Smoothing Properties of Nonlinear Dispersive Equations in Two Spatial Dimensions

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1. INTRODUCTION

In this paper we prove smoothing properties for nonlinear dispersive equations in two spatial dimensions. We consider equations of the following two general types. First, we consider the semilinear equation,

\[ u_t + au_{xxx} + bu_{xxy} + cu_{yy} + f(D^2 u, Du, u, x, y, t) = 0 \]
\[ u(x, y, 0) = \phi(x, y), \] \hspace{1cm} (1.1)

where \( a, b, c, d \) are assumed constant. We then present results for the fully nonlinear equation,

\[ u_t + f(D^3 u, D^2 u, Du, u, x, y, t) = 0 \]
\[ u(x, y, 0) = \phi(x, y). \] \hspace{1cm} (1.2)

We prove sufficient conditions on equations of type (1.1) and (1.2) for which a solution \( u \) will experience an infinite gain in regularity. Specifically, we prove conditions on (1.1) and (1.2) for which initial data \( \phi \) possessing sufficient decay at infinity and a minimal amount of regularity will lead to a unique solution \( u(t) \in C^\infty(\mathbb{R}^2) \) for \( 0 < t < T^* \) where \( T^* \) is the existence time of the solution. In particular, we show that singularities in the initial data instantly disappear.

A number of results have appeared regarding smoothing properties of nonlinear dispersive equations. Cohen [2] considered the KdV equation,
showing that “box-shaped” initial data $\phi \in L^2(\mathbb{R})$ with compact support lead to a solution $u(t)$ which is smooth for $t > 0$. Using different methods, Kato [10] generalized this result, showing that if the initial data $\phi$ are in $L^2((1+e^{sx}) \, dx)$, then the unique solution $u(t) \in C^\omega(\mathbb{R})$ for $t > 0$. Kruzhkov and Faminski [12] replaced the exponential weight function with a polynomial weight function, quantifying the gain in regularity of the solution in terms of the decay at infinity of the initial data. In a separate sequence of articles, Hayashi et al. [7–9] proved similar types of smoothing properties for certain nonlinear Schrodinger equations. Other gain in regularity results for linear and nonlinear dispersive equations include the works of Constantin and Saut [3], Ponce [14], Ginibre and Velo [6], and Kenig et al. [11]. In [4], Craig and Goodman considered an equation in which the dispersive term has a variable coefficient. They proved infinite smoothing results for equations of the form $u_t + a(x, t) u_{xxx} = 0$ which satisfy $a(x, t) \geq c > 0$. Craig et al. [5] generalized this idea to fully nonlinear equations of KdV-type in one spatial dimension of the form $u_t + f(u_{xxx}, ...) = 0$. In particular, their main assumption on the dispersive term is that $f_{xxx} \geq c > 0$. Under this condition, they classified a set of equations for which initial data with minimal regularity and sufficient decay as $x \to +\infty$ lead to a solution $u(t) \in C^\omega(\mathbb{R})$ for $t > 0$. Here we consider infinite gain in regularity results for nonlinear third-order equations in more than one spatial dimension.

We note that while [3, 11] include local smoothing results for some $m$th-order dispersive equations in $n$ spatial dimensions, their results and techniques are different from those presented here. First, they consider equations with only mild nonlinearities. Here, we consider equations with very general nonlinearities including a fully nonlinear equation of the form (1.2). Second, their results indicate local gains in finite regularity. Here, we prove complementary results, showing the relationship between the decay at infinity of the initial data and the amount of gain in regularity. More specifically, we prove conditions under which an equation of form (1.1) or (1.2) will have a vector $(r, s)$ such that initial data decaying appropriately as $rx+sy \to +\infty$ leads to a solution $u(t) \in C^\omega(\mathbb{R}^2)$. We remark that even with no additional decay assumption on the initial data as $rx+sy \to +\infty$, the method used here proves local gains in finite regularity for initial data in $H^s(\mathbb{R}^2)$ (for an appropriate value of $s$). Thus, the results here complement, and in the sense of allowing more general nonlinearities, generalize some of the local finite gain in regularity results in [3, 11].

In studying propagation of singularities, it is natural to consider the bicharacteristics associated with the differential operator. For the KdV equation, it is known that the bicharacteristics all point to the left for $t > 0$, and thus all singularities travel in that direction. Kato [10] makes use of this uniform dispersion, choosing a nonsymmetric weight function decaying
as \( x \to -\infty \) and growing as \( x \to +\infty \) to prove the infinite gain in regularity. In [5], Craig et al. also make use of a unidirectional propagation of singularities in their results on infinite smoothing properties for generalized KdV-type equations for which \( f_{xxx} \geq c > 0 \).

For the two-dimensional case, we are interested in classifying equations for which similar gains in regularity hold. Generically, an equation of the form (1.1) or (1.2), may have bicharacteristics pointing in any direction in \( \mathbb{R}^2 \) for \( t > 0 \). In [13], the author studies the KP-II equation, a two-dimensional analogue of the KdV equation, showing that although singularities travel along bicharacteristics throughout the open left-half plane, it suffices to choose a weight function depending only on \( x \) and not on \( y \). More specifically, she shows that for initial data in a certain Sobolev space with sufficient decay as \( x \to +\infty \), there exists a unique solution \( u(t) \in C^\infty(\mathbb{R}^2) \) for \( 0 < t < T \) where \( T \) is the existence time of the solution. In particular, no extra assumption is made on the decay of the initial data in the \( y \) direction.

Here we extend this type of result to a large class of nonlinear dispersive equations in two spatial dimensions. In particular, our main assumption on the dispersive nature of the equation is that the bicharacteristics all point into some half-plane \( \{(x, y) : rx + sy < 0\} \) for \( t > 0 \). Under this assumption and an additional assumption on the parabolic terms, we are able to show that for initial data decaying appropriately as \( rx + sy \to +\infty \) and with a minimal amount of regularity, the solution \( u(t) \) is infinitely differentiable for \( t > 0 \). In particular, although singularities may travel in a range of directions, we are able to choose a weight function depending only on \( rx + sy \), one spatial variable.

In consideration of the above comments, we will look at dispersive equations (1.1) and (1.2) satisfying the following main assumptions.

**Main Assumptions.** (i) The homogeneous polynomial

\[
p_s(\omega_1, \omega_2) \equiv -(a\omega_1^3 + b\omega_2^3 + c\omega_1\omega_2 + d\omega_2^2) \tag{1.3}
\]

associated with the leading-order symbol for (1.1) has exactly one real root and two non-real roots. By this, we mean that there exist real constants \( k_1, ..., k_5 \) such that

\[
p_s(\omega_1, \omega_2) = (k_1\omega_1 + k_2\omega_2)(k_3\omega_1^2 + k_4\omega_1\omega_2 + k_5\omega_2^2), \tag{1.4}
\]

where the quadratic form

\[
k_5\omega_1^2 + k_4\omega_1\omega_2 + k_5\omega_2^2
\]
is positive definite. As will be detailed later, this assumption forces the bicharacteristics associated with (1.1) to point into one half-plane for \( t > 0 \). In particular, this assumption implies that for \( \rho \), there exists a vector \((r, s) \in \mathbb{R}^2\) such that \( \nabla \rho \cdot (\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \((\omega_1, \omega_2) \neq (0, 0) \in \mathbb{R}^2\).

Similarly, we assume the symbol associated with (1.2),

\[
p_n(\omega_1, \omega_2) \equiv - (f_{x\omega_1\omega_1 + f_{y\omega_1\omega_2} + f_{x\omega_2\omega_2} + f_{y\omega_2\omega_2}^3),
\]

has exactly one real root and two non-real roots for each value of \( x, y, t, u \) and its derivatives. As described above and will be proven in Section 2, this assumption implies there exists a vector \((r, s)\), which may depend on \( x, y, t, u \) and its derivatives, for which \( \nabla \rho \cdot (r, s) < 0 \) for all \((\omega_1, \omega_2) \neq (0, 0)\). As will be described later (see Section 6), we further assume that a vector \((r, s)\) may be chosen independent of \( x, y, t \) and \( u \) and its derivatives for which \( \nabla \rho \cdot (r, s) < 0 \) for all \((\omega_1, \omega_2) \neq (0, 0)\), thus allowing us to find a half-plane \( \{ (x, y): rx+sy < 0 \} \) into which all bicharacteristics point for \( t > 0 \).

(ii) The nonlinearity \( f \) associated with each of these equations satisfies

\[
f_{xx}, f_{yy} \leq 0
\]

\[
(f_{xy})^2 - 4f_{xx}f_{yy} \leq 0.
\]

This assumption prevents a backwards parabolic term from overpowering the dispersive effects.

**Remark.** It may be possible to weaken this second assumption; see [1]. For our purposes, however, we will focus on the smoothing effects of the dispersive terms, and, thus not look at weakening this assumption.

As will be proven later, these assumptions, along with two minor, technical assumptions on the nonlinearity \( f \), allow us to prove a smoothing effect for (1.1) and (1.2).

**Remark.** We refer the reader to the Zakharov–Kuznetsov equation [15], a model for the nonlinear evolution of ion-acoustic waves in a magnetized plasma, as an example of an equation satisfying the above assumptions and thus the necessary hypotheses for the gain of regularity theorems presented in this paper.

We now state a special case of one of our main theorems.
Gain of Regularity for the Semilinear Equation. Consider an equation of form (1.1). Assume the polynomial \( p_s(\omega_1, \omega_2) \) defined in (1.3) has exactly one real root and two non-real roots and that the nonlinearity \( f \) satisfies (1.6). Let \( u \) be a solution of (1.1) in \( \mathbb{R}^2 \times [0, T] \) such that for all integers \( L \geq 1 \),

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} (1 + (rx + sy)^+)^L \sum_{|\alpha| \leq 6} (\partial^\alpha u)^2 \, dx \, dy < +\infty, \tag{1.7}
\]

where \((r, s)\) satisfies \( \nabla p_s \cdot (r, s) < 0 \) and \( z^+ = \max\{0, z\} \). Then our solution \( u(t) \in C^\infty(\mathbb{R}^2) \) for \( 0 < t \leq T \).

Remarks. (1) As will be shown later, the assumption that

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} (1 + (rx + sy)^+)^L \sum_{|\alpha| \leq 6} (\partial^\alpha u)^2 \, dx \, dy < +\infty
\]

for all integers \( L \geq 1 \) may be reduced to assuming this property holds for some integer \( L \geq 1 \). The smoothing phenomenon will still occur, but the amount of smoothing will depend on the size of \( L \), thereby showing the relationship between the decay at infinity of the initial data and the gain in regularity of the solution.

(2) As will be shown in Sections 4 and 5, (1.7) holds under natural assumptions on the initial data. More specifically, for initial data \( f \) satisfying (1.7), there exists a solution \( u \) satisfying (1.7) for a time \( T \) depending only on \( ||f||_{H^6(\mathbb{R}^2)} \).

(3) The amount of regularity assumed on the solution \( u \) may be reduced for equations of type (1.1) with mild nonlinearities. In particular, the \( H^6 \) assumption may be weakened.

(4) As mentioned above, we will prove some results for the fully nonlinear Eq. (1.2) as well (see Sections 6–8). These results are similar to the one stated above for the semilinear Eq. (1.1), but the polynomial weight function is replaced by an exponential weight function.

We will state the gain of regularity result for (1.1) in its full generality in Section 3. Here we provide some ideas of the proof. The technique uses an inductive argument combined with \textit{a priori} estimates. On each level \( \beta \) of the induction, we take \( \alpha \) derivatives of (1.1) where \( |\alpha| = \beta \), multiply the differentiated equation by \( 2 \xi_\beta (\partial^\alpha u) \) where \( \xi_\beta \) is a weight function to be specified later, and integrate over \( \mathbb{R}^2 \). Upon doing so, our identity becomes
\[ \begin{align*}
&\partial_t \int \xi(\partial^su)^2 + \int (3a\xi_x + b\xi_y)(\partial^su_u)^2 + \int (2b\xi_x + 2c\xi_y)(\partial^su_s)(\partial^su) \\
&+ \int (c\xi_x + 3d\xi_y)(\partial^su)^2 \\
&- 2\xi \{ f_{ux}(\partial^su_u)^2 + f_{uy}(\partial^su_u)(\partial^su_y) + f_{uy}(\partial^su_y)^2 \} \\
&+ \int \Theta(\partial^su)^2 + \int 2\xi(\partial^su) R_1 + \int 2\xi(\partial^su) R_2 = 0, \quad (1.8)\end{align*} \]

where \( \xi = \xi_\beta \),

\[ \Theta = \{ \partial_x[\xi f_{ux}] + \partial_y[\xi f_{uy}] + \partial_{xy}[\xi f_{xy}] \} \]

\[ - \{ \xi_t + a\xi_{xxx} + b\xi_{xxy} + c\xi_{xyy} + d\xi_{yyy} \}, \quad (1.9) \]

the terms in \( R_1 \) are of order \( \beta + 1 \), namely

\[ R_1 \equiv \{ \alpha_1 \partial_x[\xi f_{ux}](\partial^{(\gamma_1 - 1, \beta)}u_{xx}) + \ldots + \alpha_2 \partial_y[\xi f_{ux}](\partial^{(\gamma_1, \beta - 1)}u_{yy}) \} \]

\[ + f_{ux}(\partial^su_u) + f_{uy}(\partial^su_u) \quad (1.10) \]

and all the terms in \( R_2 \) are of order \( \gamma \leq \beta \). By Assumption (i), there exists a vector \( (r, s) \) such that \( Vp,(\omega_1, \omega_2) \cdot (r, s) < 0 \) and \( u \) satisfies (1.7) for this choice of \( (r, s) \) and all integers \( L \geq 1 \). Consequently, by choosing a smooth weight function \( \xi \), such that \( \xi \approx (rx+sy)^l \) for some integer \( l \geq 1 \) as \( rx+sy \to +\infty \) and \( \xi \approx e^{\sigma(rx+sy)} \) for \( \sigma > 0 \) arbitrary as \( rx+sy \to -\infty \), we can show there exist constants \( K_1, K_2 > 0 \) giving us the following bound,

\[ K_1 \int \eta(\partial^su_u)^2 + K_2 \int \eta(\partial^su_y)^2 \]

\[ \leq \int (3a\xi_x + b\xi_y)(\partial^su_u)^2 + \int (2b\xi_x + 2c\xi_y)(\partial^su_s)(\partial^su) \\
+ \int (c\xi_x + 3d\xi_y)(\partial^su)^2, \quad (1.11) \]

where \( \eta = \xi_z \) for \( z = rx + sy \). In addition, by assumption (1.6) on the nonlinear term \( f \), the fifth integral in (1.8) is non-negative. Therefore,
integrating with respect to $t$ from $t = 0$ to $t = T$, we arrive at the following inequality,

\[
\int_0^T \xi(\cdot, T)(\partial^* u)^2 + K_1 \int_0^T \eta(\partial^* u_x)^2 + K_2 \int_0^T \eta(\partial^* u_y)^2
\leq \int_0^T \xi(\cdot, 0)(\partial^* \phi)^2 + \left| \int_0^T \Theta(\partial^* u)^2 \right|
\]

\[
+ \left| \int_0^T 2\xi(\partial^* u) R_1 \right| + \left| \int_0^T 2\xi(\partial^* u) R_2 \right|.
\]

(1.12)

By choosing $\xi$ such that $\xi(\cdot, 0) = 0$, we can eliminate the first term on the right-hand side. By using our inductive hypothesis, we are able to prove \textit{a priori} bounds on the other terms on the right-hand side of (1.12) depending only on weighted $L^2$ norms of $(\partial^* u)$. Consequently, we are able to prove \textit{a priori} bounds on the terms on the left-hand side, namely the weighted $L^2$ norms of $(\partial^* u_x)$ and $(\partial^* u_y)$, a gain in regularity! If (1.7) holds for all integers $L \geq 1$, we can show this inductive argument holds for all $\beta \geq 0$, and, thus, $u(t) \in C^\infty(\mathbb{R}^2)$ for $0 < t \leq T$. As will be proven in this paper, if (1.7) holds for some integer $L \geq 1$, the inductive argument will hold for $\beta \leq L + 6$, thus, quantifying the relationship between the decay at infinity of the initial data and the gain in regularity of the solution.

We now provide an overview of this paper. In Section 2 we consider the semilinear Eq. (1.1). We prove necessary and sufficient conditions on the coefficients to ensure all bicharacteristics point into some half-plane. In particular, we prove that if the polynomial $p_s(w_1, w_2)$ associated with (1.1) has exactly one real root and two non-real roots, then all bicharacteristics for $t > 0$ point into a half-plane, \{(x, y) : rx + sy < 0 \text{ where } \forall p_s(w_1, w_2), (r, s) < 0 \forall (w_1, w_2) \neq (0, 0) \in \mathbb{R}^2\}. This lemma will lead to our choice of weight function $\tilde{\zeta}(rx + sy, t)$. Also in Section 2, we describe the notation to be used for the semilinear case as well as the assumptions made on our nonlinearity $f$.

We divide the analysis of the semilinear equation into three parts. In Section 3 we state our main results on the gain of regularity for the semilinear Eq. (1.1) and prove the \textit{a priori} estimates used in the main theorem. We provide an outline of the proof on the gain of regularity, but defer the formal proof until the end of Section 5 after proving the necessary theorems and lemmas.

In Section 4 we prove the local existence and uniqueness results for (1.1) used in the gain of regularity result in Section 3. Specifically, we show that for initial data $\phi \in H^N(\mathbb{R}^2)$, $N \geq 6$, there exists a unique solution $u \in L^\infty([0, T] ; H^N(\mathbb{R}^2))$ where the time of existence $T$ depends only on the norm of $\phi \in H^N(\mathbb{R}^2)$. 

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Finally in Section 5 we show that solutions $u$ to (1.1) also satisfy a persistence property. Specifically, we prove that if the initial data $\phi$ lies in a certain weighted Sobolev space, then the unique solution $u$ of the semilinear Eq. (1.1) lies in the same weighted Sobolev space. At the conclusion of Section 5, we give a formal proof of our gain in regularity theorem for the semilinear Eq. (1.1).

In Section 6 we study the regularity properties for the fully nonlinear Eq. (1.2). As we will discuss, we are not able to use the same type of weight function as we did in the semilinear case. Instead of using a polynomial weight function, we use an exponential weight function. In Section 6, we introduce the notation and assumptions for the fully nonlinear equation and discuss the differences in the results and proofs between the semilinear and fully nonlinear equations.

In Section 7, we state and prove our main results concerning the gain of regularity for solutions to the fully nonlinear Eq. (1.2), including the main estimates for the remainder terms.

Finally, in Section 8, we prove existence and uniqueness results for certain equations of type (1.2). For these equations, we show that for initial data $\phi$ in a particular weighted Sobolev space, there exists a unique solution $u$ in the same weighted Sobolev space.

### 2. THE SEMILINEAR EQUATION

We consider the semilinear equation

$$
\begin{align*}
u_t + au_{xxx} + bu_{xy} + cu_{yyy} + du_{yy} + f(D^2 u, Du, u, x, y, t) &= 0 \\
u(x, y, 0) &= \phi(x, y),
\end{align*}
$$

where $a, b, c, d$ are constant. We make the following assumptions on (2.1).

**Assumptions.** (A1) The homogeneous polynomial

$$
p_s(\omega_1, \omega_2) \equiv -(a\omega_1^3 + b\omega_1^2\omega_2 + c\omega_1\omega_2^2 + d\omega_2^3)
$$

associated with the leading-order symbol has one real root and two non-real roots. Under this assumption, we will show in Lemma 2.1 below that there exists a direction $(r, s)$ such that $\nabla p_s(\omega_1, \omega_2) \cdot (r, s) < 0$ for all $(\omega_1, \omega_2) \neq (0, 0)$. This implies there exists a half plane $\{(x, y): rx + sy < 0\}$ such that all bicharacteristics point into that half-plane for $t > 0$. 

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The nonlinear term \( f: \mathbb{R}^3 \times [0, T] \to \mathbb{R} \) is \( C^\infty \) and satisfies
\[
\begin{align*}
    f_{u_x}, f_{u_y} &\leq 0 \\
    f_{u_x}^2 &\leq 4 f_{u_x} f_{u_y}.
\end{align*}
\]
This prevents the existence of a backwards parabolic term.

All the derivatives of \( f(w, x, y, t) \) are bounded for \( (x, y) \in \mathbb{R}^2 \), for \( t \in [0, T] \) and \( w \) in a bounded set.

\( x^N y^M \partial_x^i \partial_y^j f(0, x, y, t) \) is bounded for all \( N, M \geq 0 \), \( i, j \geq 0 \), and \( (x, y) \in \mathbb{R}^2, t \in (0, T] \).

These assumptions imply that \( f \) has the form
\[
f = u_{xx}g_5 + u_{xy}g_4 + u_{yy}g_3 + u_xg_2 + u_yg_1 + ug_0 + h, \tag{2.3}
\]
where we define the \( g_i \) as
\[
\begin{align*}
g_5 &= \left[ f(u_{xx}, u_{xy}, \ldots) - f(0, u_{xy}, \ldots) \right]/u_{xx} \quad \text{for } u_{xx} \neq 0, \\
    &\quad \left[ \partial_{u_x} f(0, u_{xy}, \ldots) \right]/u_{xx} \quad \text{for } u_{xx} = 0, \tag{2.4}
\end{align*}
\]
\[
\begin{align*}
g_4 &= \left[ f(0, u_{xy}, u_{yy}, \ldots) - f(0, 0, u_{yy}, \ldots) \right]/u_{xy} \quad \text{for } u_{xy} \neq 0, \\
    &\quad \left[ \partial_{u_y} f(0, u_{xy}, \ldots) \right]/u_{xy} \quad \text{for } u_{xy} = 0, \tag{2.5}
\end{align*}
\]
and similarly for \( g_3, g_2, \ldots, h \). Assumption (A2) implies that
\[
g_5^4 - 4g_5g_3 \leq 0. \tag{2.6}
\]
Assumption (A3) implies that \( g_5, g_4, \ldots, h \) are \( C^\infty \) and each of their derivatives is bounded for \( w \) bounded, \( (x, y) \in \mathbb{R}^2 \) and \( t \in [0, T] \).

Under these assumptions, we claim there exists a direction \((r, s)\) such that any singularities in the initial data will propagate into the half-plane \( \{ (x, y) : rx + sy < 0 \} \) at infinite speed and instantly disappear. Consequently, we will choose our weight function depending only on \( rx + sy \) and \( t \).

Now we show that if \( p \) satisfies assumption (A1) above, there is a direction \((r, s)\) such that \( \nabla p \cdot (\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \( (\omega_1, \omega_2) \neq (0, 0) \). We will use this fact in our choice of weight function.

**Lemma 2.1.** Consider the third degree homogeneous polynomial
\[
q(\omega_1, \omega_2) = -(a\omega_1^3 + b\omega_1^2\omega_2 + c\omega_1\omega_2^2 + d\omega_2^3). \tag{2.7}
\]
The polynomial \( q(\omega_1, \omega_2) \) has one real root and two non-real roots if and only if there exists a direction \((r, s)\) such that \( \nabla q \cdot (r, s) < 0 \) for \( (\omega_1, \omega_2) \neq (0, 0) \).
Proof. First, we prove the forward implication. Assume \( q \) has exactly one real root and two non-real roots. Without loss of generality, we may assume
\[
q(\omega_1, \omega_2) = -\omega_1(a\omega_1^2 + b\omega_1\omega_2 + c\omega_2^2), \tag{2.8}
\]
where \( a\omega_1^2 + b\omega_1\omega_2 + c\omega_2^2 \) has non-real roots, and, therefore, \( b^2 - 4ac < 0 \). Therefore,
\[
(bc)^2 - 4(b^2 - 3ac)(c^2) = -3c^2(b^2 - 4ac) > 0,
\]
and there exists a vector \( (r, s) \) such that
\[
4\{(b^2 - 3ac) r^2 + bcrs + c^2 s^2\} < 0.
\]
For this choice of \( (r, s) \),
\[
(2br+2cs)^2 - 4(cr)(3ar+bs) = 4\{(b^2 - 3ac) r^2 + bcrs + c^2 s^2\} < 0.
\]
This implies that for this choice of \( (r, s) \) the polynomial
\[
\nabla q \cdot (r, s) = -(3ar+bs) \omega_1 - (2br+2cs) \omega_1 \omega_2 - (cr) \omega_2^2 < 0
\]
for all \( (\omega_1, \omega_2) \neq (0, 0) \).

Next, we prove the reverse implication. Suppose \( q \) does not have one real root and two non-real roots. Then in particular, \( q \) has three (not necessarily distinct) real roots. Therefore, \( q \) can be written as
\[
q = -K(\omega_1 + ao_1)(\omega_1 + bo_2)(\omega_1 + co_2).
\]
Let \( (r, s) \) be the vector such that \( \nabla q \cdot (r, s) < 0 \) for all \( (\omega_1, \omega_2) \neq (0, 0) \). This implies
\[
-K[3r+(a+b+c) s] \omega_1^2 + [2(a+b+c) r + 2(ac+bc+ab) s] \omega_1 \omega_2 + [(ac+bc+ab) r + 3abc s] \omega_2^2 < 0
\]
for \( (\omega_1, \omega_2) \neq (0, 0) \). Consequently,
\[
[2(a+b+c) r + 2(ac+bc+ab) s]^2
- 4[3r+(a+b+c) s]((ac+bc+ab) r + 3abc s] < 0.
\]
Therefore,
\[
[(a+b+c)^2 - 3(ac+bc+ab)] r^2 + [(a+b+c)(ac+bc+ab) - 9abc] rs
+ [(ac+bc+ab)^2 - 3abc(ac+bc+ab)] s^2 < 0.
\]

In particular, this requires that
\[
(a+b+c)^2 - 3(ac+bc+ab) < 0.
\]

However,
\[
(a+b+c)^2 - 3(ac+bc+ab) = \frac{1}{2} (a-b)^2 + \frac{1}{2} (a-c)^2 + \frac{1}{2} (b-c)^2 > 0.
\]

Therefore, we have a contradiction! Consequently, \( q \) must have one real root and two non-real roots. 

**Choice of Weight Function.** As described in the Introduction, we will be using non-symmetric weight functions. Specifically, we will use weight functions \( \xi \) which behave roughly like powers of \( rx+sy \) for \( rx+sy > 1 \) and decay exponentially for \( rx+sy < -1 \), where \( (r, s) \) is a fixed vector satisfying \( \nabla p(\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \( (\omega_1, \omega_2) \neq (0, 0) \). For each step of the induction, we will decrease our weight function by one power of \( rx+sy \), showing the relationship between the regularity of the solution and the decay at infinity of the initial data. With these ideas in mind, we define our weight classes as follows. We say that a positive \( C^\infty \) function \( \xi \) belongs to the weight class \( W_{r,s}^{\sigma,i,k} \), where \( \sigma \geq 0, k \geq 0 \), if \( \xi(rx+sy, t) \) satisfies the following:

\[
0 < c_1 e^{-t} e^{-a(rx+sy)} \xi(rx+sy, t) \leq c_2 \quad \text{for} \quad rx+sy < -1
\]

\[
0 < c_1 t^{-k} (rx+sy)^{-i} \xi(rx+sy, t) \leq c_2 \quad \text{for} \quad rx+sy > 1
\]

\[
(t |\partial_r \xi| + |\partial_s \xi| + |\partial_r \partial_s \xi|)/\xi \leq c_3 \quad \text{in} \quad \mathbb{R}^2 \times [0, T] \quad \text{for all} \quad j.
\]

Thus \( \xi \) looks like \( t^k \) as \( t \to 0 \), like \( (rx+sy)^i \) as \( rx+sy \to +\infty \) and like \( e^{a(rx+sy)} \) as \( rx+sy \to -\infty \).

In what follows, we will be fixing values \( r, s \) such that \( \nabla p(\omega_1, \omega_2) \cdot (r, s) < 0 \) for \( (\omega_1, \omega_2) \neq (0, 0) \). Consequently, we will drop the superscript notation and let

\[
W_{\sigma,i,k} = W_{r,s}^{\sigma,i,k},
\]

for \( r, s \) fixed.
Notation. We introduce the following notation for our weighted Sobolev spaces. Let $H^\beta(W_{s,i,k})$ be the space of functions with finite norm

$$\|f\|_{H^\beta(W_{s,i,k})} = \int \sum_{|\alpha| \leq \beta} (\partial^\alpha f)^2 |\xi(rx + sy, t)| \, dx \, dy$$  \hfill (2.9)

for any $\xi \in W_{s,i,k}$, $\beta \geq 0$, and $0 \leq t \leq T$. We note that although this norm depends on $\xi$, all choices of $\xi$ in this class lead to equivalent norms. With the same notation, let $L^p(H^\beta(W_{s,i,k}))$ be the space of functions with finite norm

$$\|f\|_{L^p(H^\beta(W_{s,i,k}))} = \int_0^T \left\{ \int \sum_{|\alpha| \leq \beta} (\partial^\alpha f)^2 |\xi(rx + sy, t)| \, dx \, dy \right\}^{\frac{1}{p}} \, dt$$  \hfill (2.10)

for $\beta \geq 0$, where $\xi \in W_{s,i,k}$. In addition, we define the following spaces,

$$\tilde{W}_{s,i,k} = \bigcup_{j<i} W_{s,j,k}$$

$$L^p(H^\beta(\tilde{W}_{s,i,k})) = \bigcup_{j<i} L^p(H^\beta(W_{s,j,k})).$$

The spaces $\tilde{W}_{s,i,k}$ will only be used for $i = -1$.

3. GAIN OF REGULARITY

In this section, we state our main theorem, Theorem 3.1, on the gain of regularity for the semilinear Eq. (2.1), satisfying Assumptions (A1)–(A4). Specifically, if our solution $u$ has sufficient decay at infinity and a minimal amount of regularity, then we will show that our solution $u$ is in fact smoother than its initial data. As stated in the Introduction, the proof relies on a priori estimates on smooth solutions as well as an existence theorem and a “persistence” property of the initial data. Consequently, we will structure Sections 3–5 as follows. After the statement of our main theorem, we will provide an outline of the proof. Subsequently, we will prove the main a priori estimates in Lemma 3.2. The existence and uniqueness theorems will be stated and proved in Section 4 while the “persistence” property will be handled in Section 5. After providing these pieces, we will prove Theorem 3.1 at the end of Section 5.

We now state our main theorem on the gain of regularity for (2.1).
Theorem 3.1. Assume Eq. (2.1) satisfies (A1)–(A4). Let $T > 0$ and let $u$ be the solution of (2.1) in the region $\mathbb{R}^2 \times [0, T]$ such that $u \in L^2(H^\sigma(W_{r,s}^{−1,L}))$ for some integer $L \geq 1$ and some $(r, s)$ satisfying $\nabla p(\omega_1, \omega_2) \cdot (r, s) < 0$ for all $(\omega_1, \omega_2) \neq (0, 0)$. Then

$$u \in L^2(H^{\sigma+1/2}(W_{r,s}^{−1,L−1,1})) \cap L^2(H^{\sigma+1}(W_{r,s}^{−1,L−1,1}))$$

(3.1)

for all $\sigma > 0$, $0 \leq l \leq L$, where $W_{r,s}^{−1,L}$ is replaced by $\tilde{W}_{r,s}^{−1,L}$.

In what follows, we assume $r, s$ to be fixed and thus drop the superscript notation. In particular, we let

$$W^{r,s}_{r,s} \equiv W_{r,s}^{r,s},$$

where $r, s$ satisfy the assumption in the statement of Theorem 3.1.

As stated above, we will defer the proof of Theorem 3.1 until the end of Section 5. Consequently, here we provide an outline of the pieces we will need.

Outline of Proof. We begin by approximating our initial data $\phi$ by a sequence $\{\phi^{(n)}\}$ of smooth functions. In Section 4 we show that for each smooth $\phi^{(n)}$ there exists a smooth solution $u^{(n)}$. In Section 5, we show that if our initial data $\phi^{(n)}$ also lies in a weighted Sobolev space, then our solution $u^{(n)}$ lies in the same weighted Sobolev space. Combining the results of Sections 4 and 5, we prove that

$$u^{(n)} \in L^2(H^\sigma(W_{0,L,0})) \cap L^2(H^{\sigma+1}(W_{0,L,1,0}))$$

(3.2)

with a bound that depends only on the norm of $\phi^{(n)} \in H^\sigma(W_{0,L,0})$.

We then use an inductive argument, beginning with $\beta = 7$. In particular, for all $\alpha$ such that $|\alpha| = 7$, we use the following a priori estimate, which will be proven in Lemma 3.2 below, on smooth solutions $u^{(n)}$,

$$\sup \int_{0 \leq t \leq T} \zeta((\partial^\alpha u)^2 + \int_0^t \int_0^t \eta(\partial^\alpha u)^2 + \int_0^t \int_0^t \eta(\partial^\alpha u)^2 \leq C,$n

where $C$ depends only on the norm of $u^{(n)}$ in

$$L^2(H^\sigma(W_{0,L,0})) \cap L^2(H^{\sigma+1}(W_{0,L,1,0})).$$

(3.4)

As stated above, however, this norm is bounded by the norm of $\phi^{(n)} \in H^\sigma(W_{0,L,0})$. Therefore, we conclude

$$u^{(n)} \in L^2(H^{\sigma+1}(W_{0,L,1,1})) \cap L^2(H^{\sigma+1}(W_{0,L,2,1})).$$

(3.5)
The proof is continued inductively and completed by a convergence argument.

We now provide a proof of the a priori estimates described above.

**Lemma 3.2 (Main Estimates).** For $u$, a solution of (2.1), sufficiently smooth and with sufficient decay at infinity,

$$\sup_{0 \leq t \leq T} \left( \int_0^t |\xi(t)\partial^\alpha u|^2 + \int_0^T \left| \eta(t)\partial^\beta u_x \right|^2 + \int_0^T \left| \eta(t)\partial^\gamma u_y \right|^2 \right) \leq C$$

(3.6)

for $7 \leq \beta \leq L + 6$, where $|\alpha| = \beta$, $\xi \in W^{\alpha}_{\text{L} - \beta + 6, \beta - 6}$, $\eta \in W^{\alpha}_{\text{L} - \beta + 5, \beta - 6}$, where $C$ depends only on the norms of $u$ in

$$L^\infty(W^{\alpha}_{\text{L} - \gamma + 6, \gamma - 6}) \cap L^2(W^{\gamma + 1}_{\text{L} - \beta + 5, \beta - 6})$$

(3.7)

for $6 \leq \gamma \leq \beta - 1$ and on the norm of $u$ in $L^\infty(W^\alpha_{0, 0})$.

The idea of the proof is described in the Introduction. In particular, for each $\beta \geq 7$, we take $\alpha$ derivatives of (2.1), multiply the differentiated equation by $2\xi(t)\partial^\alpha u$ where

$$\xi(t) = \int_{-\infty}^{\infty} \eta(z, t) dz$$

for $\eta \in W^{\alpha}_{\text{L} - \beta + 5, \beta - 6}$

(3.8)

and integrate over $\mathbb{R}^2 \times [0, t]$ for $0 < t \leq T$. By Assumptions (A1) and (A2) on Eq. (2.1) and the choice of weight function, we arrive at the following inequality,

$$\int \left| \xi(\cdot, t)\partial^\alpha u \right|^2 + K_1 \int_0^t \int \left| \eta(t)\partial^\beta u_x \right|^2 + K_2 \int_0^t \int \left| \eta(t)\partial^\gamma u_y \right|^2$$

$$\leq \int \left| \xi(\cdot, 0)\partial^\alpha \phi \right|^2 + \int_0^t \int \left| \Theta(t)\partial^\alpha u \right|^2$$

$$+ \int_0^t \int 2\xi(t)\partial^\alpha u \left( R_1 \right) + \int_0^t \int 2\xi(t)\partial^\alpha u \left( R_2 \right)$$

(3.9)

for some constants $K_1, K_2 > 0$, where $|\alpha| = \beta$, $\xi = \xi(t)$, $\Theta$ is given in (1.9), $R_1$ is given in (1.10) and the terms in $\int_0^t \int \xi(t)\partial^\alpha u \left( R_2 \right)$ are given in Lemma 3.3 below. In the proof of Lemma 3.2, we show that the right-hand side of (3.9) is bounded in terms of (3.7) and the norm of $u \in L^\infty(W^\alpha_{0, 0})$. Before giving the proof for Lemma 3.2, we show the form of the terms in $\int_0^t \int 2\xi(t)\partial^\alpha u \left( R_2 \right)$. 


**Lemma 3.3 (Form of remainder terms).** Every term in the integrand of

\[
\int_0^t 2\xi^d (\partial^a u) \, R_2
\]

is of the form

\[
\xi^d (D^f) \cdot (\partial^a u) \cdots (\partial^a u)(\partial^a u),
\]

where

\[
D^f \equiv \partial^p_{a_x} \partial^p_{a_y} \partial^p_{a_x} \partial^p_{a_y}
\]

for \( \partial^p_{a_x} \equiv \partial^p_{a_{xx}} \partial^p_{a_{xy}} \partial^p_{a_{yx}} \partial^p_{a_{yy}} \), \( \partial^p_{a_y} \equiv \partial^p_{a_{xx}} \partial^p_{a_{xy}} \partial^p_{a_{yx}} \partial^p_{a_{yy}} \), \( 1 \leq |v_1| \leq \ldots \leq |v_p| \leq |a| \), and

\[
p = p_2 + p_1 + p_0 \geq 0 \quad (3.11)
\]

\[
|v_1| + \cdots + |v_p| + |x| + \mu_1 + \mu_2 = 2 |x| + 2p_2 + p_1 \quad (3.12)
\]

\[
p + |v_{p-1}| + |v_p| \leq |a| + 6 \quad \text{if} \quad p \geq 2. \quad (3.13)
\]

**Proof.** Each term comes from differentiation of \( f(D^f u, \ldots) \), so it must have the form shown. Equation (3.12) is the total number of derivatives. Each differentiation with respect to \( u_{xx}, u_{xy}, \) or \( u_{yy} \) adds two extra derivatives. Each differentiation with respect to \( u_x \) or \( u_y \) adds one extra derivative. Therefore, \( |v_1| + \cdots + |v_p| \geq 3p_2 + 2p_1 + p_0 \).

In particular, this implies that \( |v_1| + \cdots + |v_{p-2}| \geq 3(p_2 - 2) + 2p_1 + p_0 \).

Using (3.12),

\[
2 |x| + 2p_2 + p_1 \geq |v_1| + \cdots + |v_p| + |x|
\]

\[
\geq 3(p_2 - 2) + 2p_1 + p_0 + |v_{p-1}| + |v_p| + |x|.
\]

Therefore,

\[
|x| + 6 \geq p_2 + p_1 + p_0 + |v_{p-1}| + |v_p|
\]

\[
\geq p + |v_{p-1}| + |v_p|.
\]

We now prove Lemma 3.2. In particular, we show that the terms on the right-hand side of (3.9) are bounded in terms of (3.7) and the norm of \( u \in L^\infty (H^q(W_{0,0,0})) \).
Proof of Lemma 3.2. As described above, for each \( \alpha \in \mathbb{Z}^+ \times \mathbb{Z}^+ \), \( |\alpha| \geq 7 \), we have the following inequality,

\[
\int_0^\tau \zeta(\cdot, t)(\partial^\alpha u)^2 + K_1 \int_0^\tau \eta(\partial^\alpha u_x)^2 + K_2 \int_0^\tau \eta(\partial^\alpha u_y)^2 \\
\leq \int_0^\tau \zeta(\cdot, 0)(\partial^\alpha \phi)^2 + \left| \int_0^\tau \Theta(\partial^\alpha u) \right| + \left| \int_0^\tau 2 \xi(\partial^\alpha u) R_1 \right| + \left| \int_0^\tau \int_0^\tau 2 \xi(\partial^\alpha u) R_2 \right|, \tag{3.14}
\]

where \( \Theta \) is defined in (1.9), \( R_1 \) is defined in (1.10) and the terms in \( \int_0^\tau \int 2 \xi(\partial^\alpha u) R_2 \) are described in Lemma 3.3 above. We need to show that the right-hand side of (3.14) is bounded by a constant \( C \) depending only on the norm of \( u \) in \( L^\infty(H^4(W_{0,L,0})) \) and the norm of \( u \) in (3.7).

First, we notice that \( \zeta(\cdot, 0) = 0 \). In addition,

\[
\left| \int_0^\tau \Theta(\partial^\alpha u) \right| \leq \left| \int_0^\tau \left\{ \partial_{xx} [\xi f_{uu_x}] + \partial_{xy} [\xi f_{uu_y}] + \partial_{yy} [\xi f_{uu_y}] \right\}(\partial^\alpha u)^2 \right| \\
+ \left| \int_0^\tau \left\{ \xi_{t} + a_{xxx} + b_{xxy} + c_{xyy} \right\}(\partial^\alpha u)^2 \right| \\
\leq C \left( \int_0^\tau \xi(\partial^\alpha u)^2 \right),
\]

where \( C \) depends at most on \( ||u||_{L^\infty(\mathbb{R}^n)} \) because of assumption (A3) on \( f \) and the fact that \( |\partial^\alpha \xi| \leq C \xi \).

Therefore, inequality (3.14) becomes

\[
\int_0^\tau \zeta(\cdot, t)(\partial^\alpha u)^2 + K_1 \int_0^\tau \eta(\partial^\alpha u_x)^2 + K_2 \int_0^\tau \eta(\partial^\alpha u_y)^2 \\
\leq C \left( \int_0^\tau \xi(\partial^\alpha u)^2 \right) + 2 \left| \int_0^\tau \xi(\partial^\alpha u) R_1 \right| + 2 \left| \int_0^\tau \xi(\partial^\alpha u) R_2 \right|, \tag{3.15}
\]

where \( C \) depends only on the norm of \( u \in L^\infty(H^4(W_{0,L,0})) \). It remains to show that the other two remainder terms above are bounded by a constant depending only on (3.7) and the norm of \( u \in L^\infty(H^4(W_{0,L,0})) \).
The remainder terms involving $R_1$ are shown in (1.10). They can be estimated as follows. For example, we estimate

$$
\left| \int_0^t \xi(\partial^su) \{a_i \partial_s [f_{w_{n_0}}(\partial^{s_1-1,s_2}u)] \} \right| \\
= C \left| \int_0^t \alpha_1 \xi \partial_s [f_{w_{n_0}}] ((\partial^su)^2) \right| \\
= C \left| \int_0^t \partial_s [\alpha_1 \xi \partial_s [f_{w_{n_0}}]] ((\partial^su)^2) \right| \\
\leq C \left( \int_0^t \xi((\partial^su)^2) \right),
$$

where again $C$ depends only on $\|u\|_{L^\infty(H^s(\mathbb{R}))}$. The other terms in (1.10) of this form can be dealt with similarly.

The other set of terms in $R_1$ can be dealt with by integrating by parts once,

$$
\left| \int_0^t 2\xi(\partial^su) f_{w_{n_0}}(\partial^su) \right| = \left| \int_0^t \xi f_{w_{n_0}} ((\partial^su)^2) \right| \\
\leq \left| \int_0^t \xi f_{w_{n_0}} ((\partial^su)^2) \right| + \left| \int_0^t \xi \partial_s [f_{w_{n_0}}] ((\partial^su)^2) \right| \\
\leq C \left( \int_0^t \xi((\partial^su)^2) \right),
$$

where $C$ depends at most on the norm of $u$ in $L^\infty(H^s(W_{0,L} \delta))$. We can deal with the term $\int_0^t 2\xi(\partial^su) f_{w_{n_0}}(\partial^su)$ similarly.

The other remainder term, denoted by $\int_0^t 2\xi(\partial^su) R_2$, contains terms which are of the form described in Lemma 3.3 above, namely

$$
\left| \int_0^t \xi(\partial^su)(D^rf) \cdot (\partial^su) \cdots (\partial^su)(\partial^su) \right|.
$$

The plan is to divide the weight function among the terms in the integral and bound each term by a constant depending only on (3.7) and the norm of $u$ in $L^\infty(H^s(W_{0,L} \delta))$. We will consider the case $rx+sy > 1$ below. The case $rx+sy < -1$ is even easier to handle. We divide the analysis of the remainder terms into various cases.
The Case $|v_{p-1}| \leq |x|-3$ (if $p \geq 2$) or $p = 1$. For smooth weight functions $\zeta$ such that $|\partial^i \zeta| \leq C\zeta$, we have the following weighted estimate,

$$|\xi(\partial^i u)^2|_{L^p(R^2)} \leq C \left( \int \xi \left\{ (\partial^i u)^2 + (\partial^i u_{xx})^2 + (\partial^i u_{yy})^2 \right\} \right).$$  \hspace{1cm} (3.16)

It follows easily from the fact that $H^2(R^2) \subset L^\infty(R^2)$.

Therefore, for $0 \leq |\gamma| \leq |x|-3$ and $\zeta_{\gamma} \in W_{p_0, L_0, \gamma}^{r, (p_0-4)^+}$,

$$\sup_{\theta \in T} \sup_{(x, y) \in T} |\zeta_{\gamma}(\partial^\gamma u)^2| \leq C,$$  \hspace{1cm} (3.17)

where $C$ depends only on (3.7) and the norm of $u \in L^\infty(H^4(W_{0, L, 0})).$ In addition, we can use the fact that

$$\int_0^T \int \zeta_{\gamma}(\partial^\gamma u)^2 \leq C \hspace{1cm} (3.18)$$

for $\zeta_{\gamma} \in W_{p_0, L_0, (p_0-6)^+}^{r, (p_0-7)^+}$ and

$$\int_0^T \int \zeta_{\gamma}(\partial^\gamma u)^2 \leq C \hspace{1cm} (3.19)$$

for $\zeta_{\gamma} \in W_{p_0, L_0, (p_0-6)^+}^{r, (p_0-7)^+}$, where $C$ depends only on (3.7). With these ideas in mind, we can split up the integral as follows. Let

$$A \equiv \{(x, y) : rx+sy \geq 1\}.$$  

Therefore,

$$\left| \int_0^T \int_A \xi_{\beta}(D\beta f) \cdot (\partial^\gamma u) \cdots (\partial^\gamma u)(\partial^\gamma u) \right| \leq CT^M \sup_A (rx+sy)^Q$$

$$\leq CT^M \sup_A (rx+sy)^Q \cdot |\zeta_{\gamma}(\partial^\gamma u)|_{L^\infty} \cdots |\zeta_{\gamma}(\partial^\gamma u)|_{L^\infty} \left( \int_0^T \int \zeta_{\gamma}(\partial^\gamma u)^2 \right)^{\frac{1}{2}} \left( \int_0^T \int \zeta_{\gamma}(\partial^\gamma u)^2 \right)^{\frac{1}{2}}.$$

By (3.17)–(3.19), the terms on the right-hand side are bounded by (3.7) and the norm of $u \in L^\infty(H^4(W_{0, L, 0})), using the appropriate weight functions $\zeta_{\gamma}, \zeta_{\alpha}$ described above. It remains to verify that $M \geq 0$ and $Q \leq 0$ so that $T^M$ and $\sup_A (rx+sy)^Q$ are bounded.
The Powers of $t$. First, we check the powers of $t$. The weight function, $\xi_p$, in the integrand is in the weight class $W_{s,L-\frac{p}{6},0,-\frac{p}{6}}$. Therefore, it contributes the power $t^{|a|-6}$. But, by our estimates above, we have that each factor $(\partial^\gamma u)$ uses up the power $t^{(|n_j|^{-4})+/2}$ for $j = 1, \ldots, p-1$, while the last two terms, $(\partial^\gamma u)$ and $(\partial^\gamma u)$ use up the powers $t^{(|n_p|^{-7})+/2}$ and $t^{(|a|^{-7})+/2}$, respectively. Let $M = \text{net powers of } t$. We want to show that $M \geq 0$ and, therefore, we can throw away the extra powers of $t$.

$$M = (|x| - 6) - \sum_{j=1}^{p-1} (|v_j| - 4)^+ - (|v_p| - 7)^+ - (|x| - 7)^+. \quad (3.20)$$

Let $q$ be the largest integer such that $|v_q| \leq 4$. Assume $q_2$ of the derivatives come from differentiation with respect to $u_{xx}, u_{xy}$, or $u_{yx}$, $q_1$ of the derivatives come from differentiation with respect to $u_x$ or $u_y$, and $q_0$ of the derivatives come from differentiation with respect to $u$. Therefore, $q = q_2 + q_1 + q_0$. For now, assume $q \leq p-1$.

The Subcase $q \leq p-1$ for $p \geq 2$. As in the proof of Lemma 3.3,

$$\sum_{j=1}^q |v_j| \geq 3q_2 + 2q_1 + q_0. \quad (3.21)$$

Combining this with (3.12), we have

$$2M = |x| - 5 - \sum_{j=1}^{p-1} (|v_j| - 4)^+ - (|v_p| - 7)^+$$

$$\geq \left\{ 3q_2 + 2q_1 + q_0 + \sum_{j=1}^{p-1} |v_j| - 2p_2 - p_1 \right\} - 5 - \sum_{j=q+1}^{p-1} (|v_j| - 4) - (|v_p| - 7)^+$$

$$= \{ 3q_2 + 2q_1 + q_0 + |v_p| - 2p_2 - p_1 \} - 5 + 4(p - q - 1) - (|v_p| - 7)^+$$

$$\geq p + 1 + |v_p| - 9 - (|v_p| - 7)^+. $$

If $|v_p| \geq 7$, then the right hand side is $\geq p - 1 > 0$. If $|v_p| = 6$, then the right hand side is $\geq p - 2$, which is $\geq 0$ as long as $p \geq 2$. If $|v_p| = 5$, then the right-hand side is $\geq p - 3$ which is $\geq 0$ as long as $p \geq 3$. If $|v_p| = 5$ and $p = 2$, it can easily be seen by (3.20) that $M > 0$ because $|x| \geq 7$.

The Subcase $p = 1$ or $p = q$. If $p = 1$, then using the fact that $|v_p| \leq |x|$ and (3.20), we easily see that $M > 0$. If $p = q$, then $|v_p| \leq 4$, and, therefore, by (3.20), $M > 0$ because $|x| \geq 7$.

After looking at the powers of $rx + sy$, we will look at the terms in which $|v_{p-1}| > |x| - 3$. 
The Powers of $rx + sy$. For now we continue to assume $|v_{p-1}| \leq |a| - 3$. We need to show that $Q \leq 0$ so any extra powers of $rx + sy$ can be thrown away. The weight function $\zeta_\beta$ contributes the power $(rx + sy)^{L - |a| + 6}$, while the other terms use up the following powers of $rx + sy$. The weight functions $\zeta_y$ behave like $(rx + sy)^{L - (|y| - 4)^+}$, for $j = 1, \ldots, p - 1$. The weight function $\zeta_y$ behaves like $(rx + sy)^{L - (|y| - 6)^+}$, while the weight function $\zeta_a$ behaves like $(rx + sy)^{L - |a| + 6}$. Therefore,

$$Q = \{L - |a| + 6\} - \frac{1}{2} \sum_{j=1}^{p-1} \{L - (|v_j| - 4)^+\} - \frac{1}{2} \{L - (|v_p| - 6)^+\} - \frac{1}{2} \{L - |a| + 6\}. \quad (3.22)$$

We need to show that $Q \leq 0$, so the extra powers of $rx + sy$ can be “thrown away”. Again, let $q$ be the largest integer such that $|v_q| \leq 4$. Assume $q \leq p - 1$.

**The Subcase** $q \leq p - 1$ for $p \geq 2$. Therefore, using (3.21), (3.12), and (3.22), we have

$$-2Q = |a| - 6 + L(p - 1) + 4(p - q - 1) - \sum_{j=q+1}^{p-1} |v_j| - (|v_p| - 6)^+$$

$$\geq 3q_2 + 2q_1 + 2q_0 + \sum_{j=q+1}^{p} |v_j| - 2p_2 - p_1 - 6$$

$$+ L(p - 1) + 4(p - q - 1) - \sum_{j=q+1}^{p-1} |v_j| - (|v_p| - 6)^+$$

$$= 3q_2 + 2q_1 + 2q_0 + |v_p| - 2p_2 - p_1 - 6$$

$$+ L(p - 1) + 4(p - q - 1) - (|v_p| - 6)^+. \quad (3.23)$$

If $|v_p| \geq 5$, it follows easily from (3.23) that

$$-2Q \geq 2p - 4 \geq 0$$

for $p \geq 2$, as desired. If $|v_p| \leq 4$, then $q = p$.

**The subcase** $p = q$ or $p = 1$. If $p = q$, then it follows easily from (3.22) that $Q \leq 0$ using the fact that $|a| \geq 7$. Also, if $p = 1$, then using the fact that $|v_p| \leq |a|$, it follows easily from (3.22) that $Q = 0$.

The last thing we need to consider is the case when $|v_{p-1}| > |a| - 3$.

**The Case** $|v_{p-1}| > |a| - 3$. Using (3.13), we have that $|a| + 6 \geq p + 2(|a| - 2)$, and, thus, that $10 \geq p + |a|$. By assumption, however, $p \geq 2$ and
Therefore, the only possibilities are \( p = 2 \) and \( |\alpha| = 8 \) or \( p = 2 \) or 3 and \( |\alpha| = 7 \).

First, we consider the case \( p = 2 \) and \( |\alpha| = 8 \). Using (3.12) and the fact that \( |\alpha| = 8 \), \( p = 2 \), we have

\[
|v_1| + |v_2| + 8 \leq 2p_2 + p_1 + 16.
\]

In particular, this means

\[
|v_1| + |v_2| \leq 12.
\]

We are also assuming that \( |\rho_{p-1}| \geq |\alpha| - 2 \). Therefore, \( |v_1| \geq 6 \). And, \( |v_2| \geq |v_1| \). Therefore, the only possibility is \( |v_1| = |v_2| = 6 \). This is a term of the form

\[
\left| \int_0^T \int_A \xi_\beta (D'f) \cdot (\partial^n u)(\partial^n u)(\partial^n u) \right|,
\]

where \( \xi_\beta \in W_{0, L^{-2}} \).

Therefore, for \( rx + sy > 1 \), \( \xi_\beta \) is of the form \( t^2(rx + sy)^{L-2} \). Thus, we can break up the integral as

\[
\left| \int_0^T \int_A t^2(rx + sy)^{L-2} (D'f) \cdot (\partial^n u)(\partial^n u)(\partial^n u) \right|
\leq C \left( \int_0^T \int_A t^2(rx + sy)^{L-2} (\partial^n u)^4 \right)^{\frac{1}{4}}
\cdot \left( \int_0^T \int_A t^2(rx + sy)^{L-2} (\partial^n u)^2 \right)^{\frac{1}{2}}.
\]

Now we can use Sobolev’s embedding theorem on the first two terms on the right hand side. Namely, \( \|u\|_{L^4(A)} \leq \|u\|_{H^4(A)} \). Using this fact, we conclude that

\[
\left| \int_0^T \int_A t^2(rx + sy)^{L-2} (\partial^n u)^4 \right| \leq C \int_0^T \left\| t^2(rx + sy)^{L-2} (\partial^n u) \right\|_{L^4(A)}^4 dt
\leq C \left\| t^2(rx + sy)^{L-2} (\partial^n u) \right\|_{L^4(H^4(A))}^4
\leq C \|u\|_{L^4(A)}^4,
\]

for \( i = 1, 2 \), which is bounded by (3.7) and the norm of \( u \in L^\infty(H^8(W_{0, L, 0})) \) as desired.
The remaining case is $p = 2$ or $3$ and $|x| = 7$. The case $p = 3$ and $|x| = 7$ is easier to handle, so we only consider the case $p = 2$, $|x| = 7$. We know by (3.13) in Lemma 3.3 above, that

$$|x| + 6 \geq p + |y| + |y_p-1|.$$ 

The first subcase is $|y_p| = 7$, but this implies $|y_p-1| \leq 4$, which has already been taken care of in the case $|y_p-1| \leq |x| - 3$. The second subcase is $|y_p| = 6$, which implies $|y_p-1| \leq 5$. Note that if $|y_p| = 5$, then $|y_p-1| = 5$ (because we are assuming here that $|y_p-1| \geq |x| - 2$). This case is even easier to handle, so we consider only the case $|y_p| = 6$, $|y_p-1| = 5$. In particular, $\xi_p \in W_{a,L-1,1}$ implies for $rx + sy > 1$, $\xi_y$ is of the form $t(rx + sy)^{L-1}$. We estimate the integral as

$$\left| \int_0^t \int_D t(rx + sy)^{L-1} (\partial^a u)(\partial^a u)(\partial^a u) \right| \leq \sup_{0 \leq t \leq T} \left( \int_D (\partial^a u)^2 \right)^{L-1} \int_0^T \left| x^{-\frac{L-1}{2}} (\partial^a u) \right|^{L-1} \left( \int_0^T \int_D x^{-\frac{L-1}{2}} (\partial^a u)^2 \right)^{L-1}.$$ 

It is easy to see that each of the terms on the right-hand side above is bounded by a constant depending only on (3.7) or the norm of $u \in L^\infty(H^q(W_{0,L,0})).$

Thus, we have handled the terms in $\int_0^t \int_D \xi_y(\partial^a u) R_2$ and using (3.15) shown that for $0 \leq t \leq T$,

$$\int \xi(\cdot, \cdot) (\partial^a u)^2 + K_1 \int_0^t \eta (\partial^a u)^2 + K_2 \int_0^t \eta (\partial^a u_y)^2 \leq C,$$

where $C$ depends only on the norm of $u$ in $L^\infty(H^q(W_{0,L,0}))$ and the norm of $u$ in (3.7). Thus our lemma is proved.

4. EXISTENCE AND UNIQUENESS

In this section we prove that for initial data $\phi \in H^N(\mathbb{R}^2)$ there exists a unique solution $u$ of (2.1) such that $u \in L^\infty([0, T]; H^q(\mathbb{R}^2))$ for a time $T$ depending only on the norm of $\phi \in H^q(\mathbb{R}^2)$. These results are used in the proof of Theorem 3.1. First we prove the uniqueness result.

**Lemma 4.1 (Uniqueness).** Let $0 < T < \infty$. Assume Eq. (2.1) satisfies Assumptions (A1)–(A4). Then for $\phi \in H^q(\mathbb{R}^2)$ there is at most one solution $u \in L^\infty([0, T]; H^q(\mathbb{R}^2))$ of (2.1) with initial data $\phi$. 


Proof. Assume \( u, v \) are two solutions of (2.1) in \( L^w([0, T]; H^4(\mathbb{R}^2)) \) with the same initial data. By Eq. (2.1), \( \partial_t u, \partial_t v \in L^w([0, T]; H^3(\mathbb{R}^2)) \), so the integrations below are justified. Therefore,

\[
\begin{align*}
\partial_t (u-v) + a(u-v)_{xxx} + b(u-v)_{xxy} + c(u-v)_{xyy} + d(u-v)_{yyy} \\
+ \sum_{j=0}^5 \{ f(\ldots, \partial^{j+1}v, \partial^j u, \partial^{j-1} u, \ldots) \\
- f(\ldots, \partial^{j+1}v, \partial^j v, \partial^{j-1} u, \ldots) \} = 0,
\end{align*}
\]

(4.1)

where \( \gamma_0 = (0, 0), \ldots, \gamma_5 = (2, 0) \). By the mean value theorem, there exist smooth functions \( h^{(0)}, \ldots, h^{(5)} \), depending on \( u_{xx}, v_{xx}, \ldots, u, x, y, t \) such that

\[
\begin{align*}
f(u_{xx}, u_{xy}, \ldots, u, x, y, t) - f(v_{xx}, v_{xy}, \ldots, v, x, y, t) \\
= h^{(0)}(u_{xx} - v_{xx}), \ldots, f(v_{xx}, \ldots, v_{y}, u, x, y, t) - f(v_{xx}, \ldots, v_{y}, v, x, y, t) \\
= h^{(0)}(u - v).
\end{align*}
\]

Therefore, (4.1) can be rewritten as

\[
\begin{align*}
\partial_t (u-v) + a(u_{xxx} - v_{xxx}) + b(u_{xxy} - v_{xxy}) + c(u_{xyy} - v_{xyy}) + d(u_{yyy} + v_{yyy}) \\
+ h^{(3)}(u_{xx} - v_{xx}) + \cdots + h^{(0)}(u - v) = 0.
\end{align*}
\]

(4.2)

Now multiplying (4.2) by \( 2(u - v) \) and integrating over \( (x, y) \in \mathbb{R}^2 \), our equation becomes

\[
\begin{align*}
\int 2(u-v) \partial_t (u-v) + \int 2a(u-v)(u-v)_{xxx} + \int 2b(u-v)(u-v)_{xxy} \\
+ \int 2c(u-v)(u-v)_{xyy} + \int 2d(u-v)(u-v)_{yyy} + \int 2(u-v) h^{(3)}(u-v)_{xx} \\
+ \int 2(u-v) h^{(3)}(u-v)_{x} + \int 2(u-v) h^{(3)}(u-v)_{y} + \int 2(u-v) h^{(3)}(u-v)_{x} \\
+ \int 2(u-v) h^{(3)}(u-v)_{y} + \int 2(u-v) h^{(0)}(u-v) = 0.
\end{align*}
\]

(4.3)

By integrating by parts, the 2nd-5th terms in (4.3) are shown to be identically zero. Integrating the sixth term in (4.3) by parts several times, we see

\[
\int 2(u-v) h^{(3)}(u-v)_{xx} = \int \partial_{xx}[h^{(3)}](u-v)^2 - \int 2h^{(3)}(u_x - v_x)^2.
\]
Similarly,
\[
\int 2(u-v) h^{(4)}(u-v)_{xv} = \int \partial_{xv}[h^{(4)}](u-v)^2 - \int 2h^{(4)}(u_x - v_x)(u_y - v_y)
\]
\[
\int 2(u-v) h^{(3)}(u-v)_{yy} = \int \partial_{yy}[h^{(3)}](u-v)^2 - \int 2h^{(3)}(u_y - v_y)^2.
\]

The next two terms are also integrated by parts. Consequently, we write
\[
\int 2(u-v) h^{(2)}(u_x - v_x) = - \int \partial_x[h^{(2)}](u-v)^2
\]
\[
\int 2(u-v) h^{(1)}(u_y - v_y) = - \int \partial_y[h^{(1)}](u-v)^2.
\]

Putting these together, (4.3) becomes
\[
\partial_t \int (u-v)^2 - \int 2h^{(5)}(u_x - v_x)^2 - \int 2h^{(4)}(u_x - v_x)(u_y - v_y) + \int 2h^{(3)}(u_y - v_y)^2
\]
\[
+ \int \{ \partial_{xx}[h^{(5)}] + \partial_{xy}[h^{(4)}] + \partial_{yy}[h^{(3)}] + 2h^{(0)} \} (u-v)^2
\]
\[
= \int \{ \partial_x[h^{(2)}] + \partial_y[h^{(1)}] \} (u-v)^2.
\]

By assumption (A2) on \( f_{xu_x}, f_{u_y}, \) and \( f_{u_y}, \) we conclude that
\[-2 \int h^{(5)}(u_x - v_x)^2 + h^{(4)}(u_x - v_x)(u_y - v_y) + h^{(3)}(u_y - v_y)^2 \geq 0.\]
Therefore,
\[
\partial_t \int \xi(u-v)^2 \leq \int \{ |\partial_x[h^{(5)}]| + |\partial_y[h^{(4)}]| + |\partial_{xx}[h^{(5)}]|
\]
\[
+ |\partial_{xy}[h^{(4)}] + |\partial_{yy}[h^{(3)}]| + 2|h^{(0)}| \} (u-v)^2.
\]

For \( C > 0 \) sufficiently large, we have
\[
\partial_t \int (u-v)^2 \leq C \int (u-v)^2. \quad (4.4)
\]

Using Gronwall’s inequality and the fact that \( u(\cdot, 0) = v(\cdot, 0) \), we conclude that \( u \equiv v \). □

We now state our main existence theorem for the semilinear Eq. (2.1).
Theorem 4.2 (Existence). Assume Eq. (2.1) satisfies (A1)–(A4). Let \( N \geq 6 \) and let \( k_0 > 0 \). Then there exists \( 0 < T < \infty \) depending only on \( k_0 \) such that for all \( \phi \in H^N(\mathbb{R}^2) \) with \( \|\phi\|_{H^N(\mathbb{R}^2)} \leq k_0 \), there exists a solution of (2.1) with \( u \in L^\infty([0, T]; H^N(\mathbb{R}^2)) \) and \( u(x, y, 0) = \phi(x, y) \).

The method of proof for Theorem 4.2 is the following. We differentiate (2.1) by applying \( \partial^2 \). Our equation becomes

\[
\partial^2 u_t + a \, \partial^2 u_{xxx} + b \, \partial^2 u_{xxy} + c \, \partial^2 u_{xyy} + d \, \partial^2 u_{xxy} + f_{u_x} \, \partial^2 u_{xx} + f_{u_y} \, \partial^2 u_{xy} \\
+ f_{u_y} \, \partial^2 u_{xx} + \{3 \partial_y [f_{u_y}] \, \partial^{(1, 0)} u_{xx} + \cdots + 3 \partial_y [f_{u_y}] \, \partial^{(0, 3)} u_{yy}\} \\
+ f_{u_x} \, \partial^2 u_{yx} + \partial^2 u_{xy} + O(\partial^6 u) = 0,
\]

where \( |\beta| = 4 \). Next we let \( u = \lambda v \) where \( \lambda = (I + \partial^2)^{-1} \). Therefore,

\[
v_t + a \, \partial^2 v_{xxx} + b \, \partial^2 v_{xxy} + c \, \partial^2 v_{xyy} + d \, \partial^2 v_{xxy} + f_{u_x} \, \partial^2 v_{xx} + f_{u_y} \, \partial^2 v_{xy} \\
+ f_{u_y} \, \partial^2 v_{xx} + \{4 \partial_y [f_{u_y}] \, \partial^{(0, 3)} v_{xx} + \cdots + 4 \partial_y [f_{u_y}] \, \partial^{(0, 3)} v_{yy}\} \\
+ f_{u_x} \, \partial^2 v_{yx} + \partial^2 v_{xy} + O(\partial^6 v) = 0.
\]

Next, we linearize the equation by introducing a new variable \( w \),

\[
v_t + a \, \partial^2 v_{xxx} + b \, \partial^2 v_{xxy} + c \, \partial^2 v_{xyy} + d \, \partial^2 v_{xxy} + f_{u_x} \, \partial^2 v_{xx} + f_{u_y} \, \partial^2 v_{xy} \\
+ f_{u_y} \, \partial^2 v_{xx} + \{4 \partial_y [f_{u_y}] \, \partial^{(0, 3)} v_{xx} + \cdots + 4 \partial_y [f_{u_y}] \, \partial^{(0, 3)} v_{yy}\} \\
+ f_{u_x} \, \partial^2 v_{yx} + \partial^2 v_{xy} + O(\partial^6 v) = 0,
\]

where \( f_{u_x} = \lambda u_x(Aw_{xx}, Aw_{xy}, Aw_{yy}, \ldots) \), etc. Finally we define a sequence of approximations by

\[
v_\ell^{(a)} + a \, \partial^2 v_{xxx}^{(a)} + b \, \partial^2 v_{xxy}^{(a)} + c \, \partial^2 v_{xyy}^{(a)} \\
+ d \, \partial^2 v_{xxy}^{(a)} + f_{u_x} \, \partial^2 v_{xx}^{(a)} + f_{u_y} \, \partial^2 v_{xy}^{(a)} \\
+ f_{u_y} \, \partial^2 v_{xx}^{(a)} + \{4 \partial_y [f_{u_y}] \, \partial^{(0, 3)} v_{xx}^{(a)} + \cdots + 4 \partial_y [f_{u_y}] \, \partial^{(0, 3)} v_{yy}^{(a)}\} \\
+ f_{u_x} \, \partial^2 v_{yx}^{(a)} + \partial^2 v_{xy}^{(a)} + O(\partial^6 v) = 0,
\]

where \( |\beta| = 4, f = f(Aw_{xx}^{(a−1)}, Aw_{xy}^{(a−1)}, Aw_{yy}^{(a−1)}, \ldots) \) and where the initial condition is given by \( v_0^{(a)}(x, y, 0) = \phi(x, y) + \partial^3 \phi(x, y) \). The first approximation is given by \( v_0(x, y, t) = \phi(x, y) + \partial^3 \phi(x, y) \).

Now Eq. (4.7) is a linear equation. By Lemma 4.4 below, we show that for initial data \( \phi \in H^6(\mathbb{R}^2) \), the approximate Eq. (4.7) has a unique solution \( v^{(a)} \), for each \( n \), defined in any time interval in which the coefficients are defined. Consequently, the time interval will depend only on \( \|\phi\|_{H^6(\mathbb{R}^2)} \).

We then use Lemma 4.3 to show that our sequence \( v^{(a)} \) is a bounded
sequence in \( L^\infty([0, T]; H^2(\mathbb{R}^2)) \) where \( T \) depends only on \( \|\phi\|_{H^4(\mathbb{R}^2)} \). Consequently, there exists a subsequence which converges to some \( v \) in \( L^\infty([0, T]; H^2(\mathbb{R}^2)) \). By using convergence arguments, we then show that \( v \) is the unique solution to (4.5) and that \( u = Av \in L^\infty([0, T]; H^4(\mathbb{R}^2)) \) is the unique solution to (2.1). We conclude the proof of Theorem 4.2 by showing that for initial data \( \phi \) in a Sobolev space \( H^N(\mathbb{R}^2) \) where \( N \geq 7 \), the unique solution \( u_0 \), obtained earlier in the proof, lies in \( L^\infty([0, T]; H^N(\mathbb{R}^2)) \) where the time \( T \) depends only on \( \|\phi\|_{H^4(\mathbb{R}^2)} \).

Before proving Theorem 4.2, we state Lemma 4.3, where we prove the main differential estimates for our sequence of solutions \( v^{(n)} \).

**Lemma 4.3.** Let \( v, w \) be a pair of functions in \( C^4([0, \infty); H^N(\mathbb{R}^2)) \) for all \( k, N \) which satisfy (4.6). For each \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \), there exist positive, nondecreasing functions \( g, h \) such that for all \( t \geq 0 \)

\[
\begin{align*}
\partial_t \int (\partial^\alpha v)^2 &\leq C(1 + \|v\|_{H^2}^2) \|v\|_{H^m}^2 + g(\|w\|_{H^m}) \|v\|_{L^2}^2 + h(\|w\|_{H^m}),
\end{align*}
\]

where \( m = \max\{2, |\alpha|\} \).

We now prove Theorem 4.2 assuming Lemma 4.3 above and Lemma 4.4, the existence lemma for the linearized equation, stated below.

**Proof of Theorem 4.2.** First we show that for initial data \( \phi \in H^4(\mathbb{R}^2) \) there exists a solution \( u \) in \( L^\infty([0, T]; H^4(\mathbb{R}^2)) \) where the time of existence \( T \) depends only on \( \|\phi\|_{H^4(\mathbb{R}^2)} \). Using the same approximation procedure as in Section 3, we can assume \( \phi \in \bigcap_{k \geq 0} H^4(\mathbb{R}^2) \).

By Lemma 4.4 below, for each \( n \), Eq. (4.7) has a unique solution for any time interval in which the coefficients are defined. In particular, the time interval will only depend on \( \|\phi\|_{H^4(\mathbb{R}^2)} \). We now show that this sequence of solutions \( \{v^{(n)}\} \) is bounded in \( L^\infty([0, T]; H^2(\mathbb{R}^2)) \) for a time \( T \) independent of \( n \). By Lemma 4.3, we have

\[
\begin{align*}
\partial_t \int (\partial^\alpha v^{(n)})^2 &\leq C(1 + \|v^{(n-1)}\|_{H^2}^2) \|v^{(n)}\|_{H^m}^2 + g(\|v^{(n-1)}\|_{H^m}) \|v^{(n)}\|_{L^2}^2 + h(\|v^{(n-1)}\|_{H^m}),
\end{align*}
\]

where \( m = \max\{2, |\alpha|\} \) and \( g, h \equiv g^{(n)}, h^{(n)} \) are smooth functions of their arguments. Let \( |\alpha| = 2 \). Let \( k_0 \geq \|\phi + A^2\phi\|_{H^2} \geq \|\phi\|_{H^2} \). For each iterate \( n \), \( \|v^{(n)}(\cdot, t)\|_{H^2} \) is continuous in \( t \) and \( \|v^{(0)}(\cdot, 0)\|_{H^2} = \|\phi + A^2\phi\|_{H^2} \leq k_0 \). Let \( c_0 = k_0 + 1 \) and let

\[
T^{(n)}_0 = \sup\{t; \|v^{(n)}(\cdot, t)\|_{H^2} \leq c_0 \text{ for } 0 \leq t \leq \bar{t} \leq t, 0 \leq k \leq n\}. 
\]
Therefore, for $t \in [0, T^*(n)]$, we have

$$
\|v^{(n)}(\cdot, t)\|_{L^2}^2 \leq \|v^{(n)}(\cdot, 0)\|_{L^2}^2 + C \int_0^t \left( 1 + \|v^{(n-1)}\|_{H^2} \right) \|v^{(n)}\|_{H^2}^2 + \int_0^t g(\|v^{(n-1)}\|_{H^2}) \|v^{(n)}\|_{H^2}^2 + h(\|v^{(n-1)}\|_{H^2})
$$

$$
\leq k_0^2 + C t (1 + c_0) \varepsilon_0^2 + C t g(c_0) \varepsilon_0^2 + C t h(c_0).
$$

(4.11)

Now choose $T$ such that

$$
CT(1 + c_0) \varepsilon_0^2 + C T g(c_0) \varepsilon_0^2 + C T h(c_0) \leq 1.
$$

(4.12)

In particular, $T$ does not depend on $n$, but only on $k_0$, and

$$
\sup_{0 \leq t \leq T} \|v^{(n)}(\cdot, t)\|_{H^2}^2 \leq \varepsilon_0^2.
$$

(4.13)

We have shown that $v^{(n)}$ is a bounded sequence in $L^\infty([0, T]; H^2(\mathbb{R}^2))$. Therefore, there is a weak* convergent subsequence, still denoted $\{v^{(n)}\}$ such that $v^{(n)} \rightharpoonup v$ in $L^\infty([0, T]; H^2(\mathbb{R}^2))$. We claim that $u = Av$ is the solution of (2.1). We first need to show that $v$ is a solution of (4.5). We do so by showing that each term in (4.7) converges to its correct limit. First, we note that $A^2 A^2 v^{(n)} \to A^2 A^2 v$ strongly in $L^2([0, T]; L^2_{loc}(\mathbb{R}^2))$. Therefore, the sequence, $v^{(n)}$, is a bounded sequence in $L^2([0, T]; H^{-1}(\mathbb{R}^2))$. By Aubin’s compactness theorem, there is a subsequence $v^{(n)} \to v$ strongly in $L^2([0, T]; H^1_{loc}(\mathbb{R}^2))$. Therefore, for a subsequence, $v^{(n)} \to v$ a.e. in $x$, $y$, and $t$. It follows that $f_{u^{(n)}} A^2 A^2 v^{(n)} \to f_{u^v} A^2 A^2 v$ strongly in $L^2([0, T]; L^2_{loc}(\mathbb{R}^2))$, because $f_{u^{(n)}}(A v^{(n)}(x, y, \cdot) \to f_{u^v}(A v(x, y, \cdot))$ strongly in $L^2([0, T]; L^2_{loc}(\mathbb{R}^2))$ and $A^2 A^2 v^{(n)} \to A^2 A^2 v$ weakly in $L^2([0, T]; L^2(\mathbb{R}^2))$. Similarly for the other nonlinear terms.

Therefore, $v^{(n)} \to v$ strongly in $L^2([0, T]; L^2_{loc}(\mathbb{R}^2))$ and $v$ is a solution to (4.5).

Applying the Eq. (4.5) we established that $A^v$ satisfies (2.1) and $v \in L^\infty([0, T]; H^4(\mathbb{R}^2))$.

Next we want to prove that there exists a solution $u \in L^\infty([0, T]; H^N(\mathbb{R}^2))$ with $N \geq 7$, where $T$ depends only on $\|\phi\|_{H^\delta}$. We will prove that the approximating sequence $v^{(n)}$ is bounded in $L^\infty([0, T_{N-6}]; H^{N-4}(\mathbb{R}^2))$, where the time of existence $T_{N-6} \geq T$. Therefore, the time of existence depends only on $\|\phi\|_{H^\delta}$. Then by the same arguments as before it can be shown that the limit of $v^{(n)} = v \in L^\infty([0, T]; H^{N-4}(\mathbb{R}^2))$, and, therefore, $u = A v \in L^\infty([0, T]; H^N(\mathbb{R}^2))$. 


Let $|\alpha| = N - 4$ for $N \geq 7$. Using the result of Lemma 4.3, we have that

$$
\partial_t \|v^{(\alpha)}\|_{H^N}^2 \leq C(1 + \|v^{(\alpha-1)}\|_{L^2}) \|v^{(\alpha)}\|_{H^N}^2 + g(\|v^{(\alpha-1)}\|_{H^N}) \|v^{(\alpha)}\|_{H^N}^2
+ h(\|v^{(\alpha-1)}\|_{H^N}).
$$

(4.14)

Let $c_{N-6}^2 = \|f\|_{H^N}^2 + 1$ and let

$$
T_{N-6}^{(\alpha)} = \sup \{ t : \|v^{(\alpha)}(\cdot, t)\|_{H^N} \leq c_{N-6} \text{ for } 0 \leq t \leq k \leq n \}.
$$

(4.15)

Therefore, for $t \in [0, T_{N-6}^{(\alpha)}]$, we have

$$
\|v^{(\alpha)}(\cdot, t)\|_{H^N}^2 \leq \|v^{(\alpha)}(\cdot, 0)\|_{H^N}^2 + C \int_0^t (1 + \|v^{(\alpha-1)}\|_{L^2}) \|v^{(\alpha)}\|_{H^N}^2
+ g(\|v^{(\alpha-1)}\|_{H^N}) \|v^{(\alpha)}\|_{H^N}^2 + h(\|v^{(\alpha-1)}\|_{H^N}) \|v^{(\alpha)}\|_{H^N}^2
\leq \|\phi\|_{H^N}^2 + C t(1 + c_{N-6}) c_{N-6} + C t g(c_{N-6}) c_{N-6} + C t h(c_{N-6}).
$$

Now by choosing $T_{N-6}^{(\alpha)}$ such that

$$
C T_{N-6}(1 + c_{N-6}) c_{N-6} + C T_{N-6} g(c_{N-6}) c_{N-6} + C T_{N-6} h(c_{N-6}) = 1,
$$

we have that $T_{N-6}^{(\alpha)} \geq T_{N-6}$. Therefore,

$$
\sup_{0 \leq t \leq T_{N-6}} \|v^{(\alpha)}(\cdot, t)\|_{H^N}^2 \leq c_{N-6}^2
$$

and $v^{(\alpha)}$ is a bounded sequence in $L^\infty([0, T_{N-6}]; H^{N-4}(\mathbb{R}^2))$ which converges weak* to $v \in L^\infty([0, T_{N-6}]; H^{N-4}(\mathbb{R}^2))$. Therefore, $u = Av$ is in $L^\infty([0, T_{N-6}]; H^N(\mathbb{R}^2))$.

Now let $T_N^*$ be the maximum time such that $u \in L^\infty([0, t]; H^N(\mathbb{R}^2))$ for $0 < t < T_N^*$. In particular, $T \leq T_N^*$, and, therefore, a time of existence can be chosen depending only on $\|\phi\|_{H^N(\mathbb{R}^2)}$.

We now prove the main inequality used in our existence theorem above. Namely, we show that the sequence of solutions $\{v^{(\alpha)}\}$ to our approximate Eq. (4.7) is \textit{a priori} bounded.

\textit{Proof of Lemma 4.3.} We begin by applying $\partial^\alpha$ to (4.6) where $\alpha = (\alpha_1, \alpha_2)$. Therefore, our equation becomes

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\[ \begin{align*}
\frac{\partial}{\partial t} v + a(\partial^2 A^2 v_{xxx}) + b(\partial^2 A^2 v_{xxy}) + c(\partial^2 A^2 v_{xyy}) + d(\partial^2 A^2 v_{yyy}) \\
+ f_{u,v_1}(\partial^2 A^2 v_{xx}) + f_{u,v_2}(\partial^2 A^2 v_{xy}) \\
+ f_{u,v_3}(\partial^2 A^2 v_{yx}) + \sum_{5 \leq i + j \leq |\alpha| + 5} h^{(i,j)} \partial^{(i,j)} A^2 v \\
+ \sum_{|b| = |\alpha| + 4} q_i(\partial^2 A w, \ldots)(\partial^2 A w) + p(\partial^2 A w, \ldots) = 0,
\end{align*}\]

where \(|\alpha| = 4\), \(|\gamma| = |\alpha| + 3\), and the \(h^{(i,j)}\) are smooth functions depending on \(\partial^k A w, \ldots\), where \(|k| = |\alpha| - i - j + 8\). In particular, for \(|\alpha| \geq 3\), the \(p(\partial^j A w, \ldots)\) depend at most linearly on \(\partial^j A w\) for \(|\gamma| = |\alpha| + 3\). For \(|\alpha| = 2\) the \(p(\partial^j A w, \ldots)\) depend at most quadratically on \(\partial^j A w\) for \(|\gamma| = |\alpha| + 3\).

We now multiply (4.16) by \(2(\partial^* v)\) and integrate over \((x, y) \in \mathbb{R}^2\), as

\[ \begin{align*}
2 \int (\partial^* v) (\partial^* v) + 2 \int a(\partial^* v)(\partial^* A^2 v_{xxx}) + 2 \int b(\partial^* v)(\partial^* A^2 v_{xxy}) \\
+ 2 \int c(\partial^* v)(\partial^* A^2 v_{xyy}) + 2 \int d(\partial^* v)(\partial^* A^2 v_{yy}) \\
+ 2 \int \{ (\partial^* v) f_{u,v_1}(\partial^* A^2 v_{xx}) + (\partial^* v) f_{u,v_2}(\partial^* A^2 v_{xy}) + (\partial^* v) f_{u,v_3}(\partial^* A^2 v_{yx}) \} + 2 \int (\partial^* v) \sum_{5 \leq i + j \leq |\alpha| + 5} h^{(i,j)} \partial^{(i,j)} A^2 v \\
+ \int 2(\partial^* v) \sum_{|b| = |\alpha| + 4} q_i(\partial^* A w, \ldots)(\partial^* A w) + \int 2(\partial^* v) p(\partial^* A w, \ldots) = 0.
\end{align*}\]

First, we write

\[ 2 \int (\partial^* v)(\partial^* v) = \partial_t \int (\partial^* v)^2. \]

Upon integrating each of the next four terms in (4.17) by part several times, we see these quantities are identically zero.

We now deal with the terms in the sixth integral of (4.17). They can be handled as
\[ 2 \int (\partial^s v) f_{\alpha\nu}(\partial^s A^2 A_{\alpha\nu}) \]
\[ = 2 \int (\partial^s Av + \partial^s A^2 Av) f_{\alpha\nu}(\partial^s A^2 A_{\alpha\nu}) \]
\[ = 2 \int f_{\alpha\nu}(\partial^s Av)(\partial^s A^2 A_{\alpha\nu}) + 2 \int f_{\alpha\nu}(\partial^s A^2 Av)(\partial^s A^2 A_{\alpha\nu}) \]
\[ \equiv I_1 + I_2. \]

Integrating \( I_1 \) by parts twice, we see that
\[ 2 \int f_{\alpha\nu}(\partial^s Av)(\partial^s A^2 A_{\alpha\nu}) \leq C(1 + \|w\|_{H^2}) \|v\|_{H^2}^2. \]

The term \( I_2 \) is also integrated by parts,
\[ 2 \int f_{\alpha\nu}(\partial^s A^2 Av)(\partial^s A^2 A_{\alpha\nu}) = \int \partial^s \left[ f_{\alpha\nu} \right](\partial^s A^2 Av)^2 - 2 \int f_{\alpha\nu}(\partial^s A^2 Av)^2 \]
\[ \leq C(1 + \|w\|_{H^2}) \|v\|_{H^2}^2 - 2 \int f_{\alpha\nu}(\partial^s A^2 Av)^2. \]

Similarly for the other terms in the sixth integral of (4.17),
\[ \int (\partial^s v) f_{\alpha\nu}(\partial^s A^2 A_{\alpha\nu}) \leq C(1 + \|w\|_{H^2}) \|v\|_{H^2}^2 - 2 \int f_{\alpha\nu}(\partial^s A^2 Av)(\partial^s A^2 A_{\alpha\nu}). \]
\[ \int (\partial^s v) f_{\alpha\nu}(\partial^s A^2 A_{\alpha\nu}) \leq C(1 + \|w\|_{H^2}) \|v\|_{H^2}^2 - 2 \int f_{\alpha\nu}(\partial^s A^2 Av)^2. \]

Next we consider the seventh integral in (4.17). These terms are given by
\[ 2 \int (\partial^s v) \sum_{s \leq i + j \leq |a| + 5} h^{(i,j)}(\partial^{(i,j)}Av) \]
where the \( h^{(i,j)} \) depend at most on \( \partial^k Av, \ldots \) where \(|k| = |a| - i - j + 8\). First, if \( i + j = |a| + 5 \), we integrate by parts, as in the following example. Consider the term of the form,
\[ 2 \int (\partial^s v)(\partial_{\alpha}[f_{\alpha\nu}](\partial^{(|a| - 1, s)} A^2 A_{\alpha\nu})) \equiv 2 \int (\partial^s v)(\partial_{\alpha}[f_{\alpha\nu}](\partial^s A^2 A_{\alpha\nu})). \]
Integrating by parts, we arrive at the estimate,

\[
2 \int (\partial^s u)(\partial_n[f_{uv}](\partial^s A^2 A v))
\]
\[
= 2 \int (\partial^s A v + \partial^s A^2 A v) \partial_n[f_{uv}](\partial^s A^2 A v)
\]
\[
= -2 \int (\partial^s A v) \partial_n[f_{uv}](\partial^s A^2 A v) - 2 \int (\partial^s A v) \partial^2_n[f_{uv}](\partial^s A^2 A v)
\]
\[
- \int \partial^2_n[f_{uv}](\partial^s A^2 A v)^2
\leq C(1 + \|w\|_{L^2}) \|v\|_{L^2}^2.
\]

Next, if \(6 \leq i + j \leq |x| + 4\), we have

\[
2 \int (\partial^s u) \sum_{6 \leq i + j \leq |x| + 4} h^{(i,j)}(\partial^s A^2 A v) \leq C \sum_{6 \leq i + j \leq |x| + 4} |h^{(i,j)}|_{L^2} \left( \int (\partial^s u)(\partial^s A^2 A v) \right)
\]
\[
\leq g(\|w\|_{L^2}) \|v\|_{L^2}^2,
\]
where \(g = g^\beta\) is a smooth, nondecreasing function. If \(i + j = 5\), we estimate as

\[
2 \int (\partial^s u) \sum_{5 = i + j} h^{(i,j)}(\partial^s A^2 A v) \leq C \left( \int (h^{(i,j)})^4 \right)^{\frac{1}{4}} \left( \int (\partial^s A^2 A v)^2 \right)^{\frac{1}{2}} \left( \int (\partial^s u)^2 \right)^{\frac{1}{2}}
\]
\[
\leq g(\|w\|_{L^2}) \|v\|_{H^2} \|v\|_{H^2},
\]
where again \(g = g^\beta\) is a smooth, nondecreasing function.

It remains to look at the last two integrals in (4.17),

\[
\int 2(\partial^s u) \sum_{|\mu| = 4} q_{\mu}(\partial^\nu A w, ...) (\partial^\gamma A w) + \int 2(\partial^s u) p(\partial^\gamma A w, ...)
\]

where \(|\mu| = 4\) and \(|\gamma| = |x| + 3\). First,

\[
2 \int (\partial^s u) \sum_{|\mu| = 4} q_{\mu}(\partial^\nu A w, ...) (\partial^\gamma A w) \leq h(\|w\|_{H^2}) \|v\|_{H^\infty} \|w\|_{L^2}.
\]
Second,
\[
2 \int (\partial^* v) \, p(\partial^* A w, \ldots) \leq h(\|w\|_{H^m}) \|v\|_{H^n}
\]
for a smooth, nondecreasing function \( h = h^\theta \).

Combining these inequalities, we have
\[
\partial_t \int (\partial^* v)^2 - 2 \int \left\{ f_{u_{xx}} (\partial^* A^2 Av_x)^2 + f_{u_{xy}} (\partial^* A^2 Av_x)(\partial^* A^2 Av_y) + f_{u_{yy}} (\partial^* A^2 Av_y)^2 \right\}
\leq C (1 + \|w\|_{H^m}) \|v\|_{H^m}^2 + g(\|w\|_{H^m}) \|v\|_{H^m}^2 + h(\|w\|_{H^m}),
\]
where \( m = \max \{2, |\alpha|\} \) and \( g, h = g^\theta, h^\theta \) are smooth, positive, nondecreasing functions. By assumption (A2) on \( f_{uxx}, f_{uxy}, f_{uyy} \), we can conclude that
\[
\partial_t (\partial^* v)^2 \leq C (1 + \|w\|_{H^m}) \|v\|_{H^m}^2 + h(\|w\|_{H^m}),
\]
where \( m = \max \{2, |\alpha|\} \), as desired.

We now prove an existence theorem for the linearized Eq. (4.6). By the same approximation method as used in Section 3, it suffices to prove the existence result for smooth initial data.

**Lemma 4.4 (Existence for linearized equation).** Given initial data \( \psi \) in \( \bigcap_{\alpha > 0} H^\alpha(\mathbb{R}^2) \), there exists a unique solution of (4.6). The solution is defined in any time interval in which the coefficients are defined.

**Proof (Sketch).** The linear equation which is to be solved at each iteration has the form
\[
\partial_t v + A^2 (a Av_{xxx} + b Av_{xxy} + c Av_{yy} + d Av_{yyy} + b_1 Av_{xx} + b_2 Av_{xy} + b_3 Av_{yy} + b_4 Av_{x} + b_5 Av_{y}) + \{b_6 \partial^{(3,0)} A v_{xx} + \cdots + b_7 \partial^{(0,3)} A v_{yy}\} + b_8 = 0, \quad (4.18)
\]
where \( b_1, \ldots, b_8 \) are smooth, bounded coefficients which satisfy
\[
b_2^2 - 4b_1 b_3 \leq 0.
\]
Fix an arbitrary time \( T > 0 \) and a constant \( M > 0 \). Let
\[
\mathcal{L} = \partial_t + A^2 (a A \partial^3_x + b A \partial^3_y + c A \partial^3_x \partial_y + d A \partial^3_y + b_1 A \partial^1_x + b_2 A \partial^1_y + b_3 A \partial^2_x + b_4 A \partial^2_y + b_5 A \partial_x + b_6 A \partial_y) + \{b_6 \partial^{(3,0)} A \partial^1_x + \cdots + b_7 \partial^{(0,3)} A \partial^1_y\}. \quad (4.19)
\]
Introduce the bilinear form
\[ \langle g, h \rangle = \int_0^T e^{-Mt} g h \, dx \, dy \, dt \] (4.20)
deﬁned on \( C_0^0(\mathbb{R}^2 \times [0, T]) \), the set of smooth functions with compact support in \( \mathbb{R}^2 \), which vanish for \( t = 0 \). Our estimates from Lemma 4.3 show that
\[ \int \mathcal{L} v \cdot v \, dx \, dy \geq \partial_t \int v^2 \, dx \, dy - C \int v^2 \, dx \, dy \] (4.21)
for some constant \( C \) suﬃciently large. Multiplying (4.21) by \( e^{-Mt} \) and integrating in time from \( t = 0 \) to \( t = T \), we obtain for \( v \in C_0^0(\mathbb{R}^2 \times [0, T]) \) with \( v(x, y, 0) = 0 \),
\[ \langle \mathcal{L} v, v \rangle \geq \langle v, v \rangle \int_0^T e^{-Mt} v^2 \, dx \, dy \, dt. \] (4.22)
Therefore, \( \langle \mathcal{L} v, v \rangle \geq \langle v, v \rangle \) provided \( M \) is chosen large enough. Similarly, \( \langle \mathcal{L}^* w, w \rangle \geq \langle w, w \rangle \) for all \( w \in C_0^0(\mathbb{R}^2 \times [0, T]) \) with \( w(x, y, T) \equiv 0 \) where \( \mathcal{L}^* \) denotes the formal adjoint of \( \mathcal{L} \). Therefore, \( \langle \mathcal{L}^* w, \mathcal{L}^* v \rangle \) is an inner product on \( D = \{ w \in C_0^0 : w(x, y, T) \equiv 0 \} \). Denote by \( X \) the completion of \( D \) with respect to this inner product. By the Riesz representation theorem, there exists a unique solution \( V \in X \) such that for any \( w \in D \),
\[ \langle \mathcal{L}^* V, \mathcal{L}^* w \rangle = -\langle b_s, w \rangle + \int \psi w(\cdot, 0). \]
Therefore,
\[ -\langle b_s, w \rangle = \langle \mathcal{L}^* V, \mathcal{L}^* w \rangle - \int \psi w(\cdot, 0) = \langle \mathcal{L} \mathcal{L}^* V, w \rangle, \]
where we have used the fact that \( b_s \in X \). Then \( v = \mathcal{L}^* V \) is a weak solution of \( \mathcal{L} v = -b_s \) with \( v \in L^2(\mathbb{R}^2 \times [0, T]) \) and \( v(\cdot, 0) = \psi \). To obtain higher regularity of the solution, we repeat the proof with higher derivatives included in the inner product.

**Corollary 4.5.** Let \( \phi \in H^N(\mathbb{R}^2) \) for some \( N \geq 6 \) and let \( \phi^{(n)} \) be a sequence converging to \( \phi \) in \( H^N(\mathbb{R}^2) \). Let \( u \) and \( u^{(n)} \) be the corresponding unique solutions given by Theorems 4.1 and 4.2, in \( L^6([0, T]; H^N(\mathbb{R}^2)) \) for a time \( T \) depending only on \( \sup_n \| \phi^{(n)} \|_{H^N(\mathbb{R}^2)} \). Then \( u^{(n)} \rightharpoonup u \) weak* in \( L^6([0, T]; H^N(\mathbb{R}^2)) \).
Proof. By assumption, \( u^{(n)} \in L^\infty([0, T]; H^N(\mathbb{R}^2)) \), and, therefore, there exists a weak* convergent subsequence, still denoted \( \{u^{(n)}\} \) such that \( u^{(n)} \rightharpoonup v \) in \( L^\infty([0, T]; H^N(\mathbb{R}^2)) \). In addition, by (2.1) \( u^{(n)} \in L^2([0, T]; H^N(\mathbb{R}^2)) \) implies \( u^{(n)}_t \in L^2([0, T]; H^{N-1}(\mathbb{R}^2)) \). Therefore, by Aubin’s compactness theorem \( u^{(n)} \to v \) strongly in \( L^2([0, T]; H^N(\mathbb{R}^2)) \). Now we just need to show that each term in (2.1) converges to its correct limit, and, thus \( u^{(n)}_t \to v_t \) for some solution \( v \in L^\infty([0, T]; H^N(\mathbb{R}^2)) \). Then by Lemma 4.1, we conclude that \( v \equiv u \).

Clearly, the linear terms converge in \( L^2([0, T]; L^1_{\text{loc}}(\mathbb{R}^2)) \). Therefore, the only thing left to show is that the nonlinear terms converge to their correct limits. In particular, we want to show that \( f(u^{(n)}_{xx}, \ldots) \to f(v_{xx}, \ldots) \) \( \in L^2([0, T]; L^1_{\text{loc}}(\mathbb{R}^2)) \). But this follows from the fact that \( u^{(n)}_t \to v_t \) strongly in \( L^2([0, T]; H^{N-1}_{\text{loc}}(\mathbb{R}^2)) \). Therefore, we conclude that \( u^{(n)}_t \to v_t \) for some solution \( v \). We also know that \( v \in L^\infty([0, T]; H^N(\mathbb{R}^2)) \). Therefore, by Theorem 4.1 on uniqueness, \( v \equiv u \).

5. WEIGHTED ESTIMATES

In this section, we prove weighted estimates on our solution \( u \) obtained in Section 4. We show that if our initial data \( \phi \in H^k(\mathbb{R}^2) \) also lie in the weighted Sobolev space \( H^k(W^{r,s}_{0,i,0}) \) for integers \( K \geq 0 \) and \( i \geq 0 \), then our solution \( u \in L^\infty([0, T]; H^k(\mathbb{R}^2)) \), found in Section 4, also lies in \( L^\infty([0, T]; H^k(W^{r,s}_{0,i,0})) \). This “persistence” property is used to start the induction in the proof of Theorem 3.1.

Theorem 5.1. Let \( i \) and \( K \) be integers such that \( i \geq 0, K \geq 0 \). Let \( 0 < T < \infty \). Assume Eq. (2.1) satisfies (A1)–(A4). Assume \( u \) is the solution to (2.1) in \( L^\infty([0, T]; H^k(\mathbb{R}^2)) \) with initial data \( \phi \in H^k(\mathbb{R}^2) \). If, in addition, \( \phi \) is in \( H^k(W^{r,s}_{0,i,0}) \), for \((r, s) \) satisfying \( \nabla_p(\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \( (\omega_1, \omega_2) \neq (0, 0) \), then

\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^2} |\nabla u(x, y, t)|^2 \, dx \, dy \, dt < \infty,
\end{align*}
\]  

(5.2)

where \( \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \), \( |\gamma| = K + 1 \), and \( \eta \) is a weight function in \( W^{r,s}_{\alpha, i-1, 0} \) for \( \sigma > 0 \) arbitrary with the exception that for \( i = 0 \), \( \eta \in W^{r,s}_{\alpha, i-1, 0} \).

Remark. In the case when \( i = 0 \), we are not assuming an additional decay condition on \( \phi \). In this case, the above theorem can be thought of as a local gain in finite regularity similar to some of the results in [3, 11].
Proof. As in the proof of Theorem 3.1, we fix \( r, s \) to satisfy the assumptions stated in Theorem 5.1, and thus will drop the superscript notation in our weight spaces and let \( W_{s, r, k} \equiv W_{s, r, k}^{r, s} \). We will use induction on \( \beta = |\alpha| \) to show that

\[
u \in L^{\infty}([0, T]; H^a(\mathbb{R}^2) \cap H^b(W_{0, i, 0})) \tag{5.3}\]

for \( 0 \leq \beta \leq K \).

In the following, we approximate general solutions by smooth solutions and general weight functions by bounded weight functions. As we have discussed approximating general solutions by smooth solutions in Section 3, here we concentrate on the approximation of general weight functions by smooth, bounded weight functions. In particular, we approximate a general weight function \( \eta \) as follows. We take a sequence of bounded weight functions \( \eta_d \), which decay as \( |rx+sy| \to +\infty \) and which approximate \( \eta \in W_{s, i-1, 0} \) from below. Let

\[
\tilde{\xi}_d(rx + sy, t) = 1 + \int_{-\infty}^{rx + sy} \eta_d(z, t) \, dz. \tag{5.4}\]

These weight functions are designed to satisfy the usual relations

\[
0 < \eta_d \leq \partial_z(\tilde{\xi}_d(z, t)). \tag{5.5}\]

The first induction step is to obtain a weighted estimate for \( \beta = 0 \). We multiply (2.1) by \( 2 \tilde{\xi}_d u \) and integrate to obtain

\[
\partial_t \int \tilde{\xi}_d u^2 + 2 \int a\tilde{\xi}_d uu_{xxx} + 2 \int b\tilde{\xi}_d uu_{xyy} + 2 \int c\tilde{\xi}_d uu_{xyy} \\
+ 2 \int d\tilde{\xi}_d uf = 0.
\]

First, by integration by parts, we obtain

\[
2 \int a\tilde{\xi}_d uu_{xxx} = -\int a(\tilde{\xi}_d)_{xxx} u^2 + 3 \int a(\tilde{\xi}_d)_x u^2.
\]

Similarly,

\[
2 \int b\tilde{\xi}_d uu_{xyy} = -\int b(\tilde{\xi}_d)_{xyy} u^2 + 2 \int b(\tilde{\xi}_d)_x u_x u_y + \int b(\tilde{\xi}_d)_y u^2,
\]

\[
2 \int c\tilde{\xi}_d uu_{xyy} = -\int c(\tilde{\xi}_d)_{xyy} u^2 + 2 \int c(\tilde{\xi}_d)_y u_y u_x + \int c(\tilde{\xi}_d)_x u_y^2,
\]

\[
2 \int d\tilde{\xi}_d uf = -\int d(\tilde{\xi}_d)_{yyy} u^2 + 3 \int d(\tilde{\xi}_d)_y u_y^2.
\]
Therefore, by the choice of $\xi$ and assumption (A1) on the coefficients $a, \ldots, d$, there exist constants $K_1, K_2 > 0$ such that

$$
\partial_t \left[ \int \xi u^2 + K_1 \int \eta u^2 + K_2 \int \eta u^2 + 2 \int \xi uf \right]
\leq \int a(\xi)_{xx} \ u^2 + \int b(\xi)_{xy} \ u^2 + \int c(\xi)_{yy} \ u^2 + \int d(\xi)_{yy} \ u^2
\leq C \int \xi u^2.
$$

(5.6)

So it remains to look at $2 \int \xi uf$. For the term $2 \int \xi uf$, we use (2.3) to write

$$
2 \int \xi uf = 2 \int \xi u(u_{ss}g_5 + u_{sy}g_4 + u_{yy}g_3 + u_{ss}s_2 + u_{s}g_1 + u_{y}g_0 + h).
$$

(5.7)

The first three terms can be dealt with as

$$
2 \int \xi u_{ss}g_5 = \int \partial_{ss} [\xi g_5] \ u^2 - 2 \int \xi g_5 u_u^2.
$$

$$
2 \int \xi u_{sy}g_4 = \int \partial_{sy} [\xi g_4] \ u^2 - 2 \int \xi g_4 u_y^2.
$$

$$
2 \int \xi u_{yy}g_3 = \int \partial_{yy} [\xi g_3] \ u^2 - 2 \int \xi g_3 u_y^2.
$$

The other four terms can be written as

$$
2 \int \xi u_{ss}g_2 = -\int \partial_{ss} [\xi g_2] \ u^2
$$

$$
2 \int \xi u_{ss}g_1 = -\int \partial_{ss} [\xi g_1] \ u^2
$$

$$
2 \int \xi u_yg_0 = 2 \int \xi g_0 u^2
$$

$$
2 \int \xi u_yh \leq C \left( \int \xi h^2 \right)^{1/2} \left( \int \xi _u u^2 \right)^{1/2} \leq C + C \left( \int \xi u^2 \right).
$$
Combining this with (5.6), we have
\[
\partial_t \int \xi \dot{u}^2 + K_1 \int \eta \dot{u}^2 + K_2 \int \eta \dot{u}^2 + \int \{ \partial_{xx} [\xi g_5] + \partial_{xy} [\xi g_4] + \partial_{yy} [\xi g_3] \} u^2 \\
- 2 \int \xi g_5 u_x^2 + g_4 u_x u_y + g_3 u_y^2 \\
+ \int \{ 2 \xi g_0 - \partial_x [\xi g_2] - \partial_y [\xi g_1] \} u^2 \leq C + C \int \xi u^2.
\]

By Assumption (A2) on \( f_{ux}, f_{uy}, f_{uyy} \), we have
\[
-2 \int \xi \{ g_5 u_x^2 + g_4 u_x u_y + g_3 u_y^2 \} \geq 0.
\]

Therefore,
\[
\partial_t \int \xi \dot{u}^2 + K_1 \int \eta \dot{u}^2 + K_2 \int \eta \dot{u}^2 \\
\leq \int \{ |\partial_{xx} [\xi g_5]| + |\partial_{xy} [\xi g_4]| + |\partial_{yy} [\xi g_3]| \} u^2 \\
+ C \int \{ |\xi g_0| + |\partial_x [\xi g_2]| + |\partial_y [\xi g_1]| \} u^2 + C + C \int \xi u^2. \quad (5.8)
\]

Notice,
\[
\int |\partial_{xx} [\xi g_5]| u^2 \leq \int \{ |(\xi)(xx) g_5| + 2 |(\xi)(x) \partial_x [g_5]| + |\xi \partial_x^2 [g_5]| \} u^2 \\
\leq C \left( \int \xi u^2 \right),
\]

where \( C \) depends at most on \( \|u\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \). Similarly for the other terms on the right-hand side of (5.8) above. Therefore, we conclude that
\[
\partial_t \int \xi \dot{u}^2 + K_1 \int \eta \dot{u}^2 + K_2 \int \eta \dot{u}^2 \leq C + C \left( \int \xi u^2 \right), \quad (5.9)
\]

where \( C \) depends at most on \( \|u\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \).
Now integrating (5.9) with respect to \( t \) for \( 0 \leq t \leq T \), we have
\[
\int_0^T \xi_u(t)u^2 + \int_0^T \eta_u u_x^2 + K_1 \int_0^T \eta u_x^2 \leq \int_0^T \xi_u(t)\phi^2 + C t + C \int_0^T \xi_u u^2.
\]

Therefore by Gronwall's inequality,
\[
\sup_{0 \leq t \leq T} \int_0^t \xi u^2 + K_1 \int_0^T \eta u_x^2 + K_2 \int_0^T \eta u_y^2 \leq C,
\]
where \( C \) depends only on \( T \) and the norm of \( \phi \) in \( L^2(W_{0,1,0}) \cap H^6(\mathbb{R}^2) \).
Consequently, we can pass to the limit as \( \delta \to 0 \) and conclude
\[
\sup_{0 \leq t \leq T} \int_0^t \xi u^2 + K_1 \int_0^T \eta u_x^2 + K_2 \int_0^T \eta u_y^2 \leq C. \tag{5.10}
\]

Now we need to prove this result for the \( \beta^a \) induction step. We begin by taking \( \alpha \) derivatives of (2.1) for \( |a| = \beta \), multiplying our differentiated equation by \( 2\xi_u u \) and integrating over \( \mathbb{R}^2 \). We will do this for each \( \alpha \) such that \( |a| = \beta \). Upon doing so, we get our familiar identity,
\[
\sum_{|\alpha| = \beta} \partial_a \left[ \xi_a(\partial^a u)^2 + \int_0^T \xi_a(\partial^a u)^2 \right.
\]
\[
+ \int_0^T \eta_a(\partial^a u_x)^2 + K_1 \int_0^T \eta_a(\partial^a u_x)^2 + K_2 \int_0^T \eta_a(\partial^a u_y)^2 \right.
\]
\[
+ \int_0^T \xi_a f_{u_x}(\partial^a u_x)^2 + c(\xi_a)(\partial^a u_x)^2 + \int_0^T \eta_a f_{u_y}(\partial^a u_y)^2
\]
\[
- \int_0^T \{ 2\xi_a f_{u_x}(\partial^a u_x)^2 + 2\xi_a f_{u_x}(\partial^a u_x)(\partial^a u_y) + 2\xi_a f_{u_y}(\partial^a u_y) \} + \int_0^T \Theta(\partial^a u)^2 + \int_0^T 2\xi_a(\partial^a u) R_1 + \int_0^T 2\xi_a(\partial^a u) R_2 = 0, \tag{5.11}
\]
where \( \Theta \) is as defined in (1.9) with \( \xi_a = \xi \), \( R_1 \) is defined in (1.10) and the terms in \( 2\xi_a(\partial^a u) R_2 \) are given by Lemma 3.3. By Assumptions (A1) and (A2) and by the choice of weight function \( \xi_a \), we can conclude that there exist constants \( K_1, K_2 > 0 \) such that
\[
\sum_{|\alpha| = \beta} \partial_a \left[ \xi_a(\partial^a u)^2 + K_1 \int_0^T \eta_a(\partial^a u_x)^2 + K_2 \int_0^T \eta_a(\partial^a u_y)^2 \right.
\]
\[
\leq \left\| \Theta(\partial^a u)^2 \right\| + \left\| 2\xi_a(\partial^a u) R_1 \right\| + \left\| 2\xi_a(\partial^a u) R_2 \right\|. 
\]
First notice that
\[
\left| \frac{\partial^s u}{\partial t^s} \right|^2 = \left| \{ \partial_{\alpha}[f_{\alpha, \beta}] + \cdots - \{ (\xi_{\beta}), + a(\xi_{\beta})_{xxx} + \cdots \} (\partial^s u)^2 \right|
\]
\[
\leq C \left( \int \xi_{\beta}(\partial^s u)^2 \right),
\]
where \( C \) depends at most on \( \|u\|_{L^\infty(H^r(\mathbb{R}^m, \gamma))} \). Therefore, it remains to show that
\[
\sum_{|\alpha| = \rho} \left| \int 2\xi_{\beta}(\partial^s u) R_j \right| \leq C \sum_{|\alpha| = \rho} \left( \int \xi_{\beta}(\partial^s u)^2 + 1 \right)
\]
for \( j = 1, 2 \), where \( C \) depends only on \( \|u\|_{H^r(\mathbb{R}^m)} \) and the norm of \( u \in L^\infty(H^r(W_{0,1,0})) \) for \( \gamma \leq \beta - 1 \), so that we can use the inductive hypothesis and Gronwall’s inequality.

We will begin by looking at the terms in \( \{ 2\xi_{\beta}(\partial^s u) R_j \} \). These terms are of order \( \beta + 1 \). They are of the form (1.10). By integration by parts, we see
\[
\left| \int 2\xi_{\beta} \partial_{\alpha}[f_{\alpha, \beta}](\partial^s u)(\partial^s u) \right| = C \left| \int \xi_{\beta} \partial_{\alpha}[f_{\alpha, \beta]}(\partial^s u)(\partial^s u) \right|
\]
\[
\leq C \left( \int \xi_{\beta}(\partial^s u)^2 \right),
\]
where \( C \) depends at most on \( \|u\|_{L^\infty(H^r(\mathbb{R}^m))} \). Similarly, for the other terms in \( \{ 2\xi_{\beta}(\partial^s u) R_j \} \).

It remains to look at terms of the form \( \{ 2\xi_{\beta}(\partial^s u) R_j \} \). It is shown in Lemma 3.3 that \( \{ 2\xi_{\rho}(\partial^s u) R_j \} \) contains terms of the form
\[
\left| \int \xi_{\rho}(D^s f) \cdot (\partial^s u) \cdots (\partial^s u)(\partial^s u) \right|.
\]
(5.12)

We divide the analysis of terms of the form (5.12) into several cases below. First, we consider the case \( p = 1 \).

The Case \( p = 1 \). If \( p = 1 \), we have
\[
\left| \int \xi_{\rho}(D^s f) (\partial^s u)(\partial^s u) \right| \leq C \left( \int \xi_{\rho}(\partial^s u)^2 \right)^{\frac{1}{2}} \left( \int \xi_{\rho}(\partial^s u)^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \sum_{|\alpha| = \rho} \left( \int \xi_{\rho}(\partial^s u)^2 + 1 \right),
\]
as desired.
The Case $|v_p| \leq |a| - 3$. We estimate as

$$\left| \int \bar{\xi}_a (D^s u) \cdot (\partial^\gamma u) \cdot (\partial^\gamma u) (\partial^\gamma u) \right| \leq C |\bar{\partial}^\gamma u|_{L^\infty} \cdots |\bar{\partial}^\gamma u|_{L^\infty} \left( \int \bar{\xi}_a (\partial^\gamma u)^2 \right)^{1/2} \leq C \left( \int \bar{\xi}_a (\partial^\gamma u)^2 \right)^{1/2},$$

where $C$ depends only on the norm of $u \in L^\infty(H^\gamma(W_{0,1,0}))$ for $\gamma \leq \beta - 1$ using the fact that $H^\gamma(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ and $|v| \leq \beta - 3$ for $1 \leq j \leq p$.

The Case $|v_p| = |a| - 2$. If $p \geq 2$, (3.13) in Lemma 3.3 implies

$$p + |v_{p-1}| \leq 8.$$

If $p = 2$, then $|v_{p-1}| \leq 6$. We will break this case up into the following subcases: $|a| \geq 9$, $|a| = 8$, $|a| = 7$ and $|a| \leq 6$. If $|a| \geq 9$, we can estimate as

$$\left| \int \bar{\xi}_a (D^s u) (\partial^\gamma u) (\partial^\gamma u) (\partial^\gamma u) \right| \leq C |\bar{\partial}^\gamma u|_{L^\infty} \left( \int \bar{\xi}_a (\partial^\gamma u)^2 \right)^{1/2} \left( \int \bar{\xi}_a (\partial^\gamma u)^2 \right)^{1/2} \leq C \left( \int \bar{\xi}_a (\partial^\gamma u)^2 + 1 \right),$$

by the inductive hypothesis that $u \in L^\infty(H^\gamma(W_{0,1,0}))$ for $\gamma \leq |a| - 1$. If $|a| = 8$, then $|v_p| = 6$. Our remainder is bounded as

$$\left| \int \bar{\xi}_a (D^s u) (\partial^\gamma u) (\partial^\gamma u) (\partial^\gamma u) \right| \leq C \left( \int \bar{\xi}_a (\partial^\gamma u)^4 \right)^{1/4} \left( \int \bar{\xi}_a (\partial^\gamma u)^4 \right)^{1/4} \left( \int \bar{\xi}_a (\partial^\gamma u)^2 \right)^{1/4} \leq C \left( \int \bar{\xi}_a (\partial^\gamma u)^2 + 1 \right),$$
because \( \|u\|_{L^\infty(H^7(\mathbb{R}))} \) is bounded on the previous step of the induction. If \( |\alpha| = 7 \), then \( |\nu_p| = 5 \) and therefore, \( |\nu_{p-1}| \leq 5 \). We proceed as above,

\[
\left| \int \xi_\beta (D'f)(\partial^{\alpha} u)(\partial^{\beta} u)(\partial^{\gamma} u) \right| \\
\leq C \left( \int \xi_\beta (\partial^{\alpha} u)^4 \right)^{\frac{1}{4}} \left( \int \xi_\beta (\partial^{\beta} u)^4 \right)^{\frac{1}{4}} \left( \int \xi_\beta (\partial^{\gamma} u)^4 \right)^{\frac{1}{4}} \\
\leq C \|u\|_{L^\infty(H^7(\mathbb{R}))} \left( \int \xi_\beta (\partial^{\beta} u)^4 \right)^{\frac{1}{4}} \\
\leq C \left( \int \xi_\beta (\partial^{\beta} u)^2 + 1 \right),
\]

because \( \|u\|_{L^\infty(H^7(\mathbb{R}))} \) is bounded on the previous step of the induction. If \( |\alpha| \leq 6 \), then \( |\nu_p| \leq 4 \) and \( |\nu_{p-1}| \leq 4 \). Therefore,

\[
\left| \int \xi_\beta (D'f)(\partial^{\alpha} u)(\partial^{\beta} u)(\partial^{\gamma} u) \right| \leq C |\partial^{\alpha} u|_{L^\infty} \left( \int \xi_\beta (\partial^{\beta} u)^4 \right)^{\frac{1}{4}} \left( \int \xi_\beta (\partial^{\gamma} u)^4 \right)^{\frac{1}{4}} \\
\leq C \left( \int \xi_\beta (\partial^{\beta} u)^2 + 1 \right),
\]

where \( C \) depends only on the norm of \( u \in L^\infty(H^6(\mathbb{R})) \) and terms bounded by the inductive hypothesis.

If \( p = 3 \), then by (3.13), \( |\nu_{p-1}| \leq 5 \). Also, by (3.12), \( |\nu_1| + |\nu_2| \leq 8 \). We consider first the case \( |\nu_2| = 5 \), which implies \( |\nu_1| \leq 3 \). Therefore, we have

\[
\left| \int \xi_\beta (D'f)(\partial^{\alpha} u)(\partial^{\beta} u)(\partial^{\gamma} u) \right| \leq C |\partial^{\alpha} u|_{L^\infty} \left( \int (\partial^{\beta} u)^4 \right)^{\frac{1}{4}} \left( \int \xi_\beta (\partial^{\beta} u)^4 \right)^{\frac{1}{4}} \\
\cdot \left( \int \xi_\beta (\partial^{\gamma} u)^4 \right)^{\frac{1}{4}} \\
\leq C \|u\|_{H^{\infty}(\mathbb{R}^5)} \|u\|_{H^{\infty}(\mathbb{R}^5)} \left( \int \xi_\beta (\partial^{\beta} u)^4 \right)^{\frac{1}{4}} \\
\leq C \left( \int \xi_\beta (\partial^{\beta} u)^2 + 1 \right),
\]
where $C$ depends only on $\|u\|_{L^{∞}(\mathbb{R}^n)}$ and $\|u\|_{L^{∞}(\mathbb{R}^{n−1}(\mathcal{W}_{i,b}))}$, which is bounded on the previous step of the induction. If $|v_2| \leq 4$, then we have

$$
\left| \int \xi_\delta(D^s f)(\partial^\nu u)(\partial^\nu u)(\partial^\nu u)(\partial^\nu u) \right|
\leq C |\partial^\nu u|_{L^∞} |\partial^\nu u|_{L^∞} \left( \int \xi_\delta(\partial^\nu u)^2 \right)^{\frac{1}{2}} \left( \int \xi_\delta(\partial^\nu u)^2 \right)^{\frac{1}{2}}
\leq C \left( \int \xi_\delta(\partial^\nu u)^2 + 1 \right),
$$

where $C$ depends only on $\|u\|_{L^{∞}(\mathbb{R}^n)}$ and the norms of terms bounded on the previous steps of the induction. If $p \geq 4$, then by (3.13), $|v_{p−1}| \leq 4$. Therefore,

$$
\left| \int \xi_\delta(D^s f)(\partial^\nu u)\cdots(\partial^\nu u)(\partial^\nu u) \right| \leq C |\partial^\nu u|_{L^∞} \cdots |\partial^{γ−1} u|_{L^∞} \left( \int \xi_\delta(\partial^\nu u)^2 \right)^{\frac{1}{2}} \cdot \left( \int \xi_\delta(\partial^\nu u)^2 \right)^{\frac{1}{2}}
\leq C \left( \int \xi_\delta(\partial^\nu u)^2 + 1 \right),
$$

where $C$ depends only on $\|u\|_{L^{∞}(\mathbb{R}^n)}$ and $\int \xi_\delta(\partial^\nu u)^2$, which is bounded on the previous step of the induction.

The Case $|v_p| = |\alpha| − 1$. If $p \geq 2$, then (3.13) of Lemma 3.3 implies

$$
p + |v_{p−1}| \leq 7. \quad (5.13)
$$

If $p = 2$, then $|v_{p−1}| \leq 5$. We will break this up into the following subcases: $|\alpha| \geq 8$, $|\alpha| = 7$, and $|\alpha| \leq 6$. If $|\alpha| \geq 8$, then we have

$$
\left| \int \xi_\delta(D^s f)(\partial^\nu u)(\partial^\nu u)(\partial^\nu u) \right| \leq C |(\partial^\nu u)|_{L^∞} \left( \int \xi_\delta(\partial^\nu u)^2 \right)^{\frac{1}{2}} \left( \int \xi_\delta(\partial^\nu u)^2 \right)^{\frac{1}{2}}
\leq C \left( \int \xi_\delta(\partial^\nu u)^2 + 1 \right),
$$
because \( \|u\|_{L^p(\mathbb{R}^n_1)} \) was bounded on the previous step of the induction and \( \|u\|_{H^\gamma(\mathbb{R}^n_1)} \leq \|u\|_{H^\gamma(\mathbb{R}^n_2)} \). If \( |\alpha| = 7 \), then we have

\[
\left| \int \xi_\alpha (D^\gamma f) (\partial_1 u)^4 (\partial_\alpha^\gamma u)^4 \right| \leq C \left( \left| \int \xi_\alpha (D^\gamma u)^4 \right| \right)^{1/2} \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 \right| \right)^{1/2} \\
\leq C \|u\|_{L^p(\mathbb{R}^n_1)} \|u\|_{L^p(\mathbb{R}^n_2)} \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 \right| \right)^{1/2} \\
\leq C \sum_{|\beta| = 7} \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 + 1 \right| \right).
\]

If \( |\alpha| \leq 6 \), then we have \( |v_1| \leq 5 \) and \( |v_2| = |\alpha| - 1 \). Therefore,

\[
\left| \int \xi_\alpha (D^\gamma f) (\partial_1 u)^4 (\partial_\alpha^\gamma u)^4 \right| \leq C \left( \left| \int (\partial_1 u)^4 \right| \right)^{1/2} \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 \right| \right)^{1/2} \\
\leq C \|u\|_{H^\gamma(\mathbb{R}^n_1)} \|\xi_\alpha \|_{L^p(\mathbb{R}^n_2)} \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 \right| \right)^{1/2} \\
\leq C \sum_{|\beta| = \beta} \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 + 1 \right| \right).
\]

If \( p \geq 3 \), then (5.13) implies that \( |v_{p-1}| \leq 4 \). Therefore, our inequality becomes

\[
\left| \int \xi_\alpha (D^\gamma f) (\partial_1 u)^4 \cdots (\partial_\alpha^\gamma u)^4 (\partial_\alpha^\gamma u)^2 \right| \leq C \|u\|_{L^p(\mathbb{R}^n_1)} \cdots \|u\|_{L^p(\mathbb{R}^n_2)} \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 \right| \right)^{1/2} \\
\cdot \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 \right| \right)^{1/2} \\
\leq C \|u\|_{H^\gamma(\mathbb{R}^n_1)} \|u\|_{H^\gamma(\mathbb{R}^n_2)} \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 \right| \right)^{1/2} \\
\leq C \left( \left| \int \xi_\alpha (\partial_\alpha^\gamma u)^2 + 1 \right| \right).
\]

The Case \( |v_p| = |\alpha| \). If \( p \geq 2 \), Lemma 3.3 implies

\[
p + |v_{p-1}| \leq 6.
\]
Therefore, for \( p \geq 2, |v_{p-1}| \leq 4 \) and, thus, we have

\[
\left\| \tilde{\xi}_a(D^p f)(\partial^\nu u) \cdots (\partial^\nu u) u \right\| \leq C \| (\partial^\nu u) \cdots (\partial^\nu-1) u \|_{L^p} \left( \int \tilde{\xi}_a(\partial^\nu u)^2 \right) \frac{1}{2} \cdot \left( \int \tilde{\xi}_a(\partial^\nu u)^2 \right) \frac{1}{2}
\]

\[
\leq C \| u \|_{H^{p-1}(\mathbb{R}^3)} \sum_{|\nu|=\beta} \left( \int \tilde{\xi}_a(\partial^\nu u)^2 \right)
\]

\[
\leq C \sum_{|\nu|=\beta} \left( \int \tilde{\xi}_a(\partial^\nu u)^2 \right).
\]

Consequently, we conclude that

\[
\sum_{|\nu|=\beta} \int \tilde{\xi}_a(\partial^\nu u) R_j \leq C \sum_{|\nu|=\beta} \left( \int \tilde{\xi}_a(\partial^\nu u)^2 + 1 \right)
\]

for \( j = 1, 2 \) where \( C \) depends only on the norm of \( u \in L^\infty(H^6(\mathbb{R}^3) \cap H^{5.5}(W_{0,1,0})), which is bounded on the previous step of the induction. Thus, we have

\[
\sum_{|\nu|=\beta} \partial_i \int \tilde{\xi}_a(\partial^\nu u)^2 + K_1 \int \eta_a(\partial^\nu u_x)^2 + K_2 \int \eta_a(\partial^\nu u_y)^2 \leq C \sum_{|\nu|=\beta} \left( \int \tilde{\xi}_a(\partial^\nu u)^2 + 1 \right).
\]

By integrating with respect to \( t \) for \( 0 \leq t \leq T \), we have

\[
\sum_{|\nu|=\beta} \int \tilde{\xi}_a(\partial^\nu u)^2 + K_1 \int_0^t \eta_a(\partial^\nu u_x)^2 + K_2 \int_0^t \eta_a(\partial^\nu u_y)^2 \leq C \sum_{|\nu|=\beta} \left( \int \tilde{\xi}_a(\partial^\nu u)^2 + 1 \right).
\]

By Gronwall’s inequality,

\[
\sup_{0 \leq t \leq T} \sum_{|\nu|=\beta} \int \tilde{\xi}_a(\partial^\nu u)^2 + K_1 \int_0^T \eta_a(\partial^\nu u_x)^2 + K_2 \int_0^T \eta_a(\partial^\nu u_y)^2 \leq C, \quad (5.14)
\]
where $C$ depends only on $T$ and the norm of $\phi \in H^b(\mathbb{R}^2) \cap H^{b_0}(W_{0,L,0})$. In particular, the constants do not depend on $\delta$, so we can take the limit as $\delta \to 0$ and conclude that

$$
\sup_{0 \leq t \leq T} \sum_{u=0}^{T} \int_0^T \xi(\partial^u u)^2 + K_1 \int_0^T \eta(\partial^u u_x)^2 + K_2 \int_0^T \eta(\partial^u u_y)^2 \leq C, \quad (5.15)
$$

and thus our theorem is proved.

We have now proven all the lemmas and theorems necessary to give a formal proof of our main theorem on the gain of regularity for semilinear equations of the form (2.1) satisfying Assumptions (A1)–(A4).

Proof of Theorem 3.1. We will use induction, beginning with $\beta = 7$. We will apply Lemma 3.2 to a smooth approximation of the solution. Let $u$ be a solution such that $u \in L^\infty(\mathbb{R}^2)$ and $u_t \in L^2(\mathbb{R}^2)$. Hence $u$ is a weakly continuous function of $t$ with values in $H^b(W_{0,L,0})$. In particular, $u(\cdot, t) \in H^b(W_{0,L,0})$ for all $t$.

Let $t_0 \in [0, T)$ and let $\{\phi^{(\alpha)}(\cdot)\}$ be a sequence of functions in $C^\infty(\mathbb{R}^2)$ which converges to $u(\cdot, t_0)$ strongly in $H^b(W_{0,L,0})$. Let $u^{(\alpha)}$ be the unique solution with initial data $\phi^{(\alpha)}$ at time $t = t_0$. By Theorem 4.2 it is guaranteed to exist in a time interval $[t_0, t_0 + \delta]$ where $\delta > 0$ does not depend on $n$. By Theorem 5.1,

$$
u^{(\alpha)} \in L^\infty([t_0, t_0 + \delta]; H^b(W_{0,L,0})) \cap L^2([t_0, t_0 + \delta]; H^b(W_{n-1,0})). \quad (5.16)
$$

with a bound that depends only on the norm of $\phi^{(\alpha)} \in H^b(W_{0,L,0})$. Theorem 5.1 also guarantees the non-uniform bounds

$$
\sup_{t \in [t_0, t_0 + \delta]} \sup_{(x,y)} (1 + (rx+sy)^+)^k |\partial^u u^{(\alpha)}(x, y, t)| < +\infty \quad (5.17)
$$

for each $n$, $k$, and $\alpha$. Therefore, the a priori estimates in Lemma 3.2 are justified for each $u^{(\alpha)}$ in the interval $[t_0, t_0 + \delta]$.

We start our induction with $\beta = 7$. Take a weight function $\eta \in W_{\alpha,L-2,1}$ and let $\xi = \int_{-\infty}^{\infty} \eta(z, t) dz$. As shown in Lemma 3.2, for all $\alpha$ such that $|\alpha| = 7$, we have the following bounds on $u^{(\alpha)}$,

$$
\sup_{t \in [t_0, t_0 + \delta]} \int_{t_0}^{t_0 + \delta} \int_0^T \xi(\partial^b u^{(\alpha)})^2 + \int_0^T \eta(\partial^b u_x^{(\alpha)})^2 + \int_0^T \eta(\partial^b u_y^{(\alpha)})^2 \leq C, \quad (5.18)
$$

where $C$ depends only on the norm of $u^{(\alpha)}$ in

$$
L^\infty([t_0, t_0 + \delta]; H^b(W_{0,L,0})) \cap L^2([t_0, t_0 + \delta]; H^b(W_{n-1,0})). \quad (5.19)
$$
As stated above, this norm of \( u^{(n)} \) is bounded by the norm of \( \phi^{(n)} \in H^2(W_{0,L,0}) \). Therefore, we conclude

\[
\| u^{(n)} \|_{L^\infty([t_0, t_0 + \delta]; H^7(W_{n,L-1,1}))} \leq C \| f^{(n)} \|_{H^6(W_{n,L-2,1})}.
\]

(5.20)

This estimate is proved by induction for \( \beta = 8, 9, \ldots, L+6 \). At each level, let \( g^{(n)} \) be defined as above, with the exception that for \( \beta = L+6 \) replace \( W_{n-1,L} \) with \( \bar{W}_{n-1,L} \), and define \( \xi^{(n)} = \int_{t_0}^{t_0 + \delta} \eta(z, t) \, dz \). Using Lemma 3.2 and the inductive hypothesis, we conclude that

\[
\| u^{(n)} \|_{L^\infty([t_0, t_0 + \delta]; H^8(W_{n,L-1,1}))} \leq C \| f^{(n)} \|_{H^6(W_{n,L-2,1})}.
\]

(5.21)

Since \( u^{(n)} \to u \) weak* in \( L^\infty([t_0, t_0 + \delta]; H^8(\mathbb{R}^2)) \) by Corollary 4.5, it follows that

\[
\| u \|_{L^\infty([t_0, t_0 + \delta]; H^{8+l}(W_{n,L-1,1}))} \leq C \| f \|_{H^8(\mathbb{R}^2)}
\]

(5.22)

for \( 0 \leq l \leq L \), with the exception that on the last level, \( |x| = L+6 \), we replace \( W_{n-1,L} \) with \( \bar{W}_{n-1,L} \). Since \( \delta \) is fixed, this result is valid over the whole interval \([0, T]\).

6. THE FULLY NONLINEAR EQUATION

We now consider the fully nonlinear equation

\[
\begin{align*}
t + f(D^4u, D^3u, D^2u, Du, u, x, y, t) &= 0 \\
u(x, y, 0) &= \phi(x, y).
\end{align*}
\]

(6.1)

We make similar assumptions to those made in the semilinear case. In particular, for Eq. (6.1), we make the following assumptions.

**Assumptions.** (B1) There exists a vector \((r, s) \in \mathbb{R}^2\) such that the symbol

\[
p_n(\omega_1, \omega_2) = -(f_{uxxx} \omega_1^3 + f_{uxx} \omega_1 \omega_2 + f_{ux} \omega_2^2 + f_{u} \omega_2^3)
\]

(6.2)

associated with the third-order terms satisfies \( \nabla \nabla p_n(\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \((\omega_1, \omega_2) \neq (0, 0)\). We recall Lemma 2.1. If \( p_n(\omega_1, \omega_2) \) has exactly one real root and two non-real roots for each value of \( x, y \), and \( t \) as well as \( u \) and
its derivatives, we know there exists a vector \((r, s)\) (which may depend on \(x, y, t, u\), etc.) for which \(\nabla p_n(\omega_1, \omega_2) \cdot (r, s) < 0\). However, we need to find a half-plane \(\{(x, y): rx + sy < 0\}\) into which all bicharacteristics point. Consequently, we need to assume not only that \(p_n\) has one real and two non-real roots for each value of \(x, y, t, u\) and its derivatives, but, moreover, that there exists a vector \((r, s)\) independent of \(x, y, t, u\) and its derivatives for which \(\nabla p_n(\omega_1, \omega_2) \cdot (r, s) < 0\) for all \((\omega_1, \omega_2) \neq (0, 0)\).

In particular, we discuss a simple example of a symbol which satisfies the above assumption. Consider

\[ p_n(\omega_1, \omega_2) = -a(x) \omega_1^3 - \omega_1 \omega_2^2, \]

where \(a(x) \geq k > 0\). Then

\[ p_n(\omega_1, \omega_2) = -\omega_1 (a(x) \omega_1^2 + \omega_2^2) \]

has exactly one real root and two non-real roots for each value of \(x\). Furthermore,

\[ \nabla p_n(\omega_1, \omega_2) = (-3a(x) \omega_1^2 - \omega_2^2, -2\omega_1 \omega_2). \]  

Therefore, by choosing \((r, s) = (1, 0)\), we notice that \(\nabla p_n \cdot (r, s) < 0\) for all \((\omega_1, \omega_2) \neq (0, 0)\).

(B2) The nonlinear term \(f: \mathbb{R}^{12} \times [0, T] \to \mathbb{R}\) is \(C^\infty\) and satisfies

\[ f_{ux}, f_{uy} \leq 0 \]
\[ f_{u^2} \leq 4 f_{ux} f_{uy}. \]

(B3) All the derivatives of \(f(w, x, y, t)\) are bounded for \((x, y) \in \mathbb{R}^2\), for \(t \in [0, T]\) and \(w\) in a bounded set.

(B4) \(x^N y^M \partial_x^i \partial_y^j f(0, x, y, t)\) is bounded for all \(N, M \geq 0, i, j \geq 0\), and \((x, y) \in \mathbb{R}^2, t \in (0, T]\).

These assumptions imply that \(f\) has the form

\[ f = u_{xxx} g_9 + u_{xx} g_8 + u_{x} g_7 + u g_6 + u_{xx} g_5 + u_{x} g_4 + u g_3 + u_{x} g_2 + u g_1 + u g_0 + h, \]  

\[ (6.4) \]
where we define the $g_j$ as

$$
g_9 = \begin{cases} 
\left[ f(u_{xxx}, u_{xxy}, \ldots) - f(0, u_{xxy}, \ldots) \right]/u_{xxx} & \text{for } u_{xxx} \neq 0 \\
\partial_{u_{xxx}} f(0, u_{xxy}, \ldots) & \text{for } u_{xxx} = 0
\end{cases} \tag{6.5}
$$

$$
g_8 = \begin{cases} 
\left[ f(0, u_{xxy}, u_{xyy}, \ldots) - f(0, 0, u_{xyy}, \ldots) \right]/u_{xxy} & \text{for } u_{xxy} \neq 0 \\
\partial_{u_{xxy}} f(0, 0, u_{xyy}, \ldots) & \text{for } u_{xxy} = 0,
\end{cases} \tag{6.6}
$$

and similarly for $g_7, g_6, \ldots, h$. Assumption (B2) implies that

$$g_9^2 - 4g_5^2 - 4g_5^2, \leq 0.$$

Assumption (B3) implies that $g_9, g_8, \ldots, h$ are $C^\infty$ and each of their derivatives is bounded for $w$ bounded, $(x, y) \in \mathbb{R}^2$ and $t \in [0, T]$.

**Differences between Semilinear and Fully Nonlinear Cases.** In order to illustrate the differences between the semilinear and fully nonlinear cases, we consider the following example,

$$u_t + a(x)u_{xxx} + u_{xyy} = 0, \tag{6.7}$$

where $a(x) \geq k > 0$ for all $x \in \mathbb{R}$. The homogeneous polynomial $p_n$ associated with this operator is given by

$$p_n(\omega_1, \omega_2) = -a(x)(\omega_1)^3 - \omega_1 \omega_2^2.$$

By Lemma 2.1 of Section 2, the assumption that $a(x) \geq k > 0$ implies there exists a vector $(r, s)$ such that $\nabla p_n(\omega_1, \omega_2) \cdot (r, s) < 0$ for all $(\omega_1, \omega_2) \neq (0, 0)$. Namely, this vector $(r, s)$ can be chosen to be $(1, 0)$. Therefore, in order to prove a gain in regularity for solutions to (6.7), we would like to choose our weight function $\zeta$ such that $\zeta \approx x^j$ as $x \to +\infty$ and $\zeta \approx e^{\alpha x}$ for $\sigma > 0$ arbitrary as $x \to -\infty$.

Proceeding as we did for the semilinear equation, we multiply (6.7) by $2\zeta u$, where $\zeta$ is some weight function, yet to be specified, and integrate over $\mathbb{R}^2 \times [0, T]$. Upon doing so, we arrive at the following *a priori* estimate,

$$\int \zeta(\cdot, T)u^2 + 3\int_0^T \left[ \zeta u \right]_x u_x^2 + 2\int_0^T \left[ \zeta u \right]_y u_y^2 + \int_0^T \zeta u_x^2 = \int \zeta(\cdot, 0)\phi^2 + \int_0^T \left[ \zeta a \right]_{xxx} u^2 + \int_0^T \zeta_{xxy} u^2.$$
If \( a(x) \equiv k > 0 \), we may choose \( \xi \approx x^j \) for \( x > 1 \). However, for \( a(x) \) variable, we need \( \xi \) to satisfy more properties. In particular, we need weight functions \( \xi, \eta > 0 \) such that there exist constants \( K_1, K_2 > 0 \) satisfying

\[
K_1 \int_0^T \int \eta \xi u_x^2 + K_2 \int_0^T \int \eta \xi u_y^2 \leq 3 \int_0^T \int [\xi a]_x \xi u_x^2 + 2 \int_0^T \int \xi \xi u_y + \int_0^T \int \xi u_y^2.
\]

In particular, we would like to find a smooth, weight function \( \xi = \xi(x, t) \) such that

\begin{enumerate}
  \item \( \xi > 0 \)
  \item \( [\xi a]_x \geq K_1 \eta \)
  \item \( \xi \xi \geq K_2 \eta \).
\end{enumerate}

For a general function \( a(x) \), we cannot find a smooth function \( \xi \) such that \( \xi \approx x^j \) as \( x \to +\infty \) and \( \xi \approx e^x \) as \( x \to -\infty \), without imposing extra restrictions on \( a(x) \). However, if we choose \( \xi \approx e^{Rx} \) for \( R > 0 \), sufficiently large, then criteria (i)–(iii) above are satisfied, and we can prove a gain in regularity. Consequently, as we will describe in more detail below, for the fully nonlinear Eq. (6.1), we will choose our weight function \( \xi \approx e^{R(rx+sy)} \) where \( R > 0 \) sufficiently large and \((r, s)\) satisfies \( V_p(\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \((\omega_1, \omega_2) \neq (0, 0)\).

Recall that for the semilinear case, the gain of regularity result relied on the existence and persistence properties of Sections 4 and 5. Before stating a special case of the gain of regularity result for the fully nonlinear Eq. (6.1), we state the main existence assumption for the fully nonlinear Eq. (6.1) under which we have proven a gain in regularity theorem.

**Assumption 6.1 (Existence Hypothesis).** Assume Eq. (6.1) satisfies Assumptions (B1)–(B4). Let \( k_0 > 0 \) and \( N \) be an integer \( \geq 8 \). Let \( (r, s) \) satisfy \( V_p(\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \((\omega_1, \omega_2) \neq (0, 0)\) and let \( R > 0 \) be chosen sufficiently large. Then for all \( \phi \in H^N \) with the weight \( 1+e^{R(rx+sy)} \) with

\[
\int (1 + e^{R(rx+sy)}) \sum_{|\alpha| \leq 8} (\partial^\alpha \phi)^2 \leq k_0^2
\]

there exists a unique solution \( u \) of (6.1) for a time interval \([0, T]\) depending only on \( k_0 \) such that \( u(x, y, 0) = \phi(x, y) \) and

\[
\sup_{0 \leq t \leq T} \int (1 + e^{R(rx+sy)}) \sum_{|\alpha| \leq N} (\partial^\alpha u)^2 + \int_0^T \sum_{|\alpha| \leq N+1} (\partial^\alpha u)^2 < +\infty.
\]
Remarks. (1) We have not proven this existence hypothesis for all equations of type (6.1) for which we have proven a gain in regularity theorem. Consequently, we state it here as an assumption. In Section 8, however, we prove sufficient conditions on Eq. (6.1) for which such an assumption holds.

(2) The above assumption is a weighted existence theorem for (6.1). If an equation of the form (6.1) is well-posed in an unweighted space, then a persistence property can be proven as in Theorem 5.1. In particular, if one can prove that for \( \phi \in H^q(\mathbb{R}^2) \) there exists a solution \( u \in L^\infty([0, T]; H^q(\mathbb{R}^2)) \) for a time \( T \) depending only on the norm of \( \phi \in H^q(\mathbb{R}^2) \), then if \( \phi \) also satisfies (6.8), then the solution \( u \) will satisfy (6.9).

Gain of Regularity for the Fully Nonlinear Equation. Assume Eq. (6.1) satisfies Assumptions (B1)–(B4) and Assumption 6.1 stated above. Let \( T > 0 \) and let \( u \) be the solution of (6.1) in the region \( \mathbb{R}^2 \times [0, T] \) such that

\[
\int (1 + e^{R(rx+sy)}) \sum_{|\xi| \leq R} (\partial^\xi u)^2 < +\infty
\]

for all \( R > 0 \), where \((r, s)\) satisfies \( \nabla p_n(\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \((\omega_1, \omega_2) \neq (0, 0)\). Then \( u(t) \in C^\infty(\mathbb{R}^2) \) for \( 0 < t \leq T \).

We will state the main theorem for the gain of regularity for the fully nonlinear Eq. (6.1) in its full generality in Section 7. First, we define the weight functions and notation to be used in the next two sections.

Choice of Weight Function. As before, we will be using weight functions which depend only on \( rx+sy \), where \((r, s)\) satisfies \( \nabla p_n(\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \((\omega_1, \omega_2) \neq (0, 0)\). As described above, we will not be using a weight function behaving like a power, but instead will be using exponential weight functions. We define our weight classes as follows. We say that a positive \( C^\infty \) function \( \zeta \) belongs to the weight class \( Y_{J,R,k}^\prime \), where \( J, R \geq 0, k \geq 0 \) is an integer, if \( \zeta(rx+sy, t) \) satisfies the following:

\[
0 < c_1 \leq t^{-R(rx+sy)^2} \zeta(rx+sy, t) \leq c_1 \quad \text{for} \quad rx+sy < -1
\]

\[
0 < c_1 \leq t^{-R(rx+sy)^2} \zeta(rx+sy, t) \leq c_1 \quad \text{for} \quad rx+sy > 1
\]

\[
(t (|\partial_x \zeta| + |\partial_y \zeta| + |\partial_x \zeta|)) / \zeta \leq c_1 \quad \text{in} \quad \mathbb{R}^2 \times [0, T] \quad \text{for all} \quad j.
\]
Thus \( \xi \) looks like \( t^k \) as \( t \to 0 \), like \( e^{R(rx+sy)} \) as \( rx+sy \to +\infty \) and like \( e^{J(rx+sy)} \) as \( rx+sy \to -\infty \).

Again we assume \( r, s \) fixed and thus in the definitions below drop the superscript notation, letting \( Y_{r,R,k} \equiv Y_{r,R,k}^\cdot \) for a given \( r, s \).

**Notation.** We introduce the following notation for our weighted Sobolev spaces. For \( \beta \geq 0 \), let \( H^\beta(Y_{r,s}J,R,k) \) be the space of functions with finite norm

\[
\|f(\cdot, t)\|^2_{H^\beta(Y_{r,s}J,R,k)} = \int \sum_{|\alpha| \leq \beta} (\partial^\alpha f)^2 |\xi(rx+sy, t)| \, dx \, dy
\]  

for any \( \xi \in Y_{r,s}J,R,k \) and \( 0 \leq t \leq T \). Again, although this norm depends on \( \xi \), all choices of \( \xi \) in this class lead to equivalent norms. With the same notation, for \( \beta > 0 \), let \( L^\beta(H^\beta(Y_{r,s}J,R,k)) \) be the space of functions with finite norm

\[
\|f\|^2_{L^\beta(H^\beta(Y_{r,s}J,R,k))} = \int_0^T \left\{ \int \sum_{|\alpha| \leq \beta} (\partial^\alpha f)^2 |\xi(rx+sy, t)| \, dx \, dy \right\}^\frac{\beta}{2} \, dt,
\]  

where \( \xi \in Y_{r,s}J,R,k \).

### 7. Gain of Regularity—Fully Nonlinear Equation

In this section we state and prove our main theorem on the gain of regularity for the fully nonlinear Eq. (6.1). We also prove the main estimates on the remainder terms.

**Theorem 7.1.** Assume Eq. (6.1) satisfies Assumptions (B1)–(B4) and Assumption 6.1. Let \( T > 0 \) and let \( u \) be the solution of (6.1) in the region \( \mathbb{R}^2 \times [0, T] \) such that \( u \in L^\infty(H^\beta(y_{0,R,0})) \) for some \( R > 0 \) sufficiently large, where \( (r, s) \) satisfies \( \nabla p_r(\omega_1, \omega_2) \cdot (r, s) < 0 \) for all \( (\omega_1, \omega_2) \neq (0, 0) \). Then

\[
u \in L^\infty(H^{\beta+l}(Y_{R,R,l}^\cdot)) \cap L^2(H^{\beta+l}(Y_{R,R,l}^\cdot)),
\]  

for \( 0 \leq l \leq L \), where \( L \) is some integer depending on \( R \) and \( f_{u,x}, \ldots, f_{u,yy} \).

**Remark.** As described in the previous section, we will choose our weight function to be \( e^{R(rx+sy)} \) for \( R > 0 \) sufficiently large. As will be shown below (and as may be expected) the number of times the inductive argument may be repeated will depend on the size of \( R \).
In what follows, we assume \( r, s \) are fixed and satisfy the assumptions in the statement of Theorem 7.1. Consequently, we drop the superscript notation and let \( Y_{R, R, k} \). As in the proof for the semilinear case, the proof of Theorem 7.1 relies on an inductive argument, where on each step of the induction, we combine \( a \) priori \( estimates with our inductive hypothesis to prove bounds on higher derivatives. We now state the Lemma for these main estimates.

**Lemma 7.2 (Main Estimates).** For \( u \), a solution of (6.1), sufficiently smooth and with sufficient decay at infinity,

\[
\sup_{\theta < r < T} \sum_{|\alpha| < \beta} \left( \int_0^T \xi_\beta (\partial^\alpha u)^2 \, dt + \int_0^T \eta_\beta (\partial^\alpha u)^2 \, dt \right) \leq C \tag{7.2}
\]

for \( 9 \leq \beta \leq L + 8 \), where \( L \) depends on \( R \), \( \xi_\beta \in Y_{R, R, \beta - 1} \), \( \eta_\beta \in Y_{R, R, \beta - 1} \), where \( C \) depends only on the norms of \( u \) in

\[
L^\alpha (H^\gamma (Y_{R, R, -1})) \cap L^2 (H^{\gamma - 1} (Y_{R, R, -1})) \tag{7.3}
\]

for \( 8 \leq \gamma \leq \beta - 1 \) and on the norm of \( u \) in \( L^\alpha (H^\gamma (Y_{0, R, 0})) \).

The proof is similar to Lemma 3.2 in Section 3. For each \( 9 \leq \beta \leq 8 + L \), where \( L \) depends on the size of \( R \), we take \( \alpha \) derivatives of (6.1) where \( |\alpha| = \beta \). We then multiply our differentiated equation by \( 2 \xi_\beta (\partial^\alpha u) \) where

\[
\xi_\beta \equiv \int_{-\infty}^{x + sy} \eta_\beta (z, t) \, dz \quad \text{for} \quad \eta_\beta \in Y_{R, R, \beta - 1} \tag{7.4}
\]

and integrate over \( \mathbb{R}^2 \). Upon doing so, we get the following identity,

\[
\sum_{|\alpha| = \beta} \int \xi (\partial^\alpha u)^2 + \{ 3 \partial_x [f_{xxx}, \xi] - 2 \alpha_1 \xi \partial_x [f_{xxx}] \\
+ \partial_y [f_{yyy}, \xi] - 2 \alpha_2 \xi \partial_y [f_{yyy}] \} (\partial^\alpha u)^2 \\
+ \{ 2 \partial_x [f_{xxx}, \xi] - 2 \alpha_1 \xi \partial_x [f_{xxx}] + 2 \partial_y [f_{xxx}, \xi] \\
- 2 \alpha_2 \xi \partial_y [f_{xxx}] \} (\partial^\alpha u_x) (\partial^\alpha u_y) \\
+ \{ \partial_x [f_{xxx}, \xi] - 2 \alpha_1 \xi \partial_x [f_{xxx}] + 3 \partial_y [f_{xxx}, \xi] \\
- 2 \alpha_2 \xi \partial_y [f_{xxx}] \} (\partial^\alpha u_y)^2 \\
- 2 \alpha_1 \xi \partial_x [f_{xxx}] (\partial^\alpha u_y) (\partial^{(s_1, s_2 - 1)} u_{xy}) \\
- 2 \alpha_2 \xi \partial_y [f_{xxx}] (\partial^\alpha u_x) (\partial^{(s_1, s_2 - 1)} u_{xx}) \}
\]
\[-\int 2\xi_1 \partial_x [f_{xxx}] (\partial^{(s_1-1, s_2)}u_{xxx}) (\partial^{(s_1-1, s_2)}u_{xxx})
\]
\[-\int 2\xi \{f_{ux} (\partial^s u_x)^2 + f_{ux} (\partial^s u_x)(\partial^s u_x) + f_{ux} (\partial^s u_x)^2\}
\]
\[+ \int \Theta_1 (\partial^s u)^2 + \int \Theta_2 + \int 2\xi (\partial^s u) S_1 + \int 2\xi (\partial^s u) S_2 = 0,
\]
\[(7.5)\]

where \(\xi = \xi_2,\)

\[\Theta_1 = \alpha_1 \partial^2_x [\xi (\partial^{(s_1-1, s_2)}u_{xxx})] + \alpha_2 \partial_{xy} [\xi (\partial^{(s_1-1, s_2)}u_{xxx})]
\]
\[+ \alpha_2 \partial_x \xi \partial_y [f_{xxx}] + \alpha_1 \partial^2_x [\xi \partial_y [f_{xxx}]]
\]
\[+ \alpha_2 \partial_{xy} \xi \partial_y [f_{xxx}] + \alpha_2 \partial_x \xi \partial_y [f_{xxx}] - \partial_x \xi \partial_y [f_{xxx}]
\]
\[+ \alpha_2 \partial_{xy} \xi \partial_y [f_{xxx}] + \partial_x \xi \partial_y [f_{xxx}]
\]
\[(7.6)\]

\[\Theta_2 = -\alpha_2 \partial_{xy} \xi (\partial^{(s_1-1, s_2)}u_{xxx}) (\partial^{(s_1-1, s_2)}u_{xxx})
\]
\[-\alpha_1 \partial_{xy} \xi (\partial^{(s_1-1, s_2)}u_{xxx}) (\partial^{(s_1-1, s_2)}u_{xxx})
\]
\[+ 2\alpha_2 \partial_x \xi \partial_y [f_{xxx}] (\partial^{(s_1-1, s_2)}u_{xxx}) (\partial^{(s_1-1, s_2)}u_{xxx})
\]
\[+ 2\alpha_1 \partial_{xy} \xi \partial_y [f_{xxx}] (\partial^{(s_1-1, s_2)}u_{xxx}) (\partial^{(s_1-1, s_2)}u_{xxx})
\]
\[(7.7)\]

the terms in \(S_1\) are of order \(\beta + 1\), namely

\[S_1 = \left(\frac{\alpha_1}{2}\right) \partial^2_x [f_{xxx}] (\partial^{(s_1-2, s_2)}u_{xxx}) + \alpha_2 \partial_{xy} [f_{xxx}] (\partial^{(s_1-2, s_2)}u_{xxx})
\]
\[+ \left(\frac{\alpha_2}{2}\right) \partial^2_x [f_{xxx}] (\partial^{(s_1-2, s_2)}u_{xxx}) + \ldots + \left(\frac{\alpha_1}{2}\right) \partial^2_x [f_{xxx}] (\partial^{(s_1-2, s_2)}u_{xxx})
\]
\[+ \alpha_2 \partial_{xy} [f_{xxx}] (\partial^{(s_1-2, s_2)}u_{xxx}) + \alpha_2 \partial_{xy} [f_{xxx}] (\partial^{(s_1-2, s_2)}u_{xxx})
\]
\[+ \left\{\alpha_1 \partial_x [f_{xxx}] (\partial^{(s_1-1, s_2)}u_{xxx}) + \alpha_2 \partial_{xy} [f_{xxx}] (\partial^{(s_1-1, s_2)}u_{xxx})\right\}
\]
\[+ f_{ux} (\partial^s u_x) + f_{ux} (\partial^s u_x),
\]
\[(7.8)\]

and the terms in \(\int \xi (\partial^s u) S_2\) are given by Lemma 7.3 below. By assumption (B2), we know the seventh integral in (7.5) can be chosen to be non-negative. Therefore, by the choice of \(\xi\) and assumption (B1), for \(9 \leq \beta \leq 8 + L\)
where \( L \geq 1 \) depends on the size of \( R \), we can find constants \( K_1, K_2 > 0 \) such that

\[
\sum_{|\alpha| = \beta} \int_{\mathbb{R}} \xi(\cdot, t)(\partial^\alpha u)^2 + K_1 \int_0^t \int_{\mathbb{R}} \eta(\partial^\beta u_x)^2 + K_2 \int_0^t \int_{\mathbb{R}} \eta(\partial^\beta u_y)^2 \leq \int \zeta(\cdot, 0)(\partial^\gamma \phi)^2
\]

\[
+ \left| \int_0^t \int \Theta_1(\partial^\alpha u)^2 \right| + \left| \int_0^t \int \Theta_2 \right| + \left| \int_0^t \int 2 \zeta(\partial^\gamma u) S_1 \right| + \left| \int_0^t \int 2 \zeta(\partial^\gamma u) S_2 \right|. \tag{7.9}
\]

Notice that the size of \( L \) (the number of times the inductive argument can be repeated) is dependent on being able to find constants \( K_1, K_2 > 0 \) satisfying (7.9) above. By assumption (B1), for each \( a \) there exists a \( S > 0 \) such that for \( t \in \mathbb{R} \), (7.5) implies there exist constants \( K_1, K_2 > 0 \) satisfying (7.9). Note that such a choice of \( S \) may depend on \( a \). Therefore, the size of \( L \) above may depend on the initial choice of \( R \).

Our goal is to bound terms on the right-hand side of (7.9) in terms of (7.3) and the norm of \( u \in H^\infty(\mathbb{R}^2) \). Before proving this, we state the form of all terms in \( \int_0^t \int 2 \zeta(\partial^\gamma u) S_2 \).

**Lemma 7.3 (Form of remainder terms).** All terms in the integrand of

\[
\int_0^t \int 2 \zeta(\partial^\gamma u) S_2
\]

are of the form

\[
\zeta(\partial^\alpha f) \cdot (\partial^\nu u) \cdot \cdots \cdot (\partial^\nu u)(\partial^\gamma u), \tag{7.10}
\]

where

\[
D^\gamma f \equiv \partial^\gamma_{\alpha_1} \partial^\gamma_{\alpha_2} \partial^\gamma_{\alpha_3} \partial^\gamma_{\alpha_4} \partial^\gamma_{\alpha_5} f
\]

\[
\partial^\gamma_{\alpha_1} \equiv \partial_{\alpha_1} \partial_{\alpha_2} \partial_{\alpha_3} \partial_{\alpha_4} \partial_{\alpha_5},
\]

\[
\partial^\gamma_{\alpha_1} \equiv \partial^{\gamma_{\alpha_1}}_{\alpha_1} \partial^{\gamma_{\alpha_2}}_{\alpha_2} \partial^{\gamma_{\alpha_3}}_{\alpha_3} \partial^{\gamma_{\alpha_4}}_{\alpha_4} \partial^{\gamma_{\alpha_5}}_{\alpha_5},
\]

and

\[
|\alpha_1| + \cdots + |\alpha_5| + |\gamma| + |\mu_1| + |\mu_2| = 2|\alpha| + 3|\beta| + 2|\gamma| + |\nu|
\]

(7.11)

\[
|\nu| + |\nu| + |\nu| + |\alpha| + 8.
\]

(7.12)

(7.13)

**Proof.** Follows as in the proof of Lemma 3.3 in Section 3.

We now give the proof of Lemma 7.2.
Proof of Lemma 7.2. We must show that the terms on the right-hand side of (7.9) are all bounded by a constant depending only on (7.3) and the norm of \( u \in L^\infty(H^s(Y_{0,R,0})) \), thus proving our lemma. This proof is similar to the proof of Lemma 3.2 for the semilinear case. Consequently, we will omit some of the details.

First, we notice that \( \xi \equiv \xi_{\beta} \in Y_{R,R,\beta-s} \). Therefore, \( \xi(\cdot, 0) = 0 \). Second, as in the proof of Lemma 3.2, it can easily be shown that
\[
\left| \int_0^t \int \Theta_1(\partial^nu)^2 \right| + \left| \int_0^t \Theta_2 \right| \leq C,
\]
where \( C \) depends at most on (7.3) and \( \|u\|_{L^\infty(L^\infty(\mathbb{R}^3))} \) because of assumption (B3) on \( f \) and the fact that \( |\partial^\alpha \xi| \leq C \xi \). The terms in \( \int_0^t \int \xi(\partial^nu) S_1 \) can also be handled as in the proof of Lemma 3.2 by integrating by parts once.

Therefore, we only need to consider the remainder terms in \( \int_0^t \int 2\xi(\partial^nu) S_2 \). It remains to show that all remainder terms of this form are bounded by constants depending only on (7.3) and the norm of \( u \in L^\infty(H^s(Y_{0,R,0})) \). By Lemma 7.3, each of these terms is of the form
\[
\int_0^t \xi_{\beta}(D^s f)(\partial^nu) \cdots (\partial^nu) u,
\] (7.14)
where \( \xi_{\beta} \in Y_{R,R,\beta-s} \). We use a similar technique as in the proof of Lemma 3.2, namely we break up terms of the form (7.14), being sure to provide each term with an appropriate amount of the weight function. The main difference in this proof is the use of the exponential weight function. In particular, for the proof of Lemma 3.2 for each \( \beta \geq 7 \) we bound (3.6) by a constant depending on (3.7) and \( \|u\|_{L^\infty(H^s(Y_{0,R,0}))} \). Notice that for \( rx + sy \leq -1 \), these weighted norms depend on a weight function \( e^{R(rx+sy)} \) for \( R > 0 \) arbitrary. Here we need to show that for \( \beta \geq 9 \), every term of the form (7.14) is bounded by a constant depending on (7.3) and \( \|u\|_{L^\infty(H^s(Y_{0,R,0}))} \). In this case, for \( rx + sy \leq -1 \), the weighted norms in (7.3) depend on \( e^{R(rx+sy)} \) for \( R > 0 \) fixed. Consequently, for terms \( \partial^nu \) in (7.14) where \( |n| \geq 8 \), we will use the weighted interpolation estimate, proven in the Appendix, of the form
\[
\|\partial^nu\|_{L^\infty(T_{\xi_1,T_2,l_3})} \lesssim \|u\|^\theta_{H^s(\mathbb{R}^3)} \|u\|_{H^s(\mathbb{R}^3)}^{1-\theta},
\] (7.15)
where
\[
|n| - \frac{2}{q} = (8-1) \theta + (|n| - 1)(1-\theta)
\]
and

\[
\frac{l_i}{q} = \frac{R}{2} (1 - \theta) \\
\frac{l_i}{q} = \frac{m_i}{2} \theta + \frac{n_i}{2} (1 - \theta), \quad i = 2, 3.
\]

First we consider the case where \(|v_i| \leq \beta - 1\) for \(1 \leq i \leq p\).

**The Case** \(|v_i| \leq \beta - 1\) for \(1 \leq i \leq p\). Let \(q\) be the largest integer such that

\(|v_i| \leq 7 \quad \text{for} \quad 1 \leq i \leq q\).

For these terms, we will use the fact that

\[
\left( \int (\partial^\nu u)^m \right)^{\frac{1}{p}} \leq C \|u\|_{W^\nu(R_\theta,x)},
\]

for all \(m\), such that \(2 \leq m, < + \infty\). For now we assume \(p \geq g + 1\).

**The Subcase** \(p \geq q + 1\). Therefore, by the assumption that \(|v_i| \leq \beta - 1\) for \(1 \leq i \leq p\),

\[
8 \leq |v_i| \leq \beta - 1 \quad \text{for} \quad q + 1 \leq i \leq p.
\]

For these terms, we use the weighted interpolation estimate (7.15) above.

With these ideas in mind, we now divide a term of the form (7.14). For ease of notation, we define

\[z \equiv rx + sy.\]

Recall that \(\xi_\beta \in Y_{R_\theta,x,\beta-\frac{8}{5}}\). We divide the integral as

\[
\left| \int_0^T \int \xi_\beta (D^\nu f)(\partial^\nu u) \cdots (\partial^\nu u)(\partial^\nu u) \right|
\leq C \int_0^T \left| T^{\nu} \left( \int \xi_\beta (\partial^\nu u)^m \right)^{\frac{1}{m}} \cdots \left( \int \xi_\beta (\partial^\nu u)^m \right)^{\frac{1}{m}} \right|
\leq \sup_{0 \leq t \leq T} T^\nu \left( \int \xi_\beta (\partial^\nu u)^m \right)^{\frac{1}{m}} \cdots \left( \int \xi_\beta (\partial^\nu u)^m \right)^{\frac{1}{m}}
\cdot \left( \int \left( t^{\nu} e^{R_\theta} (\partial^\nu u)^2 \right)^{\frac{1}{2}} \right),
\]

(7.16)
where
\[ \tilde{\zeta}_i \in Y_{\beta, \frac{q+1}{2}, 0} \quad \text{for} \quad 1 \leq i \leq q \]
\[ \tilde{\zeta}_i \in Y_{\beta, \frac{q+1}{2}, k_i} \quad \text{for} \quad q + 1 \leq i \leq p \]
and \( m_i, J_i, k_i \) are defined as follows. If \( q \geq 1 \), meaning there exists at least one factor of \( \tilde{\alpha}_i u \) in (7.14) such that \( |\tilde{\alpha}_i| \leq 7 \), we let \( m_i \) be chosen sufficiently large. In particular, choosing \( \varepsilon \) such that \( 0 < \varepsilon < 1/(q - 1) \), we let
\[ m_i = \frac{1}{\varepsilon} \quad \text{for} \quad 1 \leq i \leq q \]  
(7.17)
and let
\[ m_i = \frac{2(p-q)}{1 - 2\varepsilon} \quad \text{for} \quad q + 1 \leq i \leq p. \]  
(7.18)
If \( |\tilde{\alpha}_i| \geq 8 \) for all \( i \), we let
\[ m_i = 2(p-q) \quad \text{for} \quad 1 \leq i \leq p. \]  
(7.19)
Under these choices for \( m_i \), we have
\[ \sum_{i=1}^{p} \frac{1}{m_i} + \frac{1}{2} = 1, \]
as desired. In addition, in the estimate above, we let the power of \( t \) be chosen such that
\[ k_i = m_i \left( \frac{\beta - 9}{2} \right) (1 - \theta_i) \]  
(7.20)
where
\[ \theta_i = \frac{\beta - 2 - |\tilde{\alpha}_i| + 2/m_i}{\beta - 9} \quad \text{for} \quad \beta \geq 10, \]  
(7.21)
\[ \theta_i = 0 \quad \text{for} \quad \beta = 9. \]  
(7.22)
With these choices, we define \( J_i \) as follows. Let
\[ J_i = m_i \frac{R}{2} (1 - \theta_i). \]
We claim that each term on the right-hand side of (7.16) is bounded by a constant depending only on (7.3) and the norm of $u$ in $L^\infty(\mathcal{H}^8(Y_0,R,0))$. In addition, we claim that $M, Q \geq 0$ and that $Q_2 \leq 0$ and therefore the terms $T^M, \sup_{n \geq 1} e^{Q_n}t$ and $\sup_{n \leq -1} e^{Q_2}t$ are all bounded.

First, we will show that each term on the right-hand side of (7.16) is bounded by a constant depending only on (7.3) and the norm of $u$ in $L^\infty(\mathcal{H}^8(Y_0,R,0))$.

For $1 \leq i \leq q$, by assumption $|n_i| \leq 7$. Therefore, we use the estimates

$$\sup_{0 \leq i \leq T} \left( \int t^n e^{\frac{\gamma}{n}} (\partial^n u)^m \right)^{\frac{1}{m}} \leq C \sup_{0 \leq i \leq T} \left( \int (1 + e^{\frac{\gamma}{n}}) (\partial^n u)^m \right)^{\frac{1}{m}}$$

$$\leq C \sup_{0 \leq i \leq T} \left( \int (1 + e^{\gamma}) (\partial^n u)^m \right)^{\frac{1}{m}}$$

$$\leq C \sup_{0 \leq i \leq T} \left. \left[ \frac{1}{\|u\| H^8(Y_0,R,0)} \right. \right]^{\frac{1}{m}}.$$

For $q+1 \leq i \leq p$, we use the weighted interpolation estimate (7.15) discussed above. For ease of notation, we define the following spaces. Let

$$A = \{(x, y): z \equiv rx + sy \geq 1\}$$

$$B = \{(x, y): z \equiv rx + sy \leq -1\}.$$

Therefore, we estimate

$$\sup_{0 \leq i \leq T} \left( \int t^n e^{\frac{\gamma}{n}} (\partial^n u)^m \right)^{\frac{1}{m}} = \left| \sup_{0 \leq i \leq T} \left( \int t^n e^{\frac{\gamma}{n}} (\partial^n u)^m \right)^{\frac{1}{m}} \right|^{1-\theta} \|u\|^{\frac{1}{\theta}} H^8(Y_0,R,0) \|u\|^{\frac{1-\theta}{\theta}} H^8(Y_0,R,0).$$

Similarly,

$$\sup_{0 \leq i \leq T} \left( \int t^n e^{\frac{\gamma}{n}} (\partial^n u)^m \right)^{\frac{1}{m}} \leq \sup_{0 \leq i \leq T} \left. \left[ \frac{1}{\|u\| H^8(Y_0,R,0)} \right. \right]^{\frac{1}{m}} \left. \left[ \frac{1}{\|u\| H^8(Y_0,R,0)} \right. \right]^{\frac{1-\theta}{\theta}} H^8(Y_0,R,0).$$

where each of the terms above are bounded by (7.3) and $\|u\|_{L^\infty(\mathcal{H}^8(Y_0,R,0))}$.

Last, we have

$$\int_0^T \left( \int t^{\frac{\gamma}{n}} e^{\frac{\gamma}{n}} (\partial^n u)^2 \right)^{\frac{1}{m}} \leq C \int_0^T \left( \int t^{\frac{\gamma}{n}} e^{\frac{\gamma}{n}} (\partial^n u)^2 \right)^{\frac{1}{m}},$$

which is bounded by (7.3).
We have shown that by dividing a term of the form (7.14) in this way, the resulting terms are all bounded by the norms stated in Lemma 7.2. Next we need to show that $M$, $Q_2 \geq 0$ and $Q_1 \leq 0$ so that $T^M$ and $(\sup \, e^{\delta t \cdot z} + \sup \, e^{\delta t \cdot z})$ are all bounded. First, we consider $M$. For $M$, we have

\[
M = (\beta - 8) - \sum_{j=q+1}^{p} \frac{k_j}{m_j} - \frac{(\beta - 9)}{2}.
\]

Therefore,

\[
2M = (2\beta - 16) - (\beta - 9) \sum_{j=q+1}^{p} \left( 1 - \left( \frac{\beta - 2 - |v_j| + \frac{2}{m_j}}{\beta - 9} \right) \right) - (\beta - 9) = (\beta - 7) - (\beta - 9) \sum_{j=q+1}^{p} \left( \frac{|v_j| - 7 - \frac{2}{m_j}}{\beta - 9} \right)
\]

\[
\geq \beta - 7 - \sum_{j=q+1}^{p} |v_j| + 7(p-q) 
\]

\[
\geq \left\{ 4q_3 + 3q_2 + 2q_1 + q_0 + \sum_{j=q+1}^{p} |v_j| - 3p_3 - 2p_2 - p_1 \right\}
\]

\[
-7 - \sum_{j=q+1}^{p} |v_j| + 7(p-q) \geq p + 3(p-q) - 7 \geq p - 4 \geq 0
\]

for $p \geq 4$, or $p \geq 2$ if $p \geq q + 2$. We will consider the case $p \leq 3$, $p = q + 1$ later.

Now we will consider $Q_1$. In particular, we have

\[
Q_1 = R - \frac{R}{2} p - \frac{R}{2} \leq 0
\]

as long as $p \geq 1$, as desired.
Now we will show that $Q_2 \geq 0$, and consequently $\sup_B e^{Q_2(rx+sy)}$ is bounded. In particular,

$$Q_2 = R - \sum_{j=q+1}^p \frac{J_j}{m_j}$$

$$= R - \sum_{j=q+1}^p \left(1 - \theta_j\right)$$

$$= R - \sum_{j=q+1}^p \left(\frac{\beta - 2 - |v_j| + \frac{2}{m_j}}{\beta - 9}\right).$$

Therefore,

$$\frac{2}{R} Q_2 (\beta - 9) = (\beta - 9) - \sum_{j=q+1}^p \left(|v_j| - 7 - \frac{2}{m_j}\right)$$

$$\geq 4q_1 + 3q_2 + 2q_1 + q_0 + \sum_{j=q+1}^p |v_j| - 3p_1 - 2p_2$$

$$- p_1 - 9 - \sum_{j=q+1}^p |v_j| + 7(p - q)$$

$$\geq p + 3(p - q) - 9 \geq p - 6 > 0$$

for $p \geq 6$ by the assumption that $p \geq q + 1$. We also note that $Q_2 \geq 0$ for $p \geq 3$ if $p \geq q + 2$. So it remains to consider the cases $p = 1$ or $2$ or $3 \leq p \leq 5$ if $p = q + 1$.

The Subcase $p = 1, 2$ or $3 \leq p \leq 5$ if $p = q + 1$. Recall that in the region $A = \{(x, y); z \equiv rx+sy \geq 1\}$, we proved that $Q_1 \leq 0$ for all $p \geq 1$, and therefore, the term $e^{Q_1(rx+sy)}$ is bounded. Therefore, we only need to consider the region $B = \{(x, y); z \equiv rx+sy \leq -1\}$ here.

First we consider the case $p = q + 1$. The fact that $p = q + 1$ implies that $|v_i| \leq 7$ for $1 \leq i \leq p - 1$. For these cases we break up the integral as

$$\left| \int_0^T \int_B t^{\beta - 8} e^{R \sum_j (\partial^j f)(\partial^{\nu_1} u) \cdots (\partial^{\nu_{\nu_j}} u)(\partial^\nu y)} \right|$$

$$\leq \sup_{0 \leq t \leq T} T^M \sup_B e^{Q_2} C \left(\int_B (\partial^{\nu_1} u)^{m_1}\right)^{\frac{1}{\nu_1}} \cdots \left(\int_B (\partial^{\nu_{\nu_j}} u)^{m_{\nu_j}}\right)^{\frac{1}{\nu_{\nu_j}}}$$

$$\cdot \left(\int_B t^{\beta - 9} e^{R J} (\partial^\nu y)^{\nu} \right)^{\frac{1}{\nu}} \left(\int_B t^{\beta - 9} e^{R J} (\partial^\nu y)^{\nu} \right)^{\frac{1}{2}}.$$
If $p \geq 2$, choose $\epsilon$ such that $0 < \epsilon < 1/2q$. If $p = 1$, let $\epsilon = 0$. With these choices, we let

$$m_i = \frac{1}{\epsilon} \quad \text{for} \quad 1 \leq i \leq p-1$$

$$m_p = \frac{2(p-q)}{1-2q\epsilon}.$$ 

In addition, we let

$$k_p = m_p \left( \frac{\beta-9}{2} \right) \frac{(1-\theta_p)}{(\beta-1-|\nu_p|+m_p)}$$

$$\theta_p = \frac{\beta-8}{\beta-1-|\nu_p|+m_p}$$

$$R_p = m_p \frac{R}{2} \left( 1-\theta_p \right).$$

Now for $1 \leq i \leq p-1$, we have

$$\sup_{0 \leq t \leq T} \left( \int_B (\partial_t^nu)^{m_i} \right)^{\frac{1}{m_i}} \leq \sup_{0 \leq t \leq T} \|\partial_t^nu\|_{H^1(B)}$$

$$\leq \sup_{0 \leq t \leq T} \|u\|_{H^s(\gamma_0,\gamma_0)},$$

while for the $\nu_p$ term, by our weighted interpolation estimate from the Appendix, we have

$$\int_0^T \left( \int_B t^{\gamma_p} e^{R\nu_p}(\partial_t^nu)^{m_p} \right)^{\frac{1}{m_p}} \leq \int_0^T \|u\|_{H^s(B)}^{\gamma_p} \|u\|^{1-\gamma_p}_{H^s(\gamma_0,\gamma_0,\gamma_0)}.$$ 

It remains to show for this case that $M \geq 0$ and $Q_2 \geq 0$. First,

$$M = (\beta-8) - \frac{(\beta-9)}{2} \left( 1-\theta_p \right) - \frac{(\beta-9)}{2}. $$
Therefore,

\[
2M = \beta - 7 - (\beta - 9) \left( \frac{\beta - 1 - |v_p| + \frac{2}{m_p}}{1 - \frac{2}{\beta - 8}} \right)
\]

\[
= \beta - 7 - \left( |v_p| - 7 \frac{2}{m_p} \right) + \left( \frac{|v_p| - 7 - \frac{2}{m_p}}{\beta - 8} \right)
\]

\[
= \beta - |v_p| + \frac{2}{m_p} + \left( \frac{|v_p| - 7 - \frac{2}{m_p}}{\beta - 8} \right) > 0,
\]

by the assumption that \( \beta \geq |v_p| \geq 8, \beta \geq 9, \) and \( \frac{2}{m_p} \leq 1. \)

Now for \( Q_2, \) we have

\[
Q_2 = R - \frac{R_p}{m_p} - \frac{R}{2}
\]

\[
= \frac{R}{2} - \frac{R}{2} (1 - \theta_p)
\]

\[
= \frac{R}{2} - \frac{R}{2} \left( \frac{|v_p| - 7 - \frac{2}{m_p}}{\beta - 8} \right).
\]

Therefore,

\[
\frac{2}{R} Q_2 (\beta - 8) = \beta - 8 - |v_p| + 7 + \frac{2}{m_p} > 0
\]

by the assumption that \( |v_p| \leq \beta - 1. \)

Now we need to consider the case when \( p = 2, \ p = q + 2. \) In this case, \( 8 \leq |v_1|, |v_2| \leq \beta - 1. \) Consequently, we break up the integral as

\[
\left| \int_{0}^{T} \int_{B} t^{\beta - 8} e^{R(\partial^{\gamma} u)(\partial^{\gamma} u) + (\partial^{\gamma} u)^{\gamma}} \right| \leq \sup_{0 \leq t \leq T} T^{M} e^{Q_2} \int_{0}^{T} \int_{B} t^{k} e^{\xi(\partial^{\gamma} u)^{\gamma}} \left( \int_{B} t^{k} e^{\xi(\partial^{\gamma} u)^{\gamma}} \right)^{\gamma}.
\]
where
\[ k_i = 2(\beta - 9)(1 - \theta_i) \]
\[ \theta_i = \frac{\beta - \frac{1}{2} - |v_i|}{\beta - 8} \]
\[ J_i = 2R(1 - \theta_i). \]

Now for \( i = 1, 2 \), we use the interpolation estimate,
\[ \left( \int t^{k_i} e^{\beta t} (\partial^n u)^m \right)^{\frac{1}{m}} \leq \|u\|_{H^8(B)}^{\theta_i} \|u\|_{H^8(B)(\mathbb{R}, \mathbb{R}, \beta - 9)}^{1 - \theta_i}. \]

Therefore, we have
\[ \left| \int_0^T \int_B t^{\beta - 8} e^{Rz} (\partial^n u)(\partial^n u)(\partial^n u) \right| \leq C \sup_{0 \leq t \leq T} T^M \sup_B e^{O(t)} \left( \int_0^T \|u\|_{H^8(B)}^{\theta_1 + \theta_2} \|u\|_{H^8(B)(\mathbb{R}, \mathbb{R}, \beta - 9)}^{1 - \theta_1 + 1 - \theta_2} \right)^{\frac{1}{2}} \]
\[ \cdot \left( \int_0^T \int_B t^{\beta - 8} e^{Rz} (\partial^n u)^2 \right)^{\frac{1}{2}}. \]

Notice that \( 4 - 2\theta_1 - 2\theta_2 \leq 2 \) because
\[ \theta_1 + \theta_2 = \frac{2\beta - 1 - |v_1| - |v_2|}{\beta - 8} \geq 1 \]
by (7.13), namely the fact that \( p + |v_{p-1}| + |v_p| \leq \beta + 8 \). Therefore,
\[ \left( \int_0^T \|u\|_{H^8(B)(\mathbb{R}, \mathbb{R}, \beta - 9)}^{4 - 2\theta_1 - 2\theta_2} \right)^{\frac{1}{2}} \leq \left( \int_0^T \|u\|_{H^8(B)(\mathbb{R}, \mathbb{R}, \beta - 9)}^2 \right)^{\frac{1}{2}}, \]
which is bounded by (7.3), as desired. Also notice that
\[ \left( \int_0^T \int_B t^{\beta - 8} e^{Rz} (\partial^n u)^2 \right)^{\frac{1}{2}} \leq C, \]
where $C$ depends only on (7.3). It remains to show that $M, Q \geq 0$. First,

$$M = \beta - 8 - \frac{k_1}{4} - \frac{k_2}{4} - \frac{\beta - 9}{2}$$

$$= \beta - 8 - \frac{\beta - 9}{2} (1 - \theta_1) - \frac{\beta - 9}{2} (1 - \theta_2) - \frac{\beta - 9}{2}.$$

Therefore,

$$2M = \beta - 7 - (\beta - 9)(2 - \theta_1 - \theta_2) \geq 2 > 0$$

by the fact that $\theta_1 + \theta_2 \geq 1$, proven above. Similarly,

$$Q = R - \frac{J_1}{4} - \frac{J_2}{4} - \frac{R}{2}$$

$$= \frac{R}{2} - \frac{R}{2} (1 - \theta_1) - \frac{R}{2} (1 - \theta_2) \geq 0,$$

by the fact that $\theta_1 + \theta_2 \geq 1$.

Thus far we have assumed $p \geq q + 1$. We now consider the case $p = q$.

The Subcase $p = q$. Therefore, $|v| \leq 7$ for $1 \leq i \leq p$. In the region

$$A = \{(x, y); \ z \equiv rx + sy \geq 1\},$$

we have

$$\left| \int_0^t \int_A e^{\beta - \frac{9}{2} (\partial_i^nu)} (\partial_i^nu) \cdot (\partial_i^nu) \right|$$

$$\leq C \sup_{0 \leq i \leq T} \int_A e^{Qz_i} \left( \int_A e^{\frac{9}{2} (\partial_i^nu)} \right)^{\frac{1}{2}} \cdot \left( \int_A e^{\frac{9}{2} (\partial_i^nu)} \right)^{\frac{1}{2}}.$$

In the region

$$B = \{(x, y); \ z \equiv rx + sy \leq -1\},$$

we have

$$\left| \int_0^t \int_B e^{\beta - \frac{9}{2} (\partial_i^nu)} (\partial_i^nu) \cdot (\partial_i^nu) \right|$$

$$\leq C \sup_{0 \leq i \leq T} \int_B e^{Qz_i} \left( \int_B (\partial_i^nu)^{m_i} \right)^{\frac{1}{2}} \cdot \left( \int_B (\partial_i^nu)^{m_i} \right)^{\frac{1}{2}}.$$
For 1 ≤ i ≤ p,
\[
\left( \int_A e^{\nu_i x} \nu_i^2 (\partial^i u)^m \right)^{\frac{1}{m}} + \left( \int_B (\partial^i u)^m \right)^{\frac{1}{m}} \leq C \| u \|_{L^\infty(H^2(\mathbb{R}, \mathbb{R}))}.
\]

It is straightforward to check that \( M_0, Q_1 \geq 0 \) and \( Q_2 \leq 0 \).

The Case \(|v| = \beta\). By (7.13), \( p + |v_{p-1}| \leq 8 \) and thus \( |v_{p-1}| \leq 6 \). We estimate as
\[
\left| \int_0^T \int (t^{m-1} e^{Rx} (D'f)(\partial^i u) \cdots (\partial^i u) (\partial^i u)) \right|
\leq \sup_{0 \leq t \leq T} T |\partial^i u|_{L^\infty} \cdots |\partial^i u|_{L^\infty} 
\left( \left. \left( \int_0^T \int t^{m-1} e^{Rx} (\partial^i u)^2 \right)^{\frac{1}{2}} \left( \int_0^T \int t^{m-1} e^{Rx} (\partial^i u)^2 \right)^{\frac{1}{2}} \right) \right)
\leq T \| u \|_{L^{p+1}} \left( \int_0^T \int t^{m-1} e^{Rx} (\partial^i u)^2 \right)^{\frac{1}{2}} \left( \int_0^T \int t^{m-1} e^{Rx} (\partial^i u)^2 \right)^{\frac{1}{2}},
\]

and these terms are bounded by (7.3) and the norm of \( u \) in \( L^\infty(H^2(Y_0, \mathbb{R}, 0)) \), as desired.

We now prove our main theorem on the gain of regularity for the fully nonlinear Eq. (6.1).

**Proof of Theorem 7.1.** The proof follows as in the proof of Theorem 3.1. Let \( t_0 \in [0, T) \) and let \( \{ \phi^{(n)}(\cdot) \} \) be a sequence of functions in \( C_c^\infty(\mathbb{R}^2) \) which converges to \( u(\cdot, t_0) \) strongly in \( H^2(Y_0, \mathbb{R}, 0) \). By Assumption 6.7 (or Theorem 8.2 if Eq. (6.1) satisfies an extra condition), there exists a unique solution \( u^{(n)} \) with initial data \( \phi^{(n)} \) for a time interval \([t_0, t_0 + \delta]\) where \( \delta > 0 \) does not depend on \( n \). In addition,
\[
u^{(n)}(x, t) \in L^\infty(H^2(Y_0, \mathbb{R}, 0)) \cap L^2(H^2(Y_0, \mathbb{R}, 0)) \quad (7.23)
\]
with a bound that depends only on the norm of \( \phi^{(n)} \in H^2(Y_0, \mathbb{R}, 0) \), and there exist non-uniform bounds
\[
\sup_{t \in [t_0, t_0 + \delta]} \sup_{(x, y)} (1 + e^{R(x+y)}) |\partial^4 u^{(n)}(x, y, t)| < +\infty \quad (7.24)
\]
for each $n$, $R$, and $\alpha$. Therefore, the \textit{a priori} estimates in Lemma 7.2 are justified for each $u^{(a)}$ in the interval $[t_0, t_0 + \delta]$. Starting our inductive argument at $\beta = 9$, we conclude

$$u^{(a)} \in L^\infty([t_0, t_0 + \delta]; H^{3+1/2}(Y_{R,R,1})) \cap L^2([t_0, t_0 + \delta]; H^{9+1}(Y_{R,R,1})) \quad (7.25)$$

for $0 \leq l \leq L$, where $L$ depends on $R$ and $f_{uuu}$, ..., $f_{uyy}$. Using a convergence argument, we conclude that $u$ lies in the same space. Since $\delta$ is fixed, this result is valid over the whole interval $[0, T]$. 

8. EXISTENCE AND UNIQUENESS—FULLY NONLINEAR EQUATION

In the previous section, we proved a gain in regularity theorem for fully nonlinear equations of the form (6.1) which satisfy Assumptions (B1)–(B4) and Assumption 6.1. Using a technique similar to the existence proof in Section 4 for the semilinear equation, we can prove this assumption for certain equations of type (6.1). In this section, we prove sufficient conditions for which Assumption 6.1 is satisfied. Specifically, we assume the following decay condition.

\textbf{Assumption 8.1.} For $(r, s)$ satisfying $\mathbb{N} \mathbb{P}(\omega_1, \omega_2) \cdot (r, s) < 0$ for all $(\omega_1, \omega_2) \neq (0, 0)$, we assume there exist constants $K, \epsilon > 0$, such that

$$|\partial_x a|, ..., |\partial_x d| \leq \frac{K}{|rx+sy|^{1+\epsilon}} \text{ for } rx+sy < -1$$

$$|\partial_y a|, ..., |\partial_y d| \leq \frac{K}{|rx+sy|^{1+\epsilon}} \text{ for } rx+sy < -1,$$

where $a, ..., d$ denote $f_{uuu}$, ..., $f_{uyy}$, respectively.

\textit{Remark.} Assumption 8.1 allows us to prove an existence theorem of the type described in Assumption 6.1 by the same methods as used in Section 4 for the semilinear case. In particular, this assumption is used to prove a differential inequality for (6.1) analagous to the differential inequality (4.8) for the semilinear Eq. (2.1). This differential inequality used in our existence proof for (6.1) is contained in Lemma 8.3. Assumption 8.1 is also used in our uniqueness lemma below.
With this assumption on the derivatives of \( f_{xxxx}, \ldots, f_{uyyy} \), we define a smooth weight function \( \xi_B \) for \( B \leq -2 \) as follows. Let \( \eta \in Y_{R,0} \). Let

\[
\begin{align*}
\xi_B & \equiv 1 + \frac{\psi(rx+sy)}{|rx+sy|^2} \int_{-\infty}^{rx+sy} \eta(z, t) \, dz \quad \text{for} \quad rx + sy \leq B, \\
\xi_B & \equiv 2 + \int_{-\infty}^{rx+sy} \eta(z, t) \, dz \quad \text{for} \quad rx + sy \geq B + 1,
\end{align*}
\]

(8.2)

where \( \psi \) is a smooth function such that \( \psi(z) \equiv 1 \) for \( z \leq B < -2 \) and \( \psi(z) = |z|^2 \) for \( z \geq B + 1 \). Note, in particular, that \( \xi_B \in Y_{0,R,0} \).

**Lemma 8.1 (Uniqueness).** Let \( 0 < T < \infty \). Assume Eq. (6.1) satisfies Assumptions (B1)–(B4) and Assumption 8.1. Then for \( \phi \in H^s(Y_{0,R,0}) \), \( R > 0 \) sufficiently large, there is at most one solution \( u \in L^\infty([0, T]; H^s(Y_{0,R,0})) \) of (6.1) with initial data \( \phi \).

**Proof.** Assume \( u, v \) are two solutions of (6.1) in \( L^\infty([0, T]; H^s(Y_{0,R,0})) \) with the same initial data. Therefore,

\[
\begin{align*}
\partial_t (u - v) & + \sum_{j=0}^9 \left\{ f(\ldots, \partial^{j+1}v, \partial^j u, \partial^{j-1} u, \ldots) \\
& - f(\ldots, \partial^{j+1}v, \partial^j v, \partial^{j-1} u, \ldots) \right\} = 0.
\end{align*}
\]

(8.3)

Proceeding as in the proof of Lemma 4.1, we may write (8.3) as

\[
\begin{align*}
\partial_t (u - v) & + \sum_{j=0}^9 h^{(j)}(\partial^j u - \partial^j v) = 0,
\end{align*}
\]

(8.4)

where the \( h^{(j)} \) are smooth functions of \( u_{xxxx}, v_{xxxx}, \ldots \). Now multiplying (8.4) by \( 2\xi_B(u-v) \) where \( \xi_B \) is defined by (8.2) for \( B \leq -2, |B| \) sufficiently large, and integrating over \((x, y) \in \mathbb{R}^2\), performing the necessary integration by parts, our equation becomes

\[
\begin{align*}
\partial_t & \int \xi_B(u-v)^2 + \int \left\{ 3\partial_x[\xi h^{(9)}] + \partial_x[\xi h^{(8)}] \right\}(u_x - v_x)^2 \\
& + \int \left\{ 2\partial_x[\xi h^{(8)}] + \partial_x[\xi h^{(7)}] \right\}(u_x - v_x)(u_y - v_y) \\
& + \int \left\{ 2\partial_x[\xi h^{(7)}] + \partial_x[\xi h^{(6)}] \right\}(u_y - v_y)^2 \\
& - 2 \int \xi(h^{(8)}(u_x - v_x)^2 + h^{(9)}(u_x - v_x) + h^{(10)}(u_x - v_y)^2 \\
& - 2 \xi h^{(8)}(u_x - v_x)^2 + h^{(9)}(u_x - v_x) + h^{(10)}(u_x - v_y)^2)
\end{align*}
\]
For this case, we apply Theorem 1. For this case, we prove a weighted existence theorem. Assuming (B1)–(B4) and Assumption 8.1, are satisfied. We show that for Eq. (6.1),

\[ u(y, 0) = f(y), \]

where \( f \) and Gronwall’s inequality as we did in Lemma 4.1, we conclude that \( u \equiv v \).

Next we prove our main existence theorem for the fully nonlinear Eq. (6.1), Assuming (B1)–(B4) and Assumption 8.1, are satisfied. We show that for \( u \in L^2([0, T]; H^N(Y_0, R, 0)) \cap L^2([0, T]; H^{N+1}(Y_0, R, 0)) \) where the time of existence \( T \) depends only on \( ||\phi||_{H^N(Y_0, R, 0)} \).

The proof will proceed as in Theorem 4.2 for the semilinear Eq. (2.1). The main difference is that in this case we prove a weighted existence theorem. For this case, we apply \( A^1 \) to (6.1), let \( u = Av \) where \( A \equiv (I - A^1)^{-1} \) and as a result, arrive at a linear approximation of the form,

\[
v^{(1)} = f_{u_{xxx}} A^1 v_{xxx} + f_{u_{xxy}} A^1 v_{xxy} + f_{u_{yy}} A^1 v_{yxy} + f_{u_{yyy}} A^1 v_{yyy} + \{6 \partial_x [f_{u_{xxx}}] (\partial_x^2 A v_{xxx}) + \cdots + 6 \partial_y [f_{u_{yyy}}] (\partial_y^2 A v_{yyy}) \}
\]

where \( f_{u_{xxx}} = f_{u_{yyyy}} (A v_{xxx}^{(n-1)}, ...) \), etc. We let the initial condition be given by \( v^{(0)}(x, y, 0) = \phi(x, y) - A^1 \phi(x, y) \) and the first approximation be given.
by $v^{(0)}(x, y, t) = \phi(x, y) - A^3 \phi(x, y)$. Assumption 8.1 is used in proving a differential estimate, analogous to (4.8) in Lemma 4.3, on the sequence of solutions $\{v^{(n)}\} \in H^N(Y_{0,R,0})$.

We proceed as in Theorem 4.2.

**Theorem 8.2 (Existence).** Assume Eq. (6.1) satisfies Assumptions (B1)–(B4) and Assumption 8.1. Let $N \geq 8$ and let $k_0 > 0$. Let $(r, s)$ satisfy $V_p(\alpha_1, \alpha_2) \cdot (r, s) < 0$ for all $(\alpha_1, \alpha_2) \neq (0, 0)$ and let $R > 0$ be chosen sufficiently large. Then there exists $0 < T < \infty$ depending only on $k_0$ such that for all $f \in H^N(Y_{0,R,0})$, such that $||f||_{H^N(Y_{0,R,0})} \leq k_0$, there exists a solution of (6.1) with $u \in L^\infty(H^N(Y_{0,R,0})) \cap L^2([0, T]; H^{N+1}(Y_{R,0}))$ with $u(x, y, 0) = \phi(x, y)$.

**Proof.** Again we drop the superscripts and let $Y_{J,R,k} = Y_{J,R,k}^r$. We prove this result for $N = 8$. The same argument as in Theorem 4.2 shows the result holds for $N \geq 9$, with the time of existence depending only on $||f||_{H^N(Y_{0,R,0})}$. Again, it suffices to prove this result for $f \in \cap_{k \geq 0} H^N(Y_{0,R,0})$.

By Lemma 8.5, for each $n$, the linear Eq. (8.5) has a unique solution for any time interval in which the coefficients are defined. We now want to show that this sequence of solutions $\{v^{(n)}\}$ is bounded in $L^2([0, T]; H^2(Y_{0,R,0}))$ for a time $T$ independent of $n$. By Lemma 8.3, we have

$$
\begin{align*}
\|v^{(0)}\|_{H^2(Y_{0,R,0})}^2 &+ \|v^{(0)}\|_{H^{N+1}(Y_{0,R,0})}^2 \\
&\leq C(1 + \|v^{(n-1)}\|_{H^2(Y_{0,R,0})}) \|v^{(0)}\|_{H^2(Y_{0,R,0})}^2 \\
&\quad + g(\|v^{(n-1)}\|_{H^2(Y_{0,R,0})}) \|v^{(0)}\|_{H^2(Y_{0,R,0})} + h(\|v^{(n-1)}\|_{H^2(Y_{0,R,0})}),
\end{align*}
$$

where $m = \max\{2, |x|\}$ for smooth functions $g, h \equiv g^{|x|, h^{|x|}}$. Let $|x| = 2$. Let $k_0 \geq \|\phi - A^3 \phi\|_{H^2(Y_{0,R,0})} \geq \|\phi\|_{H^2(Y_{0,R,0})}$. For each iterate, $\|v^{(0)}(\cdot, t)\|_{H^2(Y_{0,R,0})}$ is continuous in $t$ and $\|v^{(0)}(\cdot, 0)\|_{H^2(Y_{0,R,0})} \leq k_0$. Let $c_0 = k_0^2 + 1$ and let

$$
T_0^{(0)} = \sup \{t; \|v^{(0)}(\cdot, t)\|_{H^2(Y_{0,R,0})} \leq c_0 \text{ for } 0 \leq t \leq t, 0 \leq k \leq n\}. \quad (8.6)
$$

Therefore, for $t \in [0, T_0^{(0)}]$, we have

$$
\begin{align*}
\|v^{(0)}(\cdot, t)\|_{H^2(Y_{0,R,0})}^2 &+ \int_0^t \|v^{(0)}(\cdot, s)\|_{H^2(Y_{0,R,0})}^2 \\
&\leq \|v^{(0)}(\cdot, 0)\|_{H^2(Y_{0,R,0})}^2 + C \int_0^t \|v^{(n-1)}(\cdot, s)\|_{H^2(Y_{0,R,0})} \|v^{(0)}\|_{H^2(Y_{0,R,0})} \\
&\quad + g(\|v^{(n-1)}\|_{H^2(Y_{0,R,0})}) \|v^{(0)}(\cdot, s)\|_{H^2(Y_{0,R,0})} + h(\|v^{(n-1)}\|_{H^2(Y_{0,R,0})}) \\
&\leq k_0^2 + C(1 + c_0) c_0^2 t + g(c_0) c_0^2 t + h(c_0) t. \quad (8.7)
\end{align*}
$$
Choosing \( T \) such that

\[
C(1 + c_0) c_0^2 T + g(c_0) c_0^2 T + h(c_0) T \leq 1, \tag{8.8}
\]

we conclude that \( T_0^* \geq T \), and,

\[
\sup_{0 \leq t < T} \|v^{(n)}(\cdot, t)\|^2_{H^2(\gamma_0, \kappa_0)} + \int_0^T \|v^{(n)}\|^2_{H^2(\gamma_0, \kappa_0)} \leq c_0^2. \tag{8.9}
\]

We have shown that \( v^{(n)} \) is a bounded sequence in \( L^\infty([0, T]; H^2(Y_{0, R, 0})) \). Therefore, \( \xi^{1/2} v^{(n)} \in L^\infty([0, T]; H^2(\mathbb{R}^3)) \), where \( \xi \in Y_{0, R, 0} \). Consequently, there exists a weak* convergent subsequence, still denoted \( \{v^{(n)}\} \) such that \( \xi^{1/2} v^{(n)} \rightharpoonup v \) in \( L^\infty([0, T]; H^2(\mathbb{R}^3)) \). By (8.5), \( \xi^{1/2} v^{(n)} \), is a sum of terms, each of which is the product of a coefficient, bounded uniformly in \( n \), and a function in \( L^1([0, T]; H^{-1}(\mathbb{R}^3)) \). Therefore, the sequence, \( \xi^{1/2} v^{(n)} \) is bounded in \( L^1([0, T]; H^{-1}(\mathbb{R}^3)) \). By Aubin’s compactness theorem, there is a subsequence such that \( \xi^{1/2} v^{(n)} \to \xi^{1/2} v \) strongly in \( L^1([0, T]; H^1_{loc}(\mathbb{R}^3)) \).

Therefore, for a subsequence, \( \xi^{1/2} v^{(n)} \to \xi^{1/2} v \) a.e. in \( x, y, \) and \( t \). It follows that \( \xi^{1/2} f_{xxx} A^3 A^{xxx} v^{(n)} \to \xi^{1/2} f_{xxx} A^3 A^{xxx} v \) strongly in \( L^1([0, T]; L^1_{loc}(\mathbb{R}^3)) \) because \( f_{xxx} (A^{xxx}, \ldots) \to f_{xxx} (A^{xxx}, \ldots) \) strongly in \( L^1([0, T]; L^1_{loc}(\mathbb{R}^3)) \) and \( \xi^{1/2} A^3 A^{xxx} v^{(n)} \to \xi^{1/2} A^3 A^{xxx} v \) weakly in \( L^2([0, T]; H^{-1}(\mathbb{R}^3)) \) Similarly, all other terms converge to their correct limits. Therefore, \( \xi^{1/2} \partial_t v^{(n)} \to \xi^{1/2} \partial_t v \) in \( L^1([0, T]; L^1_{loc}(\mathbb{R}^3)) \) and \( \partial_t v + (I - A^1) f(A^{xxx}, \ldots) = 0 \). Applying \( A \) to both sides of (8.5), we find that Eq. (6.1) is satisfied by \( u = Av \). In particular, we have that \( v \in L^\infty(H^2(Y_{0, R, 0})) \cap L^2(H^3(Y_{0, R, 0})) \) and therefore, \( u \in L^\infty(H^2(Y_{0, R, 0})) \cap L^2(H^3(Y_{0, R, 0})) \).

For \( N \geq 9 \), the above argument can be extended as in the proof of Lemma 4.2.

We now prove the differential inequality used in Theorem 8.2. Here we also make use of Assumption 8.1 and the weight function \( \xi_\beta \).

**Lemma 8.4.** Let \( v^{(n)}, v^{(n-1)} \) be a pair of functions in \( C([0, \infty); H^N(\mathbb{R}^3)) \) for all \( j, N \) which satisfy (8.5). Denote \( v \equiv v^{(n)} \) and \( w \equiv v^{(n-1)} \). For every integer \( \beta \geq 0 \) and \( R > 0 \) chosen sufficiently large, there exist positive, non-decreasing functions \( g^{(n)} \) and \( h^{(n)} \) such that for all \( t \geq 0 \)

\[
\partial_t v - \hat{v} H^N(\gamma_0, \kappa_0) + g(H^N(\gamma_0, \kappa_0)) \|v\|^2_{H^N(\gamma_0, \kappa_0)} + h(H^N(\gamma_0, \kappa_0)) \|w\|^2_{H^N(\gamma_0, \kappa_0)},
\]

where \( m = \max \{2, |\alpha| \} \).
Proof. We begin by applying $\partial^a$ to (8.5) where $a = (a_1, a_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $|a| = \beta$. Therefore,

$$\partial^a v = \{ f_{\text{xxx}}(\partial^a A^\delta v_{xxx}) + \cdots + f_{\text{yyyy}}(\partial^a A^\delta v_{yyyy}) \}$$

$$+ (a_1 \partial_x [f_{\text{xxx}}](\partial^{(\epsilon_1-1, \epsilon_2)} A^\delta v_{xxx}) + a_2 \partial_y [f_{\text{xxx}}](\partial^{(\epsilon_1, \epsilon_2-1)} A^\delta v_{xxx}))$$

$$+ \cdots + (a_1 \partial_x [f_{\text{yyyy}}](\partial^{(\epsilon_1-1, \epsilon_2)} A^\delta v_{yyyy}) + a_2 \partial_y [f_{\text{yyyy}}](\partial^{(\epsilon_1, \epsilon_2-1)} A^\delta v_{yyyy}))$$

$$+ \{ 6 \partial_x [f_{\text{xxx}}](\partial^a \delta_x^s A^\delta v_{xxx}) + \cdots + 6 \partial_y [f_{\text{yyyy}}](\partial^a \delta_y^s A^\delta v_{yyyy}) \}$$

$$+ \{ f_{\text{xxx}}(\partial^a A^\delta v_{xxx}) + f_{\text{yyyy}}(\partial^a A^\delta v_{yyyy}) \}$$

$$+ \sum_{7 \leq i+j \leq |a|+7} h^{(i,j)} \delta^{(i,j)} A^\delta v + \sum_{|\bar{a}|+|\bar{b}|+6} q_i(\partial^a A_{\bar{a}}, \ldots)(\bar{b}^\delta A_{\bar{b}}) + p(\partial^a A_{\bar{a}}, \ldots)$$

$$\equiv I_1 + \cdots + I_7,$$

where $|a| = 6$, $|\bar{a}| = |a| + 5$, and the $h^{(i,j)}$ are smooth functions depending on $\partial^a A_{\bar{a}}$, $\ldots$, where $|\bar{a}| = |a| - i - j + 12$.

We now multiply (8.10) by $2^t B(\partial^a v)$, where $\xi_B$ is defined by (8.2) for $B \ll -2$, $|B|$ sufficiently large, and integrate over $(x, y) \in \mathbb{R}^2$. For ease of notation, we drop the subscript notation and let $\xi \equiv \xi_B$ in the following estimates. We begin by looking at the terms in $I_1$. The first term in $I_1$ is handled as

$$\int 2\xi(\partial^a v)(f_{\text{xxx}})(\partial^a A^\delta v_{xxx}) = \int 2\xi f_{\text{xxx}}(\partial^a A^\delta v_{xxx} - \partial^a A^\delta v)(\partial^a A^\delta v_{xxx})$$

$$\equiv I_1(a) + I_1(b).$$

For $I_1(a)$, we integrate by parts several times, showing

$$\int 2\xi f_{\text{xxx}}(\partial^a A^\delta v)(\partial^a A^\delta v_{xxx}) \leq C(1 + \|\tilde{w}\|_{H^2(w_0, k, 0)}) \|v\|_{H^2(w_0, k, 0)}.$$ 

While for $I_1(b)$, we have

$$-2 \int \xi f_{\text{xxx}}(\partial^a A^\delta v)(\partial^a A^\delta v_{xxx})$$

$$= \int \partial_x [\xi f_{\text{xxx}}](\partial^a A^\delta v)^2 - 3 \int \partial_x [\xi f_{\text{xxx}}](\partial^a A^\delta v_x)^2$$

$$\leq C(1 + \|\tilde{w}\|_{H^2(w_0, k, 0)}) \|v\|_{H^2(w_0, k, 0)}^2 - 3 \int \partial_x [\xi f_{\text{xxx}}](\partial^a A^\delta v_x)^2.$$
Similarly, for the other terms in $I_1$, we have

\[
\int 2\xi (\partial^* v) f_{ux}, (\partial^* A^1 A_{ux}) \\
\leq C \left( 1 + \|w\|_{H^2(w_{0, R, 0})} \right) \|v\|_{H^4(w_{0, R, 0})} - \int \partial^*_x [\xi f_{ux}] (\partial^* A^1 A_{vx})^2 \\
- 2 \int \partial^*_x [\xi f_{ux}] (\partial^* A^1 A_{vx})(\partial^* A^1 A_{vy}) \\
\int 2\xi (\partial^* v) f_{uxy}, (\partial^* A^1 A_{uxy}) \\
\leq C \left( 1 + \|w\|_{H^4(w_{0, R, 0})} \right) \|v\|_{H^4(w_{0, R, 0})} - \int \partial^*_y [\xi f_{uxy}] (\partial^* A^1 A_{vy})^2 \\
- 2 \int \partial^*_y [\xi f_{uxy}] (\partial^* A^1 A_{vy})(\partial^* A^1 A_{vy}) \\
\int 2\xi (\partial^* v) f_{uyy}, (\partial^* A^1 A_{uyy}) \\
\leq C \left( 1 + \|w\|_{H^4(w_{0, R, 0})} \right) \|v\|_{H^4(w_{0, R, 0})} - 3 \int \partial^*_y [\xi f_{uyy}] (\partial^* A^1 A_{vy})^2.
\]

Now we consider the terms in $I_2$. For example, the first term in $I_2$ is integrated by parts once,

\[
\int 2\xi (\partial^* v) \alpha_1 \partial^*_x [f_{uxy}] (\partial^{(\alpha_1 - 1, \alpha_1)} A^1 A_{uxy}) \\
= \int 2\alpha_1 \xi \partial^*_x [f_{uxy}] (\partial^* A v - \partial^* A^1 A v)(\partial^* A^1 A_{vx}) \\
\leq C \left( 1 + \|w\|_{H^4(w_{0, R, 0})} \right) \|v\|_{H^4(w_{0, R, 0})} \\
+ 2 \int \alpha_1 \xi \partial^*_x [f_{uxy}] (\partial^* A^1 A_{vx})^2.
\]

The other terms of type $I_2$ are handled the same way.
Now consider terms of type $I_3$. For example, by performing several integrations by parts, we arrive at the estimate

$$
\int 2\zeta(\partial^s v) \{6 \partial_s [f_{u,xxx}]\} (\partial^* \partial^6_x A v_{xxx})
$$

$$
= 12 \int \zeta \partial_s [f_{u,xxx}] (\partial^s A v - \partial^s A^3 A v) (\partial^* \partial^6_x A v_{xxx})
$$

\[ \leq C(1 + \|w\|_{H^2(\omega_{R,0})} \|v\|_{H^2(\omega_{R,0})}^2)
$$

\[ + 12 \int \zeta \partial_s [f_{u,xxx}] (\partial^s A^3 A v) (\partial^* \partial^6_x A v_{xxx})
$$

\[ \leq C(1 + \|w\|_{H^2(\omega_{R,0})} \|v\|_{H^2(\omega_{R,0})}^2)
$$

\[ + 12 \int \zeta \partial_s [f_{u,xxx}] ((\partial^s \partial^6_x A v_{xxx})^2 + 3(\partial^s \partial^6_x A v_{xxx})^2)
$$

\[ + 3(\partial^s \partial^4 \partial^3_x A v_{xxx})^2 + (\partial^s \partial^4 \partial^3_x A v_{xxx})^2).
$$

Similarly, for the other terms of type $I_3$.

The terms of type $I_4$ are handled as in the following example. In particular,

$$
\int 2\zeta(\partial^s v) f_{u,xxx} (\partial^* A^3 A v_{xxx})
$$

$$
= \int 2\zeta(\partial^s A v - \partial^s A^3 A v) f_{u,xxx} (\partial^* A^3 A v_{xxx})
$$

\[ \leq C(1 + \|w\|_{H^2(\omega_{R,0})} \|v\|_{H^2(\omega_{R,0})}^2) + 2 \int \zeta f_{u,xxx} (\partial^* A^3 A v)\cdot
$$

We now look at the terms in $I_5$. First, if $i + j = |\alpha| + 7$, then the $h^{(i,j)}$ depend at most on $\partial^k A v$ where $k = 5$. These terms can all be handled by integrating by parts as in the following example,

$$
\int 2\zeta(\partial^s v) \left( \frac{\partial_{\tilde{z}}}{2} \right) \partial^2_s [f_{u,xxx}] (\partial^{(\tilde{z} + 2,2)} A^3 A v_{xxx})
$$

$$
= C \int \zeta \partial^2_s [f_{u,xxx}] (\partial^s A v - \partial^s A^3 A v) (\partial^* A^3 A v_{xxx})
$$

\[ \leq C(1 + \|w\|_{H^2(\omega_{R,0})} \|v\|_{H^2(\omega_{R,0})}^2).
$$
Next, for terms of type \( I_5 \), such that \( 8 \leq i + j \leq \abs{x} + 6 \), we estimate as

\[
\int 2\xi(\partial^\alpha v) h^{(\alpha, \beta)}(\partial^{(\alpha, \beta)} A v) \leq C \| h^{(\alpha, \beta)} \|_{L^\infty} \left( \int \xi(\partial^\alpha v)^2 \right)^\frac{1}{2} \left( \int \xi(\partial^{(\alpha, \beta)} A v)^2 \right)^\frac{1}{2}
\]

\[
\leq g_1(\|w\|_{H^m(\gamma_0, \gamma_0)} \|v\|_{H^m(\gamma_0, \gamma_0)}),
\]

where \( g_1 \) is a smooth function, using the fact that \( 8 \leq i + j \leq \abs{x} + 6 \) and the \( h^{(\alpha, \beta)} \) depend at most on \( \partial^\alpha w \) where \( |k| = \abs{x} - i - j + 12 \). Finally, for a term in \( I_5 \), where \( i + j = 7 \), we use the estimate

\[
\int 2\xi(\partial^\alpha v) h^{(\alpha, \beta)}(\partial^{(\alpha, \beta)} A v) \leq C \left( \int \xi(\partial^\alpha v)^2 \right)^\frac{1}{2} \left( \int \xi(\partial^{(\alpha, \beta)} A v)^2 \right)^\frac{1}{2}
\]

\[
\leq C \| w \|_{H^m(\gamma_0, \gamma_0)} \| v \|_{H^m(\gamma_0, \gamma_0)}^2,
\]

where \( m = \max\{2, \abs{x}\} \), using the fact that \( H^1(\mathbb{R}^2) \subset L^4(\mathbb{R}^2) \) and the fact that \( i + j = 7 \) implies \( h^{(\alpha, \beta)} \) depends at most on \( \partial^\alpha w \) where \( |k| = \abs{x} + 5 \).

Now for terms in \( I_6 \),

\[
\int 2\xi(\partial^\alpha v) \sum_{|\beta| = \abs{x} + 6} q_{\beta}(\partial^\beta w, \ldots)(\partial^{\beta} A w) \leq g_2(\|w\|_{H^m}) \| v \|_{H^m(\gamma_0, \gamma_0)} \| w \|_{H^m(\gamma_0, \gamma_0)}
\]

\[
\leq g_2(\|w\|_{H^m(\gamma_0, \gamma_0)} \| v \|_{H^m(\gamma_0, \gamma_0)}^2 + C \| w \|_{H^m(\gamma_0, \gamma_0)}^2,
\]

where \( g_2 \) is a smooth function. Last, for \( I_7 \), we have

\[
\int 2\xi(\partial^\alpha v) p(\partial^\beta A w, \ldots) \leq h(\|w\|_{H^m}) \| v \|_{H^m}.
\]

Combining the above inequalities, we have

\[
\sum_{|\beta| = \beta} \beta \xi(\partial^\alpha v)^2 + \sum_{|\beta| = \beta} \{ 3 \partial_{\beta}(\xi f_{uxx}) + \partial_{\beta}^{(\alpha)} A^3 A v \}^2
\]

\[
+ \sum_{|\beta| = \beta} \{ 2 \partial_{\beta}(\xi f_{uxx}) + 2 \partial_{\beta}^{(\alpha)} A^3 A v \} (\partial^\alpha A A v)
\]

\[
+ \sum_{|\beta| = \beta} \{ 3 \partial_{\beta}(\xi f_{uxx}) + 3 \partial_{\beta}^{(\alpha)} A^3 A v \} \partial^\alpha A A v)^2
\]

\[
- \left\{ 2\alpha_2 \xi \partial_{\beta}(f_{uxx}) (\partial^\alpha A A v)^2 + \cdots + 2 \right\}
\]
\[
-\left\{12\int \xi \partial_x [f_{w,x}](\partial^x \partial_y A v)^2 + \cdots + 12\int \xi \partial_y [f_{w,y}](\partial^y A v)^2\right\}
-2\int \xi [f_{w,x} (\partial^x A v)^2 + f_{w,y} (\partial^x A v)(\partial^y A v) + f_{w,y} (\partial^y A v)^2]
\leq C(1 + \|w\|_{\mathcal{H}^m(w_0, R, 0)}) \|v\|_{\mathcal{H}^m(w_0, R, 0)}^2 + g(\|w\|_{\mathcal{H}^m(w_0, R, 0)}) \|v\|_{\mathcal{H}^m(w_0, R, 0)}^2 + h(\|w\|_{\mathcal{H}^m(w_0, R, 0)}),
\]

(8.11)

where \( m = \max\{2, |\alpha|\} \) and \( g, h \) are smooth, positive, nondecreasing functions.

First, we note that by Assumption (B2) on \( f_{u,xxx}, \ldots, f_{u,yyy} \), the last integral on the left-hand side of (8.11) is positive. Second, note that the 5th and 6th integrals in (8.11) depend on \( t \) but no derivatives of \( t \). Therefore, by Assumption (B1) on \( f_{u,xxx}, \ldots, f_{u,yyy} \), Assumption 8.1, and the choice of weight function \( \xi \) defined in (8.2), we conclude there exists a constant \( K \), such that

\[
\sum_{|\alpha| \neq 0} \partial_\alpha \int \xi (\partial^\alpha v)^2 + K \sum_{|\eta| = 7} \partial^\eta (\partial^\eta A v)^2
\leq C(1 + \|w\|_{\mathcal{H}^m(w_0, R, 0)}) \|v\|_{\mathcal{H}^m(w_0, R, 0)}^2 + g(\|w\|_{\mathcal{H}^m(w_0, R, 0)}) \|v\|_{\mathcal{H}^m(w_0, R, 0)}^2 + h(\|w\|_{\mathcal{H}^m(w_0, R, 0)}),
\]

(8.12)

where \( m = \max\{2, |\alpha|\} \), thus, proving our lemma.

We now prove an existence theorem for a linear equation of form (8.5).

**Lemma 8.5.** Given initial data \( \psi \in \bigcap_{N \geq 0} H^N(\mathbb{R}^2) \), there exists a unique solution of (8.5). The solution is defined in any time interval in which the coefficients are defined.

**Proof.** The linear equation which is to be solved at each iteration has the form

\[
\mathcal{L} v = d,
\]

(8.13)

where

\[
\mathcal{L} = \partial_t - b_0 A^3 A \partial_x^3 - b_1 A^3 A \partial_x^2 \partial_y - b_2 A^3 A \partial_x \partial_y^2
- b_3 A^3 A \partial_y^3 - \{6b_6 \partial_x^5 A \partial_y + \cdots + 6b_7 \partial_x^5 A \partial_y^3\}
- \{b_5 A^5 A \partial_x^2 + b_7 A^4 A \partial_x \partial_y + b_8 A^4 A \partial_y^2\}
- \{15b_5 \partial_x^7 A \partial_y + \cdots + 15b_6 \partial_x^7 A \partial_y^3\}
- \{6b_{11} \partial_x^7 A \partial_y^3 + \cdots + 6b_{12} \partial_x^7 A \partial_y^5\} - b_{13} A \partial_x - b_{14} A^3 A \partial_y,
\]
for smooth, bounded coefficients $b$ and $d$. Fix an arbitrary time $T > 0$ and a constant $M > 0$. Introduce the bilinear form

$$\langle g, h \rangle = \int_0^T \int e^{-Mt} \xi \zeta gh \, dx \, dy \, dt, \quad (8.14)$$

where $\xi = \xi_B$ is defined by (8.2) for $B \leq -2$ such that $|B|$ is sufficiently large and $R > 0$ sufficiently large, defined on $C_0^\infty(\mathbb{R}^2 \times [0, T])$, the set of smooth functions with compact support in $\mathbb{R}^2$, which vanish for $t = 0$. By our choice of $\xi$ and our estimates from Lemma 8.4 we know that $\langle \mathcal{M} v, v \rangle \geq \langle v, v \rangle$. Similarly, $\langle \mathcal{L}^* w, w \rangle \geq \langle w, w \rangle$ for $w \in D = \{w \in C_0^\infty : w(x, y, T) \equiv 0\}$. Therefore, $\langle \mathcal{L}^* w, \mathcal{L}^* v \rangle$ is an inner product on $D$. The proof is completed as in Lemma 4.4.

**Corollary 8.6.** Let $\phi \in H^N(Y_0, R, 0)$ for some $N \geq 8$ and let $\phi^{(n)}$ be a sequence converging to $\phi$ in $H^N(Y_0, R, 0)$. Let $u$ and $u^{(n)}$ be the corresponding unique solutions, given by Theorems 8.2 and 8.3, in $L^q(H^N(Y_0, R, 0))$ for a time $T$ depending only on $\sup_n \|\phi^{(n)}\|_{H^N(Y_0, R, 0)}$. Then $u^{(n)} \rightharpoonup u$ weak* in $L^q(H^N(Y_0, R, 0))$.

**Proof.** This proof follows as in the proof of Theorem 8.3 above, using the ideas of Corollary 4.5.

**APPENDIX A: TECHNICAL LEMMA**

In Lemma A.1 we prove a weighted interpolation estimate used in Section 8.

**Lemma A.1 (Weighted Interpolation Estimates).** For $8 \leq |\nu| \leq |\gamma|$, $2 \leq q < +\infty$, we have the estimate

$$\|\nabla^\nu u\|_{L^q(Y_{l_1,l_2,l_3}^{(1)})} \leq \|u\|_{H^\theta(Y_{m_1,m_2,m_3}^{(1)})}^\theta \|u\|_{H^\gamma(Y_{n_1,n_2,n_3}^{(1)})}^{1-\theta} \quad (A.1)$$

where

$$|\nu| - \frac{2}{q} = (8 - |\gamma|) \theta + |\nu| - 1,$$

and

$$\frac{l_i}{q} = \frac{m_i}{2} \theta + \frac{n_i}{2} (1 - \theta), \quad i = 1, 2, 3.$$
Proof. Dividing the powers of \( t \) on the left-hand side of (A.1) between the terms on the right-hand side is straightforward, so we concentrate on dividing the term \( e^{il(rx+sy)} \). (Assume, without loss of generality that \( l_1 = l_2 \).) Without loss of generality, we assume \( r = 1 \), \( s = 0 \). Choose a partition of unity \( q_j \), such that \( q_j \in C^\infty(\mathbb{R}^2) \), \( \text{supp } q_j = (2(j-1), 2(j+1)) \times \mathbb{R} \), \( \sum_j q_j = 1 \) in \( \mathbb{R}^2 \) with \( \text{supp } q_j \cap \text{supp } q_j' = \emptyset \) for \( j \neq j', j+1 \).

By Sobolev’s embedding theorem, we have
\[
\|\partial^\gamma u\|_{L^q(\mathbb{R}^2)} \leq \|u\|_{H^{l_1}(\mathbb{R}^2)} \|u\|_{H^{l_2}(\mathbb{R}^2)}^{1-\theta}, \tag{A.2}
\]
where
\[
|\nu| - \frac{2}{q} = (8 - |\gamma|) \theta + |\gamma| - 1. \tag{A.3}
\]

Therefore,
\[
\left( \int e^{it(\partial^\gamma(\chi_j u))^q} \right)^{\frac{2}{q}} \leq e^{2(j+1)} \left( \int |\sum_{|\nu| \leq R} (\partial^\nu(\chi_j u))^2 | \right)^{\frac{2}{q}} \left( \int |\sum_{|\nu| \leq R} (\partial^\nu(\chi_j u))^2 | \right)^{\frac{2}{q}} \left( \int e^{it(\partial^\gamma(\chi_j u))^q} \right)^{\frac{2}{q}},
\]
where
\[
\frac{l}{q} = \frac{m+n}{2} + \theta (1 - \theta).
\]

Therefore,
\[
\|\partial^\gamma u\|_{L^q(\mathbb{R}^2)} \leq \sum_j \int e^{it(\partial^\gamma(\chi_j u))} \leq C \sum_j \left( \int e^{it(\partial^\gamma(\chi_j u))} \right)^{\frac{2}{q}} \left( \int e^{it(\partial^\gamma(\chi_j u))} \right)^{\frac{2}{q}} \leq C \|u\|_{H^{l_1}(\mathbb{R}^2)} \|u\|_{H^{l_2}(\mathbb{R}^2)}^{1-\theta}, \tag{\text{351}}
\]
REFERENCES