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Dirichlet problems in polyhedral domains II: Approximation by FEM and BEM

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Abstract

The convergence of the classical finite element method (FEM) and boundary element method (BEM) is poor due to the edge and vertex singularities of the solution of the involved Dirichlet problem relative to an elliptic operator in a polyhedron. Using the global regularity results of Lubuma and Nicaise (1994), we analyse refined FEM and BEM with optimal rates of convergence.

Keywords: Singularities; Finite elements; Refined meshes; Rate of convergence

1. Introduction and preliminaries

This extended version of the note [17] is the constructive part of the authors' work [18] where regularity properties are studied for the variational solution of the Dirichlet problem relative to an elliptic operator on a Lipschitz bounded polyhedron Ω with boundary Γ. We are mainly interested in optimal rates of convergence in the approximation of the solution by both the finite element method (FEM) and the boundary element method (BEM). To keep this part relatively self-contained, let us say a few words about those results of Part I which we need here.

Let

$$L = \sum_{\substack{|p|=m \\ |q|=m}} (-1)^m a_{pq} D^{p+q} \tag{1.1}$$

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be a homogeneous strongly elliptic operator of order $2m$ with constant coefficients. An integer $k \geqslant m$ and a distribution $f \in H^{k-m}(\Omega)$ being fixed, we are concerned with $u \in \mathring{H}^m(\Omega)$, solution of the well-posed variational problem:

$$\sum_{\substack{|p|=m\\|q|=m}} \int_\Omega a_{pq} D^p u \cdot \overline{D^q v}\, dx = \int_\Omega f \cdot \bar{v}\, dx \quad \forall v \in \mathring{H}^m(\Omega). \tag{1.2}$$

(For $s \in \mathbb{R}$, the usual Sobolev spaces $H^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and semi-norm $|\cdot|_{s,\Omega}$ are defined for example in [12].)

In the light of [10], u presents vertex, edge and vertex-edge singularities which are described in Part I. To recall this result, we briefly mention the suitable notation. Fix S in the set $\mathscr{S}(\Omega)$ of vertices of Ω and consider the edges $A_{S,j}$, $1 \leqslant j \leqslant J_S$, adjacent to S. The polyhedron Ω coinciding near S with a cone C_S of section G_S on the unit sphere centred at S, with $(r_S, \theta_S, \varphi_S)$ the associated spherical coordinates. Analogously, near each $A_{S,j}$ defining a dihedron of interior measure $0 < \omega_{S,j} < 2\pi$, we introduce the spherical coordinates $(r_S, \theta_{S,j}, \varphi_{S,j})$ also centred at S, where $\theta_{S,j}$ represents the angular distance to the edge $A_{S,j}$ and $0 < \varphi_{S,j} < \omega_{S,j}$. Let $\Lambda_S(k)$ and $\Lambda_{S,j}(k)$ denote the respective finite sets of poles λ and μ of $\lambda \to \mathscr{L}_S^{-1}(\lambda)$ and $\mu \to \mathscr{L}_{S,j}^{-1}(\mu)$ such that $m - \frac{3}{2} < \operatorname{Re}\lambda \leqslant m + k - \frac{3}{2}$, $m - 1 < \operatorname{Re}\mu \leqslant m + k - 1$, the operators $\mathscr{L}_S(\lambda) : \mathring{H}^m(G_S) \to H^{-m}(G_S)$ and $\mathscr{L}_{S,j}(\mu) : \mathring{H}^m(0, \omega_{S,j}) \to H^{-m}(0, \omega_{S,j})$ being obtained from L in the usual way (see [10]). Finally, we shall use two types of cut-off functions $\chi_S \equiv \chi_S(r_S) \in \mathscr{D}(\overline{\mathbb{R}_+})$ and $\chi_{S,j} \equiv \chi_{S,j}(\theta_{S,j}) \in \mathscr{D}([0, 2\pi])$: χ_S (resp. $\chi_{S,j}$) with support concentrated near S (resp. $A_{S,j}$), is identically equal to 1 in a neighbourhood of S (resp. $A_{S,j}$). Moreover, the support of $\chi_{S,j}$ is chosen in such a way that the functions $\theta_{S,j}$ and $\sin\theta_{S,j}$ are equivalent on supp $\chi_{S,j}$($\theta_{S,j} \cong \sin\theta_{S,j}$ in symbol).

The result of [10] can be rephrased as below.

Theorem 1.1 (Dauge [10]). *Assume that u can be extended to the infinite cone C_S, $S \in \mathscr{S}(\Omega)$, in such a way that the extension still denoted by u has a compact support and satisfies*

$$u \in \mathring{H}^m(C_S) \quad \text{and} \quad Lu \in H^{k-m}(C_S). \tag{1.3}$$

Assume, in addition, that

$$\begin{cases} \mathscr{L}_S(\lambda) \ \text{and} \ \mathscr{L}_{S,j}(\mu) \ \text{are invertible for} \\ \operatorname{Re}\lambda = k + m - \frac{3}{2} \ \text{and} \ \operatorname{Re}\mu = k + m - 1. \end{cases} \tag{1.4}$$

Then, u admits the singular representation

$$u = u_0 + \chi_S \sum_{\lambda \in \Lambda_S(k)} u^{S,\lambda} + \sum_{j=1}^{J_S} \sum_{\mu \in \Lambda_{S,j}(k)} u^{S,j,\mu}, \tag{1.5}$$

where $u_0 \in H^{k+m}(C_S)$ is the regular part, $u^{S,\lambda}$ is the vertex singular part relative to S and λ, $u^{S,j,\mu}$ includes the vertex-edge and the edge singularities related to S, $A_{S,j}$ and μ.

By analogy with the three-dimensional smooth conical corners and the two-dimensional case, one would like the assumption (1.3) to hold in such a way that the local formula (1.5) leads to a global decomposition of the solution u of (1.2) into regular and singular parts. Moreover, one expects the decomposition to permit the restoration in the FEM of the optimal order of convergence in $O(h^k)$ that would hold if u has the regularity $u \in H^{k+m}(\Omega)$. But such a global decomposition is not easy to obtain. Even, when the decomposition is direct, as in the case $m = k = 1$ considered in [17], the structure of the edge [10, Theorem 16.9] and vertex-edge [10, Theorem 17.13] singularities is very complex (only the vertex singular part $u^{S,\lambda}$ is explicit: its first term behaves like $r_S^\lambda \theta_{S,j}^\mu$ for suitable $\mu \in \Lambda_{S,j}(k)$, cf. [10, pp. 145, 151, 152]). Therefore, the extension to $\mathbb{R}^3$ of classical techniques like the dual singular function method [3, 4, 15], the singular function method [23] and the mesh refinement method [2] is not automatic. In this one respect, let us quote a few references: [1] (mesh refinement for a particular class of edges), [19] (adapted mesh excluding data of class $L^2(\Omega)$) and [22] (singular function method with doubt about the expected error estimate).

To overcome the above difficulties, the following global regularity result is established in [18].

Theorem 1.2. *r and δ being the distances to the vertices and edges of Ω, respectively, we consider the weighted function*

$$\theta := \sum_{S\in\mathscr{S}(\Omega)} \chi_S \left[1 - \sum_{j=1}^{J_S} \chi_{S,j}(\theta_{S,j}) + \sum_{j=1}^{J_S} \theta_{S,j}\chi_{S,j}(\theta_{S,j})\right] + \left(1 - \sum_{S\in\mathscr{S}(\Omega)} \chi_S\right)\delta. \tag{1.6}$$

Let α and β be nonnegative real numbers such that

$$\begin{cases} \alpha = 0 & \text{if } \Lambda_S(k) = \emptyset \quad \forall S \in \mathscr{S}(\Omega), \\ \text{otherwise } \alpha \notin \mathbb{N}^*, & \alpha > k + m - \tfrac{3}{2} - \operatorname{Re}\lambda \quad \forall \lambda \in \Lambda_S(k); \end{cases} \tag{1.7a}$$

$$\begin{cases} \beta = 0 & \text{if } \Lambda_{S,j}(k) = \emptyset \quad \forall S \in \mathscr{S}(\Omega), \quad 1 \leqslant j \leqslant J_S, \\ \text{otherwise } \beta \notin \mathbb{N}^*, & \beta > k + m - 1 - \operatorname{Re}\mu \quad \forall \mu \in \Lambda_{S,j}(k). \end{cases} \tag{1.7b}$$

Then u belongs to the weighted Sobolev space $H^{k+m,\alpha,\beta}(\Omega)$ of functions v fulfilling

$$\| v \|^2_{k+m,\alpha,\beta,\Omega} := \| v \|^2_{m,\Omega} + | v |^2_{k+m,\alpha,\beta,\Omega} < +\infty, \tag{1.8}$$

where

$$| v |^2_{k+m,\alpha,\beta,\Omega} := \sum_{|\gamma| = k+m} \| r^\alpha \theta^\beta D^\gamma v \|^2_{0,\Omega}.$$

Furthermore, if $m = 1$, then the conormal derivative $\psi := \sum_{p,q=1}^3 \nu_p a_{pq} \partial_q u_{/\Gamma}$ of u is in the space $H^{k-1/2,\alpha,\beta}(\Gamma)$ of traces defined by

$$H^{k-1/2,\alpha,\beta}(\Gamma) := \{\varphi \in H^{-1/2}(\Gamma) \colon | \varphi |_{k-1/2,\alpha,\beta,F} < +\infty \ \forall \text{face } F \text{ of } \Gamma\}, \tag{1.9}$$

where, with the convention $\theta = 1$ if $\beta = 0$,

$$| \varphi |^2_{k-1/2,\alpha,\beta,F} := \sum_{|\gamma|=k-1} \left\{ \int_F r^{2\alpha-1}\theta^{2\beta-1} | D^\gamma \varphi |^2 \,\mathrm{d}x + \iint_{F\times F} \frac{| r^\alpha(x)\theta^\beta(x) D^\gamma \varphi(x) - r^\alpha(y)\theta^\beta(y) D^\gamma \varphi(y) |^2}{| x - y |^3} \,\mathrm{d}x\,\mathrm{d}y \right\}. \tag{1.10}$$

Observe that the spectral condition (1.4) which is difficult to survey is not stated a priori in Theorem 1.2. This theorem is indeed the key to the desired numerical treatment of problem (1.2). In fact, we use the related regularity to refine the mesh size in such a way that the optimal convergence in the H^m and lower norms is restored for the FEM (Section 3). Likewise, we develop (Section 4) a refined BEM with optimal rates of convergence for second-order operators. Nevertheless, the reduction of the boundary value problem (1.3) to a boundary integral equation is always possible, see for instance [9]. We have avoided this case here since the practical solution of such problems for $m \geqslant 2$ remains difficult. However, we start with the classical FEM (Section 2) to underline its poor convergence and also to collect the standard notation we need.

2. The classical FEM

In this section, we indicate how slow the convergence is of the classical FEM. This requires some notation which we give now, following those in [6] as well as their extensions in [4]. Thus, according to the latter authors, we fix a real parameter $t \in [0, k]$ which influences both the accuracy and the rate of convergence of the FEM. The domain Ω being nonsmooth, the optimal value $t = k$ is not reached. Nevertheless, due to the noninteger version of Theorem 1.1 (see [10, Theorem 7.13]), one obtains the regularity

$$u \in H^{m+t}(\Omega) \quad \text{if } t < \sigma_0, \tag{2.1}$$

where

$$\sigma_0 := \min\{\tfrac{1}{2} + \operatorname{Re}\lambda > m-1 \ \&\ \operatorname{Re}\mu > m-1\colon \lambda \ \&\ \mu \text{ poles of } \mathscr{L}_S^{-1}(\lambda) \ \&\ \mathscr{L}_{S,j}^{-1}(\mu),\ S \in \mathscr{S}(\Omega),\ 1 \leqslant j \leqslant J_S\} - m + 1. \tag{2.2}$$

The inclusions (2.1) and (2.10) below are extensions to three-dimensional vertex and/or edge singularities of the lower and maximal regularity results in [4, Remark 9.12] relative to angular points.

Let us also fix a family $(\tau_h)_{h>0}$ of triangulations of $\bar{\Omega}$ which consist of straight elements K and which satisfy the usual properties [6, p. 38]. The family is supposed to be regular; i.e. the ratios h_K/ρ_K between the exterior diameters h_K and the interior diameters ρ_K of elements $K \in \bigcup_{h>0}\tau_h$ are uniformly bounded from above, the maximal mesh $h := \max_{K\in\tau_h} h_K$ tending to zero.

With each $K \in \bigcup_h \tau_h$, we associate a finite element (K, P_K, Σ_K) with the four properties below:

(i) P_K is a vector space of finite dimension M;

(ii) $P_{k+m-1}(K) \subset P_K \subset H^{k+m}(K)$; here for $l \in \mathbb{N}$, $P_l(K)$ denotes the finite-dimensional space of polynomials of degree at most l on K;

(iii) Σ_K is a finite set of M linearly independent continuous linear forms on $H^{m+t}(K)$;

(iv) the local interpolation operator π_K acting on $H^{m+t}(K)$ and the global interpolation operator π_h defined on $H^{m+t}(\Omega)$ fulfil the compatibility conditions:

$$(\pi_h v)_{/K} = \pi_K(v_{/K}) \quad \forall v \in H^{m+t}(\Omega), \tag{2.3}$$

$$\pi_h v \in \mathring{H}^m(\Omega) \quad \forall v \in \mathring{H}^m(\Omega) \cap H^{m+t}(\Omega). \tag{2.4}$$

An additional requirement, guaranteeing the approximation property of the space P_K, is as follows. With each finite element (K, P_K, Σ_K), $K \in \bigcup \tau_h$ is associated on one hand with an affine-equivalent finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$. The former concept means that there exists an invertible affine mapping $F_K: \mathbb{R}^3 \to \mathbb{R}^3: \hat{x} \rightsquigarrow x := F_K(\hat{x}) = B_K\hat{x} + b_K$ such that $K = F_K(\hat{K})$ and $\pi_K v = \hat{\pi}(v \circ F_K) \circ F_K^{-1}$, for any $v \in H^{m+t}(K)$ (given an object 0, the symbol $\hat{0}$ indicates its correspondence under the above-mentioned affine mapping). On the other hand, the elements $(\hat{K}, \hat{P}, \hat{\Sigma})$ relative to all $K \in \bigcup_{h>0} \tau_h$ are chosen such that

$$\sup\{h_{\hat{K}}: K \in \bigcup \tau_h\} < +\infty, \quad \sup\{\rho_{\hat{K}}^{-1}: K \in \bigcup \tau_h\} < +\infty \tag{2.5a}$$

and there exists the same constant $c > 0$ satisfying, for $t_1 \in [t, k]$,

$$\|\hat{v} - \hat{\pi}\hat{v}\|_{m+t_1,\hat{K}} \leqslant c|\hat{v}|_{m+t_1,\hat{K}} \quad \forall \hat{v} \in H^{m+t_1}(\hat{K}). \tag{2.5b}$$

A similar estimate for weighted Sobolev spaces will be stated in (3.1b) below.

Remark 2.1. When $m > 1$, the continuity requirement (2.4) between adjacent elements is so "critical" that a nonconforming FEM may be recommended to relax it (cf. [6, Ch. 6]). Analogously, conditions (2.5a) and (2.5b) which are the key to a satisfactory interpolation theory in Sobolev spaces are stated a priori to simplify the exposition. In fact, these properties are classically met for a family whose all elements are affine-equivalent to a single reference finite element [6]. But, except when $m = 1$, such families are practically rare. Therefore, one has to consider more complex situations as, for example, composite elements [6], almost-affine [6] or compact-affine [4] families of finite elements.

We define the trial and test space V_h by

$$V_h := \{v_h \in \mathring{H}^m(\Omega): v_{h/K} \in P_K \ \forall K \in \tau_h\}. \tag{2.6}$$

The Lax–Milgram lemma extends to the space V_h and guarantees the existence of a unique solution of the discrete problem: find $u_h \in V_h$ such that

$$\sum_{\substack{|p|=m\\|q|=m}} \int_\Omega a_{pq} D^p u_h \cdot \overline{D^q v_h}\, \mathrm{d}x = \int_\Omega f \bar{v}_h \mathrm{d}x \quad \forall v_h \in V_h. \tag{2.7}$$

Likewise, given an integer $v \leqslant m$ and a function g in $H^{m-v}(\Omega)'$, the dual of $H^{m-v}(\Omega)$ for the extension of the duality of $L^2(\Omega)$, there exists a unique solution $\varphi_h^g \in V_h$ of

$$\sum_{\substack{|p|=m\\|q|=m}} \int_\Omega a_{pq} D^p v_h \cdot \overline{D^q \varphi_h^g}\, \mathrm{d}x = \langle g, v_h \rangle \quad \forall v_h \in V_h. \tag{2.8}$$

Problem (2.8) is the discrete counterpart of the well-posed variational adjoint problem: find $\varphi^g \in \mathring{H}^m(\Omega)$, solution of

$$\sum_{\substack{|p|=m\\|q|=m}} \int_\Omega a_{pq} D^p v \cdot \overline{D^q \varphi^g}\, \mathrm{d}x = \langle g, v \rangle \quad \forall v \in \mathring{H}^m(\Omega). \tag{2.9}$$

Since linear functionals on $H^{m-\nu}(\Omega)$ can be applied to elements of $\mathring{H}^{m-\nu}(\Omega)$ (written by abuse of notation $H^{m-\nu}(\Omega)' \subset H^{\nu-m}(\Omega)$), Problem (2.9) has, as in (2.1), the lower regularity

$$\varphi^g \in H^{m+\sigma}(\Omega) \quad \text{for } \sigma < \tau_0 \text{ and } \sigma \leqslant \nu, \tag{2.10}$$

where

$$\tau_0 := -\max\{\tfrac{1}{2} + \operatorname{Re}\lambda < m-1 \ \&\ \operatorname{Re}\mu < m-1\colon \lambda \ \&\ \mu \text{ poles of } \mathscr{L}_S^{-1}(\lambda) \ \&\ \mathscr{L}_{S,j}^{-1}(\mu), \\ S \in \mathscr{S}(\Omega),\ 1 \leqslant j \leqslant J_S\} + m - 1 \tag{2.11}$$

is the analogue of σ_0 in (2.2).

We can now specify the convergence of the classical FEM (2.7).

Theorem 2.2 *If $t < \sigma_0$, there holds the asymptotic error estimate*

$$\| u - u_h \|_{m,\Omega} \leqslant ch^t |u|_{m+t,\Omega}, \tag{2.12}$$

where c represents here and elsewhere various constants independent of h. Moreover, if ν is an integer and σ is a real number such that $0 < \nu \leqslant m$, $\sigma \leqslant \nu$ and $\sigma < \tau_0$, then the error in the lower-order Sobolev space $H^{m-\nu}(\Omega)$ is

$$\| u - u_h \|_{m-\nu,\Omega} \leqslant ch^{t+\sigma} |u|_{m+t,\Omega}. \tag{2.13}$$

Proof (sketch). The proof of Theorem 2.2 works as that of Theorem 10.2 in [4]. In fact, because of (2.5), the classical interpolation theory in Sobolev spaces is valid. Therefore, as in this reference, (2.12) is based on the regularity (2.1) and on the Céa lemma, while (2.13) follows from (2.10) and the Aubin–Nitsche lemma. □

3. An improved FEM

As mentioned previously both convergence results in Theorem 2.2 are slow in the sense that the classical optimal rates known for smooth solutions and corresponding to the specific value $t = k$ are not valid. It is our task here to restore these optimal results. To this end, we shall refine the meshsize h_K. Let us first state the analogue of Theorem 3.1.1 in [6] about interpolation theory in Sobolev spaces.

Theorem 3.1. *For $\alpha < k - \frac{1}{2}$ and $\beta < k - \frac{1}{2}$, the weighted Sobolev space $H^{k+m,\alpha,\beta}(\Omega)$ introduced in Theorem 1.2 is compactly embedded in $H^m(\Omega)$. Moreover, there exists a constant $c > 0$ such that, for $v \in H^{k+m,\alpha,\beta}(\Omega)$,*

$$\inf_{z \in P_{k+m-1}(\Omega)} \| v - z \|_{k+m,\alpha,\beta,\Omega} \leqslant c |v|_{k+m,\alpha,\beta,\Omega}. \tag{3.1a}$$

(*The norm and semi-norm occurring in* (3.1a) *are defined in* (1.8)–(1.9).)

Proof (sketch). The compact embedding announced in Theorem 3.1 being proved in Part 1 of this paper (Theorem 6.2), the inequality (3.1a) follows, as in [6, Theorem 3.1.1] and [13, Lemma 8.4.1.3], from a classical argument of contradiction. □

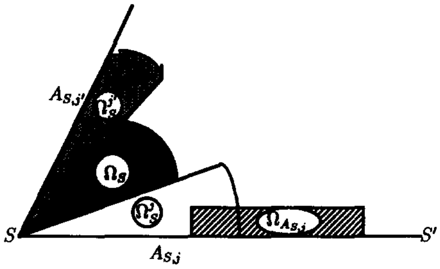

Fig. 1. Faces of the zones of influence.

Remark 3.2. Under the hypotheses of Theorem 3.1, it is worth pointing out that the constant $c > 0$ in (3.1a) depends upon the set Ω. Therefore, in the inequality

$$\| \hat{u} - \hat{\pi}\hat{u} \|_{m+k,\alpha,\beta,\hat{K}} \leqslant c |\hat{u}|_{m+k,\alpha,\beta,\hat{K}}, \tag{3.1b}$$

which is the consequence of (3.1a), we actually need, and assume a priori, following (2.5b) and the comments in Remark 2.1, that the constant c in (3.1b) is the same for all finite elements $(\hat{K}, \hat{P}, \hat{\Sigma})$. Notice that the interpolates, $\hat{\pi}\hat{u}$ and also $\pi_h u$ and $\pi_K u$, $K \in \bigcup_{h>0} \tau_h$ are meaningful because of the relations (2.1), (2.3) and (2.4). Notice also that the weighted functions of the respective spaces $H^{m+k,\alpha,\beta}(K)$ and $H^{m+k,\alpha,\beta}(\hat{K})$ are connected in a specific way described in the four cases below.

Another reason of concern is the choice of α and β. In fact, owing to Theorems 1.2 and 3.1, we fix henceforth α and β as follows:

$$\begin{cases} \alpha = 0 & \text{if } \Lambda_S(k) = \emptyset \quad \forall S \in \mathscr{S}(\Omega), \\ \text{otherwise } \alpha \notin \mathbb{N}^*, & k - \frac{1}{2} > \alpha > k + m - \frac{3}{2} - \operatorname{Re}\lambda \quad \forall \lambda \in \Lambda_S(k); \end{cases} \tag{3.2a}$$

$$\begin{cases} \beta = 0 & \text{if } \Lambda_{S,j}(k) = \emptyset \quad \forall S \in \mathscr{S}(\Omega), \quad 1 \leqslant j \leqslant J_S, \\ \text{otherwise } \beta \notin \mathbb{N}^*, & k - \frac{1}{2} > \beta > k + m - 1 - \operatorname{Re}\mu \quad \forall \mu \in \Lambda_{S,j}(k). \end{cases} \tag{3.2b}$$

The left-hand sides of the inequalities in (3.2) permit the applicability of (3.1) to u or $\hat{u}$; they are not restrictive due to Theorem 1 in [14] which states that $\operatorname{Re}\lambda > m - 1$ and $\operatorname{Re}\mu > m - \frac{1}{2}$.

Let us go back to the main purpose of this section. It is well known [6] that an asymptotic rate of convergence for the global error $\| u - u_h \|^2_{m,\Omega}$ results from a bound for each local error $|u - \pi_K u|^2_{m,K}$ associated with $K \in \tau_h$. On the other hand, the refinement of the mesh h_K is necessary only near the singularities of the domain. Therefore, we divide Ω into three zones under the influence of vertex-edge, vertex and edge singularities, respectively. More precisely, for any vertex $S \in \mathscr{S}(\Omega)$ and any edge A of Ω, we set (see Fig. 1):

$$\Omega_S^j := \operatorname{supp}\chi_S \cap \operatorname{supp}\chi_{S,j}, \quad 1 \leqslant j \leqslant J_S,$$

$$\Omega_S := B(S,\varepsilon) \setminus \textstyle\bigcup_{j=1}^{J_S} \Omega_S^j,$$

$$\Omega_A := \{x \in \Omega;\ \delta_A(x) < \varepsilon\} \setminus \textstyle\bigcup_{S \in \mathscr{S}(\Omega)} B(S,\varepsilon);$$

$\varepsilon > 0$ is small enough and δ_A is the distance to the edge A. We assume also that the triangulations τ_h are chosen in such a way that any element K contained in one of the zones Ω_S, Ω_A and Ω_S^j, $1 \leqslant j \leqslant J_S$ is arranged according to one of the four possibilities hereafter.

Case 1: *K is far away from vertices and edges.* Since $u \in H^{k+m,0,0}(K) = H^{k+m}(K)$, there holds using (2.5a) and (2.5b), the classical estimate [6, p. 133]:

$$|u - \pi_K u|_{m,K}^2 \leqslant c k_K^{2k} \sum_{|\gamma| = k+m} \int_K |D^\gamma u|^2 \,\mathrm{d}x. \tag{3.3}$$

We want to write the right-hand side of (3.3) in terms of the semi-norm of the space $H^{k+m,\alpha,\beta}(K)$. To this end, by construction (cf. (1.6)), we have, for any vertex S, the relations

$$\begin{gathered} \theta = 1 \quad \text{in } \Omega_S, \\ \theta \cong \theta_{S,j}, \quad \theta_{S,j} \cong \sin\theta_{S,j}, \quad \sin\theta_{S,j} = \frac{\delta_{A_{S,j}}}{r_S} \quad \text{in } \Omega_S^j, \end{gathered} \tag{3.4}$$

and, for any edge A,

$$\theta \cong \delta_A, \quad r_S > \varepsilon, \quad \text{in } \Omega_A. \tag{3.5}$$

Therefore, (3.4) and (3.5) transform (3.3) into

$$|u - \pi_K u|_{m,K}^2 \leqslant c h_K^{2k} |u|_{k+m,\alpha,\beta,K}^2 \times \begin{cases} (\inf_K r_S^{2\alpha})^{-1} & \text{if } K \subset \Omega_S, \\ (\inf_K r_S^{2\alpha - 2\beta} \delta_{A_{S,j}}^{2\beta})^{-1} & \text{if } K \subset \Omega_S^j, \\ (\inf_K \delta_A^{2\beta})^{-1} & \text{if } K \subset \Omega_A. \end{cases} \tag{3.6}$$

Thus, for this first case, the mesh refinement condition $(H_1^{k,\alpha,\beta})$ is

$$(H_1^{k,\alpha,\beta}) \qquad h_K \leqslant c h \times \begin{cases} (\inf_K r_S^{\alpha})^{1/k} & \text{if } K \subset \Omega_S, \\ (\inf_K r_S^{\alpha-\beta} \delta_{A_{S,j}}^{\beta})^{1/k} & \text{if } K \subset \Omega_S^j, \\ (\inf_K \delta_A^{\beta})^{1/k} & \text{if } K \subset \Omega_A. \end{cases}$$

Case 2: *K, contained in* Ω_S^j, *or* Ω_A, *is far away from vertices but meets the edge* $A_{S,j}$ *or* A. Details are provided only for $K \subset \Omega_S^j$; the other case is, owing to (3.5), analysed analogously and yields besides the same condition $(H_2^{k,\alpha,\beta})$ below.

The solution has now the regularity $u \in H^{k+m,0,\beta}(K)$. Because of the equivalences in (3.4), the transformation $\hat{u}$ of u under the affine mapping F_K (see Section 2 for the notation) is such that $\hat{u} \in H^{k+m,0,\beta}(\hat{K})$ where the weighted function to be considered is the distance $\hat{\delta}_{A_{S,j}}(\hat{x}) := d(\hat{x}, \hat{A}_{S,j})$.

As in [6, Theorems 3.1.2 and 3.1.3],

$$|u - \pi_K u|_{m,K}^2 \leqslant c \rho_K^{-2m} |\det B_K| \times |\hat{u} - \hat{\pi}\hat{u}|_{m,\hat{K}}^2.$$

From $|\hat{u} - \hat{\pi}\hat{u}|_{m,\hat{K}}^2 \leqslant \|\hat{u} - \hat{\pi}\hat{u}\|_{m+k,\alpha,\beta,\hat{K}}^2$ and (3.1b), we infer

$$|u - \pi_K u|_{m,K}^2 \leqslant c \rho_K^{-2m} |\det B_K| \times \sum_{|\gamma| = k+m} \int_{\hat{K}} |\hat{\delta}_{A_{S,j}}^{\beta} D^\gamma \hat{u}|^2 \,\mathrm{d}\hat{x}.$$

By the change of variable $x = F_K(\hat{x})$, the relation $\hat{\delta}_{\hat{A}_{S,j}}(\cdot) \leqslant c\rho_K^{-1}\, \delta_{A_{S,j}}(\cdot)$ due to [6, Theorem 3.1.3] and $\sup_K h_{\hat{K}} < +\infty$ (cf. (2.5a)) and by the assumption $\sup_K \rho_{\hat{K}}^{-1} < +\infty$ (cf. (2.5a)), the previous inequality becomes

$$|u - \pi_K u|_{m,K}^2 \leqslant c\rho_K^{-2m-2\beta} h_K^{2(k+m)} \sum_{|\gamma|=k+m} \int_K |r_S^{\alpha-\alpha} r_S^{\beta} \theta_{S,j}^{\beta} D^{\gamma} u|^2 \, dx.$$

Finally, we arrive at the formula

$$|u - \pi_K u|_{m,K}^2 \leqslant c h_K^{2k-2\beta} \left(\inf_K r_S^{2\alpha-2\beta} \right)^{-1} |u|_{k+m,\alpha,\beta,K}^2, \tag{3.7}$$

because of the regularity of the family of triangulations. Consequently, the mesh has to be adapted in the edge direction by the condition.

$$(H_2^{k,\alpha,\beta}) \qquad h_K \leqslant c h^{\frac{k}{k-\beta}} \inf_K r_S^{(\alpha-\beta)/(k-\beta)}.$$

Case 3: *$K \subset \Omega_S^j$ meets a vertex S and the associated edge $A_{S,j}$.* In this case $u \in H^{k+m,\alpha,\beta}(K)$; $\hat{u} \in H^{k+m,\alpha,\beta}(\hat{K})$ where $\hat{r}(\hat{x}) := d(\hat{S}, \hat{x})$ and $\hat{\theta}(\hat{x})$ is the angle between the lines $\hat{S}\hat{x}$ and $\hat{A}_{S,j}$. Once again, the inequality (3.1b) and arguments similar to those in the above considered second case yield the estimate

$$|u - \pi_K u|_{m,K}^2 \leqslant c h_K^{2k-2\alpha} |u|_{k+m,\alpha,\beta,K}^2. \tag{3.8}$$

The restriction on the mesh in the edge-vertex direction is

$$(H_3^{k,\alpha,\beta}) \qquad h_K \leqslant c h^{k/(k-\alpha)}.$$

Case 4: *$K \subset \Omega_S$ meets the vertex S but not the edges.* Now $u \in H^{k+m,\alpha,0}(K)$ and $\hat{u} \in H^{k+m,\alpha,0}(\hat{K})$, the weight $\hat{r}$ being defined in the previous case. Because of the fact that $\theta = 1$ in Ω_S, the adaptation of our arguments which are now familiar yields the same local error bound (3.8) as well as the same condition $(H_3^{k,\alpha,\beta})$ in the vertex direction.

In the subdomain of $\bar{\Omega}$ located far away from all the sets Ω_S, Ω_A and Ω_S^j, we keep of course classical triangulations with uniform meshsize in h. Combining the inequalities (3.6)–(3.8) and the observation in case 4, we have established the following optimal convergence result.

Theorem 3.3. *If the regular family of triangulations (τ_h) is refined according to the conditions $(H_1^{k,\alpha,\beta})$, $(H_2^{k,\alpha,\beta})$ and $(H_3^{k,\alpha,\beta})$, then there holds the asymptotic error estimate*

$$\| u - u_h \|_{m,\Omega} \leqslant c h^k |u|_{k+m,\alpha,\beta,\Omega}$$

between the solution u of (1.2) *and its approximation u_h in* (2.7).

Regarding the convergence in lower norms $\| \cdot \|_{m-\nu}$, the optimal rates are restored under more restrictive conditions as we specify now.

Theorem 3.4. *Fix an integer ν, $0 < \nu \leqslant m$ and real numbers α' and β' satisfying relations* (3.2a), (3.2b) *but with ν, α' and β' in lieu of k, α and β, respectively. In addition to the requirements $(H_1^{k,\alpha,\beta})$, $(H_2^{k,\alpha,\beta})$ and $(H_3^{k,\alpha,\beta})$, we assume that the triangulations (τ_h) are subject to $(H_1^{\nu,\alpha',\beta'})$, $(H_2^{\nu,\alpha',\beta'})$ and $(H_3^{\nu,\alpha',\beta'})$.*

Then

$$\|u - u_h\|_{m-\nu,\Omega} \leqslant ch^{k+\nu} |u|_{k+m,\alpha,\beta,\Omega}.$$

Proof. For $g \in H^{m-\nu}(\Omega)' \subset H^{\nu-m}(\Omega)$, the choice of ν, α' and β' guarantees, owing to Theorem 1.2, that the solution φ^g of the adjoint problem (2.9) belongs to the weighted Sobolev space $H^{\nu+m,\alpha',\beta'}(\Omega)$ for which the analogue of Theorem 3.1 holds. Therefore, for the Galerkin approximation $\varphi_h^g \in V_h$ of φ^g (cf. 2.8)), Theorem 3.3 yields the error bound

$$\|\varphi^g - \varphi_h^g\|_{m,\Omega} \leqslant ch^{\nu} |\varphi^g|_{\nu+m,\alpha',\beta',\Omega}. \tag{3.9}$$

As the adjoint problem (2.9) is regular, i.e., the operator $g \in H^{m-\nu}(\Omega)' \to \varphi^g \in H^{\nu+m,\alpha',\beta'}(\Omega)$ is continuous, the Aubin–Nitsche lemma combined with (3.9) and the estimate in Theorem 3.3 imply the desired result. □

Remark 3.5. Simplifications are possible in Theorem 3.4. For example, both series of refinement conditions coincide when $m = k = \nu = 1$. Furthermore, for smooth conical corners, our conditions degenerate to $(H_1^{k,\alpha,0})$ or $(H_1^{\nu,\alpha',0})$ and $(H_3^{k,\alpha,0})$ or $(H_3^{\nu,\alpha',0})$ which, in the case of second-order two-dimensional problems, are exactly those obtained in [20].

Remark 3.6. Along the lines of Section 3 of this paper, there remains the important question of generating effectively refined triangulations of the polyhedral volume Ω. It seems to the authors that this problem is so difficult that the existing schemes are, as reported by several mathematicians, obtained from computer programs or concern restrictive class of geometries [1]. However, as we explain in the next section, the situation is much easier if one is concerned with refined triangulations of the boundary Γ which is a surface.

4. A refined BEM

In this section, we consider second-order operators of the form

$$L = -\sum_{i,j=1}^{3} a_{ij}\partial_{ij}^2. \tag{4.1}$$

Such an assumption already allows to bring Section 3 closer to realistic situations as, for example, tetrahedral polynomial finite elements of class C^0. Nevertheless, here we shall use an alternative technique which simplifies the approximation analysis. Namely, we study a boundary element method which, reducing the initial three-dimensional problem (1.2) and (4.1) on Ω to a two-dimensional one (cf. (4.5) below) on Γ, provides effective refined triangulations of Γ consisting of triangles. As explained in the Introduction, practical boundary element methods for operators of order $2m$, with $m \geqslant 2$ are still difficult, that is the reason why we do not consider this case.

The BEM works here because of the specific properties of the operator L. In fact, on one hand it is well known that this operator has a fundamental solution E that is a two-sided inverse of L on the space of compactly supported distributions on $\mathbb{R}^3$; moreover, E has a weakly singular kernel, still denoted by E, such that the function $(x, y) \rightsquigarrow E(x, y)$ is C^∞ outside the diagonal of $\mathbb{R}^3 \times \mathbb{R}^3$. On the other hand, the second Green formula yields the next lemma.

Lemma 4.1. *The solution* $u \in \mathring{H}^1(\Omega)$ of (1.2) *and* (4.1) *corresponding to the right-hand side of* $f \in H^{k-1}(\Omega)$ *obeys, with obvious notation, the integral representation formula*

$$u(x) = \mathscr{U}f(x) + \mathscr{V}\psi(x)$$
$$\equiv \int_\Omega E(x, z) f(z)\,\mathrm{d}z - \int_\Gamma E(x, y)\psi(y)\,\mathrm{d}s(y), \quad x \in \Omega, \tag{4.2}$$

where

$$\psi \equiv \partial_{\tilde{\nu}}\, u_{/\Gamma} := \sum_{i,j=1}^{3} \nu_i a_{ij} \partial_j u_{/\Gamma}, \tag{4.3}$$

with $\psi \in L^2(\Gamma)$ (*cf. Theorem* 1.2 *and* [18, *Theorem* 2.5]) *is the conormal derivative of* u.

Formula (4.2) is the starting point of the so-called direct method of potentials. The functions $\mathscr{U}f(x)$ and $\mathscr{V}\psi(x)$ are, indeed, the Newtonian and single-layer potentials, respectively. We denote by $\mathscr{U}$ and $\mathscr{V}$ the corresponding operators on suitable Sobolev spaces, $\mathscr{V}_0$ being the boundary integral operator defined by the distributional trace of $x \rightsquigarrow \mathscr{V}\varphi(x)$.

Lemma 4.2 (Costabel [7]). (a) *The linear operators* $\mathscr{U}: H^{k-1}(\Omega) \to H^{k+1}(\Omega)$, $\mathscr{V}: H^{-1/2}(\Gamma) \to H^1(\Omega)$ *and* $\mathscr{V}_0: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *are continuous.*

(b) *The operator* $\mathscr{V}_0$ *is strongly elliptic, i.e., there exist a compact operator* T: $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *and a constant* $\lambda > 0$ *such that a Garding inequality holds*:

$$\langle (\mathscr{V}_0 + T)v, v \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} \geqslant \lambda \| v \|^2_{-1/2,\Gamma} \quad \forall v \in H^{-1/2}(\Gamma). \tag{4.4}$$

With part (a) of Lemma 4.2, the representation relative to the direct method (4.2) extends to points x on the boundary Γ; this permits to consider the unknown $\psi \in H^{-1/2}(\Gamma)$ in this method as a solution to the Fredholm boundary integral equation of the first kind:

$$\mathscr{V}_0\psi = g \text{ or } \langle \mathscr{V}_0\psi, z \rangle = \langle g, z \rangle \quad \forall z \in H^{-1/2}(\Gamma), \tag{4.5}$$

with $g := -\mathscr{U}f_{/\Gamma} \in H^{k+1/2}(\Gamma)$.

For the constructive treatment of (4.5), we now fix, by analogy with the notation of Section 3, a regular family (τ_h) of triangulations of Γ obtained separately for each face and made of triangles which satisfy the usual compatibility conditions in [6]. We consider triangular polynomial finite elements $(K, P_{k-1}(K), \Sigma_K)$, $K \in \tau_h$, the set Σ_K of degree of freedom being chosen, as in [5], such that

$$\int_K (\psi - \pi_K \psi)\,\mathrm{d}x = 0. \tag{4.6}$$

The trial and test space is

$$V_h := \{v_h: \Gamma \to \mathbb{R};\, v_{h/K} \in P_{k-1}(K)\ \forall K \in \tau_h\} \subset H^{-1/2}(\Gamma). \tag{4.7}$$

Notice that the family of finite elements $(\hat{K}, \hat{P}, \hat{\Sigma})$ reduces now to a single reference element; thus the analogues of (2.5) and (3.1b) are automatically valid (cf. Remark 2.1).

Theorem 4.3. (a) *Eq.* (4.5) *has exactly one solution* $\psi \in H^{-1/2}(\Gamma)$. *This equation is equivalent to the domain-variational problem* (1.2) *and* (4.1), *i.e., the solution* $u \in \mathring{H}^1(\Omega)$ *is related to* ψ *by* (4.2) *and* (4.3).

(b) *For h small enough, there exists a unique solution of the classical Galerkin BEM: find* $\psi_h \in V_h$ *such that*

$$\langle \mathscr{V}_0 \psi_h, v_h \rangle = \langle g, v_h \rangle \quad \forall v_h \in V_h. \tag{4.8}$$

(c) *The convergence of* ψ_h *to* ψ *is poor, namely,*

$$\| \psi - \psi_h \|_{-1/2,\Gamma} \leqslant c h^{1/2}. \tag{4.9}$$

Proof. (a) The uniqueness of the solution of the variational problem (1.2) and (4.1) implies that the operator $\mathscr{V}_0$ is injective (see [7, 21] for similar arguments). Thus injectivity means bijectivity since by (4.4), $\mathscr{V}_0$ is a Fredholm operator of index zero. Now, if $\varphi \in H^{-1/2}(\Gamma)$ solves (4.5), then by Lemma 4.2(a), the function $v := \mathscr{U} f + \mathscr{V} \varphi$ belongs to $\mathring{H}^1(\Omega)$ and is the unique solution of (1.2) and (4.1). Writing formula (4.2) for v and identifying, we get $\mathscr{V}_0 \varphi = \mathscr{V}_0 (\partial v / \partial \tilde{v})$. This yields the claim $\varphi = \partial v / \partial \tilde{v}_{/\Gamma}$.

(b) This part is a direct application of the results of [13] in their form extensively exploited by several authors as, for example [8, Lemma 5.2] and [7, 19, 21].

(c) The estimate (4.9) is a rephrasing of Theorem 5 in [7]. □

The convergence of the classical BEM (4.8) being slow (cf. (4.9)), we shall, as in Section 3, refine the meshsize of the triangulations in such a way that the convergence of the resulting BEM is optimally improved.

The importance of the assumption (4.6) is to deduce, as in [5] for two-dimensional problems, the relations

$$\| \psi - \psi_h \|^2_{-1/2,\Gamma} \leqslant c \sum_{K \in \tau_h} | \psi - \pi_K \psi |^2_{-1/2,K},$$

and, for any $K \in \tau_h$,

$$c_1 \| \psi - \pi_K \psi \|_{-1/2,K} \leqslant | \psi - \pi_K \psi |_{-1/2,K} \leqslant c_2 \| \psi - \pi_K \psi \|_{-1/2,K},$$

where

$$| \psi - \pi_K \psi |_{-1/2,K} := \sup \left\{ \frac{| \int_K v (\psi - \pi_K \psi) \, \mathrm{d}x}{| v |_{1/2,K}} ; v \in H^{1/2}(K), | v |_{1/2,K} \neq 0 \right\}.$$

Thus the computation of the error $\| \psi - \psi_h \|_{-1/2,\Gamma}$ is equivalent to that of the local errors $| \psi - \pi_K \psi |_{-1/2,K}$. Moreover, the weighted Sobolev spaces $H^{k-1/2,\alpha,\beta}(\Gamma)$ being compactly embedded in $H^{-1/2}(\Gamma)$ (cf. [18, Theorem 2.5]), there holds, under appropriate technical modifications (see Remark 4.6 below), the following analogue of Theorem 3.3.

Theorem 4.4. *Assume that* $\alpha = \beta$. *For all faces F of* Γ, *assume also the regular family* (τ_h) *to be such that any triangle K contained in* $F_S := \bar{\Omega}_S \cap F$, *in* $F_A := \bar{\Omega}_A \cap F$ *or in* $F_S^j := \bar{\Omega}_S^j \cap F$ *fulfils the conditions* $(H_1^{k,\alpha,\alpha})$ *and* $(H_2^{k,\alpha,\alpha})$ *or* $(H_3^{k,\alpha,\alpha})$ *simplified as follows:*

$$(H_1^{k,\alpha,\alpha}) \qquad h_K \leqslant c h \times \begin{cases} (\inf_K r_S^\alpha)^{1/k} & \text{if } K \subset F_S, \\ (\inf_K \delta_{A_{S,j}}^\alpha)^{1/k} & \text{if } K \subset F_S^j, \\ (\inf_K \delta_A^\alpha)^{1/k} & \text{if } K \subset F_A, \end{cases}$$

where it is understood that the intersection between K and vertices and edges is empty; otherwise

$$(H_{2'}^{k,\alpha,\alpha}) = (H_{3'}^{k,\alpha,\alpha}), \qquad h_K \leqslant ch^{k/(k-\alpha)}.$$

Then, we have the optimal rate of convergence:

$$\|\psi - \psi_h\|_{-1/2,\Gamma} \leqslant ch^k \left(\sum_{F \subset \Gamma} |\psi|^2_{k-1/2,\alpha,\alpha,F}\right)^{1/2}.$$

Remark 4.5. Rather than Eq. (4.5), one can, using as in [11] the indirect method of potentials, obtain a Fredholm integral equation of the second kind, the operator of which is the double-layer potential. This operator acts on the Sobolev space $H^{1/2}(\Gamma)$ (cf. [7]) and even on the space $C(\Gamma)$ of continuous functions, according to the three-dimensional version of [16]. Here again, the restoration of the optimal order of convergence is subject to a refined BEM of the above type since the constructive two-dimensional approaches in [5, 8, 16] based on explicit form of singularities are not easily extendable to $\mathbb{R}^3$.

Remark 4.6. The aforesaid technical modifications in the derivation of Theorem 4.4 are related to bounds of $|\psi - \pi_K \psi|^2_{-1/2,K}$.

As a consequence of Lemma 4.2(a) and Theorem 4.4, we have the following corollary.

Corollary 4.7. *Under the conditions of Theorem* 4.4, *the approximation u_h of u obtained by replacing in* (4.2) *ψ by ψ_h has the convergence property* $\|u - u_h\|_{1,\Omega} = \mathrm{O}(h^k)$.

In the rest of this section, we illustrate Theorem 4.4. To this end, we fix a face F as well as one of its vertices S and edges $A_{S,j}$. For convenience, we may (after translation, rotation and homothety) consider $F_S \cup F_{A_{S,j}} \cup F_S^j$ in the plane xoy as a subset of the unit triangle with vertices (0, 0), (1, 0) and (1, 1). Then the vertex S becomes the origin (0, 0), the piece of $A_{S,j}$ to be refined being the segment $P_0 P_0^0$ on the x-axis (see Fig. 2).

Let $n \geqslant 1$ be an integer and $\mu > 0$ the grading parameter to be chosen later on. In the edge direction, we proceed as in [12, 20]; namely, we consider the points $P_i^0 = (1, (i/n)^\mu)$ and $P_i = ((i/n)^\mu, (i/n)^\mu)$, $i = 0, 1, \ldots, n$.

In the vertex-direction, we argue by induction on i as follows ($i = 1$ and 2 in Fig. 2). Set
$t_0 := (1/n)^\mu$;
$t_i := (i/n)^\mu - ((i-1)/n)^\mu$, $i = 1, 2, \ldots, n-1$;
$P_i^k := P_i^0 - (kt_i, 0)$, $i = 0, 1, \ldots, n-1$, $k = 0, \ldots, N_i$ with $d(P_i, P_i^{N_i}) < t_i$.

For $i \geqslant 1$, let $P_{i-1}^{k_l}$, $l = 1, \ldots, N_i$, be the closest point to P_{i-1}^0 such that $d(P_{i-1}^0, P_{i-1}^{k_l}) \geqslant lt_i$. We adopt the convention $k_0 = 0$ and $k_{1+N_i} = N_{i-1}$. With each $l = 0, 1, \ldots, N_i$, we associate a series of triangles obtained by drawing the segments with end points P_i^0 and $P_{i-1}^{k_l+j}$, $j = 0, 1, \ldots, k_{l+1} - k_l$, as well as the segment defined by $P_i^{N_i}$ and P_{i-1}.

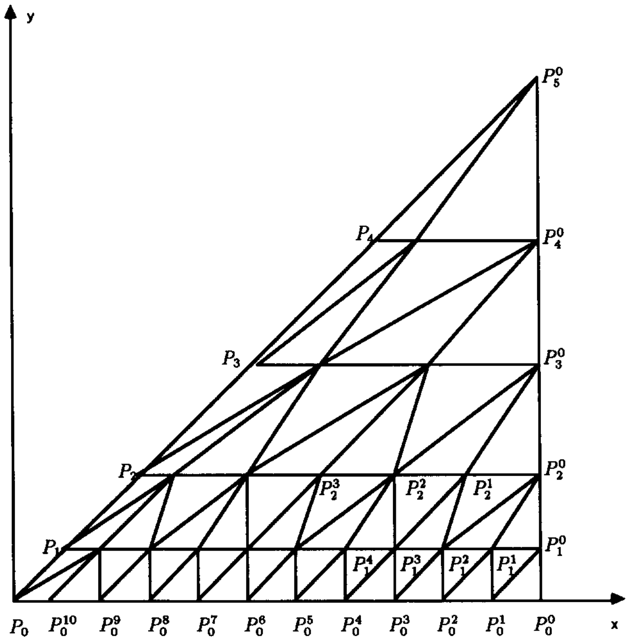

Fig. 2. Refined mesh: $n = 5$, $N_0 = 10$, $N_1 = 9$, $N_2 = 5$, $P_1^{k_1} = P_1^2$, $P_1^{k_2} = P_1^3$.

It is easy to show that the above constructed family (in n) of triangulations is regular and satisfies in fact the equivalences

$$h \cong \frac{1}{n}, \qquad t_i \cong \frac{(i-1)^{\mu-1}}{n^\mu} \text{ for } i \geqslant 2, \qquad h_K \cong t_i \tag{4.10}$$

for any triangle K between the lines $y = (i/n)^\mu$ and $y = ((i-1)/n)^\mu$, $i = 1, \dots, n$. Furthermore, for $i \geqslant 1$ and x in such a triangle the distances $r(x)$ to the vertex $(0, 0)$ and $\delta(x)$ to the edge are such that

$$r(x) \geqslant \delta(x) \geqslant \left(\frac{i-1}{n}\right)^\mu. \tag{4.11}$$

From (4.10) and (4.11), we infer that this triangulation fulfils the conditions $(H_1^{k,\alpha,\alpha})$ and $(H_2^{k,\alpha,\alpha})$ if μ is chosen such that

$$\mu \geqslant \frac{k}{k-\alpha}. \tag{4.12}$$

Finally, the number of triangles in the zone of the face subject to the refinement condition is of order n^μ; this may be seen easily on counting the triangles relative to each generation i ($i = 1, 2, \dots, n$). Furthermore, the measure of the segment $P_0P_0^0$ representing the edge being 1, it follows that, for any refined triangulation, the number of triangles meeting the edge is at least $cn^{k/(k-\alpha)}$ since for such triangles $h_K \leqslant ch^{k/(k-\alpha)}$. Therefore, the total number of elements in our construction is of optimal order n^μ for the choice $\mu = k/(k-\alpha)$.

Remark 4.8. The optimal rates restored in Theorem 4.4 are obtainable neither by the approach in [19] where data of class $L^2(\Omega)$ are omitted nor by that in [11] based on trace spaces with integer exponents.

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