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Polynomial and rational solutions of holonomic systems

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Abstract

In this article, we give two new algorithms to find the polynomial and rational function solutions of a given holonomic system associated to a set of linear differential operators in the Weyl algebra $D = \mathbf{k}\langle x_1,\ldots,x_n,\partial_1,\ldots,\partial_n\rangle$, where $\mathbf{k}$ is a computable subfield of the complex numbers. Both algorithms are based on the theory of D-modules – the first algorithm obtains degree bounds on the solutions through Gröbner deformations and b-functions while the second algorithm evaluates the dimension of the solutions through duality and restriction. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Polynomial and rational solutions for linear ordinary differential equations can be obtained by algorithmic methods. For instance, the maple package `DEtools` provides efficient functions `polysols` and `ratsols` to find polynomial and rational solutions for a given linear ordinary differential equation with rational function coefficients.

A natural analogue of the notion of linear ordinary differential equation in the several variables case is the notion of holonomic system. A holonomic system is a system of linear partial differential equations whose characteristic variety is middle dimensional.

Chyzak [4] gave an algorithm to find the rational solutions of holonomic systems by using elimination in the ring of differential operators with rational function coefficients combined with Abramov's algorithm for rational solutions of ordinary differential

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equations with parameters. To the authors, solving holonomic systems is analogous to solving systems of algebraic equations of zero-dimensional ideals. Under this analogy, the method of Chyzak corresponds to the elimination method for solving systems of algebraic equations.

The aim of this paper is to give two new algorithms based on the theory of D-modules to find polynomial and rational solutions for a given holonomic system associated to a set of linear differential operators in the Weyl algebra

$$D = \mathbf{k}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle,$$

where $\mathbf{k}$ is a subfield of $\mathbf{C}$.

Polynomial and rational solutions can be obtained, if they exist, by using an exhaustive search. For instance, when $f = 0$ is the singular locus of a holonomic system $M = D/I$, any rational solution has the form g/f^r. If we have upper bounds for the degree of the polynomial g and for r, then we can construct all rational solutions by solving linear equations satisfied by the coefficients of g. Alternatively, if we know the dimension of rational solutions, then we can obtain all rational solutions by increasing the degree of g and r. Hence, the problem reduces to finding effective bounds for these numbers.

In Sections 2 and 3, we give algorithms for upper bounds on the degree of g and on r. The main techniques we use are Gröbner deformations in D as introduced in the book [11] and the b-function for D/I and f.

In Section 4, we give an algorithm to evaluate the dimension of polynomial and rational solutions. Our method is an analog in D of a question studied by Singer [12], who gave an algorithm to compute $\mathrm{Hom}_R(M,N)$ for left $R{:=}k(x_1)\langle\partial_1\rangle$-modules M and N and studied its relation to factorizations of ordinary differential operators. The theory of D-modules translates our problem on polynomial and rational solutions to constructions in the ring of differential operators D. For example, the $\mathbf{k}$-vector space

$$\mathrm{Hom}_D(D/I, \mathbf{k}[\mathbf{x}]) \simeq H^{-n}(\Omega \otimes_D^L \mathbf{D}(D/I))$$

is the space of the polynomial solutions of the left ideal I. Here, Ω is the module of the top dimensional differential forms and $\mathbf{D}$ is the dualizing functor. See, e.g., the book of Björk [3] on this translation. We evaluate the dimension of the right-hand side by recent developments of computational algebra such as construction of free resolutions in the ring D and restrictions of D-modules [8,9,14,17]. Our approach also allows us to evaluate the dimension of solutions to a holonomic system inside any holonomic module. For instance, we can find the dimension of the delta function solutions to I.

Throughout the paper, we refer to the book [11] for fundamental facts on the algorithmic treatment of D. Also, the algorithms which appear in the paper have been implemented in either `kan` [13] or Macaulay 2 [5].

2. Polynomial solutions by Gröbner deformations

How can we obtain all polynomial solutions for ordinary differential equations? One method is to compute the indicial polynomial at infinity, find an upper bound on the degrees of polynomial solutions, and determine the coefficients of polynomials. The analogous method works for holonomic systems by using Gröbner deformations.

A D-module M, where D is the nth Weyl algebra, is said to be holonomic if its characteristic variety has dimension n. For the reader's convenience, we recall the definition of characteristic variety for a cyclic D-module D/I. Given an element $\ell = \sum_{\beta\in\mathbf{N}^n} f_\beta(\mathbf{x})\partial^\beta \in D$, the initial term of ℓ with respect to the order filtration is the subsum $\mathrm{in}_{(0,e)}(\ell) = \sum_{\{\beta\in\mathbf{N}^n:\ f_\beta(\mathbf{x})\neq 0,\ |\beta| \text{ maximum}\}} f_\beta(\mathbf{x})\xi^\beta$ in the polynomial ring $\mathbf{k}[x_1,\dots,x_n,\xi_1,\dots,\xi_n]$. The characteristic variety of D/I is the zero locus of the initial ideal $\mathrm{in}_{(0,e)}(I) = \{\mathrm{in}_{(0,e)}(\ell) \colon \ell \in I\} \subset \mathbf{k}[\mathbf{x},\xi]$. We abuse notation and say that an ideal $I \subset D$ is holonomic if its quotient module D/I is holonomic. The holonomic D-modules form a full subcategory and satisfy many nice properties which we shall use.

Let us now recall the notion of Gröbner deformation. For $\ell \in D$ and the weight vector $w \in \mathbf{R}^n$, we denote by $\mathrm{in}_{(-w,w)}(\ell)$ the initial form of ℓ with respect to the weight $(-w,w)$. Explicitly, if $\ell = \sum_{\alpha,\beta\in\mathbf{N}^n} c_{\alpha\beta}\mathbf{x}^\alpha\partial^\beta$, then

$$\mathrm{in}_{(-w,w)}(\ell) := \sum_{\{\alpha,\beta\in\mathbf{N}^n:\ c_{\alpha\beta}\neq 0,\, -w\cdot\alpha+w\cdot\beta \text{ maximum}\}} c_{\alpha\beta}\mathbf{x}^\alpha\partial^\beta$$

(see, e.g., [11, Section 1.1] for more details). The following proposition now follows from the definition of $\mathrm{in}_{(-w,w)}$.

Proposition 2.1. *Suppose that $f(x_1,\dots,x_n)$ is a polynomial solution of a left ideal $I \subset D$. Take $w \in \mathbf{Z}^n$ and expand the function $f(t^{w_1}x_1,\dots,t^{w_n}x_n)$ as a Laurent polynomial in t as*

$$f_w(\mathbf{x})t^p + \mathrm{O}(t^{p+1}).$$

Then for any $\ell \in I$, we have

$$\mathrm{in}_{(-w,w)}(\ell) \bullet f_w = 0.$$

The initial ideal $\mathrm{in}_{(-w,w)}(I)$ is the ideal of D spanned as a $\mathbf{k}$-vector space by the initial forms $\mathrm{in}_{(-w,w)}(\ell)$ for all elements $\ell \in I$. It is sometimes called the Gröbner deformation of I with respect to $(-w,w)$.

Theorem 2.2 (Assi et al. [2]). *There exist only finitely many Gröbner deformations of I.*

The Newton polytope of a polynomial solution f is defined as the convex hull of the exponent vectors of f. For generic w, the polynomial f_w of Proposition 2.1 is a monomial cx^a and the point a is a vertex of the Newton polytope of f.

Let $R=\mathbf{k}(x_1,\ldots,x_n)\langle\partial_1,\ldots,\partial_n\rangle$ and $\theta_i=x_i\partial_i$. Since w is generic and I is holonomic, the indicial ideal

$$\widetilde{\mathrm{in}_{(-w,w)}}(I):=R\cdot\mathrm{in}_{(-w,w)}(I)\cap\mathbf{k}[\theta_1,\ldots,\theta_n],$$

is an Artinian commutative ideal (see e.g. [11, Section 2.3] for a proof), and $a\in\mathbf{Z}^n$ belongs to its zero set. Thus, we can construct a polytope that contains the Newton polytopes of the polynomial solutions by taking the convex hull of all the non-negative integral roots of all possible indicial ideals with respect to generic weight vectors w.

It is not necessary to find all Gröbner deformations to obtain polynomial solutions. Let $b(s)$ be the generator of $\mathrm{in}_{(-w,w)}(I)\cap\mathbf{k}[s]$, $s=\sum_{i=1}^n w_i\theta_i$. The polynomial $b(s)$ is called the b-function of I with respect to $(-w,w)$. When I is holonomic, the b-function is non-zero for any weight vector w (here w need not be generic). The next proposition follows from the definition of $b(s)$.

Proposition 2.3. *Let w be a strictly negative weight vector, i.e. $w_i<0$ for all i. Consider the b-function $b(s)$ of I with respect to $(-w,w)$ and let $-k_1$ be the smallest integer root of $b(s)=0$. The polynomial solutions of I have the form*

$$\sum_{p_i\geq 0,\ p\cdot w\leq k_1} c_p\mathbf{x}^p. \tag{1}$$

Algorithm 2.4. (Finding the polynomial solutions by a Gröbner deformation)
INPUT: a holonomic left ideal I.
OUTPUT: the polynomial solutions of I.

1. Take a strictly negative weight vector w, compute the Gröbner deformation $\mathrm{in}_{(-w,w)}(I)$, and compute the smallest non-positive integer root $-k_1$ of the b-function with respect to $(-w,w)$. See, e.g. [11, Algorithm 5.15] for these procedures.
2. If we do not have such a root, then there is no polynomial solution other than 0.
3. If there is a minimal integer root, then determine the coefficients c_p of (1) by solving linear equations for the coefficients.

Example 2.5. The following system of differential equations in two variables is called the Appell differential equation $F_1(a,b,b',c)$ [1]:

$$\theta_x(\theta_x+\theta_y+c-1)-x(\theta_x+\theta_y+a)(\theta_x+b),$$
$$\theta_y(\theta_x+\theta_y+c-1)-y(\theta_x+\theta_y+a)(\theta_y+b'),$$
$$(x-y)\partial_x\partial_y-b'\partial_x+b\partial_y,$$

where a,b,b',c are complex parameters. Let us demonstrate how Algorithm 2.4 works for the system of parameter values $(a,b,b',c)=(2,-3,-2,5)$. First, we choose a strictly negative weight vector $w=(-1,-2)$ and compute the b-function $b(s)$, $s=-\theta_x-2\theta_y$, which is the generator of the principal ideal $\mathrm{in}_{(-w,w)}(I)\cap\mathbf{Q}[-\theta_x-2\theta_y]$. We can use the V-homogenization or the homogenized Weyl algebra to get the generator (see, e.g. [11, Section 1.2]). Second, we need to find the integer roots of the b-function $b(s)=0$.

In our example, these are

$$-7, 0, 4.$$

From Proposition 2.1, the highest $(-w)$-degree monomial $cx^p y^q$ in a polynomial solution gives rise to an integer solution $w_1 p + w_2 q = -p - 2q$ of the b-function. Hence, the polynomial solutions are of the form

$$f = \sum_{p,q \geq 0, p+2q \leq 7} c_{pq} x^p y^q.$$

Finally, we determine the coefficients c_{pq} by applying the differential operators to f and putting the results to 0. In our example, we have only one polynomial solution

$$\begin{aligned}&(-\tfrac{1}{21}y^2 + \tfrac{1}{7}y - \tfrac{4}{35})x^3 + (\tfrac{3}{14}y^2 - \tfrac{24}{35}y + \tfrac{3}{5})x^2 \\ &\quad + (-\tfrac{12}{35}y^2 + \tfrac{6}{5}y - \tfrac{6}{5})x + \tfrac{1}{5}y^2 - \tfrac{4}{5}y + 1.\end{aligned}$$

The rank of a left ideal $I \subset D$ is by definition the dimension of $R/R \cdot I$ as a vector space over $\mathbf{k}(x_1,\ldots,x_n)$, where as before $R = \mathbf{k}(x_1,\ldots,x_n)\langle\partial_1,\ldots,\partial_n\rangle$. Ideals which have finite rank are closely related to holonomic ideals by work of Kashiwara – namely, an ideal $I \subset D$ has finite rank if and only if its Weyl closure, which is the ideal $\mathrm{Cl}(I) := R \cdot I \cap D$, is holonomic (see e.g. [11, Section 1.4]). We can thus adapt Algorithm 2.4 to compute the polynomial solutions of finite rank ideals. The only potential problem is that the b-function, which we recall is the monic generator of $\mathrm{in}_{(-w,w)}(I) \cap \mathbf{k}[s]$ might be zero, where $s = \sum_i w_i \theta_i$. Instead, we should therefore replace the b-function by the monic generator of

$$\widetilde{\mathrm{in}_{(-w,w)}}(I) \cap \mathbf{k}[s] = \mathrm{Cl}(\mathrm{in}_{(-w,w)}(I)) \cap \mathbf{k}[s],$$

which is guaranteed to be non-zero. For generic weight vectors w, the indicial ideal may be computed efficiently by methods of [11, Section 2.3]. For the reader's convenience, let us summarize the steps of the finite rank algorithm. We use the notation $[\theta]_\beta = \prod_{i=1}^n \prod_{j=1}^{b_i - 1} (\theta_i - j)$ for $\beta \in \mathbf{N}^n$.

Algorithm 2.6. (Polynomial solutions of a finite rank ideal)
 INPUT: a finite rank left ideal I.
 OUTPUT: the polynomial solutions of I.

1. Take a strictly negative weight vector w, and compute the Gröbner deformation $\mathrm{in}_{(-w,w)}(I)$. If w is generic, then a set of $(-w,w)$-homogeneous generators $\{L_i\}_{i=1}^r$ for $\mathrm{in}_{(-w,w)}(I)$ may be written as $L_i = x^{\alpha(i)} p_i(\theta) \partial^{\beta(i)}$, and under this representation the indicial ideal is generated by the set $\{[\theta]_{\beta(i)} p(\theta - \beta(i))\}_{i=1}^r$. If w is not generic, pick a different w.
2. Compute the monic generator $B(s)$ of the intersection $\widetilde{\mathrm{in}_{(-w,w)}}(I) \cap \mathbf{k}[s]$ where $s = \sum_i w_i \theta_i$, and compute the smallest non-positive integer root $-k_1$ of $B(s)$ with respect to $(-w,w)$.
3. Follow steps 2 and 3 from Algorithm 2.4.

3. Rational solutions by Gröbner deformations

The singular locus of a D-ideal I is the Zariski closure of the projection of the characteristic variety of I minus the zero section from the cotangent bundle to the coordinate base space. In other words, it is the zero set

$$\mathrm{Sing}(I) = V(\langle \mathrm{in}_{(0,e)}(I) : (\xi_1, \ldots, \xi_n)^\infty \rangle \cap \mathbf{k}[x_1, \ldots, x_n]).$$

We note that $\langle \mathrm{in}_{(0,e)}(I) \colon (\xi_1, \ldots, \xi_n)^\infty \rangle \cap \mathbf{k}[x_1, \ldots, x_n]$ can be computed by a single Gröbner basis computation in the Weyl algebra, followed by saturation and intersection computations in a commutative polynomial ring (see e.g. [11, Section 1.4]).

Any rational solution to I has its poles contained inside the singular locus. Thus for our purposes, let $f(\mathbf{x})$ define the codimension 1 component of $\mathrm{Sing}(I)$. Then we may limit our search for rational solutions to $\mathbf{k}[\mathbf{x}][1/f]$.

We will present a method to obtain an upper bound of the order of the poles along $f = 0$ for each rational solution. For this purpose we use the notion of the b-function for f and a section u of a holonomic system, which was introduced by Kashiwara [7]: Let $\mathscr{D}$ be the sheaf of algebraic differential operators on $X = \mathbf{C}^n$. For a holonomic $\mathscr{D}$-module $\mathscr{M} = \mathscr{D}/\mathscr{D}I$ and a polynomial f, consider the tensor product

$$\mathscr{N} = \mathscr{O}[f^{-1}, s] f^s \otimes_{\mathscr{O}_X} \mathscr{M}. \tag{2}$$

This $\mathscr{N}$ has a structure of a left $\mathscr{D}$-module via the Leibniz rule. Let u be a section of $\mathscr{M}$. Then the b-function for f and u (or for $f^s u$) at $p \in \mathbf{C}^n$ is the minimum degree monic polynomial $0 \neq b(s) \in \mathbf{C}[s]$ such that

$$b(s) f^s \otimes u \in \mathscr{D}[s](f^{s+1} \otimes u) \tag{3}$$

holds in $\mathscr{N}$ at p (i.e., as a germ of $\mathscr{N}$ at p). This b-function depends on the point p. As a function of p, there is a stratification of $\mathbf{C}^n$ for which the b-function does not change on each stratum (see e.g. [8] for an algorithmic proof of this fact). In definitions (2) and (3) for b-function, if we replace $\mathscr{O}$ by the polynomial ring $\mathbf{k}[\mathbf{x}]$, $\mathscr{D}$ by the Weyl algebra D, and $\mathscr{M}$ by a holonomic D-module $M = D/I$, then we obtain the global b-function for f and u. It is the least common multiple of b-functions at every point.

Theorem 3.1. *Let u be the residue class of* 1 *in $\mathscr{D}/\mathscr{D}I$, and let $b(s)$ be the b-function for f and u at a point $p \in \mathbf{C}^n$ where $f(p) = 0$. Assume that I admits an analytic solution of the form gf^r around p, where $r \in \mathbf{C}$, g is a holomorphic function on a neighborhood of p, and $g(p) \neq 0$. Then $s + r + 1$ divides $b(s)$.*

Proof. Let $\mathscr{D}^{\mathrm{an}}$ and $\mathscr{O}^{\mathrm{an}}$ be, respectively, the sheaf of analytic differential operators and the sheaf of holomorphic functions on $\mathbf{C}^n$. We may define the analytic b-function by replacing $\mathscr{O}$ by $\mathscr{O}^{\mathrm{an}}$, $\mathscr{D}$ by $\mathscr{D}^{\mathrm{an}}$, and $\mathscr{M}$ by a $\mathscr{D}^{\mathrm{an}}$-module $\mathscr{M}^{\mathrm{an}}$ in definitions (2) and (3). Since the b-function is an analytic invariant and the analytic and the algebraic b-functions coincide (see e.g. [8, Section 8]), we may work in the analytic category.

We do this to consider solutions gf^r where g is holomorphic at p. If we only wish to consider solutions gf^r where g is a polynomial, then we may work in the algebraic category.

In general, given a map of left $\mathscr{D}^{\mathrm{an}}$-modules $\phi : \mathscr{M}_1^{\mathrm{an}} \to \mathscr{M}_2^{\mathrm{an}}$ and a section u of $\mathscr{M}_1^{\mathrm{an}}$, the b-function for $f^s u$ at a point p is divisible by the b-function for $f^s \phi(u)$ at p. We apply this basic fact to the following map φ. Let J^{an} be the annihilating ideal of gf^r in $\mathscr{D}^{\mathrm{an}}$. Since $J^{\mathrm{an}} \supseteq I^{\mathrm{an}} := \mathscr{D}^{\mathrm{an}} I$ and $g(p) \neq 0$, we have a left $\mathscr{D}^{\mathrm{an}}$-homomorphism

$$\varphi : \mathscr{D}^{\mathrm{an}}/I^{\mathrm{an}} \to \mathscr{D}^{\mathrm{an}} g f^r = \mathscr{D}^{\mathrm{an}} f^r \hookrightarrow \mathscr{O}^{\mathrm{an}}[f^{-1}] f^r,$$

which sends u to gf^r. This map extends to a left $\mathscr{D}^{\mathrm{an}}[s]$-homomorphism

$$\begin{aligned} 1 \otimes \varphi : & \ \mathscr{O}^{\mathrm{an}}[f^{-1}, s] f^s \otimes_{\mathscr{O}^{\mathrm{an}}} \mathscr{D}^{\mathrm{an}}/I^{\mathrm{an}} \\ & \to \mathscr{O}^{\mathrm{an}}[f^{-1}, s] f^s \otimes_{\mathscr{O}^{\mathrm{an}}} \mathscr{O}^{\mathrm{an}}[f^{-1}] f^r = \mathscr{O}^{\mathrm{an}}[f^{-1}, s] f^{s+r}, \end{aligned}$$

which sends $f^s \otimes u$ to gf^{s+r}. By the definition of $b(s)$, there exists a germ $P(s)$ of $\mathscr{D}[s]$ at p such that

$$(P(s)f - b(s))(f^s \otimes u) = 0.$$

Since $1 \otimes \varphi$ is a left $\mathscr{D}^{\mathrm{an}}$-homomorphism, applying it to the above equation gives the equation $(P(s)f - b(s))(gf^{s+r}) = 0$, or in other words,

$$g^{-1} P(s) g f^{s+r+1} = b(s) f^{s+r}.$$

Thus, we see that the Bernstein–Sato polynomial $b_f(s)$ of f at p divides $b(s-r)$. Note that $s+1$ divides $b_f(s)$ since $f(p) = 0$ (cf. [6]). In conclusion, we have proved that $s+1$ divides $b(s-r)$. This completes the proof. □

By virtue of the above theorem, we can obtain upper bounds by computing the b-function for $f^s u$ at a smooth point of each irreducible component of the singular locus of I. From now on, let us also take $f \in \mathbf{k}[\mathbf{x}]$ to be a square-free polynomial defining the codimension one component of the singular locus, and let $f = f_1 \cdots f_m$ be its irreducible decomposition in $\mathbf{k}[\mathbf{x}]$.

Theorem 3.2. *Let $b_i(s)$ be the b-function for $f_i^s u$ at a generic point of $f_i = 0$. Denote by r_i the maximum integer root of $b_i(s) = 0$. Then any rational solution (if any) to I can be written in the form $g f_1^{-r_1-1} \cdots f_m^{-r_m-1}$ with a polynomial $g \in \mathbf{C}[\mathbf{x}]$. If some $b_i(s)$ has no integral root, then there exist no rational solutions to I other than zero.*

Proof. An arbitrary rational solution to M is written in the form $g f_1^{-v_1} \cdots f_m^{-v_m}$ with integers $v_1, \ldots, v_m$ and $g \in \mathbf{C}[\mathbf{x}]$. Since the space of the rational solutions with coefficients in $\mathbf{C}$ is spanned by those with coefficients in $\mathbf{k}$, we may assume $g \in \mathbf{k}[x]$, and f and g are relatively prime in $\mathbf{k}[\mathbf{x}]$. Let p be a generic point of $f_i = 0$. We may assume that f_i is smooth at p, $g(p) \neq 0$, and $f_j(p) \neq 0$ for $j \neq i$. It follows from Theorem 3.1 that $b_i(v_i - 1) = 0$. This implies $v_i \leq r_i + 1$. □

Since b-functions divide the global b-function, an upper bound can also be obtained from the global b-function.

Corollary 3.3. *Let $b_i(s)$ be the global b-function for $f_i^s u$, and denote by r_i the maximum integer root of $b_i(s)=0$. Then any rational solution (if any) to I can be written in the form $gf_1^{-r_1-1}\cdots f_m^{-r_m-1}$ with a polynomial $g \in \mathbf{C}[\mathbf{x}]$.*

We mention the corollary because as we shall see shortly, the algorithm to compute global b-functions is simpler than the algorithm to compute b-functions. However, the b-function offers finer information. For instance, the well-known example $f = x^2 + y^2 + z^2 + w^2$ has Bernstein–Sato polynomial $(s+1)(s+2)$ coming from the functional equation, $\frac{1}{4}(\partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2)\cdot f^{s+1} = (s+1)(s+2)f^s$. Now consider the module $M = D\cdot f^{-1}$ and let u be the element f^{-1}. The global b-function for $f^s u$ is $s(s+1)$, and hence Corollary 3.3 implies that rational solutions of M all have the form gf^{-1} or gf^0, where g is a polynomial not divisible by f. On the other hand, the Bernstein–Sato polynomial of f at any nonsingular point p of $f=0$ (i.e. except for the origin) is $s+1$. It follows that the b-function for $f^s u$ equals s at the generic point of $f=0$ and hence Theorem 3.2 implies that all rational solutions actually have the form gf^{-1}.

An algorithm to compute the b-function and the global b-function for $f^s u$ was first given in [8] based upon tensor product computation, which is slow and memory intensive. Shortly thereafter, Walther introduced in [16] a more efficient method to compute the global b-function for $f^s u$. Both methods give the global b-function exactly, under the condition that I is f-saturated. Otherwise, we get a multiple of the global b-function. Similarly, the method of [8] gives the b-function exactly if I is f-saturated and additionally a certain primary decomposition in $\mathbf{C}[\mathbf{x}]$ is known. If primary decomposition is only available in $\mathbf{k}[\mathbf{x}]$, we again get a multiple of the b-function.

Let us now describe an algorithm to compute the b-function for $f^s u$ at a generic point of $f = 0$ by combining the method of [16] and the primary decomposition as was used in [8].

Algorithm 3.4. (Computing an upper bound of the b-function at a generic point)

Input: a finite set G_0 of generators of a holonomic D-ideal I and an irreducible polynomial $f \in \mathbf{k}[\mathbf{x}]$.

Output: $b'(s) \in \mathbf{k}[s]$, which is a multiple of the b-function $b(s)$ for $f^s u$ at a generic point of $f = 0$, where u is the residue class of 1 in D/I.

1. Introducing a new variable t, put $\vartheta_i = \partial_i + (\partial f/\partial x_i)\partial_t$. Let $\tilde{I}$ be the left ideal of D_{n+1}, the Weyl algebra on the variables $x_1,\dots,x_n,t$, that is generated by
$$\{P(\mathbf{x},\vartheta_1,\dots,\vartheta_n)\,|\,P(\mathbf{x},\partial_1,\dots,\partial_n)\in G_0\}\cup\{t-f(\mathbf{x})\}.$$
2. Let G_1 be a finite set of generators of the left ideal $\mathrm{in}_{(-1,0,\dots,0;1,0,\dots,0)}(\tilde{I})$ of D_{n+1}. Here, -1 is the weight for t and 1 is the weight for ∂_t.

3. Rewrite each element P of G_1 in the form

$$P = \partial_t^{\mu} P'(t\partial_t, \mathbf{x}, \partial_1, \dots, \partial_n) \quad \text{or} \quad P = t^{\mu} P'(t\partial_t, \mathbf{x}, \partial_1, \dots, \partial_n)$$

with a non-negative integer μ, and define $\psi(P)$ by,

$$\psi(P) := t^{\mu}\partial_t^{\mu} P' = t\partial_t \dots (t\partial_t - \mu + 1) P'(t\partial_t, \mathbf{x}, \partial_1, \dots, \partial_n)$$

$$\text{or} \quad \psi(P) := \partial_t^{\mu} t^{\mu} P' = (t\partial_t + 1) \dots (t\partial_t + \mu) P'(t\partial_t, \mathbf{x}, \partial_1, \dots, \partial_n).$$

Put

$$G_2 := \{\psi(P)(-s-1, \mathbf{x}, \partial_1, \dots, \partial_n) \mid P \in G_1\}.$$

4. Compute the elimination ideal $J := \mathbf{k}[s, \mathbf{x}] \cap D[s]G_2$. (The global b-function can be obtained at this stage by computing the monic generator of the ideal $J \cap \mathbf{k}[s]$.)
5. Compute a primary decomposition of J in $\mathbf{k}[s, \mathbf{x}]$ as

$$J = Q_1 \cap \dots \cap Q_v.$$

6. For each $i = 1, \dots, v$, compute $Q_{ix} := Q_i \cap \mathbf{k}[\mathbf{x}]$, which is a primary ideal of $\mathbf{k}[\mathbf{x}]$.
7. Let $b'(s)$ be the monic generator of the ideal

$$\bigcap \{Q_i \cap \mathbf{k}[s] \mid \sqrt{Q_{ix}} \subset \mathbf{k}[\mathbf{x}]f\}$$

of $\mathbf{k}[s]$. (Note that $\sqrt{Q_{ix}} \subset \mathbf{k}[\mathbf{x}]f$ implies that $\sqrt{Q_{ix}}$ equals $\mathbf{k}[\mathbf{x}]f$ or $\{0\}$.)

Theorem 3.5. *In the above algorithm, the polynomial $b'(s)$ is precisely the b-function for $f^s u$ at a generic point of $f = 0$ if I is f-saturated (i.e., $I : f^{\infty} = I$) and each $\mathbf{C}[s, \mathbf{x}]Q_i$ remains primary in $\mathbf{C}[s, \mathbf{x}]$. Otherwise, the polynomial $b'(s)$ is a multiple of the b-function for $f^s u$ at a generic point of $f = 0$.*

Proof. Using essentially the same method as the proof of Lemma 4.1 in [16], we can prove that $\tilde{I}$ is precisely the annihilator ideal for $\delta(t - f(\mathbf{x})) \otimes u$ in

$$\tilde{M} := (D_{n+1}\delta(t - f(\mathbf{x}))) \otimes_{\mathbf{C}[\mathbf{x}]} D/I,$$

where $\delta(t - f(\mathbf{x}))$ denotes the residue class of $(t - f(\mathbf{x}))^{-1}$ in $\mathbf{k}[\mathbf{x}, (t - f(\mathbf{x}))^{-1}]/\mathbf{k}[\mathbf{x}]$. Let $b_t(s)$ be the indicial polynomial for $\delta(t - f(\mathbf{x})) \otimes u$ along $t = 0$ at a point $(0, p)$ with $f(p) = 0$. Then by Theorem 6.14 of [8], the b-function $b(s)$ for $f^s u$ at p divides out, and if I is f-saturated, coincides with $b_t(-s-1)$.

It follows from the definition that $b_t(-s-1)$ is a generator of the ideal $\mathcal{O}_p[s]J \cap \mathbf{C}[s]$ of $\mathbf{C}[s]$, where $\mathcal{O}_p$ denotes the stalk of $\mathcal{O}$ at p. If $\mathbf{C}[s, \mathbf{x}]Q_i$ are primary in $\mathbf{C}[s, \mathbf{x}]$, $b'(s)$ generates the above ideal in view of Theorem 4.7 of [8] (cf. also Lemma 4.4 of [9]). In general, although Q_i is primary in $\mathbf{k}[s, \mathbf{x}]$, the extension $\mathbf{C}[s, \mathbf{x}]Q_i$ is no longer primary in $\mathbf{C}[s, \mathbf{x}]$ and admits a primary decomposition

$$\mathbf{C}[s, \mathbf{x}]Q_i = Q_{i1} \cap \dots \cap Q_{i\mu_i}.$$

In this case, $b'(s)$ is the least common multiple of the generators of the ideals $\mathcal{O}_p[s]Q_{ij} \cap \mathbf{C}[s]$ for $j = 1, \dots, \mu_i$, while $b_t(-s-1)$ is the generator of $\mathcal{O}_p[s]Q_{ij} \cap \mathbf{C}[s]$ for some

j (such that p belongs to the zero set of $Q_{ij} \cap \mathbf{C}[\mathbf{x}]$ which is also the zero set of a factor of f). This completes the proof. $\square$

Remark 3.6. In the notation in the above proof, the linear factors of $b'(s)$ and those of $b_t(-s-1)$ coincide. In particular the set of integer roots of $b'(s) = 0$ is the same as that of $b_t(-s-1) = 0$. In fact, this follows from the fact that the linear factors of $b'(s)$ in $\mathbf{k}[s]$ are invariant under the action of the Galois group of $\bar{\mathbf{k}}$ over $\mathbf{k}$.

Remark 3.7. I is f-saturated if and only if the -1th cohomology group of the restriction of $\tilde{M}$ in the proof of Theorem 3.5 to $t = 0$ vanishes (see Theorem 6.4 and Proposition 6.13 of [8]), which is computable by Algorithm 5.10 of [8] or by Algorithm 5.4 of [9].

Remark 3.8. The f-saturation of I, which is the ideal $D[1/f] \cdot I \cap D$, may be computed by using the localization algorithm of [10] (if I is specializable along f) or by using the less efficient algorithm of [14] (if I is general). By replacing I with its f-saturation, we may then compute the local b-function exactly. However, since saturation is often an expensive algorithm, we avoid making this replacement in practice.

Once we have determined the integers $r_1, \ldots, r_m$ of Theorem 3.2, we can use Gröbner deformations to obtain the rational solutions. Put $k_i = -r_i - 1$. Then by virtue of Theorem 3.2, we have only to determine rational solutions of the form $g f_1^{k_1} \ldots f_m^{k_m}$ for some polynomial g. This amounts to computing polynomial solutions of some twisted ideal $I_{(k_1,\ldots,k_m)}$ of I.

Lemma 3.9. *Let $I \subset D$ be a left ideal generated by $\{P_1, \ldots, P_r\}$, let $f = f_1 \ldots f_m \in \mathbf{k}[\mathbf{x}]$, and let $\{k_1, \ldots, k_m\} \subset \mathbf{C}$. For each generator P_i, let $a_i \in \mathbf{N}$ be sufficiently large so that $f^{a_i} P_i$ may be expressed as*

$$f^{a_i} P_i = p_i(x_1, \ldots, x_n, f\partial_1, \ldots, f\partial_n) \in \mathbf{k}\langle x_1, \ldots, x_n, f\partial_1, \ldots, f\partial_n \rangle.$$

Now consider the ideal

$$I_{(k_1,\ldots,k_m)} := D\{p_i(x_1, \ldots, x_n, L_1, \ldots, L_n)\}_{i=1}^{m},$$

where

$$L_i = f\partial_i + \sum_{j=1}^{m} k_j \frac{f}{f_j} \frac{\partial f_j}{\partial x_i}. \tag{4}$$

Then the space V of polynomial solutions of $I_{(k_1,\ldots,k_m)}$ is isomorphic to the space W of solutions of I inside the $\mathbf{C}[\mathbf{x}]$-module $\mathbf{C}[\mathbf{x}] f_1^{k_1} \ldots f_m^{k_m}$ by the map $V \to W$ sending $g \mapsto g f_1^{k_1} \ldots f_m^{k_m}$. Moreover, $\mathrm{rank}(I) = \mathrm{rank}(I_{(k_1,\ldots,k_m)})$.

Proof. Consider how $f\partial_i$ acts on an element $gf_1^{k_1}\dots f_m^{k_m} \in \mathbf{C}[\mathbf{x}]f_1^{k_1}\dots f_m^{k_m}$:

$$f\partial_i \bullet (gf_1^{k_1}\dots f_m^{k_m}) = \left(f\frac{\partial g}{\partial x_i} + \sum_{j=1}^{m} k_j \frac{f}{f_j}\frac{\partial f_j}{\partial x_i} g \right) f_1^{k_1}\dots f_m^{k_m}.$$

In other words, $f\partial_i$ acts on the polynomial part g as the differential operator L_i of (4), and the part of the lemma on solutions follows.

Given a point p of I away from both the singular locus of I and the zero locus of f, then the map $V \to W$ also extends to a map between the holomorphic solution spaces of $I_{(k_1,\dots,k_m)}$ and I at p (here a branch of $f_1^{k_1}\dots f_m^{k_m}$ at p is chosen so that it may be regarded as a holomorphic function at p). Since rank of an ideal is generically equal to the dimension of the holomorphic solution space, the part of the lemma on rank follows. This can also be shown algebraically by observing that

$$\mathrm{in}_{(0,e)}(p(x_1,\dots,x_n,f\partial_1,\dots,f\partial_n)) = \mathrm{in}_{(0,e)}(p(x_1,\dots,x_n,L_1,\dots,L_n)).$$

We also remark that the definition of $I_{(k_1,\dots,k_m)}$ is ambiguous but can be made well-defined by applying the Weyl closure operation. □

Algorithm 3.10. (Computing the rational solutions of a holonomic ideal)
INPUT: generators of a holonomic D-ideal I.
OUTPUT: A basis of the rational solutions $h \in \mathbf{k}(\mathbf{x})$ of $I \bullet h = 0$.

1. Compute a polynomial f in $\langle \mathrm{in}_{(0,e)}(I)\colon (\xi_1,\dots,\xi_n)^\infty \rangle \cap \mathbf{k}[x_1,\dots,x_n]$.
2. Compute the irreducible decomposition $f = f_1 \cdots f_m$ in $\mathbf{k}[\mathbf{x}]$ (optional step).
3. For each $i = 1,\dots,m$, compute the output $b'(s)$ of Algorithm 3.4 with I and f_i as input. Let r_i be the maximum integer root of $b'(s) = 0$ and put $k_i = -r_i - 1$. If $b'(s)$ has no integral root for some i, then there exists no rational solution other than zero.
4. Compute the twisted ideal $I_{(k_1,\dots,k_m)}$ of Lemma 3.9.
5. Compute a basis $\{g_1,\dots,g_k\}$ of the polynomial solutions of $I_{(k_1,\dots,k_m)}$ using Algorithm 2.4 or Algorithm 2.6.
6. Output: $\{g_1 f_1^{k_1}\cdots f_m^{k_m},\dots,g_k f_1^{k_1}\cdots f_m^{k_m}\}$, a basis of the rational solutions of I.

Example 3.11. Let I be the left ideal generated by

$$L_1 = \theta_x(\theta_x + \theta_y) - x(\theta_x + \theta_y + 3)(\theta_x - 1)$$

and

$$L_2 = \theta_y(\theta_x + \theta_y) - y(\theta_x + \theta_y + 3)(\theta_y + 1).$$

The Appell function $F_1(3,-1,1,1;x,y)$ is a solution of this system. The singular locus of I is $xy(x-y)(1-x)(1-y) = 0$. We can compute the local indicial polynomial of u, the modulo class of 1 in D_2/I, along $x = 0$ directly by the algorithm of [8, Section 4]: It is $s(s-1)$ on $\{(0,y) \mid y \neq 0\}$, and $s(s-1)^2$ at $(0,0)$. In the same way, the indicial polynomial of u along $y = 0$ is $s(s+1)$ on $\{(x,0) \mid x \neq 0\}$, and $s(s+1)(s-1)$ at $(0,0)$.

Now let us compute the b-function for $(1-y)^s u$. The local indicial polynomial of $\delta(t+y-1)\otimes u$ along $t=0$ is $s(s+3)$ at any point of $t=0$. Hence, the b-function for $(1-y)^s u$ divides $(s+1)(s-2)$. In the same way, the local indicial polynomial of $\delta(t+x-1)\otimes u$ along $t=0$ is $s(s+1)$ at any point of $t=0$. Finally, the indicial polynomial of $\delta(t-x+y)\otimes u$ is $s(s-1)$ on $\{(x,x)\,|\,x\neq 0\}$, and $s(s-1)(s-2)$ at $(0,0)$.

Therefore, we conclude that any rational solution to I, if it exists, can be written in the form $g(x,y)y^{-1}(1-x)^{-1}(1-y)^{-3}$ with a polynomial g. Now we may compute the twisted ideal $I_{(0,-1,0,-1,-3)}$, where $f_1=x$, $f_2=y$, $f_3=x-y$, $f_4=x-1$, $f_5=y-1$, and f is the product. Multiplying by f^2, we get the expressions,

$$\begin{aligned} f^2L_1 &= (x^2-x^3)(f\partial_x)^2 + x\left((1-3x)f-(1-x)y\frac{\partial f}{\partial y}-(1-x)x\frac{\partial f}{\partial x}\right)(f\partial_x) \\ &\quad + x(1-x)y(f\partial_y)(f\partial_x) + xyf(f\partial_y) + 3xf^2, \\ f^2L_2 &= (y^2-y^3)(f\partial_y)^2 + y\left((1-5y)f-(1-y)x\frac{\partial f}{\partial x}-(1-y)y\frac{\partial f}{\partial y}\right)(f\partial_y) \\ &\quad + y(1-y)x(f\partial_x)(f\partial_y) - yxf(f\partial_x) - 3yf^2, \end{aligned}$$

and we set T_1 and T_2 to be the operators obtained from the substitution of L_i into $f\partial_i$ as defined by (4). We remark that the ideal $I_{(0,-1,0,-1,-3)}$ generated by T_1 and T_2 is neither holonomic nor specializable with respect to the weight vector $(1,1,-1,-1)$, hence we use Algorithm 2.6 to get its polynomial solutions. Our Macaulay 2 script finds,

```
i1 : RatSols(I,{x,y,x-1,y-1,x-y} , {10,1} )

                                        3          2
          - x + y             -x*y + 3*x*x*y - 3*x*x*y + 4*x - 3*y
o1 = {--------------------, -------------------------------------}
          4    3     2                  4       3       2
      - y + 3y - 3y + y             -y  + 3*y  - 3*y  + y
```

Here, the second argument to the function `RatSols` is a list of factors of the singular locus, and the third argument is a weight vector for the Gröbner deformation in Algorithm 2.6. After some simplification, we find that the rational function solutions are x/y and $(xy^2-3xy+3x-1)/(y-1)^3$.

Remark 3.12. As mentioned in the introduction, an algorithm to compute rational solutions was given by Chyzak in [4], and furthermore, the components of his algorithm have been implemented in Maple. For Example 3.11, the first step of Chyzak's algorithm is to eliminate ∂_x from I to produce the equation,

$$\begin{aligned} &(xy^4-y^5-xy^3+y^4)\partial_y^3 + (8xy^3-9y^4-3xy^2+4y^3)\partial_y^2 \\ &\quad + (10xy^2-18y^3+2y^2)\partial_y - 6y^2. \end{aligned}$$

This equation is then interpreted as an ordinary differential equation in the variable y over the field $\mathbf{k}(x)$, and its solutions can be computed by Abramov's algorithm to be

$$\mathrm{Span}_{\mathbf{k}(x)}\left\{\frac{-y+x}{y(y-1)^3}, \frac{1}{y}\right\}.$$

Similarly, eliminating ∂_y from I produces an ordinary differential equation in the variable x over the field $\mathbf{k}(y)$,

$$(x^5 - x^4y - x^4 + x^3y)\partial_x^3 + (5x^4 - 4x^3y - 2x^3 + x^2y)\partial_x^2$$

and its solutions are

$$\mathrm{Span}_{\mathbf{k}(y)}\{1, x\}.$$

From these solutions, we get degree bounds and can conclude that a rational solution to the entire system I can be expressed in the form $(axy^3 + by^3 + cxy^2 + dy^2 + exy + fy + gx + h)/y(y-1)^3$. Solving for the coefficients a, b, c, d, e, f, g, h then gives the solutions we found earlier.

We remark that we have not made a comparison between the performance of Chyzak's algorithm and our algorithm. One reason is that the algorithms are implemented in different computer algebra systems (Maple versus Macaulay 2 or kan). Another reason is that we do not have any complexity results for our algorithm. Both of these reasons would make for good subjects for future work.

Remark 3.13. Based on Theorem 3.1 and Lemma 3.9, the Algorithm 3.10 can also be adjusted to find all solutions of an ideal I which have the form $gf_1^{a_1}\cdots f_m^{a_m}$ for $g, f_1, \ldots, f_m \in \mathbf{k}[x_1, \ldots, x_n]$ and $a_1, \ldots, a_m$ computable complex numbers.

4. Solutions by duality

For holonomic M and N, it is well known [7] that

$$\mathrm{Ext}_D^i(M, N) \simeq H^{n-i}(\Omega \otimes_D^L (\mathbf{D}(M) \otimes_{\mathbf{k}[x]}^L N)), \tag{5}$$

where $\Omega := (D/\{x_1, \ldots, x_n\} \cdot D)$, and $\mathbf{D}(M)$ is the holonomic dual,

$$\mathbf{D}(M) := \mathrm{Hom}_{\mathbf{k}[x]}(\Omega, \mathrm{Ext}_D^n(M, D)).$$

The spaces $\mathrm{Ext}_D^i(M, N)$ are finite-dimensional $\mathbf{k}$-vector spaces and correspond to the solutions of M in N when $i=0$. For example, if $N=\mathbf{k}[\mathbf{x}]$, then we obtain the polynomial solutions of M, whereas if $N = D/D \cdot \{x_1, \ldots, x_n\}$, then we obtain the delta function solutions of M with support at the origin.

In this section, we explain how (5) can be used to compute the dimensions of $\mathrm{Ext}_D^i(M, N)$. We first discuss how to compute the holonomic dual, next discuss the special cases $N = \mathbf{C}[\mathbf{x}]$ and $N = \mathbf{C}[\mathbf{x}][\frac{1}{f}]$, and last discuss the general case of holonomic N. A method to extend these algorithms to compute an explicit basis of $\mathrm{Hom}_D(M, N)$ and $\mathrm{Ext}_D^i(M, N)$ is the subject of the forthcoming paper [15].

Notation. Let us explain the notation we will use to write maps of left or right D-modules. As usual, maps between finitely generated modules will be represented by matrices, but some care has to be given to the order in which elements are multiplied due to the noncommutativity of D.

Given an $r\times s$ matrix $A=[a_{ij}]$ with entries in D, we get a map of free left D-modules,

$$D^r \xrightarrow{\cdot A} D^s \quad [g_1,\ldots,g_r] \mapsto [g_1,\ldots,g_r]\cdot A,$$

where D^r and D^s are regarded as modules of row vectors, and the map is matrix multiplication. Under this convention, the composition of maps $D^r \xrightarrow{\cdot A} D^s$ and $D^s \xrightarrow{\cdot B} D^t$ is the map $D^r \xrightarrow{\cdot AB} D^t$ where AB is usual matrix multiplication. In general, suppose M and N are left D-modules with presentations D^r/M_0 and D^s/N_0. Then the matrix A induces a left D-module map between M and N, denoted $(D^r/M_0)\xrightarrow{\cdot A}(D^s/N_0)$, precisely when $\boldsymbol{g}\cdot A\in N_0$ for all row vectors $\boldsymbol{g}\in M_0$. Conversely, any map of left D-modules between M and N can be represented by some matrix A in the manner above.

Now let us discuss maps of right D-modules. The $r\times s$ matrix A also defines a map of right D-modules in the opposite direction,

$$(D^s)^{\mathrm{T}} \xrightarrow{A\cdot} (D^r)^{\mathrm{T}} \quad [h_1,\ldots,h_s]^{\mathrm{T}} \mapsto A\cdot[h_1,\ldots,h_s]^{\mathrm{T}},$$

where the superscript-T means to regard the free modules $(D^s)^{\mathrm{T}}$ and $(D^r)^{\mathrm{T}}$ as consisting of column vectors. This map is equivalent to the map obtained by applying $\mathrm{Hom}_D(-,D)$ to $D^r \xrightarrow{\cdot A} D^s$, thus $(D^s)^{\mathrm{T}}$ may be regarded as the dual module $\mathrm{Hom}_D(D^s,D)$. We will suppress the superscript-T when the context is clear. As before, the matrix A induces a right D-module map between right D-modules $N'=(D^s)^{\mathrm{T}}/N_0'$ and $M'=(D^r)^{\mathrm{T}}/M_0'$ when $A\cdot\boldsymbol{g}\in M_0'$ for all column vectors $\boldsymbol{g}\in N_0'$. We denote the map by $(D^s)^{\mathrm{T}}/N_0' \xrightarrow{A\cdot}(D^r)^{\mathrm{T}}/M_0'$.

Left–right correspondence and Ω: As is well known, a standard use for Ω is to establish a correspondence between the categories of left and right D-modules. The correspondence can be expressed through the adjoint operator τ, which is the algebra involution

$$\tau: D\to D, \quad x^\alpha\partial^\beta \mapsto (-\partial)^\beta x^\alpha.$$

Namely, given a left D-module $M\simeq D^r/M_0$, the corresponding right D-module is $\Omega\otimes_{\mathbf{k}[\mathbf{x}]} M \simeq D^r/\tau(M_0)$. Conversely, given a right D-module $N\simeq D^s/N_0$, the corresponding left D-module is $\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega,N)\simeq D^s/\tau(N_0)$. Similarly, given a homomorphism of left D-modules $\phi:(D^r/M_0)\to(D^s/N_0)$ defined by right multiplication by the $r\times s$ matrix $A=[a_{ij}]$, the corresponding homomorphism of right D-modules $\tau(\phi):(D^r/\tau(M_0))\to(D^s/\tau(N_0))$ is defined by left multiplication by the $s\times r$ matrix $\tau(A):=[\tau(a_{ij})]^{\mathrm{T}}$.

Let us explain details of the above correspondence for the non-specialist. Given a left D-module M, there is a corresponding right D-module $\Omega\otimes_{\mathbf{k}[\mathbf{x}]} M$ where the structure is given by extending the actions,

$$(w \otimes m)f = wf \otimes m \quad (w \otimes m)\xi = w\xi \otimes m - w \otimes \xi m$$

for $f \in \mathbf{k}[\mathbf{x}]$ and $\xi \in \mathrm{Der}(\mathbf{k}[\mathbf{x}])$. Given a presentation D^r/M_0 for M with generators denoted $\{e_i\}_{i=1}^r$, then in $\Omega \otimes_{\mathbf{k}[\mathbf{x}]} M$ we have

$$(1 \otimes e_i)x^\alpha \partial^\beta = (1 \otimes x^\alpha e_i)\partial^\beta = 1 \otimes (-\partial)^\beta x^\alpha e_i = 1 \otimes \tau(x^\alpha \partial^\beta)e_i.$$

It follows that $\Omega \otimes_{\mathbf{k}[\mathbf{x}]} M$ is generated by $\{1 \otimes e_i\}_{i=1}^r$ and gets the presentation $D_n^r/\tau(M_0)$.

Conversely, given a right D-module N, there is a corresponding left D-module $\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$ where the structure is given by extending the action,

$$(f\varphi)(w) = \varphi(w)f \quad (\xi\varphi)(w) = \varphi(w\xi) - \varphi(w)\xi$$

for $\varphi \in \mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$, $w \in \Omega$, $f \in \mathbf{k}[\mathbf{x}]$, and $\xi \in \mathrm{Der}(\mathbf{k}[\mathbf{x}])$. A morphism $\varphi \in \mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$ can be identified with its image $\varphi(1) \in N$. Since

$$\begin{aligned}(x^\alpha \partial^\beta \varphi)(1) &= (x^\alpha(\partial^\beta \varphi))(1) = (\partial^\beta \varphi)(1)x^\alpha = \varphi(1)(-\partial)^\beta x^\alpha \\ &= \varphi(1)\tau(x^\alpha \partial^\beta),\end{aligned}$$

the morphism $x^\alpha \partial^\beta \varphi$ gets identified with $\varphi(1)\tau(x^\alpha \partial^\beta)$. In particular, given a presentation D^s/N_0 of N, then $\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$ is generated as a left D-module by the morphisms $\{\varphi_i\}_{i=1}^s$ such that $\varphi_i(1) = e_i$. By the computation above, a relation $\sum_i e_i g_i = 0$ in N corresponds to a relation $\sum_i \tau(g_i)\varphi_i$ in $\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$ because $(\sum_i \tau(g_i)\varphi_i)(1) = \sum_i e_i \tau(\tau(g_i)) = \sum_i e_i g_i$. It follows that $\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$ is generated by $\{\varphi_i\}_{i=1}^s$ and gets the presentation

$$\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N) \simeq D_n^s/\tau(N_0). \tag{6}$$

4.1. Holonomic dual

Let us discuss how $\mathbf{D}(M)$ can be computed.

Algorithm 4.1. (Computing the holonomic dual)
INPUT: $D^{r_0}/D \cdot \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_{r_1}\}$, a presentation of a holonomic left D-module M.
OUTPUT: The holonomic dual $\mathbf{D}(M)$.

1. Compute the first $n+1$ steps of any free resolution of M. Let the nth part of the resolution be $D^p \xrightarrow{\cdot P} D^q \xrightarrow{\cdot Q} D^r$.
2. Dualize and apply the adjoint operator (recall if $P = [p_{ij}]$, then $\tau(P) = [\tau(p_{ij})]^{\mathrm{T}}$) to get $D^p \xleftarrow{\cdot\tau(P)} D^q \xleftarrow{\cdot\tau(Q)} D^r$.
3. Return $\ker(\cdot\tau(P))/\mathrm{Im}(\cdot\tau(Q))$.

Proof. Let the first $n+1$ steps of a free resolution of M be denoted,

$$F^\bullet : D^{r_{n+1}} \xrightarrow{\cdot P} D^{r_n} \xrightarrow{\cdot Q} D^{r_{n-1}} \to \cdots \to D^{r_0} \to 0.$$

Applying $\mathrm{Hom}_D(D,-)$ yields a complex of right D-modules,

$$\mathrm{Hom}_D(D,F^\bullet):(D^{r_{n+1}})^{\mathrm{T}}\xleftarrow{P\cdot}(D^{r_n})^{\mathrm{T}}\xleftarrow{Q\cdot}(D^{r_{n-1}})^{\mathrm{T}}\leftarrow\cdots\leftarrow(D^{r_0})^{\mathrm{T}}\leftarrow 0,$$

and by definition,

$$\mathrm{Ext}^n_D(M,D)\simeq\frac{\ker(D^{r_{n+1}}\xleftarrow{P\cdot}D^{r_n})}{\mathrm{Im}(D^{r_n}\xleftarrow{Q\cdot}D^{r_{n-1}})}.$$

Since $\mathbf{D}(M){=}\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega,\mathrm{Ext}^n_D(M,D))$, it only remains to determine the effect of applying $\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega,-)$. Using Eq. (6), if $\{\boldsymbol{L}_1,\dots,\boldsymbol{L}_k\}$ are generators of $K{=}\ker(D^{r_{n+1}}\xleftarrow{P\cdot}D^{r_n})$, and $\sum_i \boldsymbol{L}_i g_i\in I=\mathrm{Im}(D^{r_n}\xleftarrow{Q\cdot}D^{r_{n-1}})$ is a relation, then the corresponding relation $\sum_i\tau(g_i)\varphi_i$ in $\mathrm{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega,\mathrm{Ext}^n_D(M,D))$ can be realized as the relation $\sum_i\tau(\boldsymbol{L}_ig_i)=\tau(g_i)\tau(\boldsymbol{L}_i)\in\tau(I)$. It follows that

$$\mathbf{D}(M)\simeq\frac{\ker(D^{r_{n+1}}\xleftarrow{\cdot\tau(P)}D^{r_n})}{\mathrm{Im}(D^{r_n}\xleftarrow{\cdot\tau(Q)}D^{r_{n-1}})},$$

which is the output of step 3. □

Example 4.2. The Appell differential equation $F_1(2,-3,-2,5)$ of Example 2.5 has the resolution $0\to D^1\xrightarrow{\cdot Q_1}D^2\xrightarrow{\cdot Q_0}D^1\to 0$, where

$$Q_0=\begin{bmatrix}(\theta_x-3)\partial_y-(\theta_y-2)\partial_x\\(y^2-y)(\partial_x\partial_y+\partial_y^2)-2(y+x)\partial_x+4y\partial_y+2\partial_x-8\partial_y-4\end{bmatrix}^{\mathrm{T}},$$

$$Q_1=\begin{bmatrix}(y^2-y)(\partial_x\partial_y+\partial_y^2)-2x\partial_x+6y\partial_y+\partial_x-9\partial_y\\-(\theta_x-3)\partial_y+(\theta_y-1)\partial_x\end{bmatrix}.$$

The holonomic dual $\mathbf{D}(F_1(2,-3,-2,5))$ is the cokernel of $\tau(Q_1)$ and is the Appell differential equation $F_1(-1,4,2,-3)$.

4.2. Polynomial and rational solutions by duality

When $N=\mathbf{k}[\mathbf{x}]$, the isomorphism (5) specializes to

$$\mathrm{Ext}^i_D(M,\mathbf{k}[\mathbf{x}])\simeq H^{n-i}(\Omega\otimes^L_D\mathbf{D}(M)).\tag{7}$$

The right-hand side is equivalently the $(n-i)$th integration of $\mathbf{D}(M)$ to the origin. An algorithm to compute integration is given in [9]. Using it, we can evaluate the dimensions of $\mathrm{Ext}^i_D(M,\mathbf{k}[\mathbf{x}])$ and in particular $\mathrm{Hom}_D(M,\mathbf{k}[\mathbf{x}])$.

Algorithm 4.3. (Evaluating dimensions of polynomial solution spaces)
INPUT: a holonomic left D-module M.
OUTPUT: dimensions of $\mathrm{Ext}^i_D(M,\mathbf{k}[\mathbf{x}])$.

1. Compute the dual $\mathbf{D}(M)$ using Algorithm 4.1
2. Compute the integrations of $\mathbf{D}(M)$ to the origin using the algorithm in [9]. They are finite dimensional vector spaces.
3. Return the dimensions.

The dimensions of rational solution spaces can be evaluated in a similar way. When $N=\mathbf{k}[\mathbf{x}][1/f]$, the isomorphism (5) specializes to

$$\mathrm{Ext}^i_D(M,\mathbf{k}[\mathbf{x}][1/f]) \simeq H^{n-i}(\Omega \otimes^L_D \mathbf{D}(M)[1/f]). \tag{8}$$

The right-hand side is now equivalently the $(n-i)$th integration of $\mathbf{D}(M)[1/f]$ to the origin. An algorithm to compute localization is given in [10]. Using it and the integration algorithm, we can evaluate the dimensions of $\mathrm{Ext}^i_D(M,\mathbf{k}[\mathbf{x}][1/f])$ and $\mathrm{Hom}_D(M,\mathbf{k}[\mathbf{x}][1/f])$. To get the dimension of all rational solutions, take f to be any polynomial vanishing on the singular locus.

We summarize how to compute the integration of a module N to the origin according to [9] in a slightly more general way. The generalization sometimes gives a more efficient strategy than [9]. We need to recall some definitions. Given any strictly positive $w\in\mathbf{Z}^n_{>0}$, we get an integration filtration F_w of D defined by $F^i_w(D)=\mathrm{Span}_k\{x^\alpha\partial^\beta \mid w\cdot\alpha-w\cdot\beta\le i\}$. More generally, for $\boldsymbol{m}\in\boldsymbol{Z}^r$, we also get a shifted filtration $F_w[\boldsymbol{m}]$ of the free module D^r defined by $F^i_w[\boldsymbol{m}](D^r)=\mathrm{Span}_k\{x^\alpha\partial^\beta e_j \mid w\cdot\alpha-w\cdot\beta-m_j\le i\}$. We will often write $D^r[\boldsymbol{m}]$ for the free module D^r equipped with the shifted filtration $F_w[\boldsymbol{m}]$ when the context is clear. The filtrations $F_w[\boldsymbol{m}]$ induce filtrations on subquotients of D^r in the natural way. Now we may sum up the steps of the integration algorithm. First, compute a $(w,-w)$-strict free resolution $G^\bullet$ of N of length $n+1$. This is a resolution of N by free modules $D^{r_j}[\boldsymbol{m}_j]$ with the property that the differentials preserve the filtration and moreover induce a resolution on the associated graded level. Second, compute the integration b-function of N with respect to $(w,-w)$, and find its minimal and maximal integral roots k_0 and k_1. The integration b-function is the monic polynomial $b(s)$ of least degree satisfying $b(\sum_i w_i\partial_i x_i)\cdot F^0(N)\subset F^{-1}(N)$. Third, compute the cohomology of the complex $F_{-k_0}(\Omega\otimes_D G^\bullet)/F_{-k_1-1}(\Omega\otimes_D G^\bullet)$, which is a complex of finite-dimensional vector spaces. The dimensions of the cohomology groups are equal to the dimensions of the integration modules of N.

Example 4.4. Let us evaluate the dimension of polynomial solutions to the Appell differential equation $M=F_1(2,-3,-2,5)$ of Example 2.5. Choose the weight vector $w=(1,2)$. The resolution of Example 4.2, after dualizing, applying the adjoint operator, and shifting,

$$0\to D^1[0]\xrightarrow{\cdot\tau(Q_0)} D^2[-1,1]\xrightarrow{\cdot\tau(Q_1)} D^1[0]\to 0,$$

preserves filtrations but does not induce a resolution on the associated graded level. On the other hand, if we adjust the resolution to

$$G^\bullet: 0\to D^1[1]\xrightarrow{\cdot P_0} D^2[0,1]\xrightarrow{\cdot P_1} D^1[0]\to 0,$$

where

$$P_0 = \begin{bmatrix} -(\theta_x+5)\partial_y + (\theta_y+2)\partial_x \\ (x^2-x)(\partial_x^2+\partial_x\partial_y) + 4x\partial_x + 2(3x+2y)\partial_y + 4\partial_x - 5\partial_y - 2 \end{bmatrix}^{\mathrm{T}}$$

$$P_1 = \begin{bmatrix} (x^2-x)(\partial_x^2+\partial_x\partial_y) + 2x\partial_x + 4(x+y)\partial_y + 5\partial_x - 4\partial_y - 4 \\ (\theta_x+4)\partial_y - (\theta_y+2)\partial_x \end{bmatrix},$$

then we do obtain a $(w,-w)$-strict resolution of $\mathbf{D}(M) = F_1(-1,4,2,-3)$.

The integration b-function with respect to $(w,-w)$ is $(s+5)(s-2)(s-5)$, hence the integration complex for $\mathbf{D}(M)$ is quasi-isomorphic to the truncated complex $F_w^5(\Omega \otimes_D G^\bullet)/F_w^{-6}(\Omega \otimes_D G^\bullet)$, which is a complex of finite-dimensional vector spaces with dimensions,

$$0 \to \mathbf{Q}^{16} \xrightarrow{\cdot P_0} \mathbf{Q}^{28} \xrightarrow{\cdot P_1} \mathbf{Q}^{12} \to 0.$$

For instance, $F_w^5(\Omega[0])$ consists of the 12 monomials,

$$\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, x^4, xy^3, x^5\},$$

and so on. Note that $\tau(P_1)$ is a $(w,-w)$-Gröbner basis of $F_1(2,-3,-2,5)$ and hence for this case, the duality method essentially coincides with the Gröbner deformation method of Section 2 at the level of $\mathrm{Hom}_D(M, \mathbf{k}[\mathbf{x}])$. The above computations were made in Macaulay 2, where we get the output,

```
i2 : PolyExt(M)
                   1
o2 = HashTable{0 => QQ }
                   2
          1 => QQ
                   1
          2 => QQ
```

Here, the output `i => QQ`$^{\mathtt{j}}$ means that $\dim \mathrm{Ext}_D^i(M, \mathbf{k}[\mathbf{x}]) = j$.

Example 4.5. Let us now evaluate the dimension of rational solutions to $M = F_1(2,-3,-2,5)$. The singular locus is $xy(x-y)(x-1)(y-1)$. We will search for solutions in $\mathbf{k}[x,y][1/x]$ first. From Example 4.2, $\mathbf{D}(M)$ has the presentation $D/\tau(Q_1)$. Let u be the section corresponding to the residue class of $\bar{1}$ in this presentation. Then the localization $\mathbf{D}(M)[1/x]$ is generated by $u \otimes 1/x^7$ and gets the presentation D/J, where

$$J = D \cdot \left\{ \begin{array}{c} (\theta_x\theta_y + \theta_y^2 + 8\theta_y + 2\theta_x + 12) - (\theta_x + \theta_y + 4)\partial_y \\ (\theta_x\theta_y + 2\theta_x + 7\theta_y + 14) - (\theta_x + 10)x\partial_y \end{array} \right\}.$$

The natural localization map can be written as $\varphi : D/\tau(Q_1) \to D/J$, where $\varphi(1) = x^7$. Choose the integration weight vector $w = (1,2)$. Then $\mathbf{D}(M)[\frac{1}{x}]$ has a $(w,-w)$-strict resolution

$$G^\bullet : 0 \to D^1[-1] \xrightarrow{\cdot[v_1,v_2]} D^2[0,-1] \xrightarrow{\cdot[u_1,u_2]^{\mathrm{T}}} D^1[0] \to 0,$$

where

$$u_1 = -x^2\partial_x\partial_y + xy\partial_x\partial_y + 2x\partial_x - 11x\partial_y + 7y\partial_y + 14,$$

$$u_2 = x^3\partial_x^2 + x^3\partial_x\partial_y - x^2\partial_x^2 - x^2\partial_x\partial_y + 16x^2\partial_x + 11x^2\partial_y$$
$$+4xy\partial_y - 9x\partial_x - 11x\partial_y + 52x - 7,$$

$$v_1 = x^3\partial_x^2 + x^3\partial_x\partial_y - x^2\partial_x^2 - x^2\partial_x\partial_y + 16x^2\partial_x + 12x^2\partial_y$$
$$+4xy\partial_y - 8x\partial_x - 11x\partial_y + 52x - 6,$$

$$v_2 = x^2\partial_x\partial_y - xy\partial_x\partial_y - 2x\partial_x + 11x\partial_y - 6y\partial_y - 12.$$

The integration b-function is $(s+12)(s+5)(s+2)$, hence we want the cohomology of the complex $F_w^{12}(\Omega \otimes_D G^\bullet)/F_w^1(\Omega \otimes_D G^\bullet)$, which has the shape,

$$0 \to \mathbf{Q}^{41} \to \mathbf{Q}^{86} \to \mathbf{Q}^{45} \to 0.$$

By evaluating the dimensions of the cohomology groups in Macaulay 2, we find

```
i3:RatExt(M,x)
                    2
o3 = HashTable{0 => QQ }
                    5
               1 => QQ
                    3
               2 => QQ
```

Since we already computed a polynomial solution, this means there is one rational solution with pole along x. Similarly, we get the exact same dimensions for $\mathrm{Ext}_D^i(M, \mathbf{k}[x,y][1/y])$, which means that there is also one rational solution with pole along y. The rank of the system is 3, therefore we have found all the solutions. We could also compute,

```
i4:RatExt(M,f)
                    1
o4 = HashTable{0 => QQ }
                    3
               1 => QQ
                    2
               2 => QQ
```

where f is any of the polynomials $x-y$, $x-1$, or $y-1$. As expected, there are no rational solutions with poles along $x-y$, $x-1$, or $y-1$, but in all cases there are new Ext^1 and Ext^2. We have not computed Ext with respect to any products of poles since it is computationally too intensive for now.

Once we have evaluated the dimension of the solution spaces, we can compute the solutions by a brute force method.

1. For a given holonomic system M, compute its singular locus. Let f be a polynomial such that $f = 0$ contains the singular locus.
2. Evaluate the dimension d of the rational solutions by the homological duality method.
3. Try to find rational solutions of the form $r(x)/f^k$, degree$(r) = p$. Increase $p + k$ until we find d linearly independent solutions.

4.3. Holonomic solutions by duality

Isomorphism (5) can also be expressed (see e.g. [3]) as

$$\mathrm{Ext}^i_D(M,N) \simeq \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M,D),N).$$

To compute the right-hand-side, it is also well known that for all M', and in particular $M' = \mathrm{Ext}^n_D(M,D)$,

$$\mathrm{Tor}^D_{n-i}(M',N) \simeq H^i(K^\bullet((M' \otimes_{\mathbf{k}[\mathbf{x}]} \Omega^{-1})\hat{\otimes} N; \{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n)),$$

where $K^\bullet$ denotes the Koszul complex and $\hat{\otimes}$ denotes the external tensor product into the category of $D_{2n} = \mathbf{k}\langle x_1, y_1, \dots, x_n, y_n, \partial_1, \delta_1, \dots, \partial_n, \delta_n\rangle$-modules. Combining these isomorphisms leads to

$$\mathrm{Ext}^i_D(M,N) \simeq H^i(K^\bullet((\mathbf{D}(M)\hat{\otimes} N; \{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n)).$$

By an automorphism of D, we can transform $\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n$ into $\{x_i, y_i\}_{i=1}^n$, for which the Koszul complex computes the derived restriction to the origin.

Algorithm 4.6. (Evaluating dimensions of holonomic solution spaces)
INPUT: holonomic left D-modules M and N
OUTPUT: dimensions of $\mathrm{Ext}^i_D(M,N)$.

1. Compute the dual $\mathbf{D}(M)$ using Algorithm 4.1
2. Form the D_{2n}-module $\mathbf{D}(M)\otimes_k N$ and apply the change of coordinates $\eta : D_{2n} \to D_{2n}$ where η maps,
$$\begin{aligned} x_i &\mapsto \tfrac{1}{2}x_i - \delta_i, & \partial_i &\mapsto \tfrac{1}{2}y_i + \partial_i,\\ y_i &\mapsto -\tfrac{1}{2}x_i - \delta_i, & \delta_i &\mapsto \tfrac{1}{2}y_i - \partial_i. \end{aligned}$$
3. Compute the restrictions of $\eta(\mathbf{D}(M)\otimes_k N)$ to the origin using the algorithm in [9]. They are finite dimensional vector spaces.
4. Return the dimensions.

Example 4.7. Let $M = F_1(2,-3,-2,5)$ be the Appell differential equation of Example 2.5, and let $N = \mathbf{k}[x,y][1/x]/\mathbf{k}[x,y]$. It has presentation $D/D\cdot\{x, \partial_y\}$, where the generator

1 corresponds to $1/x$. Using the above algorithm, we compute

```
i5 : DExt(M, N)
                          1
o5 = HashTable {0 => QQ}
                          3
                 1 => QQ
                          2
                 2 => QQ
```

Similarly, let $N = \mathbf{k}[\partial_x, \partial_y] \simeq D/D \cdot \{x, y\}$, the module of the delta functions with the support $(0, 0)$. Then we compute

```
i6 : DExt(M, N)
                          0
o6 = HashTable {0 => QQ}
                          1
                 1 => QQ
                          2
                 2 => QQ
```

As before, once we know the dimension of $\mathrm{Hom}_D(M, N)$, we can compute the solutions of M in N by a brute force method.

1. For given holonomic systems M and N, evaluate the dimension d of $\mathrm{Hom}_D(M, N)$ by the homological duality method.
2. Filter N by finite-dimensional vector spaces $F^i(N)$ and search for solutions in $F_i(N)$ for increasing i until d linearly independent solutions are found.

For instance in step 2, if $N = D/J$, then we can use the induced Bernstein filtration B where $B^i(D/J)$ consists of residues of elements $L \in D$ whose total degree is less than or equal to i.

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