.$$  \hspace{1cm} (1.12.1)

We call $(\lambda_1, \ldots, \lambda_s)$ the type of $(L_i)_{i \in \mathbb{Z}}$ of origin $L_0$ (see also [4, 1.7]).
1.13. Lemma. Let $\theta \subseteq \hat{\Pi}$ be (a), (b), (c), or (d) in 1.10. Then a solution $p = (p_i) \in V = \mathfrak{A} = \mathfrak{P}^n$ satisfying $p \in \mathfrak{P}$ and $\langle \beta, p \rangle = 0$ for only all $\beta \in \theta$ is given as follows: Putting

$$p_1 = \cdots = p_{n_1} = c_1,$$
$$p_{n_1 + \cdots + n_{i-1} + 1} = \cdots = p_{n_1 + \cdots + n_i} = c_i, \quad 2 \leq i \leq t,$$

$$c_i = \begin{cases} 
\frac{2(t - i) + 1}{4t}, & 1 \leq i \leq t \text{ for (a)}, \\
\frac{t - i}{2t - 1}, & 1 \leq i \leq t \text{ for (b), (d)}, \\
\frac{t - i + 1}{2t}, & 1 \leq i \leq t \text{ for (c)}. 
\end{cases}$$

Proof. We can check this directly.

1.14. Lemma. Let $\theta \subseteq \hat{\Pi}$ be (a), (b), (c), or (d) with the partition $n = n_1 + \cdots + n_t$ in 1.10. Then the period $s = s(\theta)$ and the type $(\lambda_1, \ldots, \lambda_s)$ of $(L_{e_i})_{e_i \in Z}$ of origin $L_0$ by shifting the indices (if necessary) are given in terms of $t$ and $n_i$ as follows: For (a), $s = 2t; \lambda_i = n_i, \lambda_{i+1} = n_{i-1} + 1, 1 \leq i \leq t,$

$$L_0 = e_{-1} \oplus \cdots \oplus e_{-n} \oplus e_n \oplus \cdots \oplus e_1.$$  

For (b), $s = 2t - 1; \lambda_i = n_i, \lambda_{i+1} = n_{i-1}, 1 \leq i \leq t - 1, \lambda_t = 2n_t,$

$$L_0 = e_{-1} \oplus \cdots \oplus e_{-n} \oplus e_n \oplus \cdots \oplus e_1.$$  

For (c), $s = 2t - 1; \lambda_i = n_{i+1}, \lambda_{i+1} = n_{i-1}, 1 \leq i \leq t - 1, \lambda_{2t-1} = 2n,$

$$L_0 = e_{-1} \oplus \cdots \oplus e_{-n} \oplus e_n \oplus \cdots \oplus e_{n_1} \oplus \cdots \oplus e_1.$$  

For (d), $s = 2t - 2; \lambda_i = n_{i+1}, \lambda_{i+1} = n_{i-1}, 1 \leq i \leq t - 2, \lambda_{t-1} = 2n_t,$

$$L_0 = e_{-1} \oplus \cdots \oplus e_{-n} \oplus e_n \oplus \cdots \oplus e_{n_1} \oplus \cdots \oplus e_1.$$  

Here $\{e_i\}$ is the Witt basis of $X$ in 1.1.

Proof. We get slices of the flags of $\mathfrak{P}$-lattices $\mathcal{L}_0 = (L_{e_i})_{e_i \in Z}$ in $X = F^{2n}$, determined by Lemmas 1.11 and 1.13 for $\theta \subseteq \hat{\Pi}$ in 1.10, as follows: Putting $f_i = e_i, f_{2n+1-i} = e_{-i}, 1 \leq i \leq t,$

$$L_0 = f_{2n} \oplus \cdots \oplus f_1,$$

$$L_i = f_{2n} \oplus \cdots \oplus f_{\lambda_{i}+\lambda_{i+1}} \oplus f_{\lambda_{i}+\lambda_{i+2}} \oplus \cdots \oplus f_1, \quad 1 \leq i < s.$$
for \( \theta \subseteq \tilde{\Pi} \) of (a) and (b), and

\[
L_i = \mathcal{O} f_{2n} \oplus \cdots \oplus \mathcal{O} f_{\lambda_1 + \ldots + \lambda_{i+1} + 1} \oplus \mathcal{O} f_{\lambda_1 + \ldots + \lambda_{i+1}} \oplus \cdots \oplus \mathcal{O} f_1,
\]

\[
0 \leq i \leq s - 2,
\]

\[
L_{s-1} = \mathcal{O} f_{2n} \oplus \cdots \oplus \mathcal{O} f_{\lambda_1 + \ldots + \lambda_{i+1} + \lambda_{i+1}/2 + 1} \oplus \mathcal{O} f_{\lambda_1 + \ldots + \lambda_{i+1} + \lambda_{i+1}/2} \oplus \cdots \oplus \mathcal{O} f_1
\]

for \( \theta \subseteq \tilde{\Pi} \) of (c) and (d), where \( s \) and \( (\lambda_1, \ldots, \lambda_{i}) \) are in the assertion.

1.15. Note that the type \( (\lambda_1, \ldots, \lambda_{i}) \) of \( (L_i)_{i \in \mathbb{Z}} \) of origin \( L_0 \) in Lemma 1.14 determines uniquely the slice \( L_0 \supseteq L_1 \supseteq \ldots \supseteq L_{s-1} \supseteq \omega L_0 \) of the forms as above by Lemma 1.11, and hence \( (L_i)_{i \in \mathbb{Z}} \).

Further it follows that this flag depends only on the set \( \mathcal{F} \) defined in 1.8. Since \( \mathcal{F} \) is uniquely determined by \( \theta \subseteq \tilde{\Pi} \), we may write

\[
\mathcal{L}_\theta = \{ L_i \}_{i \in \mathbb{Z}}
\]

and \( s = s(\theta) \) the period of \( \mathcal{L}_\theta \).

1.16. For a \( \mathcal{O} \)-lattice \( L \) in \( X \), we define the \( \mathcal{O} \)-dual lattice \( L^* \) in \( X \) by

\[
L^* = \{ x \in X \mid f(x, L) \subseteq \mathcal{O} \}.
\]

Then we can directly see that the dual \( L_i^* \) of \( L_i \) in \( \mathcal{L}_\theta \) again belongs to \( \mathcal{L}_\theta \), by the explicit forms of \( L_0 \supseteq L_1 \supseteq \ldots \supseteq L_{s-1} \supseteq \omega L_0 \) in the proof of Lemma 1.14.

1.17. Let \( \theta \subseteq \tilde{\Pi} \) be as in 1.10. The flag of \( \mathcal{O} \)-lattices \( \mathcal{L}_\theta = \{ L_i \}_{i \in \mathbb{Z}} \) in \( X = F^{2n} \) defines the hereditary \( \mathcal{O} \)-order \( \mathcal{U}_\theta \) in \( M_{2n}(F) \) (see [23]). Let \( \mathcal{B}_\theta \) be the Jacobson radical of \( \mathcal{U}_\theta \). Then we have

\[
\mathcal{B}_\theta^u = \{ g \in M_{2n}(F) \mid \operatorname{End}_F(X) \mid gL_i \subseteq L_{i+n}, i \in \mathbb{Z} \}
\]

for \( u \in \mathbb{Z} \). In particular, \( \mathcal{U}_\theta = \mathcal{B}_\theta^0 \) and \( \mathcal{B}_\theta = \mathcal{B}_\theta^1 \). We have

\[
\mathcal{B}_\theta^0 = \mathcal{O} \mathcal{U}_\theta,
\]

where \( s = s(\theta) \).

1.18. Proposition. Let the notations and assumptions be as in 1.17. Then

\[
P(\theta) = G \cap \mathcal{U}_\theta.
\]

Proof. Take \( p \in V = \mathcal{A} = \mathbb{R}^n \) in Lemma 1.13 for \( \theta \subseteq \tilde{\Pi} \). Then \( P(\theta) = \hat{P}_p = \{ g \in G \mid g \cdot p = p \} \) by Proposition 1.9. Hence the assertion follows from [5, 3.9].
1.19. By using $\mathfrak{U}_\theta^i$, $i \geq 1$, in 1.17, we can also define a natural filtration on $P(\theta) = G \cap \mathfrak{U}_\theta$ by

$$P_i = P(\theta)_i = G \cap (1 + \mathfrak{U}_\theta^i), \quad i \geq 1$$

(cf. [8, 6, 18]), and $P_0 = P(\theta)_0 = P(\theta)$.

1.20. **Proposition.** For $\theta \not\subseteq \tilde{\Pi}$, set $P = P(\theta)$. Then the filtration $(P_i)_{i \geq 0}$ satisfies the following conditions:

(i) $[P_i, P_j] \subset P_{i+j}$ for $i, j \geq 0$, where $[\ , \ ]$ denotes the commutator.

(ii) $P_i$ is a normal subgroup of $P = P(\theta)$ for $i \geq 1$.

(iii) For $i \geq j \geq 1$, $P_i/P_{i+j}$ is an abelian group.

**Proof.** By definition, this can be directly checked (cf. [22, Sect. 2]).

1.21. **Remark.** Another filtration on $P = P(\theta)$ is introduced in [22, Sect. 2] (cf. [12]). It is defined by using a function on the affine root system $\Phi$, which is called the affine height function. We can prove that this filtration coincides with ours in 1.19. But we do not discuss this here.

1.22. For $\theta \not\subseteq \tilde{\Pi}$, we define the $\mathfrak{g}$-lattices of $\mathfrak{g}$ by

$$\mathfrak{g}_{\theta, i} = \mathfrak{g} \cap \mathfrak{U}_\theta^i, \quad i \in \mathbb{Z}.$$  

For $x \in \mathfrak{g}$ such that $1 + x$ is invertible, the Cayley map $C$ is defined by

$$C(x) = (1 - x)(1 + x)^{-1}.$$  

**Proposition [17, 2.13].** The Cayley map $C$ induces a bijection

$$\mathfrak{g}_{\theta, i} = P_i, \quad i \geq 1.$$  

1.23. We define a canonical inner product $\langle \ , \ \rangle$ on $M_{2n}(F)$ by

$$\langle x, y \rangle = \text{tr}(xy), \quad x, y \in M_{2n}(F),$$  

where $\text{tr}$ denotes the usual trace in $M_{2n}(F)$. For an $\mathfrak{g}$-lattice $L$ of $M_{2n}(F)$, put

$$L^\ast = \{ x \in M_{2n}(F) | \langle x, L \rangle \subseteq \mathfrak{g} \}$$  

and, for $\mathfrak{g}_{\theta, i}$,

$$(\mathfrak{g}_{\theta, i})^\ast = \{ x \in \mathfrak{g} | \langle x, \mathfrak{g}_{\theta, i} \rangle \subseteq \mathfrak{g} \}.$$  


1.24. Proposition (cf. [18, Lemma 4.18]). Let \( \theta \subsetneq \tilde{\Pi} \) and put \( s = s(\theta) \). Then, for \( i \in \mathbb{Z} \),
\[
\left( \Psi^i_{\theta} \right)^* = \Psi^1_{\theta} 1^{-i} \quad \text{and} \quad \left( \Omega^i_{\theta, i} \right)^* = \Omega_{\theta, 1^{-i} - s}.
\]
We define the function \( \lambda: \mathbb{Z} \to \mathbb{Z} \) by
\[
\lambda(i) = 1 - i - s, \quad i \in \mathbb{Z}.
\]

1.25. Let \( \psi \) be an additive character of \( F \) with conductor \( \varnothing \), that is, \( \psi |_{\varnothing} = 1 \) and \( \psi |_{\varnothing^{-1}} \neq 1 \). From now on, fix this \( \psi \).

Proposition. Let the notations and assumptions be as above. For integers \( i \geq j \geq 1 \), the mapping \( \alpha + \Omega_{\theta, i} \to \Omega_{\theta, j} \), defined by
\[
\Omega_{\theta}(C(x)) = \psi(\langle x, -\alpha \rangle/2), \quad x \in \Omega_{\theta, i},
\]
induces an isomorphism
\[
\Omega_{\theta, i+j}/\Omega_{\theta, i} \simeq \left( P_{i+j} / P_{i+j} \right)^*,
\]
where the right-hand side is the Pontrjagin dual of \( P_{i+j} / P_{i+j} \).

Proof. This follows from Propositions 1.22 and 1.24 (cf. [18, Sect. 4]).

2. Very Cuspidal Elements

2.1. In this section, we first consider particular elements \( \alpha \) in \( \mathfrak{g} \) which are elliptic regular semisimple; that is, the centralizers \( Z_c(\alpha) \) of the \( \alpha \) in \( G \) are compact Cartan subgroups of \( G \), and have simple block forms. We determine \( \theta \subsetneq \tilde{\Pi} \) and an integer \( M \) such that \( \Omega_{\theta, M} \) contains such elements \( \alpha \), and then define the very cuspidal elements in \( \Omega_{\theta, M} \).

2.2. We start by determining \( \theta \subsetneq \tilde{\Pi} \) and an integer \( M \) which satisfy the condition:

(P) \( \Omega_{\theta, M} \) contains a nonsingular element \( x \) with \( x \notin \Omega_{\theta, M+1} \) such that \( x^{-1} \in \Psi^M_{\theta} \) and \( x^{-1} \notin \Psi^{M+1}_{\theta} \).

2.3. For \( \theta \subsetneq \tilde{\Pi} \), let \( \mathcal{X}_\theta = \{ L_i \} \subset \mathbb{Z} \) with period \( s = s(\theta) \), \( \mathcal{V}_\theta \) and \( \mathcal{P}_\theta \) be as in 1.17. Then there is a natural ring isomorphism
\[
R_{1}: \mathcal{V}_\theta / \mathcal{P}_\theta = \prod_{i=0}^{s-1} \text{End}_F(L_i / L_{i+1})
\]
(cf. [8, Sect. 3]). Let \((\lambda_1, \ldots, \lambda_s)\) be the type of \(L_\theta = (L_i)_{i \in \mathbb{Z}}\) of origin \(L_0\) (see (1.12.11)). Then

\[2n = \lambda_1 + \lambda_2 + \cdots + \lambda_s.\]

The set \(\mathcal{U}_\theta\) is \(GL_{2n}(F)\)-conjugate to the order of \(s \times s\) block matrices

\[
\begin{pmatrix}
\tilde{\mathcal{C}} & \tilde{\mathcal{C}} & \cdots & \tilde{\mathcal{C}} \\
\cdots & \cdots & \cdots & \cdots \\
\mathcal{C} & \mathcal{C} & \cdots & \mathcal{C}
\end{pmatrix}, \tag{2.3.1}
\]

where the \((i, j)\)-block \(\tilde{\mathcal{C}}\) or \(\mathcal{C}\) has size \(\lambda_i \times \lambda_j, 1 \leq i, j \leq s\), and entries in \(\mathcal{C}\) if \(i \leq j\), in \(\mathcal{P}\) otherwise.

2.4. Let \(\theta \subseteq \Pi\) be \((a)\) or \((b)\) in 1.10. Then it follows from the form of the slice of \(L_\theta = (L_i)_{i \in \mathbb{Z}}\) in the proof of Lemma 1.14 that elements of \(\mathcal{U}_\theta\) have the block forms (2.3.1).

Let \(\theta \subseteq \Pi\) of \((c)\) and \((d)\) in 1.10. Again by the proof of Lemma 1.14, elements of \(\mathcal{U}_\theta\) do not have the block forms (2.3.1) in this case. We define an element \(w_0\) of \(GL_{2n}(F)\) by

\[w_0 = \begin{pmatrix} 0_{2n-n_1,n_1} & 1_{2n-n_1} \\ \tilde{\mathcal{A}} & \mathcal{A} \end{pmatrix}, \tag{2.4.1}\]

where \(0_{m,n}\) denotes the zero matrix of size \(m \times n\). Then it follows easily that \(w_0L_\theta = (w_0L_i)_{i \in \mathbb{Z}}\) has a slice of \((a)\) or \((b)\) in the proof of Lemma 1.14. Thus elements of the hereditary \(\mathcal{C}\)-order \(\mathcal{U}\) in \(M_{2n}(F)\) defined by \(w_0L_\theta\) have the block forms (2.3.1) as well, and we have \(\mathcal{U} = w_0 \mathcal{U}_\theta w_0^{-1}\).

2.5. By the isomorphism \(R_1\) in 2.3, \(\mathcal{U}_0/\mathcal{U}_1 = \mathcal{U}_\theta/\mathcal{U}_\theta\) always contains an invertible element so that \(M = 0\) and an arbitrary \(\theta \subseteq \Pi\) satisfy condition \(P\) in 2.2. Thus it follows from the periodicity of \((\mathcal{U}_\theta)_{i \in \mathbb{Z}}\) in 1.17 that an integer \(M\) divisible by \(s\), say \(s \mid M\), and an arbitrary \(\theta \subseteq \Pi\) also satisfy condition \(P\). Therefore we must consider the case where \(s\) does not divide \(M\), denoted \(s \nmid M\).

2.6. Lemma. Let \((\lambda_i)\) be as in Lemma 1.14 for \(L_\theta = (L_i)_{i \in \mathbb{Z}}\) with period \(s = s(\theta)\). Let \(M\) be an integer not divisible by \(s\). Let \(c = (s, M)\) be the greatest common divisor of \(s\) and \(M\). Then \(\theta \subseteq \Pi\) and \(M\) satisfy condition \((P)\) in 2.2 if and only if

\[\lambda_k = \lambda_{k+c} = \cdots = \lambda_{k+s-c} \quad \text{for } k = 1, \ldots, c.\]
Proof. Extending (1.12.1), we may put
\[ \lambda_i = \lambda_i + s_j \quad \text{for any integer } j \text{ and } i = 1, \ldots, s. \]

For any integer \( N \) we have a bijection
\[ \Psi^N_\theta / \Psi^{N+1}_\theta = \prod_{i=0}^{s-1} \text{Hom}_F(L_i/L_{i+1}, L_{i+N}/L_{i+N+1}). \]

If condition (P) is satisfied, it follows from this bijection that
\[ \lambda_i = \lambda_i + M \quad \text{for all } i = 1, \ldots, s. \]

For each positive integer \( k \), we define the subset \( S(k) \) of \( \mathbb{Z} \) by
\[ S(k) = \{ k, k + M, k + 2M, \ldots, k + (s/c - 1)M \}, \tag{2.6.1} \]
where, for any integer \( m, \overline{m} \) denotes the integer defined by
\[ m \equiv \overline{m} \mod s \quad \text{and} \quad 1 \leq \overline{m} \leq s. \tag{2.6.2} \]

Then by elementary number theory
\[ S(k) = \{ k, k + c, k + 2c, \ldots, k + (s-c) \} \tag{2.6.3} \]
for each \( k, 1 \leq k \leq c \), and
\[ \{1, 2, \ldots, s\} = S(1) \cup S(2) \cup \cdots \cup S(c) \quad \text{(disjoint union).} \]

Hence the above condition on \( \{ \lambda_i \} \) gives the one of this lemma. Conversely, assume that it is satisfied. For the integer \( M \), let \( M', l \) be the integers satisfying \( M = M's + l \) and \( 0 < l < s \). We remark that \( c = (s, M) = (s, M's + l) = (s, l) \) and that \( \Psi^N_\theta = \omega \mathcal{M}_\theta \). Thus it follows from the assumption on \( \{ \lambda_i \} \) and the forms of the slices of \( \mathcal{L}_\theta \) in the proof of Lemma 1.14 that for each \( \theta \subseteq \Pi \) there is a nonsingular element \( \alpha \in \Psi^M_\theta \) with \( \alpha \notin \Psi^{M+1}_\theta \), which has the following block form: For \( \theta \subseteq \Pi \) of (a) or (b) in 1.10,

\[ \alpha = \begin{pmatrix}
\omega^M
& a_1 & \cdots & a_{s-l} \\
\omega a_{s-l+1} & \cdots & \omega a_s
\end{pmatrix}, \tag{2.6.4} \]
where \( a_i \in \text{GL}_s(\mathcal{O}) \), \( 1 \leq i \leq s \), and the others are 0. For \( \theta \subseteq \mathfrak{II} \) of (c) or (d) in 1.10,

\[
\alpha = \omega^M \begin{pmatrix}
\tilde{\omega} a_s^{(2)} \\
\tilde{\omega} a_{s-1}^{(1)} \\
\tilde{\omega} a_{s-1}^{(1)} \\
\vdots \\
\tilde{\omega} a_2^{(2)} \\
\tilde{\omega} a_1^{(2)} \\
1
\end{pmatrix}
\]

(2.6.5)

where \( a_i \in \text{GL}_s(\mathcal{O}) \), \( 1 \leq i \leq s \), \( i \neq s-1 \), \( a_i^{(1)}, a_i^{(2)} \in M_{\lambda_i/2, \lambda_i}(\mathcal{O}) \), \( a_{s-1}^{(1)}, a_{s-1}^{(2)} \in M_{\lambda_{s-1}/2, \lambda_{s-1}/2}(\mathcal{O}) \) with \( (a_{s-1}^{(1)}, a_{s-1}^{(2)}) \in \text{GL}_{\lambda_{s-1}}(\mathcal{O}) \) and

\[
\begin{pmatrix}
\tilde{a}_s^{(1)} \\
\tilde{a}_s^{(2)}
\end{pmatrix} \in \text{GL}_{\lambda_s}(\mathcal{O}),
\]

and the others are 0. In this case, \( \lambda_i \) is even by Lemma 1.14 and the nonsingularity of \( \alpha \) follows from that of \( w_0 a w_0^{-1} \), where \( w_0 \) is the element of (2.4.1). Hence \( \alpha^{-1} \) is in \( \mathcal{O}_M^{-1} \) and has a similar block form, whence \( \alpha^{-1} \notin \mathcal{O}_M^{-1} \). This shows that the integer \( M \) and \( \theta \subseteq \mathfrak{II} \) satisfy condition (P) in 2.2. This completes the proof.

2.7. Remark. Let \( M \) be an integer with \( s \mid M \), that is, \( M = M's \). Then, for \( \theta \subseteq \mathfrak{II} \) of (a) or (b) in 1.10, such an element \( \alpha \in \mathcal{O}_M^{-1} \) as in the proof of Lemma 2.6 has a diagonal form, and for \( \theta \subseteq \mathfrak{II} \) of (c) or (d) in 1.10,

\[
\alpha = \omega^M \begin{pmatrix}
\tilde{\omega} a_s^{(3)} & \tilde{\omega}^{-1} a_s^{(2)} \\
\tilde{\omega} a_s^{(4)} & a_{s-1} \\
\tilde{\omega} a_{s-1}^{(2)} & a_{s-1} \\
\vdots \\
\tilde{\omega} a_3^{(2)} & a_2 \\
\tilde{\omega} a_2^{(3)} & a_1 \\
1
\end{pmatrix}
\]

(2.7.1)

where \( a_i^{(3)} \in M_{\lambda_i/2}(\mathcal{O}) \), \( 1 \leq i \leq 4 \), \( a_i \in M_{\lambda_i}(\mathcal{O}) \), \( 1 \leq i \leq s-1 \), and the others are 0.
2.8. Proposition. Let \( M \) be an integer, and put \( c = (s, M) \). Then a nonsingular element \( \alpha \) in \( \mathfrak{M}^M_\mathfrak{p} \) of the block form as in 2.6 or 2.7 has the monic eigenpolynomial

\[
\alpha^{M/c} \prod_{i=1}^{c} p_i(\alpha^{-M/c} X^{s/c}),
\]

where each \( p_i(X) \) is a monic polynomial in \( \mathfrak{o}[X] \) of degree \( \lambda_i \) (see (1.12.1)) over \( F \), and \( \lambda = \sum_{i=1}^{c} \lambda_i \).

Proof. Let \( M', l \) be the integers satisfying \( M = M's + l \) and \( 0 \leq l < s \). Let \( \omega \) be the permutation on \( \{1, 2, \ldots, n\} \) defined by

\[
\omega = \begin{pmatrix} 1 & 2 & \cdots & s-1 & s \\ 2 & 3 & \cdots & s & 1 \end{pmatrix}.
\]

Then we have

\[
\omega^M = \omega' = \begin{pmatrix} 1 & 2 & \cdots & s & s \\ 1+l & 2+l & \cdots & s+l & 1 \end{pmatrix},
\]

where we recall (2.6,2) for the definition of \( \overline{m} \).

At first, let \( \theta \subseteq l_{\mathfrak{p}} \) be (a) or (b) in 1.10. Then \( \alpha \) has the form (2.6.4).

A direct computation shows

\[
\alpha^{r/c} = \alpha^{M/c} \text{diag}(b_2, b_3, \ldots, b_s)
\]

and

\[
b_i = a_1 a_{\omega(i)} a_{\omega^2(i)} \cdots a_{\omega^{s/c-1}(i)}, \quad 1 \leq i \leq s.
\]

Since \( \alpha \) is invertible, so are all the \( a_i \). Let \( S(k) \) be the set (2.6.1). Then, by definition,

\[
S(i) = \{ i, \omega(i), \omega^2(i), \ldots, \omega^{s/c-1}(i) \}
\]

and \( a_i \in \text{GL}_k(\mathfrak{o}) \) for all \( j \in S(i) \), since \( M \) can be replaced by \( l \). Let \( p_i(X) \) be the monic eigenpolynomial of \( b_i \) for each \( i, 1 \leq i \leq c \). Then it has coefficients in \( \mathfrak{o} \) and the degree \( \lambda_i \) over \( F \). If \( k = m \mod c \),

\[
\{ a_k, a_{\omega(k)}, a_{\omega^2(k)}, \ldots, a_{\omega^{s/c-1}(k)} \} = \{ a_m, a_{\omega(m)}, a_{\omega^2(m)}, \ldots, a_{\omega^{s/c-1}(m)} \}.
\]

Since the monic eigenpolynomials of \( xy \) and \( yx \) for \( x, y \in \text{GL}_{v}(\mathfrak{o}) \) are identical, it follows from the above equations for the \( b_i \) that

\[
p_k(X) = p_m(X)
\]
if $k \equiv m \mod c$. Hence by the Cayley–Hamilton theorem

$$\prod_{i=1}^{c} p_i(\tilde{\omega}^{-M/c}x^{i/c}) = 0.$$ 

Put

$$p(X) = \tilde{\omega}^{M/c} \prod_{i=1}^{c} p_i(\tilde{\omega}^{-M/c}x^{i/c}),$$

where $\lambda$ is as above. Then $p(X)$ is a monic polynomial of degree $2n$ and $p(\alpha) = 0$. This implies that $p(X)$ is the monic eigenpolynomial of $\alpha$. Hence we have proved this proposition in case $\theta \subseteq \Pi$ is (a) or (b) in 1.10.

Let $\theta \subseteq \Pi$ be (c) or (d) in 1.10. Let $w_0$ be the matrix (2.4.1). Then we can apply the above argument to $w_0\alpha w_0^{-1}$ (cf. (2.6.4), (2.6.5), and (2.7.1)) and prove this proposition in the cases of (c) and (d). This completes the proof.

2.9. Let $\alpha \in \mathfrak{g}$ be a regular semisimple element. Then it generates the subalgebra $F[\alpha]$ of $M_{2n}(F)$ of dimension $2n$ over $F$. For the involution $\sigma$ on $M_{2n}(F)$ in 1.1, $\sigma \alpha = -\alpha$, so $\sigma$ leaves $A = F[\alpha]$ stable. Denote the restriction to $A$ again by $\sigma$. We have $A = \oplus_{i=1}^{n} E_i$, where each $E_i$ is a field extension of $F$. The involution $\sigma$ permutes the $E_i$. Put

$$T = A \cap G.$$ 

Then this is the centralizer $Z_G(\alpha)$ of $\alpha$ in $G$ and is a Cartan subgroup of $G$. Further, it generates the maximal $F$-torus $T$ of the algebraic group $G$.

As noted in 2.1, $\alpha \in \mathfrak{g}$ is called elliptic if $T$ is compact in $G$; equivalently, $T$ is $F$-anisotropic.

**Lemma.** Let $\alpha \in \mathfrak{g}$ be regular semisimple and $A = F[\alpha] = \oplus_{i=1}^{n} E_i$ be as above. Then $\alpha$ is elliptic if and only if $\sigma E_i = E_i$ and $\alpha | E_i \neq 1$ for all $i$.

**Proof.** See [17, 1.3] for a proof.

2.10. **Proposition.** Let $M$ be an integer. Assume that $s \not| M$ and that $\mathfrak{g}_{s,M} = \mathfrak{g}_{s} \cap \mathfrak{g}$ contains an elliptic regular semisimple element of the block form as in 2.6. Then $(s, M) = 1$ if $\theta \subseteq \Pi$ is (a) or (b) in 1.10, and $(s, M) = 1$ or 2 if $\theta \subseteq \Pi$ is (c) or (d) in 1.10.

**Proof.** Put $c = (s, M)$ and let $M = M's + l$ be as in Proposition 2.8. Let $\alpha$ be an elliptic regular semisimple element in $\mathfrak{g}_{s,M}$ of the block form as in 2.6. Let $p_i(X), 1 \leq i \leq c$, be the polynomials defined in the proof of Proposition 2.8 for this $\alpha$, and $A = F[\alpha]$ be the subalgebra of $M_{2n}(F)$ generated by $\alpha$ over $F$. Then, since $\alpha$ is regular, by Proposition 2.8, the algebra $A$ decomposes into the direct sum $\oplus_{i=1}^{c} A_i$, where $A_i = F[X]/(p_i(\tilde{\omega}^{-M/c}X^{i/c}))$. We describe the action on $A$ of the involution $\sigma$: $\alpha \mapsto -\alpha$. 

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At first, let $\theta \subseteq \tilde{\Pi}$ be (a) or (b) in 1.10. For an integer $\lambda \geq 1$, $I = I_{\lambda}$ denotes the matrix of size $\lambda \times \lambda$ whose $(i, \lambda - i + 1)$-entries, $1 \leq i \leq \lambda$, are 1 and the others are 0. If $\lambda$ is even, denote the matrix of size $\lambda \times \lambda$ defined in 1.1 by $J = J_{\lambda}$. Then, for $\alpha$ with the block form in 2.6, by a direct computation, we can show that

$$\sigma\alpha = \tilde{\omega}^{M'} \begin{pmatrix} a'_{1} \\ \vdots \\ a'_{s-l} \\ \vdots \\ \tilde{\omega}a'_{s} \end{pmatrix}$$

(2.10.1)

and $a'_{1}, a'_{2}, \ldots, a'_{s}$ are given as follows: Let $\sigma'$ be the permutation on $(1, 2, \ldots, s)$ defined by

$$\sigma' = \begin{pmatrix} 1 & 2 & \ldots & s-l \\ s-l & s-l-1 & \ldots & 1 \\ s-l+1 & s-l+2 & \ldots & s \\ s & s-1 & \ldots & s-l+1 \end{pmatrix}.$$

If $s$ is even, $\theta \subseteq \tilde{\Pi}$ is (a) in 1.10 by Lemma 1.14 and

$$a'_i = \varepsilon_i I \cdot a^\varepsilon_{\sigma'(i)} \cdot I, \quad 1 \leq i \leq s,$$

where $\varepsilon_i = \pm 1$ is determined as follows: When $l < s/2$,

$$\varepsilon_i = \begin{cases} 1 & \text{if } 1 \leq i \leq s/2 - l \text{ or } s/2 + 1 \leq i \leq s - l, \\ -1 & \text{otherwise}, \end{cases}$$

when $l = s/2$, $\varepsilon_i = 1$ for all $i$, and when $l > s/2$,

$$\varepsilon_i = \begin{cases} 1 & \text{if } s - l + 1 \leq i \leq s/2 \text{ or } 3s/2 - l + 1 \leq i \leq s, \\ -1 & \text{otherwise}. \end{cases}$$

If $s$ is odd, $\theta \subseteq \tilde{\Pi}$ is (b) by Lemma 1.14 and

$$a'_{j_0} = J^{-1} \cdot a^\varepsilon_{\sigma'(j_0)} \cdot I, \quad a'_{k_0} = I \cdot a^\varepsilon_{\sigma'(k_0)} \cdot J$$

and

$$a'_i = \varepsilon_i I \cdot a^\varepsilon_{\sigma'(i)} \cdot I \quad \text{for } i \neq j_0, k_0,$$
where \( j_0, k_0, \) and \( \epsilon_i = \pm 1 \) are determined as follows: When \( l < (s - 1)/2, \)
\( j_0 = (s + 1)/2 - l, \) \( k_0 = (s + 1)/2, \) and
\[
\epsilon_i = \begin{cases} 
1 & \text{if } 1 \leq i \leq (s - 1)/2 - l \text{ or } (s - 1)/2 + 2 \leq i \leq s - l, \\
-1 & \text{otherwise},
\end{cases}
\]
when \( l = (s - 1)/2, j_0 = 1, k_0 = (s + 1)/2, \) and \( \epsilon_i = 1 \) for all \( j \neq j_0, k_0, \)
and when \( l \geq (s + 1)/2, j_0 = (s + 1)/2, k_0 = (3s + 1)/2 - l, \) and
\[
\epsilon_i = \begin{cases} 
1 & \text{if } l + 1 \leq i \leq (s - 1)/2 \text{ or } (3s + 3)/2 - l \leq i \leq s, \\
-1 & \text{otherwise}.
\end{cases}
\]

We show that \( \sigma \)' permutes the sets \( S(1), S(2), \ldots, S(c), \) defined by (2.6.3).
In fact, it is sufficient to show that if \( j, k \in \{1, 2, \ldots, s\} \) and \( j \equiv k \mod c, \)
then \( \sigma'(j) \equiv \sigma'(k) \mod c. \) We first remark that \( c = (s, M) = (s, l) \) and
that the above integers \( j_0, k_0 \) belong to a single set \( S(m). \) We may set \( j, k \)
to satisfy \( j < k, k - j = h \) and \( c \mid h. \) By the definition of \( \epsilon', \) if \( j < k \leq s - l, \)
or if \( s - l < j < k, \) then clearly \( \epsilon'(k) - \epsilon'(j) = h \equiv 0 \mod c. \) So,
assume \( j \leq s - l < k. \) Then \( \epsilon'(k) - \epsilon'(j) = \epsilon'(j + h) - \epsilon'(j) = (2s - j - h - l + 1) - (s - l - j + 1) = s - h \equiv 0 \mod c. \) Hence \( \sigma' \) acts on
\( \{S(1), S(2), \ldots, S(c)\} \) as a permutation, as claimed. Note that \( \sigma'(S(i)) = \{\sigma'(k) \mid k \in S(i)\}. \) Assume now that \( c = (s, M) = (s, l) > 1. \) Then
\[
\sigma' S(1) = S(s - l) \neq S(1).
\]

As in the proof of Proposition 2.8, we have
\[
(\sigma \alpha)^{s/c} = \tilde{\omega}^{M/c} \text{diag}(b'_1, b'_2, \ldots, b'_1).
\]
In particular, \( b'_1 \) is the product of the \( a'_i, \) \( i \in S(1), \) in a certain order. It follows from \( \epsilon_i = \pm 1 \) above that the product of the \( \epsilon_i, \) \( i \in S(1), \) equals 1.
Therefore, by the definition of \( a'_i \) and \( S(s - l) = \sigma' S(1), \) we can show that the
eigenpolynomial of \( b'_1 \) is equal to \( p_{s-l}(X), \) as in the proof of Proposition 2.8.
On the other hand, since \( \sigma \alpha = -\alpha, \) \( (\sigma \alpha)^{s/c} = (-1)^{s/c} \alpha^{s/c}. \) By
the proof of Proposition 2.8, \( b'_1 = (-1)^{s/c} b_1. \) Thus we get
\[
p_{s-l}(X) = (-1)^{s/c} p_1((-1)^{s/c} X),
\]
where \( d \) denotes the degree of \( p_1(X), \) whence
\[
p_{s-l}(\tilde{\omega}^{M/c} X^{s/c}) = (-1)^{s/c} p_1(\tilde{\omega}^{M/c} (-X)^{s/c}).
\]
This implies \( \alpha A_1 = A_{s-l}, \) by definition. By the regularity of \( \alpha, \) \( \alpha A_1 = A_{s-l} \neq A_1. \) By Lemma 2.9, this contradicts that \( \alpha \) is elliptic. Hence we must have \( c = (s, M) = 1 \) in case \( \theta \subseteq \Pi \) is (a) or (b) in 1.10.
For $\theta \subseteq \hat{\Pi}$ of (c) and (d) in 1.10, we apply the above argument to $w_{0}(\sigma \alpha_{0})w_{0}^{-1}$ to get the fact that $\sigma A_{1} = A_{1} - 1 \neq A_{1}$ if $c = (s, M) \geq 3$. This is a contradiction as above, and we hence must have $c = (s, M) \leq 2$. We leave the details to the reader. This completes the proof.

2.11. PROPOSITION. Let the notations and assumptions be as in Proposition 2.10. Put $c = (s, M)$. Then $\theta \subseteq \hat{\Pi}$, $s = s(\theta)$, and $c$ are given as follows:

1. $n = \delta r$ with $\delta, r$ integers $\geq 1$,
   $$\theta = \hat{\Pi} - \{ \xi_{0}, \xi_{r}, \xi_{2r}, \dots, \xi_{\delta r} = \xi_{n} \},$$
   $s = s(\theta) = 2\delta$, and $c = 1$.

2. $n = (2\delta + 1)(r/2)$ with $\delta$ an integer $\geq 1$, $r$ an even integer $\geq 2$,
   $$\theta = \hat{\Pi} - \{ \xi_{0}, \xi_{2(r/2)}, \xi_{4(r/2)}, \dots, \xi_{2\delta(r/2)} = \xi_{n} \},$$
   $s = s(\theta) = 2\delta + 1$, and $c = 1$.

3. $n = (2\delta + 1)(r/2)$ with $\delta$ an integer $\geq 1$, $r$ an even integer $\geq 2$,
   $$\theta = \hat{\Pi} - \{ \xi_{r/2}, \xi_{3(r/2)}, \xi_{(2\delta + 1)(r/2)} = \xi_{n} \},$$
   $s = s(\theta) = 2\delta + 1$, and $c = 1$.

4. $n = (2\delta + 2)(r/2)$ with $\delta$ an integer $\geq 1$, $r$ an even integer $\geq 2$,
   $$\theta = \hat{\Pi} - \{ \xi_{r/2}, \xi_{3(r/2)}, \xi_{(2\delta + 1)(r/2)} \},$$
   $s = s(\theta) = 2\delta + 2$, and $c = 1$.

5. $n = (\delta + 1)(r_{1} + r_{2})/2$ with $\delta$ an even integer $\geq 2$, $r_{1}, r_{2}, r_{1} \neq r_{2}$, even integers $\geq 2$,
   $$\theta = \hat{\Pi} - \{ \xi_{r_{1}/2}, \xi_{r_{2}/2 + r_{1}}, \xi_{3r_{1}/2 + r_{2}}, \xi_{3r_{1}/2 + 2r_{2}}, \dots, \xi_{(\delta + 1)r_{1}/2 + \delta r_{2}/2} \},$$
   $s = s(\theta) = 2\delta + 2$, and $c = 2$.

6. $n = (\delta + 1)(r_{1} + r_{2})/2$ with $\delta$ an odd integer $\geq 1$, $r_{1}$ an even integer $\geq 2$, and $r_{2} \neq r_{1}$ an integer $\geq 1$,
   $$\theta = \hat{\Pi} - \{ \xi_{r_{1}/2}, \xi_{r_{2}/2 + r_{1}}, \xi_{3r_{1}/2 + r_{2}}, \xi_{3r_{1}/2 + 2r_{2}}, \xi_{(\delta + 1)r_{1}/2 + (\delta + 1)r_{2}/2} \},$$
   $s = s(\theta) = 2\delta + 2$, and $c = 2$.

Proof. Put $c = (s, M)$ as before. We first assume $c = 1$, that is,
$$\lambda_{1} = \lambda_{2} = \cdots = \lambda_{s}.$$
Then, by Lemma 1.14, according to (a), (b), (c), and (d) in 1.10, we easily get

1. \( t = \delta \geq 1 \); \( n_1 = n_2 = \cdots = n_t = r \geq 1 \),
2. \( t = \delta + 1 \geq 2 \); \( n_1 = \cdots = n_{t-1} = 2n_t = r \geq 2 \),
3. \( t = \delta + 1 \geq 2 \); \( 2n_1 = n_2 = \cdots = n_t = r \geq 2 \),
4. \( t = \delta + 2 \geq 2 \); \( 2n_1 = n_2 = \cdots = n_{t-1} = 2n_t = r \geq 2 \)

for some integers \( \delta, r \). We remark that \( t = \delta + 1 \geq 2 \) in (2) follows from the assumption \( \epsilon \not\in \mathcal{M} \). By 1.10, we can see at once that these (1), (2), (3), and (4) are the desired.

We next assume \( c = 2 \), that is,

\[
\lambda_k = \lambda_{k+2} = \cdots = \lambda_{k+r-2} \quad \text{for } k = 1, 2
\]

by Lemma 2.6. Then, by Proposition 2.10, \( \theta \subseteq \overline{I} \) is (c) or (d) in 1.10. Let \( S(k) \) be the set (2.6.1). Then, by the proof of Lemma 2.6, \( (1, 2, \ldots, s) = S(1) \cup S(2) \), \( S(1) = \{ \text{odd} \} \), and \( S(2) = \{ \text{even} \} \). Let \( \theta \subseteq \overline{I} \) be (c) in 1.10.

Then, by the assumption, we have \( \lambda_{r-1} \neq \lambda_r \). But \( \lambda_{r-1} = n_t = \lambda_t \) by Lemma 1.14. This is a contradiction. Thus (c) does not occur in this case.

So, let \( \theta \subseteq \overline{I} \) be (d) in 1.10. Then, again by Lemma 1.14, if \( \delta \) is even, we can put

5. \( t = \delta + 2 \geq 2 \); \( 2n_1 = n_3 = \cdots = n_{\delta+1} = r_1 \geq 2 \), and

\[
n_2 = n_4 = \cdots = n_{\delta+2} = 2n_{\delta+2} = r_2 \geq 2,
\]

and, if \( \delta \) is odd,

6. \( t = \delta + 2 \geq 3 \); \( 2n_1 = n_3 = \cdots = n_\delta = 2n_{\delta+2} = r_1 \geq 2 \), and

\[
n_2 = n_4 = \cdots = n_{\delta+1} = r_2 \geq 1
\]

for some integers \( \delta, r_1 \), and \( r_2 \). By assumption, \( r_1 \neq r_2 \). In fact, if \( r_1 = r_2 \), we have \( c = 1 \). Similarly, we can see at once that (5) and (6) are the desired, by 1.10. Hence the proof is complete.

2.12. In the proof of Proposition 2.11, for \( \theta \subseteq \overline{I} \) of (1), (2), (3), and (4), we have seen

\[
\lambda_1 = \lambda_2 = \cdots = \lambda_s.
\]

By [6, 1.18], this is equivalent to

\[
\Psi_{\theta} = \Psi_{\mathcal{M}} z = z \Psi_{\mathcal{M}} \quad \text{for some } z \in \mathcal{M}_{2n}(F).
\]
On the other hand, for \( \theta \subset \tilde{\Pi} \) of (5) and (6) in Proposition 2.11, we have

\[
\dim_F(L_i/L_{i+2}) = \lambda_1 + \lambda_2 = r_1 + r_2
\]

for all \( i \) (cf. Lemma 1.14). This shows

\[
\Psi^2 = \Psi z = z \Psi \quad \text{for some } z \in M_{2n}(F)
\]

again by [6, 1.18].

Therefore, if \( c = (s, M) = 1 \), \( \Psi \) is principal, and if \( c = (s, M) = 2 \), \( \Psi \) is principal.

2.13. The Dynkin diagrams of \( \theta \subset \tilde{\Pi} \) in Proposition 2.11 are \( \delta A_{r-1} \) for (1), \( \delta A_{r-1} + C_{r/2} \) for (2), and so on, where \( \alpha X_\delta \) denotes the direct sum of \( a \) times of \( X_\delta \) type. We can express these types multiplicatively as follows:

- \( (A_{r-1}^\delta) \) for (1),
- \( (A_{r-1}^\delta C_{r/2}^1) \) for (2), (3),
- \( (A_{r-1}^\delta C_{r/2}^2) \) for (4),
- \( (A_{r-1}^{\delta/2} C_{r/2}^1 A_{r-2}^{\delta/2} C_{r/2}^1) \) for (5),
- \( (A_{r-1}^{\delta/2} C_{r/2} C_{r/2} C_{r/2}^2) \) for (6).

To distinguish \( \theta \subset \tilde{\Pi} \) of (2) and (3), we further say those by

- \( (A_{r-1}^\delta C_{r/2}^1) \) for (2),
- \( (A_{r-1}^\delta C_{r/2}^1)^{-1} \) for (3).

2.14. We have described \( \theta \subset \tilde{\Pi} \) when \( s \notdiv M \). So we now consider the case where \( s \div M \).

**Proposition.** Let \( M \) be an integer. Assume that \( s \div M \) and that \( \Psi_{\theta, M} \) contains an elliptic regular semisimple element of the block form in 2.7. Then we have (1) \( \theta = \tilde{\Pi} - \{ \xi_n \} = \Pi \) and \( s = s(\theta) = 1 \), (2) \( \theta = \tilde{\Pi} - \{ \xi_n \} \) and \( s = s(\theta) = 1 \), or (3) \( \theta = \tilde{\Pi} - \{ \xi_n \} \) with \( 1 \leq n_1 \leq n - 1 \) and \( s = s(\theta) = 2 \).

**Proof.** Since the proof is analogous to that of Proposition 2.10, we will only give an outline of the proof.

Let \( \alpha \) be the element in the assertion. Then we have \( A = F[\alpha] = \Phi_{i=1}^s A_i \) (cf. the proof of Proposition 2.10). Then the involution \( \sigma \) acts on \( A \):

\[
\sigma A_i = A_{\sigma(i)}, \quad 1 \leq i \leq s,
\]
where, for each \( \theta \in \Phi \) in 1.10, the permutation \( \sigma' \) is given as follows:

\[
\begin{pmatrix}
1 & \cdots & t & t + 1 & \cdots & 2t \\
2t & \cdots & t + 1 & t & \cdots & 1
\end{pmatrix}
\]

for (a) in 1.10,

\[
\begin{pmatrix}
1 & \cdots & t - 1 & t & t + 1 & \cdots & 2t - 1 \\
2t - 1 & \cdots & t + 1 & t & \cdots & 1
\end{pmatrix}
\]

for (b),

\[
\begin{pmatrix}
1 & \cdots & t - 1 & t & \cdots & 2t - 2 & 2t - 1 \\
2t - 2 & \cdots & t & t - 1 & \cdots & 2t - 1
\end{pmatrix}
\]

for (c),

\[
\begin{pmatrix}
1 & 2 & \cdots & t - 1 & t & t + 1 & \cdots & 2t - 2 \\
2t - 2 & \cdots & t + 1 & t & \cdots & 2
\end{pmatrix}
\]

for (d).

Hence, by Lemma 2.9, \( \theta \in \Phi \) and \( s = s(\theta) \) are determined as in the assertion. The proof is completed.

2.15. Remarks. (i) The Dynkin diagrams of \( \theta \) in Proposition 2.14 are of type \( \mathrm{C}_n \) and \( \mathrm{C}_n \), respectively. Putting \( n_2 = r_1 / 2 \) and \( n - n_2 = r_2 / 2 \), we can rewrite the latter type by \( \mathrm{C}_n \). Hence these types are contained in those \( \delta = 0 \) and \( r_1 = r_2 \) are admitted.

(ii) In the case of \( s \mid M \), we have \( (s, M) = s \). Thus, if \( c = (s, M) = 1 \), \( \mathfrak{s}^s = \mathfrak{s} \mathfrak{s} \), and if \( c = (s, M) = 2 \), \( \mathfrak{s}^s = \mathfrak{s} \mathfrak{s} \), as in the case of \( s \not\mid M \) (see 2.12).

2.16. From now on, we assume that \( \theta \in \Phi \) is one of those in Propositions 2.11 and 2.14. We call \( \theta \in \Phi \) Type I if its Dynkin diagram is \( \mathrm{A}_1 \), \( \mathrm{A}_1 \), or \( \mathrm{A}_1 \), and Type II if it is \( \mathrm{A}_1 \), \( \mathrm{A}_1 \), or \( \mathrm{A}_1 \).

2.17. We generalize the definition of [8, 3.2] for \( GL_n \) to get that for \( Sp_{2n} \) naturally as follows:

**Definition.** Let \( s = s(\theta) \) and let \( M \) be an integer. An element \( \alpha \in \mathfrak{g}(s, M) \) with \( \alpha \not\in \mathfrak{g}(s, M + 1) \) is called very cuspidal if the following conditions are satisfied:

(a) \( c = (s, M) = 1 \) (resp. 2) if \( \theta \in \Phi \) is of Type I (resp. Type II) (cf. Propositions 2.11 and 2.14);

(b) the element \( \alpha \) is semisimple in \( M_{2n}(F) \) such that the \( F \)-algebra \( A = F[\alpha] \) and \( \alpha \) are decomposed into \( A = \bigoplus_{i=1}^{u} E_i \) and \( \alpha = (a_1, \ldots, a_u) \) with \( a_i \neq 0 \), \( 1 \leq i \leq u \) (see 2.9) which satisfy

1. \( \sigma a_i = -a_i \),
2. \( E_i = F[a_i] \) is a field,
3. the order of \( a_i \) in \( E_i \) equals \( M/c \),
4. the order of ramification of \( E_i \) over \( F \) equals \( s/c \),
(5) $\tilde{\omega}^{M/c}a_i^{x/c}$ mod $\mathcal{P}_i$ generates the residue class field $\mathcal{E}_i = \mathcal{O}_{E_i}/\mathcal{P}_i$ over $F$, where $\mathcal{O}_{E_i}$ and $\mathcal{P}_i$ denote the maximal order of $E_i$ and its maximal ideal, respectively;

(c) the dimension of the subspace $Z_{\alpha}(\alpha) = \{x \in \mathfrak{g} | x\alpha = \alpha x\}$ of $\mathfrak{g}$ over $F$ is equal to $2n$.

2.18. Remarks. (i) The very cuspidal elements of $\mathfrak{g}$ are elliptic regular semisimple (see 2.9).

(ii) Since $(s/c, M/c) = 1$, conditions (3), (4), and (5) in Definition 2.17 for $a_i$ of $E_i$ are nothing but the definitions of "cuspidal relative to $E_i/F$" in [8, 3.2] and [27, 1.1], and "minimal over $F$" in [7, 1.4.14].

2.19. We will study the properties of very cuspidal elements of Definition 2.17. To do so, we prepare the following.

Lemma. Let $\theta \in \mathfrak{g}$ and let an integer $M$ be as in Definition 2.17. Let $\alpha \in \mathfrak{g}_0$ with $\alpha \notin \mathfrak{g}_0[M] + 1$ be semisimple (not always regular) and $A = F[\alpha]$ which is stable under the involution $\sigma$. Let $\mathcal{O}_A$ and $\mathcal{P}_A$ be the maximal $\mathcal{O}$-order and the maximal ideal, respectively. Then

$\mathcal{O}_A \cap \mathfrak{g}_0 = \mathcal{O}_A$ and $\mathcal{O}_A \cap \mathfrak{g}_0 \subset \mathcal{P}_A$.

If $\alpha$ is very cuspidal, the latter is also equal.

Proof. We define the function $w: M_{2n}(F) \to \mathbb{Z} \cup \{\infty\}$ by

$$w(x) = \sup\{u \in \mathbb{Z} | x \in \mathfrak{g}_0^u\}, \quad x \in M_{2n}(F).$$

Put $c = (s, M)$. Then $w$ is an $F$-form on the $F$-algebra $A$ which satisfies the following conditions:

$$w(x + y) \geq \inf\{w(x), w(y)\}, \quad x, y \in A,$n$$
$$w(xy) \geq w(x) + w(y), \quad x, y \in A^\times,$n$$
$$w(x) = \infty \quad \text{if and only if } x = 0,$n$$
$$w(\lambda x) = w(x) + \nu_F(\lambda), \quad \lambda \in F, x \in A,$n$$

where $\nu_F$ is the normalized valuation of $F$. The set $\mathcal{O}_A = \bigoplus_{i=1}^n \mathcal{O}_{E_i}$ is the unique maximal $\mathcal{O}$-order in $A$ (cf. [9]). Put

$$\Lambda = \{x \in A | w(x) \geq 0\}.$n$$

Then $\Lambda = A \cap \mathfrak{g}_0$. We claim $\Lambda = \mathcal{O}_A$. First assume $(s(\theta), M) = 1$, that is, $\mathfrak{g}_0 = z_1 \mathfrak{g}_0 = \mathfrak{g}_0 z_1$. Then we further have

$$w(xy) = w(x) + w(y), \quad x, y \in A^\times,$n$$
$$w(x + y) = w(x) \quad \text{if } w(x) < w(y) \text{ for } x, y \in A.$$
The eigenpolynomial of every element \( x \) of \( \Lambda \) has coefficients in \( \mathcal{O} \), because \( x \in \mathcal{O}_\theta \) (cf. 2.3). Thus \( \Lambda \) is integral over \( \mathcal{O} \) so that \( \Lambda \) is contained in \( \mathcal{O}_\theta \) by [9, Prop. 26.10]. This shows the first inclusions. We show that \( \Lambda \) is integral closure in \( A \). This fact shows \( \Lambda = \mathcal{O}_\theta \) again by [9, Prop. 26.10].

In fact, let \( x \in A \) be integral over \( \mathcal{O} \), that is,

\[
x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0,
\]

where \( a_i \in \mathcal{O} \), \( 0 \leq i \leq n-1 \), and \( a_0 \neq 0 \). Then

\[
w(x) + w(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1) \geq 0.
\]

Now assume \( w(x) < 0 \). Then

\[
w(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1) > 0.
\]

On the other hand, the left-hand side equals \( w(x^{n-1}) = (n-1)w(x) < 0 \). This is a contradiction. Hence \( w(x) \geq 0 \), that is, \( x \in \Lambda \), which hence shows \( \mathcal{O}_\theta \cap A = \Lambda = \mathcal{O}_\theta \) in the case of \( (s(\theta), M) = 1 \).

Next assume \( (s(\theta), M) = 2 \), that is, \( \mathcal{O}_\theta^2 = z_2 \mathcal{O}_\theta = \mathcal{O}_\theta z_2 \). From \( \mathcal{L}_\theta = (L_i)_i \in \mathbb{Z} \), we obtain the flags of \( \mathcal{O} \)-lattices in \( X = F^{2n} \): \( \mathcal{L}_\theta^1 = (L_{2j+1})_j \in \mathbb{Z} \) and \( \mathcal{L}_\theta^2 = (L_{2j})_j \in \mathbb{Z} \) as in the proof of Proposition 2.23. Let \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) be the hereditary \( \mathcal{O} \)-order of \( M_{2n}(F) \) and its Jacobson radical defined by \( \mathcal{L}_\theta^1 \), \( j = 1, 2 \). Then, by definition, \( \mathcal{O}_1 = \mathcal{O}_\theta \cap \mathcal{O}_2 \) and \( \mathcal{O}_2 = z_2 \mathcal{O}_1 \), \( j = 1, 2 \). Thus, by the above argument in the case of \( (s(\theta), M) = 1 \), we get \( A \cap \mathcal{O}_j = \mathcal{O}_\theta \), \( j = 1, 2 \), which shows \( A \cap \mathcal{O}_1 = (A \cap \mathcal{O}_2) \cap (A \cap \mathcal{O}_2) = \mathcal{O}_\theta \) as required. The second inclusion \( A \cap \mathcal{O}_\theta \subseteq \mathcal{O}_\theta \) is easily seen.

Finally, if \( \alpha \) is very cuspidal, in the case of \( (s(\theta), M) = 1 \), it follows from Definition 2.17 that \( A \cap \mathcal{O}_\theta = \mathcal{O}_\theta \) by comparing the dimensions of the algebra \( (A \cap \mathcal{O}_\theta)/(A \cap \mathcal{O}_\theta) \) and the algebra \( \mathcal{O}_A/\mathcal{O}_\theta \) over \( F \). In the case of \( (s(\theta), M) = 2 \), similarly, we get \( A \cap \mathcal{O}_\theta = A \cap \mathcal{O}_1 \cap (A \cap \mathcal{O}_2) = \mathcal{O}_\theta \) and \( A \cap \mathcal{O}_\theta^2 = (A \cap \mathcal{O}_1) \cap (A \cap \mathcal{O}_2) = \mathcal{O}_\theta \). Consequently, \( A \cap \mathcal{O}_\theta = A \cap \mathcal{O}_1 \cap \mathcal{O}_\theta = \mathcal{O}_\theta \). The proof is completed.

2.20. Assume that \( \theta \subseteq \Pi \) and \( M \) are as in Definition 2.17. Put \( c = (s, M) \). We define the canonical injective ring homomorphism

\[
R_c: \mathcal{O}_\theta/\mathcal{O}_\theta \rightarrow \prod_{i=0}^{s-c-1} \text{End}_F(L_{e_i}/L_{e_i+c})
\]

as follows: When \( c = 1 \), let \( R_1 \) be the canonical isomorphism in 2.3. When \( c = 2 \), put \( R' = R_2 \). We have the canonical injection

\[
\text{End}_F(L_{2i}/L_{2i+1}) \times \text{End}_F(L_{2i+1}/L_{2i+2}) \rightarrow \text{End}_F(L_{2i}/L_{2i+2})
\]
so that the image is a Levi subgroup, for each \(0 \leq i \leq s/2 - 1\). Let \(R^n\) be the product of these injections over \(i\), \(0 \leq i \leq s/2 - 1\), that is,

\[
R^n: \prod_{i=0}^{s-1} \text{End}_F(L_{c_i}/L_{c_{i+1}}) \to \prod_{i=0}^{s/2-1} \text{End}_F(L_{2i}/L_{2i+2}).
\]

Define \(R_2 = R^n \circ R^r\) the composition of \(R^n\) and \(R^r\). By 2.12.

\[
\dim_F(L_{c_i}/L_{c_{i+c}}) = 2nc/s, \quad 0 \leq i \leq s/c - 1.
\]

Write \(z_c\) for the element \(z \in GL_{2n}(F)\) in 2.12 which satisfies \(\Psi_\theta = z_c \Psi_\theta = \Psi_\theta z_c = \Psi_\theta z^s = \Psi_\theta 1_{2n}\).

2.21. Lemma. Put \(s = s(\theta)\). Let \(\alpha \in \mathfrak{z}_{\theta, M}\) be a very cuspidal element, and \(c = (s, M)\). Then \(R(\tilde{\omega}^{-M}/\tilde{\omega}^s/c)\) generates a commutative semisimple algebra of dimension \(2nc/s\) over \(\overline{F}\).

Proof. Denote by \(r\) the dimension of the subalgebra of \(\prod \text{End}_F(L_{c_i}/L_{c_{i+c}})\) generated by \(R(\tilde{\omega}^{-M}/\tilde{\omega}^s/c)\) over \(\overline{F}\). We have \(R(\tilde{\omega}^{-M}/\tilde{\omega}^s/c) = (x_1, \ldots, x_{s/c})\), and the \(x_i\) can be regarded as the matrices in \(M_{2nc/s}(\overline{F})\) (cf. 2.20). Since \((s/c, M/c) = 1\), by [8, 3.1], the \(x_i\) have the same eigenpolynomial over \(\overline{F}\) (cf. the proof of Proposition 2.8). Thus \(r \leq 2nc/s\). Assume that \(r < 2nc/s\). Then, by Definition 2.17(3)-(5),

\[
\sum_{i=1}^{\mu} [E_i: \overline{F}] = r < 2nc/s,
\]

where \(A = F[\alpha] = \bigoplus_{i=1}^{\mu} E_i\) is as in Definition 2.17 and \([E_i: \overline{F}]\) denotes the extension degree of \(E_i\) over \(\overline{F}\) for each \(i\). This is the modular degree of \(E_i\) over \(\overline{F}\). Since \(\alpha\) is regular semisimple (cf. Remark 2.18(i)),

\[
2n = \sum_{i=1}^{\mu} [E_i: \overline{F}] = \sum_{i=1}^{\mu} [E_i: \overline{F}]s/c < (2nc/s)(s/c) = 2n.
\]

This is a contradiction. Thus \(r = 2nc/s\), which completes the proof.

2.22. Let \(\alpha \in \mathfrak{z}_{\theta, M}\) be a very cuspidal element and \(A = F[\alpha] = \bigoplus_{i=1}^{\mu} E_i\) be as above. Then the base space \(X\) can be regarded as an \(A\)-module and be identified with \(A\) itself, because \(\dim_F X = \dim_F A\) . Thus there are canonical isomorphisms

\[
Z_{M_2(F)}(\alpha) \simeq \text{End}_A(X) = A,
\]

where \(Z_{M_2(F)}(\alpha)\) denotes the centralizer of \(\alpha\) in \(M_2(F)\). By [25, IV, Sect. 2], associated with \(\alpha\) in \(\mathfrak{z}\), there is a \(\alpha\)-sesquilinear form \(F_\alpha\).
$A \times A \rightarrow A$ such that

$$f(ax, y) = \text{tr}_{A/F}(aF_a(x, y)), \quad x, y \in X = A, \quad a \in A,$$

where $\text{tr}_{A/F}$ denotes the trace in $A$ over $F$. According to the decomposition $A = \bigoplus_{i=1}^u E_i$, there is an element $\mu = (\mu_i) \in A$ with $\sigma \mu_i = -\mu_i$ for all $i$ such that

$$f((x_i), (y_i)) = \sum_{i=1}^u \text{tr}_{E_i/F}(\mu_i x_i \sigma y_i)$$

for $(x_i), (y_i) \in A = \bigoplus_{i=1}^u E_i$, where $\text{tr}_{E_i/F}$ denotes the trace map of $E_i$ into $F$ for each $i$ (cf. [17, 1.18]).

22.3. Proposition. Let $\alpha \in \mathfrak{u}_{\theta, M}$ be a very cuspidal element and $A = F[\alpha] = \bigoplus_{i=1}^u E_i$ be as above. Then

$$A \cap \mathfrak{u}_{\theta, k} = \begin{cases} \bigoplus_{i=1}^u (\mathcal{P}_E^k) & \text{if } (s, M) = 1, \\ \bigoplus_{i=1}^u (\mathcal{P}_E^{(k+1)/2}) & \text{if } (s, M) = 2 \end{cases}$$

for $k \in \mathbb{Z}$, where $(\mathcal{P}_E^k)_- = \{x \in \mathcal{P}_E^k \mid \sigma x + x = 0\}$ and $[(k + 1)/2]$ denotes the greatest integer among the integers less than or equal to $(k + 1)/2$.

Proof. Put $c = (s, M)$. $\mathcal{P}_A = \bigoplus_{i=1}^u \mathcal{P}_E^i$ is the maximal ideal of $\mathcal{O}_A = \bigoplus_{i=1}^u \mathcal{O}_E$. There is a prime element $\omega_a$ of $\mathcal{O}_A$ and $\mathcal{P}_A = \omega_a \mathcal{O}_A$. Let $w$ be the function (2.19.1). Then the properties of $w$ in Lemma 2.19 easily show $w(\omega_a) = c$. If $c = (s, M) = 1$, $\mathfrak{u}_{\theta} = z_2 \mathfrak{u}_{\theta} = \mathfrak{u}_{\theta} z_2$ as before. Thus, by the identification $X = A$ in 2.22, we may set $\mathcal{L}_0 = \{\mathcal{P}_A^k\}_{k \in \mathbb{Z}}$ to determine $A \cap \mathcal{P}_A^k$, $k \in \mathbb{Z}$ by Lemma 2.19 (cf. [8, 3.3]). Since the elements of $A$ act on $X = A$ via the regular representation, we have $A \cap \mathfrak{u}_{\theta}^k = \mathcal{P}_A^k$, $k \in \mathbb{Z}$, which shows this proposition in the case of $(s, M) = 1$.

Let $c = (s, M) = 2$. Then, by 2.16, $\theta \in \Pi$ is of Type II, that is, $(A_{r_1-1/2}^{2s/2}C_{r_2-1/2}^{2s/2})$ with $\delta \geq 0$ or $(A_{r_1}^{\delta-1/2}C_{r_2}^{\delta+1/2})$ with $\delta \geq 1$, where $r_1 = r_2$ is admitted. We note that the corresponding flag of $\mathfrak{c}$-lattices $\mathcal{L}_0 = \{L_i\}_{i \in \mathbb{Z}}$ comes from (d) in Lemma 1.14, as was seen in the proof of Proposition 2.11. By 2.12, $\mathfrak{u}_{\theta} = z_2 \mathfrak{u}_{\theta} = \mathfrak{u}_{\theta} z_2$. Thus we can apply the above argument to $\{L_{2k}\}_{k \in \mathbb{Z}}$ so that we may set

$$L_{2k} = \mathcal{P}_A^k = \bigoplus_{i=1}^u \mathcal{P}_E^k, \quad k \in \mathbb{Z}.$$
By the form of (d) in the proof of Lemma 1.14, we can see at once that, for each \( m \in \mathbb{Z} \), there is an integer \( k = k(m) \) such that

\[
(L_{2m})^* = L_{2k+1}
\]

(see 1.16 for the definition). On the other hand, by 2.22 and [17, 2.4],

\[
(L_{2m})^* = \left( \mathcal{P}_A^m \right)^* = \bigoplus_{i=1}^{u} \mathcal{P}_{E_i}^{-\delta_i - m},
\]

where the \( \delta_i \) are the integers determined by \( \mu = (\mu_i) \) in 2.22 and the differential exponents of the \( E_i \) over \( F \). Thus we have

\[
L_{2k+1} = \bigoplus_{i=1}^{u} \mathcal{P}_{E_i}^{-\delta_i - m}.
\]

By \( L_{2k} \supseteq L_{2k+1} \supseteq L_{2k+2} \) and by the forms of \( L_{2k} \) and \( L_{2k+2} \), we get

\[
\{ -\delta_i - m \mid 1 \leq i \leq t \} = \{ k, k + 1 \}.
\]

Since \( L_{2k+1} = z_L L_{2k-1}, k \in \mathbb{Z} \), rearranging the \( E_i \), we hence have

\[
L_{2k+1} = \bigoplus_{i=1}^{d} \mathcal{P}_{E_i}^{k} \oplus \bigoplus_{i=d+1}^{u} \mathcal{P}_{E_i}^{k+1}
\]

for a fixed integer \( d, 1 < d < u \) (cf. [17, 2.5]). Hence it follows that \( A \cap \mathfrak{g}_{\theta, k}, k \in \mathbb{Z} \), have the forms of this proposition in case \( (s, M) = 2 \). This completes the proof.

2.24. Let \( \alpha \in \mathfrak{g}_{\theta, M} \) and \( A = F[\alpha] \) be as in Proposition 2.23. Put

\[
T = A \cap G
\]

as in 2.9.

**Corollary.** Let the notations and assumptions be as above. Then

\[
T \subset P = P(\theta).
\]

**Proof.** We can prove this immediately from the fact that \( T = A \cap G \subset \mathcal{O}_A^\times \) by the proof of Proposition 2.23.

2.25. **Proposition.** Let \( \theta \subset \mathfrak{h} \) and let an integer \( M \) be as in Definition 2.17. Let \( \beta \in \mathfrak{g}_{\theta, M} \) be a very cuspidal element. Then there is a very cuspidal element \( \alpha \in \mathfrak{g}_{\theta, M} \) which has the block form as in 2.6 or 2.7 and is congruent to \( \beta \) mod \( \mathcal{P}_{\theta}^{M+1} \).
Proof. (cf. [8, Proof of Prop. 3.3]). We claim that if $\alpha \in \beta(1 + \mathfrak{F}_0)$ and $\alpha \in \mathfrak{g}_{0, \mathfrak{m}}$, then $\alpha$ is also very cuspidal. To prove this, we start by proving that the $\mathfrak{F}$-algebra $A = \mathfrak{F}[\alpha]$ generated by this $\alpha$ is commutative and semisimple. By [2, Sect. 2, n°3; Sect. 5, n°5, Prop. 9], it is sufficient to prove that $A = \mathfrak{F}[\alpha]$ has no nonzero nilpotent element. Let $\nu_p$ be the normalized valuation on $\mathfrak{F}$ and let $w$ be the function defined by (2.19.1). Then $w(\alpha) = w(\beta) = M$. Put $s = s(\theta)$ and $c = (s, M)$. By the assumption $\alpha \in \beta(1 + \mathfrak{F}_0)$,

$$
\tilde{\omega}^{-M/c^{s/c}} \text{ mod } \mathfrak{F}_0 = \tilde{\omega}^{-M/c^{s/c}} \text{ mod } \mathfrak{F}_0,
$$

(2.25.1)

and $\mathfrak{F}_0^{\alpha^{s/c}} \tilde{\omega}^{-M/c} \text{ mod } \mathfrak{F}_0$ is a commutative semisimple $\mathfrak{F}$-algebra of dimension $2nc/s$ in $\mathfrak{g}_0/\mathfrak{g}_0$, because $\beta$ is very cuspidal. Let $x = \sum_{i=0}^{2n-1} x_i \alpha_i$, $x_i \in \mathfrak{F}$ with some $x_i \neq 0$. Assume that $x^m = 0$ for some integer $m \geq 1$. If $x_i = 0$, put

$$
\nu(i) = w(x_i \alpha_i) = s \nu_p(x_i) + Mi,
$$

where we note that $w(\alpha_i) = w(\alpha)$ because $\mathfrak{g}_0$ or $\mathfrak{g}_0^2$ is principal. If $x_i = 0$, put $\nu(i) = \infty$. Let $n_0 = \min(\nu(i) \mid 0 \leq i < 2n)$ and $i_0 = \min(\nu(i) = n_0)$. Since $(s/c, M/c) = 1$, $\nu(i) = n_0$ implies $i = i_0 \text{ mod } s/c$, that is, $i = i_0 + ks/c, 0 \leq k < 2nc/s$. Thus

$$
x_i \alpha_i = x_{i_0} \alpha_{i_0} y_k \tilde{\omega}^{-M/c^{s/c}}^k,
$$

where $y_k = \tilde{\omega}^{kM/c} x_{i_0 + ks/c} x_{i_0}^{-1} x_{i_0}$. We can easily check $\nu_p(y_k) = 0$, whence $y_k \in \mathfrak{g}_0$. Put

$$
g = \sum x_i \alpha_i = x_{i_0} \alpha_{i_0} \sum_{k=0}^{2n-1} y_k \tilde{\omega}^{-M/c^{s/c}}^k,
$$

where the indices $i$ in the first summation run over $i = i_0 \text{ mod } s/c$. Then $g$ can be expressed by $g = x_{i_0} \alpha_{i_0} d$ with $d \in \mathfrak{g}_0$. If $i \neq i_0 \text{ mod } s/c$, then $x_i \alpha_i \in \mathfrak{g}_0^{n_0+1}$. It follows that $x = g + h$ for some $h \in \mathfrak{g}_0^{n_0+1}$. We note that $x, g \in \mathfrak{g}_0^{n_0}$ and $x \notin \mathfrak{g}_0^{n_0+1}$. By the assumption $x^m = 0$,

$$
0 = x^m \equiv g^m \text{ mod } \mathfrak{g}_0^{n_0+1}.
$$

Since $x_{i_0} \alpha_{i_0}$ commutes with $d$, we further have $g^m \equiv (x_{i_0} \alpha_{i_0})^m d^m \text{ mod } \mathfrak{g}_0^{n_0+1}$, whence $d^m \in \mathfrak{g}_0$, because $w(x_{i_0} \alpha_{i_0}) = n_0$. Hence $d^m = 0$ in $\mathfrak{F}[\tilde{\omega}^{-M/c^{s/c}} \text{ mod } \mathfrak{g}_0] \subset \mathfrak{g}_0/\mathfrak{g}_0$. It follows that $d = 0$, that is, $d \in \mathfrak{g}_0$, so that $w(g) \geq n_0 + 1$. This contradicts $w(g) = n_0$. Consequently, $x_i = 0$ for all $i$, that is, $x = 0$. This shows that $A = \mathfrak{F}[\alpha]$ is reduced. Hence it is
commutative semisimple as claimed above. We have $A = \bigoplus_{i=1}^{u} E_i$, where each $E_i$ is a field extension of $F$, and $\alpha = (a_i) \in A = \bigoplus_{i=1}^{u} E_i$. We prove that $\alpha$ is very cuspidal. To do so, it is sufficient to prove that each $a_i$ satisfies conditions (3), (4), and (5) in (b) and (c) of Definition 2.17. By Lemma 2.19, $A \cap \mathcal{S}_\theta \subset \mathcal{S}_\theta^*$ and $A \cap \mathcal{S}_\psi = \bigoplus_{i=1}^{u} \mathcal{P}_E \subset \mathcal{P}_E$. Denote by $\omega_{E_i}$ a prime element of $E_i$ with $\omega_{E_i}^c = \omega_{E_i}$, where $\omega_{E_i}$ is in $\mathcal{S}_E^\times$. This implies that the order of ramification of $E_i$ over $F$ is equal to $e_i$. Let $M'_i$ be the order of $a_i$ in $E_i$. Then $a_i$ can be written in the form $a_i = \omega_{E_i}^{M'_i} u_i$, for some $u_i \in \mathcal{S}_E^\times$. Hence

$$\omega_{E_i}^{M'_i/c - M_{E_i}/c} u_i^{1/c} \in \mathcal{S}_E^\times,$$

which shows $M'_i = Me_i/c$. Since $(s/c, M/c) = 1$, it follows that $s/c$ divides $e_i$, whence $e_i = sk_i/c$ for some integer $k_i \geq 1$. By (2.25.1) and by the very cuspidality of $\beta$, $\omega_{E_i}^{-M'/c} a_i^{1/c} = (\omega_{E_i}^{-M'/c} a_i^{1/c}) \mod \mathcal{P}_E$ generates a commutative semisimple $\mathcal{F}$-algebra of dimension $2nc/s$. Put

$$K_i = \mathcal{F} \left[ \omega_{E_i}^{-M'/c} a_i^{1/c} \mod \mathcal{P}_E \right].$$

Then $K_i \subset E_i$ and hence

$$\sum_{i=1}^{u} \left[ E_i : \mathcal{F} \right] \geq \sum_{i=1}^{u} \left[ K_i : \mathcal{F} \right] = 2nc/s.$$

This gives

$$s/c \sum_{i=1}^{u} \left[ E_i : \mathcal{F} \right] \geq 2n = \sum_{i=1}^{u} e_i \left[ E_i : \mathcal{F} \right],$$

whence $s/c \geq e_i = sk_i/c$. Hence $k_i = 1$ so that $e_i = s/c$ and $M'_i = M/c$. Further $[E_i : \mathcal{F}] = [K_i : \mathcal{F}]$ for all $i$ and so

$$\sum_{i=1}^{u} [E_i : \mathcal{F}] = 2n.$$

Hence conditions (3), (4), and (5) in (b) and (c) of Definition 2.17 are satisfied, whence $\alpha$ is very cuspidal as claimed first.

Finally, we can decompose the very cuspidal element $\beta \in \mathcal{S}_\theta, M$ as $\beta = \beta_0 + \beta_1$, where $\beta_1 \in \mathcal{S}_\theta, M+1$, $\beta_0 \in \mathcal{S}_\theta, M$, $\beta_0 \notin \mathcal{S}_\theta, M+1$, and $\beta_0$ has such a block form as (2.6.4), (2.6.5), or (2.7.1). We note that this decomposition is not always unique. Since $\mathcal{S}_\theta^c$, with $c = (s, M)$, is principal, $\beta^{-1} \in \mathcal{S}_\psi^{-M}$ and $\beta^{-1} \notin \mathcal{S}_\psi^{-M+1}$ by Definition 2.17. Thus we also have $\beta^{-1} = \gamma_0 + \gamma_1$, where $\gamma_1 \in \mathcal{S}_\psi^{-M+1}, \gamma_0 \in \mathcal{S}_\psi^{-M}, \gamma_0 \notin \mathcal{S}_\psi^{-M+1}$, and $\gamma_0$ has the block form as above. Thus $1 = \beta \beta^{-1} = \beta_0 \gamma_0 \mod \mathcal{S}_\theta$. Hence $\beta_0$ is
nonsingular and
\[ \beta = \beta_0(1 + \beta_0^{-1}\beta_1), \]
whence \( \beta_0 \in \beta(1 + \mathfrak{g}_\beta), \) because \( \beta_0^{-1} \in \mathfrak{g}_\beta^{-M} \) by the block form of \( \beta_0. \) Put \( \alpha = \beta_0. \) Then we have proved the very cuspidality of this \( \alpha. \) This completes the proof.

2.26. Let \( g(X) = X^m + \sum_{i=0}^{m-1} b_i X^i \) be a polynomial over a finite Galois extension \( F' \) of \( F. \) Put
\[ N_{F'/F}(g(X)) = \prod_{\gamma} \gamma g(X), \]
where \( \gamma \) in the product runs over the Galois group of \( F'/F \) and \( \gamma g(X) = X^m + \sum_{i=0}^{m-1} \gamma(b_i) X^i. \)

2.27. PROPOSITION. Let \( \alpha \in \mathfrak{g}_\beta M \) be a very cuspidal element of the block form in 2.6 or 2.7. Put \( c = (s(\theta), M). \) Then the monic eigenpolynomial of \( \alpha \) has the form
\[ \prod_{i=1}^{u} N_{F_i/F}(X^{s/c} - \tilde{\omega}^{M/c} \zeta_i), \]
where \( F_i = F[\zeta_i] \) is an unramified extension of degree \( f_i \) over \( F, \) \( 1 \leq i \leq u. \) Further each \( g_i(X) = N_{F_i/F}(X^{s/c} - \tilde{\omega}^{M/c} \zeta_i) \) is irreducible over \( F \) and satisfies
\[ g_i(-X) = (-1)^{s/c} g_i(X). \]

Proof. Let \( p(X) \) be the monic eigenpolynomial of \( \alpha. \) Then, by Proposition 2.8,
\[ p(X) = \tilde{\omega}^{\lambda/c} \prod_{j=1}^{c} p_j(\tilde{\omega}^{M/c} X^{s/c}). \]
We remark that \( c = 1 \) or 2 by Propositions 2.10 and 2.14. Since the \( p_i(X) \) have coefficients in \( \mathfrak{g}, \) by Hensel's lemma, there are polynomials \( h_i(X) \), \( 1 \leq i \leq t, \) in \( \mathfrak{g}[X] \) such that
\[ \prod_{j=1}^{c} p_j(X) = \prod_{i=1}^{u} h_i(X) \]
and the \( h_i(X) = h_i(X) \mod \mathfrak{g} \) are irreducible over \( F. \) Let \( \zeta_i \) be a root of \( h_i(X) = 0 \) for each \( i. \) Then it is well known by local field theory that \( F[\zeta_i] \) is an unramified extension of \( F. \) Thus each \( h_i(X) \) can be expressed in the
following form:

\[ h_t(X) = N_{F_t/F}(X - \zeta_t), \]

where \( F_t = F[\zeta_t]. \) Hence

\[ p(X) = \tilde{\omega}^{\lambda/M/c} \prod_{i=1}^{\mu} N_{F_t/F}(\tilde{\omega}^{-\lambda/M/c} X^{s_i/c} - \zeta_i) = \prod_{i=1}^{\mu} N_{F_t/F}(X^{s_i/c} - \tilde{\omega}^{\lambda/M/c} \zeta_i). \]

Since \((s/c, M/c) = 1,\) by [12, 4.1], the polynomials \( N_{F_t/F}(X^{s_i/c} - \tilde{\omega}^{\lambda/M/c} \zeta_i)\) are irreducible over \( F. \) Thus the last assertion follows from Lemma 2.9 (cf. [25, IV, 2.10]). Hence the proof is complete.

2.28. Remarks. (i) It follows from Propositions 2.25 and 2.27 that every class of \( \beta_{\theta, M}/\beta_{\theta, M+1} \) for some integer \( M, \) which contains a very cuspidal element is obtained from a monic polynomial with coefficients in \( F \) as in Proposition 2.27 (cf. [12, 4.6]).

(ii) \( \beta_{\theta, M} \) for \( \theta \subseteq \Pi \) other than those of Definition 2.17 seems to contain no very cuspidal elements, as Propositions 2.10 and 2.14 suggest.

2.29. We recall principal elements of \( \mathfrak{g} \) defined by Morris [17]. We consider only such elements which are regular semisimple in \( \mathfrak{g} = M_{2n}(F). \)

By [17, 2.22 and 2.23], a regular semisimple element \( \alpha \) (in \( \mathfrak{g} \)) is called principal if the following conditions are satisfied:

(i) \( F[\alpha] = \bigoplus_{i=1}^{u} E_i, \) where \( E_i \) is a tamely ramified extension field of \( F \) for each \( i, \)

(ii) \( \sigma(E_i) = E_i \) and \( \sigma|_{E_i} \neq \text{id} \) for \( 1 \leq i \leq u, \)

(iii) there exists an integer \( e \geq 1 \) such that \( \mathcal{P}_A^e = \tilde{\omega}\mathcal{P}_A, \) where \( \mathcal{P}_A \) and \( \mathcal{P}_A \) are as before for \( A = F[\alpha], \)

(iv) there exists an integer \( s \) such that \( \alpha = (\tilde{\omega}^s b_i) \in A = \bigoplus_{i=1}^{u} E_i, \)

where \( \tilde{\omega} \) is a prime element of \( E_i \) and \( b_i \) is a root of unity in \( E_i^{\times} \) prime to \( p, \) for each \( i. \)

2.30. Proposition. Every regular semisimple and principal element of \( \mathfrak{g} \) is very cuspidal.

Proof. Since \( \alpha \) is regular semisimple in \( \mathfrak{g} \) and belongs to \( \mathfrak{g}, \) it follows from 2.22 that there exists an element \( \mu = (\mu_i) \in A = \bigoplus_{i=1}^{u} E_i \) such that

\[ f((x_i), (y_i)) = \sum_{i=1}^{u} \text{tr}_{E_i/F}(\mu_i x_i \sigma y_i) \]

for \((x_i), (y_i) \in A = \bigoplus_{i=1}^{u} E_i, \) where we identify \( X \) with \( A \) as vector spaces over \( F. \) Thus a self-dual lattice chain \( \mathcal{L} \) in \( X = A \) is obtained by this
\( \mu = (\mu_i) \) as follows:

\[
\mathcal{L} = \{ \mathcal{R}_A^n \}_{n \in \mathbb{Z}} \quad \text{or} \quad \{ \mathcal{R}_A^n \}_{n \in \mathbb{Z}} \cup \{ (\mathcal{R}_A^n)^q \}_{n \in \mathbb{Z}}
\]

as in the proof of Proposition 2.23 (cf. [17, 2.3–2.5]). By Lemma 1.11, we can see that \( \mathcal{L} \) corresponds to a unique subset \( \theta \) of \( \hat{\Pi} \) (cf. [17, 2.14]). This implies \( \mathcal{L} = \mathcal{L}_\theta \) (see 1.15). Let \( \mathfrak{U}_\theta \) and \( \Psi_\theta \) be as in 1.17 and let \( \mathfrak{U}_{\theta,i} \) be as in 1.22. Then, if \( \mathcal{L}_\theta = \{ \mathcal{R}_A^n \}_{n \in \mathbb{Z}} \), we have \( s(\theta) = e \) and \( \alpha \in \mathfrak{U}_{\theta,M}, \alpha \notin \mathfrak{U}_{\theta,M+1} \) with \( M = m \), and if \( \mathcal{L}_\theta = \{ \mathcal{R}_A^n \}_{n \in \mathbb{Z}} \cup \{ (\mathcal{R}_A^n)^q \}_{n \in \mathbb{Z}} \), we have \( s(\theta) = 2m \) and \( \alpha \in \mathfrak{U}_{\theta,M}, \alpha \notin \mathfrak{U}_{\theta,M+1} \) with \( M = 2m \), where \( m \) is the integer as in 2.29. Further it can be easily seen that \( \mathfrak{U}_{\theta,M} \) satisfies condition (P) of 2.2, and so that \( \theta \) is a subset of \( \hat{\Pi} \) in Propositions 2.11 and 2.14. Hence it follows by [27, Theorem I.4] that \( (m, e) = 1 \) and that \( \alpha = (\check{\omega}^m b_\ell) \) satisfies all the conditions of Definition 2.17. The proof is complete.

3. VERY CUSPIDAL REPRESENTATIONS

Throughout this section, let \( \theta \subseteq \hat{\Pi} \) be as in Propositions 2.11 and 2.14, and put \( s = s(\theta) \) and \( P = P(\theta) \).

3.1. Let \( m \) be an integer greater than or equal to 2. By Proposition 1.20, \( P_{m-1}/P_m \) is an abelian group, and, by Proposition 1.25, a character \( \chi \) of \( P_{m-1}/P_m \) corresponds to a unique class \( \alpha + \mathfrak{U}_{\theta,\lambda(m)+1} \) of \( \mathfrak{U}_{\theta,\lambda(m)}/\mathfrak{U}_{\theta,\lambda(m)+1} \) via

\[
\chi(C(x)) = \psi(\langle x, -\alpha \rangle/2), \quad x \in \mathfrak{U}_{\theta,m-1},
\]

where \( \lambda, C, \) and \( \psi \) are as in 1.22–1.25. We say that this \( \chi \) is represented by \( \alpha + \mathfrak{U}_{\theta,\lambda(m)+1} \).

3.2. Definition (cf. [8, 4.1]). A representation \( \rho \) of \( P \) is called very cuspidal of level \( m \geq 1 \) if the following conditions are satisfied:

(i) It is trivial on \( P_m \).

(ii) In case \( m = 1 \), \( P \) is a maximal parahoric subgroup of \( G \) (cf. [21, Prop. 6.8]) and, on \( P \), \( \rho \) is the inflation to \( P \) of a cuspidal representation, in the sense of [10], of \( P/P_1 \) viewed as the group of \( \mathbb{F}_q \)-rational points in a reductive algebraic group over \( \mathbb{F}_q \), where we recall \( \mathbb{F} = \mathbb{F}_q \).

In case \( m \geq 2 \), on \( P_{m-1}/P_m \), \( \rho \) is a sum of characters represented by \( \alpha + \mathfrak{U}_{\theta,\lambda(m)+1} \)'s, where the \( \alpha \)'s are very cuspidal elements in \( \mathfrak{U}_{\theta,\lambda(m)} \).

In case \( m = 1 \), we remark that \( P = P(\theta) \) is a maximal parahoric subgroup of \( G \) if and only if \( \theta = \hat{\Pi} - \{ \xi \} \) for some \( \xi \in \hat{\Pi} \) (cf. Proposition 2.14). In case \( m \geq 2 \), the classes \( \alpha + \mathfrak{U}_{\theta,\lambda(m)+1} \)'s in this definition are mutually conjugate under elements of \( P \) (cf. [12, 3.3]).
3.3. Proposition. Let \( \alpha \in \mathcal{U}_{\alpha, M} \) be a very cuspidal element and let \( x \in \mathcal{X}_0^k \) for some integer \( k \). Suppose that \( x\alpha - \alpha x \in \mathcal{X}_0^{M+k+1} \). Then \( x \in F[\alpha] + \mathcal{X}_0^{k+1} \).

Proof. In the case of \( c = 1 \), this follows directly from [8, Lemma 3.5]. When \( c = 2 \), \( M \) is even, by Lemma 2.21, we can apply the proof of [8, Lemma 3.5] to the injective homomorphism \( R_2 \) in 2.20 and prove the assertion. So, assume that \( c = 2 \) and \( k \) is odd. For the \( \mathcal{E} \)-lattice chain \( \mathcal{D}_0 = \{ L_i \}_{i \in Z} \), let \( \mathcal{D}_i^j \), \( \mu_i \), and \( \mathcal{Y}_i \), \( j = 1, 2 \), be as in the proof of Lemma 2.19. We have the canonical isomorphism

\[
R_{(j)}: \mathcal{U}_j / \mathcal{Y}_j = \prod_{i=0}^{s/2-1} \text{End}_F(L_{2i+2-j}/L_{2i+4-j}), \quad j = 1, 2,
\]

as in 2.3. By definition,

\[
\mathcal{X}_0^{2i} = \mathcal{X}_1 \cap \mathcal{X}_2, \quad i \in Z.
\]

In particular, \( \mathcal{X}_0 = \mathcal{X}_1 \cap \mathcal{X}_2 \) and \( \mathcal{X}_0^{2} = \mathcal{X}_1 \cap \mathcal{X}_2 \).

Thus we get the canonical injections \( \mathcal{U}_0 \rightarrow \mathcal{U}_j, j = 1, 2 \), and the canonical homomorphism

\[
\varphi_j: \mathcal{U}_0 / \mathcal{X}_0^2 \rightarrow \mathcal{U}_j / \mathcal{X}_j, \quad j = 1, 2.
\]

Similarly to Lemma 2.21, we can prove

(i) \( R_{(j)}(\bar{\omega}^{-M/2}z_0^{s/2}) \) generates a commutative semisimple algebra of dimension \( 4n/s \) over \( \overline{F} \), \( j = 1, 2 \).

Let \( \bar{\omega}^{-M/2}z_0^{s/2} \) be the canonical image of the \( \bar{\omega}^{-M/2}z_0^{s/2} \) in \( \mathcal{U}_0 / \mathcal{X}_0^2 \). Then by Proposition 2.23 we can prove

(ii) The restrictions of \( \varphi_j, j = 1, 2 \), to \( \overline{F[\bar{\omega}^{-M/2}z_0^{s/2}]} \) are isomorphisms as \( \overline{F} \)-algebras.

By the assumption, \( \alpha = z_0^{M/2}z_0, \alpha_0 \in \mathcal{U}_0, \) and \( x = z_0^{(k-1)/2}x_0, x_0 \in \mathcal{X}_0 \).

Since \( s/2, M/2 \equiv 1 \), there is a unique integer \( l \), \( 0 \leq l \leq s/2 - 1 \), such that \( lM/2 = (k - 1)/2 \mod s/2 \). For this integer \( l \), define the element \( c \) of \( \mathcal{U}_0 \) by

\[
c = (z_0^{(l-1)/2}z_0^{l/2})^{M/2} \cdots (z_0^{-M/2}z_0^{l/2}z_0^{M/2})^l \alpha_0
\]

(see [8, p. 199]). Then by fact (i) we can also apply the proof of [8, Lemma 3.5] to get some polynomials \( p_j(t) \) and \( p_j(t) \) in \( \overline{F[t]} \) satisfying

\[
R_{(j)}(c^{-1}x_0) = p_j(R_{(j)}(\bar{\omega}^{-M/2}z_0^{s/2})), \quad j = 1, 2.
\]
By fact (ii), we can further get $p_1(t)$ and $p_2(t)$ so that $p_1(t) = p_2(t)$. Thus, by the proof of [8, Lemma 3.5], there is a polynomial $p(t)$ in $F[t]$ such that $x - p(\alpha)$ belongs to both $\Psi_\phi^{(k+1)/2}$ and $\Psi_\phi^{(k+1)/2}$, which shows $x \in p(\alpha) + \Psi_\phi^{k+1}$. The proof is complete.

3.4. Corollary. Let $\alpha \in \Omega_{\phi, M}$ be a very cuspidal element. If $g \in G$ intertwines $\alpha + \Psi_\phi^{M+1}$, that is,

\[(\alpha + \Psi_\phi^{M+1}) \cap g(\alpha + \Psi_\phi^{M+1})g^{-1} \neq \emptyset,
\]

then $g \in P$.

Proof. Put $A = F[\alpha]$. Let $g \in \Psi_\phi^k$ and $g \in \Psi_\phi^{k+1}$ for some integer $k$. By the assumption, there are elements $x, y \in \Psi_\phi^{M+1}$ such that $\alpha + x = g(\alpha + y)g^{-1}$. Then

\[\alpha g - g \alpha = g \delta - (g - y)g \in \Psi_\phi^{M+1} + \Psi_\phi^{M+1}g \subset \Psi_\phi^{M+k+1}.
\]

By Proposition 3.3, $g = a + b$ for $a \in A = F[\alpha]$ and $b \in \Psi_\phi^{k+1}$. We have $g^{-1} \in \Psi_\phi^{-k}$, In fact, if $\Psi_\phi$ is principal, or if $\Psi_\phi^k$ is principal and $k$ is even, it is trivial. So we must prove it in the case where $\Psi_\phi^2$ is principal and $k$ is odd. We have $a \in \Psi_\phi^k \cap F[\alpha]$, because $g = a + b \in \Psi_\phi^k$ and $b \in \Psi_\phi^{k+1}$. By Proposition 2.23, it follows that

\[a \in \Psi_\phi^k \cap F[\alpha] = \Psi_\phi^{k+1} \cap F[\alpha],
\]

whence $g = a + b \in \Psi_\phi^{k+1}$ which contradicts $g \notin \Psi_\phi^{k+1}$. This implies that $g$ does not intertwine $a + \Psi_\phi^{M+1}$ in this case. Hence $g^{-1} \in \Psi_\phi^{-k}$. It follows that $ag^{-1} = 1 - bg^{-1} \in 1 + \Psi_\phi \subset \Psi_\phi^k$. This shows that $a \in \Psi_\phi^k$ and that it is invertible. Thus $g = a(1 + a^{-1}b)$, where $a \in A \times$ and $1 + a^{-1}b \in 1 + \Psi_\phi$. Put $z = a^{-1}b$. Then $g = a(1 + z)$. Since $a \in A \times$ (cf. 1.1), we have

\[1 = \sigma a \cdot a + \sigma a \cdot az + \sigma z \cdot \sigma a \cdot a + \sigma z \cdot \sigma a \cdot az.
\]

By the forms of the slices of $\mathcal{L}_\phi$ in the proof of Lemma 1.14, we have $\sigma a \in \Psi_\phi^k$. Hence, since $z \in \Psi_\phi^k$, we must have $k = 0$, that is, $a \in \mathcal{L}_\phi = A \cap \Psi_\phi$ and $\sigma a \cdot a = 1$ mod $\Psi_\phi$, so $\mathcal{L}_A = \mathcal{L}_\phi \cap A$. Hence, by the approximation theorem [17, 2.11], there is an element $a' \in T = A \cap G$ such that $a = a' \mod \mathcal{L}_A = A \cap \Psi_\phi$, and so $a = a'(1 + z')$, $z' \in \mathcal{L}_A = A \cap \Psi_\phi$. Hence $g = a'(1 + z')$ for some $z' \in \Psi_\phi$. Since $g \in G$ and $a' \in T = A \cap G$, we also have $1 + z' \in G$ and further $1 + z' \in P_1$. By Corollary 2.24, $a' \in P$. Hence $g = a'(1 + z') \in P$, which completes the proof.

3.5. Corollary. Let $\alpha \in \Omega_{\phi, M}$ be a very cuspidal element and let $k$ be an integer greater than or equal to $1$. If $g \in P$ intertwines $\alpha + \Psi_\phi^{M+k}$, then $g \in TP_k$. 


Proof. We imitate the proof of [8, Prop. 3.6]. Let \( g \) be an element of \( G \) which intertwines \( \alpha + \Psi^i_{\theta} \). Then, since \( k \geq 1 \), \( g \in TP_g \), seen in the proof of Corollary 3.4. So we may set \( g \in P_i \). We now assume that \( g \in P_i \) for an integer \( i \) with \( 0 < i < k \). Then we show that \( g = tg' \), \( t \in T_i \), \( g' \in P_{i+1} \). In fact, we have \( g = 1 + x, x \in \Psi^i_{\theta}, \) and
\[
g^{-1}a g - \alpha \equiv -x \alpha + a x \mod \Psi^{M+i+1}_{\theta}.
\]
It follows that \( ax - x \alpha \in \Psi^{M+i+1}_{\theta} \), although \( ax - x \alpha \in \Psi^i_{\theta} \), whence, by Proposition 3.3, \( x = y + z \), \( y \in F[\alpha] \cap \Psi^i_{\theta} \), \( z \in \Psi^{i+1}_{\theta} \). Thus
\[
g = 1 + x = (1 + y)(1 + (1 + y)^{-1}z).
\]
Since
\[
1 = \sigma g \cdot g \equiv \sigma (1 + y) \cdot (1 + y) \mod \Psi^{i+1}_{\theta},
\]
we get \( y + \sigma y \equiv 0 \mod F[\alpha] \cap \Psi^{i+1}_{\theta} \). By [17, Lemma 8], there exists \( y' \in F[\alpha] \cap \Psi^{i+1}_{\theta} \) such that \( y + \sigma y = y' + \sigma y' \), which implies that \( 2^{-1}(y' - y) \in F[\alpha] \cap g_{\theta, i} \), because \( 2 \) is invertible in \( \sigma \). Hence
\[
C((y' - y)/2) \in T_i = T \cap P_i,
\]
where \( C \) is the Cayley map, and
\[
1 + y \in C((y' - y)/2)(1 + \Psi^{i+1}_{\theta}).
\]
It follows that
\[
g = 1 + x = tg', \quad t \in T_i, \quad g' \in 1 + \Psi^{i+1}_{\theta},
\]
and \( g' \in P_{i+1} \) automatically. Therefore, by induction on \( i \), we get \( g \in TP_g \).
The proof is completed.

3.6. Theorem. Let \( \rho \) be an irreducible very cuspidal representation of \( P \) of level \( m \geq 1 \). Then the representation \( \pi \) of \( G \) compactly induced from \( \rho \) is irreducible and supercuspidal.

Proof. The case of \( m = 1 \) follows from [16, Corollary 5.3]. Let \( m \geq 2 \). Assume that \( \rho \mid P_g \cap gP_{g-1} \) and \( \rho \mid P_g \cap g^{-1}P_{g-1} \) have a common constituent, for \( g \in G \). If \( g \in P \) were showed, the theorem in the case of \( m \geq 2 \) follows by [8, Prop. 1.5] which is due to Howe [13]. By the assumption, there exists a character \( \chi \) of \( P_{m-1}/P_m \) such that \( \chi \) and \( \delta \chi \) coincide on \( g_{\theta, \lambda(m)+1} \cap g_{\theta, \lambda(m)+1} \). By 3.1, the character \( \chi \) corresponds to a coset \( \alpha + g_{\theta, \lambda(m)+1} \), where \( \alpha \) is a very cuspidal element in \( g_{\theta, \lambda(m)+1} \), and
\[
\chi(C(x)) = \psi((x, -\alpha)/2), \quad x \in g_{\theta, m-1}.
\]
Thus, for any \( x \in \mathfrak{g}_{\theta, \lambda(m)+1} \cap g \mathfrak{g}_{\theta, \lambda(m)+1} g^{-1} \),
\[
\psi(\langle g^{-1}xg, -\alpha \rangle / 2) = \psi(\langle x, -\alpha \rangle / 2).
\]
It follows
\[
\psi(\langle x, \alpha - g\alpha g^{-1} \rangle / 2) = 1,
\]
whence by Proposition 1.24 and [18, 4.19]
\[
\alpha - g\alpha g^{-1} \in \mathfrak{g}_{\theta, \lambda(m)+1} + g \mathfrak{g}_{\theta, \lambda(m)+1} g^{-1}.
\]
This implies that \( g \in G \) intertwines \( \alpha + \mathfrak{g}_{\lambda(m)+1} \). Hence \( g \in P \) follows by Corollary 3.5. The proof is complete.

3.7. Remark. The proofs of [21, Corollary 4.9 and Prop. 6.6] give an alternate proof of Theorem 3.6. However we note that Lemma 3.3, used to prove Theorem 3.6, is needed for constructing the very cuspidal representation \( \rho \) in Section 4 as well.

3.8. We recall the notion of unrefined minimal \( K \)-type introduced by Moy and Prasad [20, 21].

Let \( m \) be an integer greater than or equal to 1. By [20, 5.1], a pair \( (P, \chi) \) is called an unrefined minimal \( K \)-type if \( \chi \) is a representation of \( P_{m-1} \) trivial on \( P_m \) and

(i) if \( m = 1 \), \( \chi \) is a cuspidal representation of \( P/P_1 \) inflated to \( P \),

(ii) if \( m \geq 2 \), \( \chi \) is a nondegenerate character of \( P_{m-1}/P_m \), that is, the corresponding coset \( \alpha + \mathfrak{g}_{\lambda(m)} \) as in Proposition 1.25 contains no nilpotent elements. Further, by [21, 4.8], the refined minimal \( K \)-type \( (P, \chi) \) is called anisotropic if every element of the coset \( \alpha + \mathfrak{g}_{\lambda(m)} \) is semisimple and its centralizer in \( G \) is anisotropic, equivalently, compact in the usual topology on \( G \) (note that \( G = Sp_{2n} \) is semisimple).

3.9. Proposition. Let \( \rho \) be a very cuspidal representation of \( P \) with level \( m \geq 1 \). If \( m = 1 \), put \( \chi = \rho \), and if \( m \geq 2 \), let \( \chi \) be any one character of \( P_{m-1}/P_m \) which occurs in the restriction of \( \rho \) to \( P_{m-1}/P_m \). Then the pair \( (P, \chi) \) is an unrefined minimal \( K \)-type and in case \( m \geq 2 \), it is moreover anisotropic.

Proof. In case \( m = 1 \), this is nothing but the definition of [20, 5.1(i)]. In case \( m \geq 2 \), the proof of Proposition 2.25 shows directly that \( (P, \chi) \) is an anisotropic unrefined minimal \( K \)-type by [21, 4.8]. The proof is complete.

3.10. Let \( c_i \) be a regular semisimple element of \( \mathfrak{g} \). Put \( A = F[c_i] \). Assume that \( A \) decomposes into the direct sum of tamely ramified extension fields \( E_i, 1 \leq i \leq u \), of \( F \). We say that such \( A \) is tamely ramified over \( F \).
In [17, 3.18] and [18, 5.3], a system \( \Psi = ((\psi_1, c_1, c'_1, f_1)) \) is called a cuspidal datum of rank 1 associated to \( T = A \cap G \) if \( \psi_1 \) is a character of \( T \) with conductorial exponent \( f_1 \geq 2 \) such that
\[
\psi_1(C(x)) = \psi(\langle x, -c_1 \rangle/2), \quad \text{for } x \in A \cap \mathcal{G}_{\theta, f_1},
\]
and
\[
\psi_1(C(x)) = \psi(\langle x, -c'_1 \rangle/2), \quad \text{for } x \in A \cap \mathcal{G}_{\theta, f_1}.
\]
where \( c_1' \in \mathcal{G}_{\theta, f_1} \) is principal and \( c_1' \in A \) is \( \{x \in A \mid x + \sigma x = 0\} \), where \( f_1' = [(f_1 + 1)/2] \) and \( \theta \subseteq \Pi \) corresponds uniquely to \( A = F[c_1] \) as in the proof of Proposition 2.30 (cf. [17, 3.15]).

3.11. We recall the construction of an irreducible representation \( \rho_{\Psi} \) of a subgroup \( TP_i \) of \( P = P(\theta) \) associated to a cuspidal datum \( \Psi = ((\psi_1, c_1, c'_1, f_1)) \) of rank 1 in [17, Sect. 4].

Let \( T_i = T \cap P_i, i \geq 1 \). Put \( K_{\Psi} = T_1P_i \) and \( K_{\Psi}^* = T_1P_i^* \). We can define a character \( \varphi_1 \) of \( P_i \) trivial on \( P_i^* \) by
\[
\varphi_1(C(x)) = \psi(\langle x, c'_1 \rangle/2), \quad x \in \mathcal{G}_{\theta, f_1}.
\]
Then \( \varphi_1 \) extends naturally to a character of \( K_{\Psi}^* = T_1P_i^* \). If \( f_1 \) is even, it further extends to a character \( \rho_{\Psi} \) of \( TK_{\Psi} = TP_i \). So we assume that \( f_1 \) is odd, that is, \( f_1 = 2i_1 + 1 \). Let \( \mathcal{H}_1 = K_{\Psi}/\text{Ker}(\varphi_1) \) and \( \mathcal{Z}_1 = K_{\Psi}^*/\text{Ker}(\varphi_1) \).

Then it is seen in [17, 4.5–4.7] that
\[
1 \to \mathcal{Z}_1' \to \mathcal{H}_1' \to \mathcal{Z}_1' /\mathcal{Z}_1' \to 1
\]
is a central extension and that \( \mathcal{H}_1' \) is a nondegenerate Heisenberg group. Morris [17, 4.8] defined a certain subgroup \( \mathcal{H}_1' \) of \( \mathcal{H}_1' \). Put \( \mathcal{Z}_1 = \mathcal{Z}_1' \cap \mathcal{H}_1' \).

Then we can easily see that \( \mathcal{H}_1 /\mathcal{Z}_1 \) is isomorphic to \( \mathcal{H}_1' /\mathcal{Z}_1' \) and \( \mathcal{H}_1 = T_1\mathcal{Z}_1 \) by the definition of \( \mathcal{H}_1 \) in [17, 4.8]. Thus \( \mathcal{H}_1 \) is also a Heisenberg group. It is well known that there exists an irreducible representation \( \delta_1 \) of \( \mathcal{H}_1 \) associated to the character \( \varphi_1 \) of \( \mathcal{Z}_1 \) and that \( \delta_1 \) extends to a unique irreducible representation \( \rho_{\Psi} \) of \( TK_{\Psi} \) (see [17, 4.10]).

3.12. Proposition. Let \( \Psi = ((\psi_1, c_1, c'_1, f_1)) \) be a cuspidal datum of rank 1 associated to \( T = A \cap G \), where \( A = F[c_1] \) is tamely ramified over \( F \). Then the irreducible representation \( \rho_{\Psi} \) of \( TP_i \), with \( i_1 = [f_1/2] \), constructed above, induces a very cuspidal representation \( \rho \) of \( P \).

Proof. The condition on \( c_1 \) and \( c'_1 \) in 3.9 shows that \( \psi(\text{tr}(c_1 - c'_1)x) = 1 \) for all \( x \in A \cap \mathcal{G}_{\theta, f_1} \) (cf. 1.23). By Proposition 2.23, there exists \( x \in A \cap \mathcal{G}_{\theta, f_1} \) such that \( x^{-1} \in A \cap \mathcal{G}_{\theta, f_1} \) and \( \text{tr}(c_1 - c'_1)x) \in \mathcal{O} \). Here we remark that if \( \theta \subseteq \Pi \) is of type II (cf. 2.17), \( f_1 - 1 \) is even, because \( s = s(\theta) \) and \( M = M(f_1) \) are even. Since \( A = F[c_1] \) is tamely ramified over \( F \), again by Proposition 2.23,
\[
(c_1 - c'_1)x \in A \cap \mathcal{G}_{\theta, f_1}^{1-x}.
\]
Thus
\[
c_1 - c_1' \in x^{-1} Q_\theta^{1-s} \subseteq Q_\theta^{2-f_1-s} = Q_\theta^{\lambda(f_1^{-1})}
\]
and so \(c_1' \in c_1 + Q_\theta^{\lambda(f_1^{-1})}.\) By Proposition 2.30, \(c_1\) is very cuspidal. Hence it follows that \(c_1'\) is also very cuspidal, as was seen in the proof of Proposition 2.25. By the construction of \(\rho_\psi,\) the restriction to \(J_1\) of the induced representation \(\rho\) contains the character \(\varphi_\lambda\) which corresponds to the very cuspidal element \(c_1'\) in 3.11. This shows that \(\rho\) is very cuspidal by Definition 3.2. The proof is complete.

4. DEGREES

From now on, let \(\theta \subseteq \tilde{\Pi}\) and \(s = s(\theta)\) be as in Propositions 2.11 and 2.14, and put \(P = P(\theta).\) Let \(\rho\) be an irreducible very cuspidal representation of \(P\) of level \(m \geq 2.\)

4.1. Put
\[
p' = \lfloor m/2 \rfloor, \quad p = m - p',
\]
where we recall that \(\lfloor m/2 \rfloor\) denotes the integral part of \(m/2.\) Since \(P_n/P_m\)

is an abelian group as in 3.1, on \(P_n/P_m,\) \(\rho\) is isomorphic to a finite sum of characters of \(P_n/P_m.\) By Definition 3.2, the restriction to \(P_{m-1}/P_m\) of any cuspidal element \(\alpha \in \varnothing_{\theta, \lambda(m)}\). We will fix this character \(\chi.\)

PROPOSITION. Let the notations and assumptions be as above. Then, for each type \(\theta \subseteq \tilde{\Pi},\) the level \(m\) satisfies the following condition:

\[
(A_{r_{-1}}^\delta), \delta \geq 1: \quad (s, m - 1) = 1 \text{ and } m \text{ is even,}
\]
\[
(A_{r_{-1}}^\delta C_{r_{-1}/2}^1), \delta \geq 0: \quad (s, m - 1) = 1 \text{ and } m \text{ is even or odd,}
\]
\[
(A_{r_{-1}}^\delta C_{r_{-1}/2}^2), \delta \geq 1: \quad (s, m - 1) = 1 \text{ and } m \text{ is even,}
\]
\[
(A_{r_{1-1}}^{\delta/2} C_{r_{1-1}/2} A_{r_{1-1}}^{\delta/2} C_{r_{1-1}/2}), \delta \geq 0: \quad (s, m - 1) = 2 \text{ and } m \text{ is odd,}
\]
\[
(A_{r_{1-1}}^{(\delta-1)/2} C_{r_{1-1}/2} A_{r_{1-1}}^{(\delta-1)/2}), \delta \geq 1: \quad (s, m - 1) = 2 \text{ and } m \text{ is odd,}
\]

where \(r, r_1,\) and \(r_2\) are as in Proposition 2.11 except \(A_{r_{1-1}}^{\delta/2} C_{r_{1-1}/2} A_{r_{1-1}}^{\delta/2} C_{r_{1-1}/2},\)

and, in this case, \(r_1 = r_2\) is admitted.

Proof. By Propositions 2.11 and 2.14, this follows easily from the fact \(\lambda(m) = 1 - m - s.\)
4.2. Put
\[ A = F[\alpha] \quad \text{and} \quad T = A \cap G \]
as before. We denote the centralizer of \( \chi \) in \( P \) by \( Z_P(\chi) \), that is,
\[ Z_P(\chi) = \{ g \in P \mid \chi(gxg^{-1}) = \chi(x), x \in P/P_m \}. \]

**Lemma.** Let the notations and assumptions be as above. Then
\[ Z_P(\chi) = TP'. \]

**Proof.** By \( \chi(C(x)) = \psi(\langle x, -\alpha \rangle/2), x \in \langle \theta, \mu \rangle \) in 3.1, it follows that
\( TP' \subset Z_P(\chi) \) by Proposition 1.20(i). The reverse inclusion follows from Proposition 1.24 and Corollary 3.5. Hence the proof is complete.

4.3. By [8, 5.1], \( \rho \) is induced from a representation \( \rho_1 \) of \( Z_P(\chi) \) which will be constructed below. Thus, for the degrees of \( \rho \) and \( \rho_1 \),
\[ \deg(\rho) = (P, TP') \deg(\rho_1), \]
where \((A, B)\) denotes the index of a subgroup \( B \) in a group \( A \). Put
\[ T_i = T \cap P_i, \quad i \geq 1. \]
Further, put \( \overline{P} = P/P_1 \) and \( \overline{T} = T/T_1 \). Since \( T \subset P \) by Corollary 2.24, we have \( \overline{T} \subset \overline{P} \).

**Proposition.** Let the notations and assumptions be as above. Then \( \deg(\rho) \) is given as follows:
\[ \deg(\rho) = (\overline{P}, \overline{T}) (\langle \theta, 1 \rangle, \langle \theta, p \rangle)/(T_1, T_p) \quad \text{if } m \text{ is even}, \]
\[ \{(\overline{P}, \overline{T}) (\langle \theta, 1 \rangle, \langle \theta, p \rangle)/(T_1, T_p-1)\} \{((\theta, p-1), \langle \theta, p \rangle)/(T_{p-1}, T_p)\}^{1/2} \quad \text{if } m \text{ is odd}. \]

**Proof.** We construct \( \rho_1 \) by using the method of [8] and give the formula for \( \deg(\rho) \) (cf. [15, Prop. 5.6]).

(i) Let \( m \) be even. Then \( p' = p = m/2 \). Thus, by Lemma 4.2, the character \( \chi \) of \( P'/P_m \) extends to \( \rho_2 \) trivially (cf. [8, 5.6]). Hence \( \deg(\rho_2) = 1 \). By the above equality,
\[ \deg(\rho) = (P, TP') = (P, TP_1)(TP_1, TP') \].
For \((P, TP_1)\), we get the exact sequence
\[ 1 \to \bar{T} \to \bar{P} \to P/TP_1 \to 1, \]
and so \((P, TP_1) = (\bar{P}, \bar{T})\). Further
\[ (TP_1, TP_0) = (P_1, P_0)/ (T_1, T_0). \]
By Proposition 1.22, \((P_1, P_0) = (\mathbb{A}_{\partial, 1}, \mathbb{A}_{\partial, p})\). Hence, since \(p' = p\),
\[ \deg(\rho) = (\bar{P}, \bar{T})(\mathbb{A}_{\partial, 1}, \mathbb{A}_{\partial, p})/(T_1, T_0). \]

(ii) Let \(m\) be odd. Then \(p = (m + 1)/2\) and \(p' = p - 1\). We use the method of [8, 5.3] (cf. 3.11). We denote the kernel of \(\chi\) in \(P_p\) by \(N = \text{Ker}(\chi)\). Then \(P_m \subset N \subset P_p\). Put \(H = T_1P_p/N\) and \(Z = T_1P_p/N\). Then \(Z \subset H\) and \(Z\) is central in \(H\). The character \(\chi\) can be regarded as a character of \(P_p/N\) and extends to a character of \(Z = T_1P_p/N\) trivially as in (i). We again denote this extension by \(\chi\). Put \(Q = H/Z\). Then the function \((x, y) \mapsto \chi(xyx^{-1}y^{-1}); Q \times Q \to \mathbb{C}^*\) defines an alternating form, where \(\mathbb{C}^*\) denotes the set of nonzero complex numbers. We get a central extension
\[ 1 \to Z \to H \to Q \to 1 \]
and by Proposition 1.24 and Corollary 3.5 the alternating form is nondegenerate, as in [8, 5.2]. Thus the character \(\chi\) of \(Z\) gives a unique equivalence class of an irreducible representation \(\eta\) of the Heisenberg group \(H\). Put \(H_1 = TP_p/N\) and \(B = H_1/H\). Then \(B\) is isomorphic to \(\bar{T} = T/T_1\) and is an abelian group. We also get an extension
\[ 1 \to H \to H_1 \to B \to 1 \]
as in [8, 5.3]. From the exact sequence
\[ 1 \to T_p/T_p \to P_p/P_p \to Q \to 1, \]
we get
\[ |Q| = (P_p, P_p)/(T_p, T_p) = (\mathbb{A}_{\partial, p}, \mathbb{A}_{\partial, p})/(T_p, T_p) \]
as above. Since \(Q\) is a vector space over \(\mathbb{F}_q\), \(|Q|\) is a power of \(q\). On the other hand, it will be seen in 4.6 below that \(|B| = |\bar{T}|\) is prime to \(q\). Thus \(|Q|\) and \(|B|\) are relatively prime. Hence, by [8, 5.3], the representation \(\eta\) of \(H\) can be extended to a representation \(\rho_1\) of \(H_1\). Since by the definition of \(\eta\), \(\deg(\eta) = |Q|^{1/2}\) (cf. [8, 5.2]), we get
\[ \deg(\rho_1) = |Q|^{1/2}. \]
By (i),
\[(P, TP') = (P, \bar{T}')(\theta, 1, \theta, p')/(T_1, T').\]

Hence
\[\deg(\rho) = (P, TP')\deg(\rho_1) = \frac{(P, \bar{T}')(\theta, 1, \theta, p')/(T_1, T')}{((T_1, T')/(T_1, T'))^{1/2}}.\]

Since \(p' = p - 1\), we get the formula for \(\deg(\rho)\) in the case of (ii). Hence the proof is complete.

4.4. We proceed to compute \(\deg(\rho)\) of Proposition 4.3. We first compute \(\bar{T} \cdot \bar{P}\).

**Proposition.** For each type of \(\theta \subseteq \Pi, |\bar{P}|\) is given as follows:

\[
\begin{align*}
(A_{n-1}^e) &: a(r)^6, \\
(A_{n-1}^e, C_{r/2}) &: a(r)^6c(r/2), \\
(A_{n-1}^e, C_{r/2}^2) &: a(r)^6c(r/2)^2, \\
(A_{n-1}^e, C_{r/2}^2) &: a(r)^6c(r/2)^2, \\
(A_{n-1}^{e/2}C_{r/2}^{1/2}) &: a(r_1)^{e/2}a(r_2)^{e/2}c(r_1/2)c(r_2/2), \\
(A_{n-1}^{(e+1)/2}C_{r/2}^{1/2}) &: a(r_1)^{(e+1)/2}a(r_2)^{(e+1)/2}c(r_1/2)^2,
\end{align*}
\]

where \(a(i) = |GL_i(F)|\) and \(c(i) = |Sp_i(F)|, i \geq 1\). Note that \(\delta\) is as in Proposition 4.1.

**Proof.** By 1.16, the involution \(\sigma\) in 1.1 leaves \(\Pi\) and \(\Psi\) stable. By [17, Prop. 2.12], there is the canonical isomorphism
\[P/P_1 = \{x \in \Pi/\Psi | x \cdot \bar{x} = 1\},\]
where \(\bar{x}\) denotes the involution on \(\Pi/\Psi\) defined by \(\sigma\). Hence by the isomorphism \(R_1\) in 2.3, we get natural isomorphisms of \(P/P_1\) onto the \(F\)-rational points of the connected reductive group with Dynkin diagram \(\theta\). Hence this shows the proposition.

4.5. For \(a \in \theta_{\lambda(m)}\) in 4.1, let \(A = F[a] = \bigoplus_{i=1}^u E_i\) be as in Definition 2.17. Put
\[E_{i0} = \{x \in E_i | \sigma x = x\}, \quad 1 \leq i \leq u.\]

Since each \(\sigma\) is nontrivial on \(E_i\), each \(E_i\) is a quadratic extension of \(E_{i0}\). For \(1 \leq i \leq u\), denote the modular degree of \(E_i\) over \(F\) (resp. \(E_{i0}\)) by \(f_i\) (resp. \(f_{i0}\)). Then \(f_{i0} = 1\) or 2 and it divides \(f_i\).
If $\theta \subseteq \tilde{\Pi}$ is of type $(A_{r_1-1}^{\delta/2}C_{r_2}^1A_{r_1}^{\delta/2}C_{r_2}^1)$ or $(A_{r_1-1}^{\delta-1/2}C_{r_2}^2A_{r_1}^{\delta+1/2})$, put
\[ r = (r_1 + r_2)/2. \]

Then, for every $\theta \subseteq \tilde{\Pi}$, we have
\[ \sum_{i=1}^{\mu} f_i = rc. \quad (4.51) \]

For, by Definition 2.17,
\[ 2n = \sum_{i=1}^{\mu} [E_i : F] = \left( \sum_{i=1}^{\mu} f_i \right) s/c. \]

By the values of $s$, $c$, and $n$ in Propositions 2.11 and 2.14, a direct computation shows $2nc/s = rc$ for every $\theta \subseteq \Pi$. Hence we get the desired equality.

4.6. Lemma. Let the notations and assumptions be as above. If $\theta \subseteq \tilde{\Pi}$ is of type $(A_{r_1-1}^{\delta-1}), (A_{r_2-1}^{\delta}C_{r_2}^2), (A_{r_1-1}^{\delta-1/2}C_{r_2}^1A_{r_2}^{\delta+1/2})$, then $\sum_{i=1}^{\mu} f_i = 1$. If $\theta \subseteq \Pi$ is of type $(A_{r_2-1}^{\delta}C_{r_2}^2)$ or $(A_{r_1-1}^{\delta-1/2}C_{r_2}^1A_{r_2}^{\delta+1/2})$, then $\sum_{i=1}^{\mu} f_i = \sum_{i=1}^{\mu} f_i = 2$.

Proof. By Propositions 2.11 and 2.14, $s/c$ is even if and only if $\theta \subseteq \Pi$ is of type $(A_{r_1-1}^{\delta-1}), (A_{r_2-1}^{\delta}C_{r_2}^2), (A_{r_1-1}^{\delta-1/2}C_{r_2}^1A_{r_2}^{\delta+1/2})$.

Let $A = F[\alpha] = \bigoplus_{i=1}^{\mu} E_i$, and $\alpha = (a_i)$ be as in Definition 2.17. We first let $s/c$ be even. Assume that some $E_i$ is unramified over $E_{i0}$. Then the involution $\sigma$ on $E_i$ induces the nontrivial $\tilde{E}_{i0}$-isomorphism $\overline{\sigma}$ of $E_i$. On the other hand, by Definition 2.17, $\sigma(a_i) = -a_i$, and $\tilde{E}_{i0}$ generates $E_i$ over $F$. Since $s/c$ is even, we thus have $\sigma(\tilde{E}_{i0}) = -\tilde{E}_{i0}$, whence $\overline{\sigma}$ is trivial on $E_i$. This is a contradiction. This shows that the $E_i$ are all ramified over $F$, that is, $f_{i0} = \cdots = f_{u0} = 1$. Hence we get the first assertion.

For the second, let $s/c$ be odd. By Definition 2.17, $s/c$ is the order of ramification of $E_i$ over $F$ for every $i$. If some $E_i$ is ramified over $E_{i0}$, then the order of ramification of $E_i$ over $E_{i0}$ is equal to 2 and so it divides $s/c$. This is a contradiction. Hence the $E_i$ are all unramified over $E_{i0}$, that is, $f_{i0} = \cdots = f_{u0} = 2$, which completes the proof.

4.7. By Lemma 4.6, we can get the order $|\tilde{\Pi}|$ as follows:

Proposition. Let the notations and assumptions be as in Lemma 4.6. If $\theta \subseteq \Pi$ is of type $(A_{r_1-1}^{\delta-1}), (A_{r_2-1}^{\delta}C_{r_2}^2), (A_{r_2-1}^{\delta+1/2})$, then $|\tilde{\Pi}| = 2^u$. 

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If \( \Theta \subseteq \tilde{\Pi} \) is of type \( (\mathcal{A}_{r_1}, \mathcal{A}_{r_2}) \) or \( (\mathcal{A}_{r_1-1}, \mathcal{A}_{r_2-1}, \mathcal{A}_{r_1}, \mathcal{A}_{r_2}) \),

\[
|\tilde{T}| = \prod_{i=1}^{u} (q^{f_i/2} + 1).
\]

Proof. Put \( Q = \{1 \leq i \leq u \mid f_i = 2\} \) and \( u_1 = |Q| \). We prove

\[
|\tilde{T}| = 2^{u-u_1} \prod_{i \in Q} (q^{f_i/2} + 1). \tag{4.7.1}
\]

Let \( u = 1 \). Then \( A = F[\alpha] = E \) is a field extension of degree \( 2n \) over \( F \) and with the nontrivial involution \( \sigma \). We have \( T = A \cap \mathcal{G} = \{x \in E^\times \mid x \cdot \sigma x = 1\} = \{x \in E^\times \mid N_{E/E_0}(x) = 1\} \), where \( E_0 = \{x \in E^\times \mid \sigma x = x\} \) and \( N_{E/E_0} \) denotes the norm map of \( E \) into \( E_0 \). For simplicity, put \( N = N_{E/E_0} \), \( U_E = U_E^0 = E^\times \), and \( U_E^i = 1 + \mathcal{P}_E^i, i \geq 1 \). Similarly, define \( U_{E_0} \) and \( U_{E_0}^i \), \( i \geq 1 \). Thus, by Proposition 2.23, we get the following isomorphisms:

\[
\tilde{T} = T/T_1 = \{x \in U_E \mid N(x) = 1\}/\{x \in U_E^1 \mid N(x) = 1\}.
\]

We compute the order of the last quotient. At first, assume that \( E \) is unramified over \( E_0 \). By [24, Chap. V, Sect. 2], we have \( N(U_E^i) = U_{E_0}^i, i \geq 0 \). Thus \( N \) induces a surjective homomorphism

\[
N_0: U_E/U_E^1 \rightarrow U_{E_0}/U_{E_0}^1.
\]

Further, we get the exact sequences

\[
1 \rightarrow \{x \in U_E^a \mid N(x) = 1\} \rightarrow U_E^a \rightarrow U_{E_0}^a \rightarrow 1, \quad a = 0, 1,
\]

and hence

\[
1 \rightarrow \tilde{T} \rightarrow U_E/U_E^1 \rightarrow U_{E_0}/U_{E_0}^1 \rightarrow 1.
\]

We have \( U_E^1/U_E^1 = E^\times \) and \( U_{E_0}/U_{E_0}^1 = E_0^\times \). We denote the modular degree of \( E \) over \( F \) by \( f_I \). Therefore we get

\[
|\tilde{T}| = |E_0^\times|/|E_0^\times| = (q^{f_1} - 1)/(q^{f_1/2} - 1) = q^{f_1/2} + 1.
\]

Next, assume that \( E \) is ramified over \( E_0 \). Again by [24, Chap. V, Sect. 2, Prop. 4], we have

\[
N(U_E^{2i}) \subset U_{E_0}^{i+1}, \quad N(U_E^{2i+1}) \subset U_{E_0}^{i+1}, \quad i \geq 0.
\]

Thus \( N \) induces a homomorphism

\[
N_0: U_E/U_E^1 \rightarrow U_{E_0}/U_{E_0}^1.
\]
as above. By [24, Chap. V, Sect. 2, Prop. 5(i)], \( N_0 \) is given by \( N_0(\xi) = \xi^2 \) for \( \xi \in E^\times = U_E/U_E^1 \). Since \( E = E_0 \), we get the following exact sequence:

\[
1 \to T \to E^\times \to (E^\times)^2 \to 1.
\]

Hence \( |T| = |E^\times|/(E^\times)^2| = 2 \), because \( 2 \not\mid q \). Therefore we have proved (4.7.1) in case \( u = 1 \).

In case \( u \geq 2 \), again by Proposition 2.23, we can reduce this case to that of \( u = 1 \) and prove (4.7.1). Hence by Lemma 4.6 we get the assertion of this proposition.

4.8. Proposition. Let the notations and assumptions be as in Lemma 4.6. Let \( a, b \) be integers, \( b > a \geq 1 \). If \( \theta \not\subseteq \tilde{\Pi} \) is of type \((A_{r_1-1}^\delta, A_{r_1}^\delta C_{r_1/2}^2)\), or \((A_{r_1-1}^{\delta-1/2} C_{r_1/2} A_{r_1-1}^{\delta+1/2})\),

\[
(T_a, T_b) = 1.
\]

If \( \theta \not\subseteq \tilde{\Pi} \) is of type \((A_{r_1-1}^\delta C_{r_1/2}^1)\),

\[
(T_a, T_b) = q^{(b-a)r/2}.
\]

If \( \theta \not\subseteq \tilde{\Pi} \) is of type \((A_{r_1-1}^{\delta/2} C_{r_1/2} A_{r_1-1}^{\delta/2} C_{r_1/2}^1)\),

\[
(T_a, T_b) = \begin{cases} 
q^{(b-a-1)/2} & \text{if } a \text{ is even}, \\
q^{(b-a)/2} & \text{if } a \text{ is odd}, 
\end{cases}
\]

where \( r = (r_1 + r_2)/2 \) (cf. 4.5).

Proof. Let \( Q \) and \( u_1 \) be as in the proof of Proposition 4.7. We prove

\[
(T_a, T_{a+1}) = q^d \quad \text{with } d = \sum_{i \in Q} f_i/2 \quad (4.8.1)
\]

if \((s, m - 1) = 1\), and

\[
(T_a, T_{a+1}) = \begin{cases} 
q^d \text{ with } d = \sum_{i \in Q} f_i/2 & \text{if } a \text{ is even}, \\
1 & \text{if } a \text{ is odd} 
\end{cases} \quad (4.8.2)
\]

if \((s, m - 1) = 2\), where we set \( d = 0 \) if \( Q = \emptyset \).

Let \( u = 1 \). Then \( A = F[\alpha] = E \) is a field extension of degree \( 2n \) over \( F \) with the nontrivial involution \( \sigma \). By Proposition 1.22, we get an isomorphism

\[
T_a/T_{a+1} = (A \cap \mathfrak{g}_{\theta, a})/(A \cap \mathfrak{g}_{\theta, a+1}).
\]
Putting $M = \lambda(m)$ in Proposition 2.23, we get the following isomorphisms:

If $(s, \lambda(m)) = (s, m - 1) = 1$,

$$T_a/T_{a+1} = (\mathcal{P}_E^a)_-/(\mathcal{P}_E^{a+1})_-,$$

and if $(s, m - 1) = 2$,

$$T_a/T_{a+1} = \begin{cases} (\mathcal{P}_E^{a/2})_-/(\mathcal{P}_E^{a/2+1})_- & \text{if } a \text{ is even,} \\ 0 & \text{if } a \text{ is odd.} \end{cases}$$

Here $(\mathcal{P}_E^k)_- = \{ x \in \mathcal{P}_E^k \mid x + \sigma x = 0 \} = \{ x \in \mathcal{P}_E^k \mid \text{Tr}(x) = 0 \}$ for any integer $k \geq 1$, where $E_0$ is the fixed field of $E$ under $\sigma$ and $\text{Tr} = \text{Tr}_{E/E_0}$ denotes the trace map of $E$ into $E_0$. Thus we must compute $|\mathcal{P}_E^k|/(\mathcal{P}_E^{k+1})_-$, $k = a, a/2$.

Assume now that $E$ is unramified over $E_0$. The following is well known:

$$\text{Tr}(\mathcal{P}_E^k) = \mathcal{P}_{E_0}^k$$

for any integer $k \geq 1$. Thus we get the exact sequence

$$0 \to (\mathcal{P}_E^k)_-/(\mathcal{P}_E^{k+1})_- \to \mathcal{P}_E^k/\mathcal{P}_E^{k+1} \to \mathcal{P}_{E_0}^k/\mathcal{P}_{E_0}^{k+1} \to 0.$$}

Thus

$$\left| (\mathcal{P}_E^k)_-/(\mathcal{P}_E^{k+1})_- \right| = \left| \mathcal{P}_E^k/\mathcal{P}_E^{k+1} \right|/\left| \mathcal{P}_{E_0}^k/\mathcal{P}_{E_0}^{k+1} \right|$$

$$= |E|/|E_0| = q^{f_1}/q^{f_1/2} = q^{f_1/2},$$

where $f_1$ is the modular degree of $E$ over $F$. Hence, if $(s, m - 1) = 1$,

$$(T_a, T_{a+1}) = q^{f_1/2},$$

and, if $(s, m - 1) = 2$,

$$(T_a, T_{a+1}) = \begin{cases} q^{f_1/2} & \text{if } a \text{ is even,} \\ 1 & \text{if } a \text{ is odd.} \end{cases}$$

Hence we get (4.8.1) and (4.8.2) in the unramified case.

Next, assume that $E$ is ramified over $E_0$. Then, by [24, Chap. V, Sect. 3, Lemma 4],

$$\text{Tr}(\mathcal{P}_E^k) = \mathcal{P}_{E_0}^{k+1/2}$$
for any integer \( k \geq 1 \). Hence, since \( E = E_0 \), similarly, we can show
\[
\left| \left( \mathbb{P}^k - \mathbb{P}^{k+1} \right) / \left( \mathbb{P}^{k+1} - \mathbb{P}^{k+2} \right) \right| = \left| \mathbb{P}^k / \mathbb{P}^{k+1} / \mathbb{P}^{k+2} / \mathbb{P}^{k+3} \right| = 1
\]
as above. Thus
\[
(T_a, T_{a+1}) = 1,
\]
which shows (4.8.1) and (4.8.2).

In case \( u \geq 2 \), by Proposition 2.23, we can reduce this case to that of \( u = 1 \) to get (4.8.1) and (4.8.2).

We can easily get the values of \( (T_a, T_b) \) in the assertion from (4.8.1) and (4.8.2) case by case by using Lemmas 4.1 and 4.6.

4.9. We compute \( (\theta_{b,a}, \theta_{a,b}) \), \( b > a \geq 1 \), of Proposition 4.3.

**Proposition.** Let \( l \) be an integer with \( 1 \leq l \leq s \). Then, for each type of \( \theta \in \Pi \), \( (\theta_{b,a}, \theta_{a,b}) \) is given as follows:

\[
\begin{align*}
(A_{r-1})_q & : \begin{cases} q^{r^2} & \text{if } l \text{ is even,} \\ q^{r^2 + r^2} & \text{if } l \text{ is odd,} \end{cases} \\
(A_{r-1}C_r) & : q^{r^2 + r^2}, \\
(A_{r-1}C_{2r}) & : \begin{cases} q^{r^2 + r^2 + r} & \text{if } l \text{ is even,} \\ q^{r^2 + r^2} & \text{if } l \text{ is odd,} \end{cases} \\
(A_{r-1}/2C_{r+1}) & : \begin{cases} q^{(s+1)l^2(r_1 + r_2)}/r_1 & \text{if } l = 0 \mod 4, \\ q^{(s+1)l^2(r_1 + r_2)}/r_1 & \text{if } l = 2 \mod 4, \\ q^{(s+1)l^2(r_1 + r_2)}/r_1 & \text{if } l = 1, 3 \mod 4. \end{cases}
\end{align*}
\]

**Proof.** (cf. [22, 2.16]). The matrix \( J \) in 1.1 also defines the involution on \( M_{2n}^d(F) \). Denote it by \( \alpha \). We call the form which \( \hat{\omega} \) and \( \tilde{\omega} \) are deleted from (2.6.4) by (2.6.4'), and similarly (2.6.5') and (2.7.1). By 2.6 and 2.7, \( \theta_{b,a} \) and \( \theta_{a,b} \) isomorphic to the space \( \hat{\omega} \) as vector spaces over \( F \), for each type of \( \theta \in \Pi \): For \( (A_{r-1})_q \) or \( (A_{r-1}C_{r+1})_q \) (cf. 2.12), \( \hat{\omega} \) consists of \( \alpha \in M_{2n}^d(F) \) of the form (2.6.4'), where \( a_1, \ldots, a_s \in M_{2n}(F) \) and \( \alpha = -\alpha \). For \( (A_{r-1}/2C_{r+1})_q \) or \( (A_{r-1}/2C_{r+1}A)^{(s+1)/2} \) (2.7.1), where \( a_1, \ldots, a_s \in M_{2n}(F) \) for odd \( i, i \neq s, a_i \in M_{2n}^d(F) \) for odd \( i, i \neq s \), \( a_i \) for even \( i, i \neq s, a_i \in M_{2n}^d(F) \) for odd \( i, i \neq s \), \( a_i \) for even \( i, i \neq s \), and \( \alpha = -\alpha \).
If $l$ is even and $l < s$, $q'$ consists of $\alpha \in \mathbb{M}_{2n}(\mathbb{F}_q)$ of the form (2.6.5'), where $\alpha_i \in \mathbb{M}_{r_1}(\mathbb{F}_q)$ for even $i$, $i \neq s$, $s - l$, $\alpha_i \in \mathbb{M}_{r_2}(\mathbb{F}_q)$ for odd $i$, $i \neq s$, $s - l$, $\alpha^{(1)}_i, \alpha^{(2)}_i \in \mathbb{M}_{r_1, r_1/2}(\mathbb{F}_q)$, $\alpha^{(1)}_{s-l}, \alpha^{(2)}_{s-l} \in \mathbb{M}_{r_1, r_2}(\mathbb{F}_q)$, and $\alpha = -\alpha$. If $l$ is odd and $l < s$, $q'$ consists of $\alpha \in \mathbb{M}_{2n}(\mathbb{F}_q)$ of the form (2.6.5'), where $\alpha_i \in \mathbb{M}_{r_1}(\mathbb{F}_q)$ for even $i$, $i \neq s$, $s - l$, $\alpha_i \in \mathbb{M}_{r_2}(\mathbb{F}_q)$ for odd $i$, $i \neq s$, $s - l$, $\alpha^{(1)}_i, \alpha^{(2)}_i \in \mathbb{M}_{r_1, r_1/2}(\mathbb{F}_q)$, $\alpha^{(1)}_{s-l}, \alpha^{(2)}_{s-l} \in \mathbb{M}_{r_1, r_2}(\mathbb{F}_q)$, and $\alpha = -\alpha$. Finally, for $(A^5_{r-1}C^2_{1/2})$ and $(A^5_{r-1}C^2_{1/2})$, the $q'$ are the ones with $r_1 = r_2 = r$ in the second cases.

By these forms of the elements of the space $q'$, we can directly compute the dimensions of $q'$ over $\mathbb{F}_q$. We leave the computations to the reader.

4.10. For the integer $p'$ in 4.1, let $(p'/s)$ and $\overline{p}'$ denote the positive integers determined uniquely by

$$p' = (p'/s)s + \overline{p}' \quad \text{and} \quad 1 \leq \overline{p}' \leq s.$$ 

**Proposition.** Let the notations and assumptions be as above. Then $(\mathcal{Q}_{0, \overline{p}', \overline{q}_{0, \overline{p}'}})$ is given as follows:

\[
(A^5_{r-1}) : \begin{cases} 
q^{(p'-1)(\delta + 1)(r_1+r_2)^2 + (r_1+r_2)} / 4 + (\delta + 1)r_{12} & \text{if } \overline{p}' \text{ is even,} \\
q^{(p'-1)(\delta + 1)(r_1+r_2)^2 + (r_1+r_2)} / 4 & \text{if } \overline{p}' \text{ is odd.}
\end{cases}
\]

\[
(A^5_{r-1}C^1_{1/2}) : \begin{cases} 
q^{(p'-1)(\delta + 1)(r_1+r_2)^2 + (r_1+r_2)} / 4 & \text{if } \overline{p}' \text{ is even,} \\
q^{(p'-1)(\delta + 1)(r_1+r_2)^2 + (r_1+r_2)} / 4 & \text{if } \overline{p}' \text{ is odd.}
\end{cases}
\]

\[
(A^5_{r-1}C^2_{1/2}) : \begin{cases} 
q^{(p'-1)(\delta + 1)(r_1+r_2)^2 + (r_1+r_2)} / 4 & \text{if } \overline{p}' \text{ is even,} \\
q^{(p'-1)(\delta + 1)(r_1+r_2)^2 + (r_1+r_2)} / 4 & \text{if } \overline{p}' \text{ is odd.}
\end{cases}
\]

\[
(A^5_{r-1}C^1_{1/2}A^5_{r-1}C^1_{1/2}) : \begin{cases} 
q^{(p'-1)(\delta + 1)(r_1+r_2)^2 + (r_1+r_2)} / 4 & \text{if } \overline{p}' = 1 (4), \\
q^{(p'-1)(\delta + 1)(r_1+r_2)^2 + (r_1+r_2)} / 4 & \text{if } \overline{p}' = 3 (4),
\end{cases}
\]

where $\overline{p}' = 0, 2 (4)$ means $\overline{p}' = 0, 2 \mod 4$, respectively.

**Proof.** By definition,

\[
(\mathcal{Q}_{0, \overline{p}', \overline{q}_{0, \overline{p}'}}) = (\mathcal{Q}_{0, 1, \overline{p}', \overline{q}_{0, 1, \overline{p}'}}) (\mathcal{Q}_{0, 1, \overline{p}', \overline{q}_{0, 1, 2\overline{p}'}}) \cdots \times (\mathcal{Q}_{\overline{p}'-\delta, \overline{p}'-\delta}) (\mathcal{Q}_{\overline{p}'-\delta, \overline{p}'-\delta (p'/s)}) (\mathcal{Q}_{\overline{p}'-\delta, \overline{p}'-\delta (p'/s)}) (\overline{q}_{0, \overline{p}', \overline{p}'}). 
\]
By the periodicity of \( \{ \mathcal{G}_\theta, \vartheta \}_{\iota \in \mathbb{Z}} \),
\[
( G_{\theta, 1 + i\iota}, G_{\theta, 1 + (i + 1)\iota} ) = ( G_{\theta, 1}, G_{\theta, 1 + \iota} )
\]
for \( 1 \leq i \leq \lfloor p'/s \rfloor - 1 \). Since \( G_{\theta, 1}/G_{\theta, 1 + \iota} \) is isomorphic to the algebra of \( \mathbb{F}_q \)-rational points in the Lie algebra of \( \text{Sp}_{2n} \) defined over \( \mathbb{F}_q \), we hence get
\[
( G_{\theta, 1 + i\iota}, G_{\theta, 1 + (i + 1)\iota} ) = q^{m(2n + 1)}
\]
for \( 1 \leq i \leq \lfloor p'/s \rfloor - 1 \). Similarly,
\[
( G_{\theta, 1 + \lfloor p'/s \rfloor \iota}, G_{\theta, \iota} ) = ( G_{\theta, 1}, G_{\theta, \iota} )
\]
Hence
\[
( G_{\theta, 1}, G_{\theta, \iota} ) = q^{(p'/s)m(2n + 1)}( G_{\theta, 1}, G_{\theta, \iota} )
\]
Further,
\[
( G_{\theta, 1}, G_{\theta, \iota} ) = ( G_{\theta, 1}, G_{\theta, 2} ) \cdots ( G_{\theta, \iota - 1}, G_{\theta, \iota} )
\]
Using Propositions 2.11 and 4.9, we can compute these case by case and prove this proposition. We leave the details to the reader.

4.11. By Proposition 4.1, there is a possibility that \( m \) is odd when \( \theta \subset \Pi \) is of type \( (A_{r-1}^2C_{r/2}^2), (A_{r-1}^{\delta/2}C_{r/2}^2A_{r-1}^{\delta/2}C_{r/2}^1), \) or \( (A_{r-1}^{\delta/2}C_{r/2}^1A_{r-1}^{\delta/2}C_{r/2}^1) \).

**Proposition.** Let the notations and assumptions be as above. The factors \( ((G_{\theta, \iota - 1}, G_{\theta, \iota})/(G_{\theta, \iota - 1}, G_{\theta, \iota}))^{1/2} \) of \( \deg(\rho) \) of Proposition 4.3 are given as follows:

\[
\begin{align*}
& (A_{r-1}^2C_{r/2}^2): q^{r^2/2 + r^2/4}, \\
& (A_{r-1}^{\delta/2}C_{r/2}^1A_{r-1}^{\delta/2}C_{r/2}^1): \begin{cases} q^{(\delta + 1)r^2/2} & \text{if } m = 1 \mod 4, \\ q^{(\delta + 1)r^2/2} & \text{if } m = 3 \mod 4, \end{cases}
& (A_{r-1}^{\delta/2}C_{r/2}^1A_{r-1}^{\delta/2}C_{r/2}^1): q^{(\delta + 1)r^2/2}.
\end{align*}
\]

**Proof.** This is a direct consequence of Propositions 4.8 and 4.9.

4.12. By Definition 3.2, a very cuspidal representation \( \rho \) of \( P \) of level \( m \geq 2 \) corresponds uniquely to a class \( \alpha + \mathcal{G}_{\theta, \iota(m) + 1} \in ( G_{\theta, m+1} )/G_{\theta, m+1} \), up to conjugate by elements of \( P \). So we can assign this class to a system

\[
\{ \text{type of } \theta, m, (f_1, \ldots, f_u) \},
\]
where \( f_1, \ldots, f_u \) are the modular degrees over \( F \) of the simple factors \( E_1, \ldots, E_u \) of \( \rho \) respectively as in 4.5.
4.13. THEOREM. Let \( \rho \) be an irreducible very cuspidal representation of \( P = P(\theta) \) of level \( m \geq 2 \) corresponding to a class \( \alpha + \mathbb{N}_{\theta, \lambda} + 1 \). Then, for each system for \( \rho \), \( \deg(\rho) \) is given as follows:

For \( (A_{r-1}^5, m, (f_1, \ldots, f_u)) \) with \( (2 \delta, m - 1) = 1 \),

\[
\begin{cases}
(1/2^n)a(\delta)q^{(m/2-1)(r^2+1)+r/2} & \text{if } m \equiv 0 \mod 4, \\
(1/2^n)a(\delta)q^{(m/2-1)(r^2+1)+r/2} & \text{if } m \equiv 2 \mod 4.
\end{cases}
\]

For \( (A_{r-1}^5C_r^1, m, (f_1, \ldots, f_u)) \) with \( (2 \delta + 1, m - 1) = 1 \),

\[
a(\delta)c(\delta/2)q^{(m-2)(r^2+1)+r/2}/2 \prod_{i=1}^u(q^{f_i/2} + 1).
\]

For \( (A_{r-1}^5C_r^1, m, (f_1, \ldots, f_u)) \) with \( (2 \delta + 2, m - 1) = 1 \),

\[
\begin{cases}
(1/2^n)a(\delta)c(\delta/2)^2q^{(m/2-1)(r^2+1)+r/2}/2 & \text{if } m \equiv 0 \mod 4, \\
(1/2^n)a(\delta)c(\delta/2)^2q^{(m/2-1)(r^2+1)+r/2}/2 & \text{if } m \equiv 2 \mod 4.
\end{cases}
\]

For \( (A_{r-1/2}^5C_{r-1/2}^1, m, (f_1, f_2, \ldots, f_u)) \) with \( (2 \delta + 2, m - 1) = 2 \),

\[
a(r_1)^{\delta/2}a(r_2)^{\delta/2}c(r_1/2)c(r_2/2)q^{(m-1)(\delta+1)(r_1+r_2)^2 + (r_1+r_2)/8 + (\delta+1)r_1r_2/2}/2 \prod_{i=1}^u(q^{f_i/2} + 1)
\]

if \( m \equiv 1 \mod 4,

\[
a(r_1)^{\delta/2}a(r_2)^{\delta/2}c(r_1/2)c(r_2/2)q^{(m-3)((\delta+1)(r_1+r_2)^2 + (r_1+r_2)/8 + (\delta+1)r_1r_2/2)/2} \prod_{i=1}^u(q^{f_i/2} + 1)
\]

if \( m \equiv 3 \mod 4.

For \( (A_{r-1/2}^5C_{r-1/2}^1, m, (f_1, \ldots, f_u)) \) with \( (2 \delta + 2, m - 1) = 2 \),

\[
(1/2^n)a(r_1)(\delta-1/2)a(r_2)(\delta+1/2)c(r_1/2)^2 \times q^{(m-3)(\delta+1)(r_1+r_2)^2 + (r_1+r_2)/8 + (\delta+1)r_1r_2/2}/2
\]

if \( m \equiv 3 \mod 8,

\[
(1/2^n)a(r_1)(\delta-1/2)a(r_2)(\delta+1/2)c(r_1/2)^2 \times q^{(m-3)(\delta+1)(r_1+r_2)^2 + (r_1+r_2)/8 + (\delta+1)r_1r_2/2}/2
\]

if \( m \equiv 7 \mod 8.

Then

\[
\Prod_{i=1}^u(q^{f_i/2} + 1).
\]
Here

\[ a(i) = (q^i - 1)(q^{i-1} - q) \cdots (q - 1), \]
\[ c(i) = (q^{2i} - 1)(q^{2i-1} - 1)(q^{2i-2} - 1)q^{2i-3} \cdots (q^2 - 1)q. \]

**Proof.** Let \( \theta \subsetneq \Pi \) be of type \((A_{r-1}^\delta, \cdot)\). Then, by Proposition 4.1, \( m \) is even and so \( p' = p = m/2 \). By Propositions 4.3–4.10, we get

\[ \deg(\rho) = \begin{cases} 
(1/2^n)a(r)q^{(m/2-1)(r^2+2)+r/2} & \text{if } \bar{p} \text{ is even}, \\
(1/2^n)a(r)q^{(m/2-1)(r^2+2)} & \text{if } \bar{p} \text{ is odd}.
\end{cases} \]

By definition,

\[ \bar{p} = p - (p/s)s = m/2 - 2(p/s)\delta. \]

Thus \( \bar{p} \) is even if and only if \( m = 0 \mod 4 \), and \( \bar{p} \) is odd if and only if \( m = 2 \mod 4 \). Hence we get the value of \( \deg(\rho) \) as in the assertion for \( \theta \subsetneq \Pi \). Similarly, we can check the remainders. We leave the proofs to the reader.

For \( a(i) = |GL(F)| \) and \( c(i) = |Sp_2(F)| \), see [11].

4.14. **Remark.** Theorem 4.13 shows that the very cuspidal representations of the parahoric subgroups \( P = P(\theta) \), for \( \theta \subsetneq \Pi \) of types \((A_{r-1}^\delta, C_{r/2}^1)\) and \((A_{r-1}^\delta, C_{r/2}^1)\) (cf. 2.13), have the same degrees. By [12, 2.2] (cf. [12, 4.6] and [19, Sect. 1]), they are conjugate in \( GSp_{2n}(F) \). But they are not conjugate in \( G = Sp_{2n}(F) \) by [3, Sect. 1].

4.15. **Theorem.** Let \( \pi \) be the representation of \( G \) compactly induced from an irreducible very cuspidal representation \( \rho \) of \( P = P(\theta) \) of level \( m \geq 2 \). Then, with the volume of \( P \) having one, the formal degree \( d(\pi) \) of \( \pi \) is equal to \( \deg(\rho) \) and so is given in Theorem 4.13.

**Proof.** By Theorem 3.6, \( \pi \) is irreducible and supercuspidal. Thus the formal degree \( d(\pi) \) is defined and

\[ d(\pi) = \text{vol}(P)\deg(\rho) \]

(cf. [8, 5.9]). Hence we get \( d(\pi) = \deg(\rho) \).

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REFERENCES

