A Cohomological Conley Index in Hilbert Spaces and Applications to Strongly Indefinite Problems

Marek Izydorek

Department of Technical Physics and Applied Mathematics, Technical University of Gdańsk, 80-952 Gdańsk, ul. Gabriela Narutowicza 11/12, Poland

Received October 7, 1998; revised November 5, 1999

A cohomological Conley index is defined for flows on infinite dimensional real Hilbert spaces generated by vector fields of the form \( f: H \to H \), \( f(x) = Lx + K(x) \), where \( L: H \to H \) is a bounded linear operator satisfying certain technical assumptions and \( K \) is a completely continuous perturbation. Generalized Morse inequalities for Morse decompositions of isolated invariant sets are proved. Simple examples are presented to show how the theory can be applied to strongly indefinite problems.

Key Words: (co)homology groups; Morse inequalities; critical points; the Conley index; Hilbert space; Hamiltonian system.

1. INTRODUCTION

A new infinite dimensional extension of the classical Conley index has been recently presented in [12]. The purpose of this paper is to define the cohomological version of that topological invariant and next to show how it can be applied to strongly indefinite problems. The Conley index theory, introduced by Charles Conley in the late 1960s, has proved to be a valuable tool for the study of dynamical systems in a number of settings. Recall that a compact and isolated invariant set \( S \) of a continuous flow \( \varphi: \mathbb{R} \to Z \) on a locally compact space \( Z \) possesses an index \( h(S) \), which is the homotopy type of a pointed compact space. Conley's monograph "Isolated Invariant Sets and the Morse Theory" [9] is the standard reference. That theory has had numerous applications to the qualitative study of ordinary differential equations in finite dimensional spaces. However, infinite dimensional problems can only be treated in those special cases in which a finite dimensional reduction is possible. This is due to the fact that local compactness of a phase space is crucial in the classical setting. Therefore, finding an extension of the Conley index to semiflows on
metric spaces which are not necessarily locally compact was a natural
task. The first such extension was made by Rybakowski in [22, 23]
(see also the paper [24] by Rybakowski and Zehnder). The theory has
been fruitfully applied to nonlinear elliptic and parabolic equations (see
[25]). However, Rybakowski’s theory is applicable only for isolated
invariant sets having finite dimensional unstable manifold.

Later on the Conley index theory was further generalized by Benci [5].
His approach even if applicable to sets with infinite dimensional unstable
manifold gives in that case only trivial index. Applying variational methods
to certain existence and multiplicity problems for periodic solutions of
Hamiltonian systems or wave equation one is led to consider so-called
strongly indefinite functionals, i.e., both stable and unstable manifolds at a
critical point are of infinite dimension. The fundamental paper concerning
that subject was written by H. Amann and E. Zehnder [2] The literature on
that subject is vast. Let me only mention the articles [1–3, 7, 8, 10, 13–17,
20, 21, 27, 28] and books by Bartsch [4], Mawhin and Willem [19], and
Chang [6].

In the work [10] by Conley and Zehnder multiplicity results for periodic
solutions of asymptotically linear Hamiltonian equations were proved.
Assuming on the Hessian of a Hamiltonian function \( h: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) to be
bounded on \( \mathbb{R} \times \mathbb{R}^n \) the authors were able to reduce a strongly indefinite
problem to a variational problem on a finite dimensional space. In other
words, they made a reduction on the “level” of potentials. In contrast to
that, the construction of the \( L^2 \)-index in [12] is based on finite dimen-
sional approximations of vector fields (see Subsection 2.3). Consequently,
the resulting invariant applies to a larger class of flows than that con-
sidered in [10]. In particular, the Conley and Zehnder approach cannot be
used in examples discussed in Section 5. However, if applicable gives a
similar topological information to that obtained by methods developed in
this paper.

Our aim is to define the cohomological Conley index in a Hilbert space
and prove generalized Morse inequalities for Morse decompositions of an
isolated invariant sets. Unlike Rybakowski and Benci extensions of Conley
theory in our theory an isolated invariant set with infinite dimensional
unstable manifold can have a nontrivial index. Thanks to that property we
obtain new results concerning the existence of periodic solutions of certain
Hamiltonian systems. The gradient nature of the problem is used in an
essential way. This means that in cases considered in the paper Morse
inequalities give more information about the existence of stationary points
of flows than the topological degree of the Leray–Schauder type. Besides,
functionals corresponding to those systems except being strongly indefinite
admit also degenerate critical points. As we will see in examples this is the
right situation to present how efficiently our theory works.
After this Introduction the paper is organized as follows. In Section 2 we recall basic definitions and facts concerning the extension of the Conley index introduced in [12]. Section 3 is devoted to an abstract definition of the cohomology of spectra. Subsequently in Section 4 we prove that the Morse decomposition of an isolated invariant set is “robust” in our theory. Next we prove the generalization of Morse inequalities for Morse decomposition in a Hilbert space. Finally, in Section 5 we give a few examples to illustrate how our theory works when asymptotically linear and strongly indefinite functionals are considered. We would like to point out that our examples are not covered by theorems which we have found in the literature for that kind of problems.

2. \( \mathcal{L} \mathcal{S} \)-FLOWS

The homotopy Conley index for a class of flows generated by \( \mathcal{L} \mathcal{S} \)-vector fields in Hilbert spaces has been recently defined in [12]. These kind of flows appear, for instance, when one applies variational methods to prove the existence or multiplicity results for periodic solutions of some types of Hamiltonian systems (see [2, 6, 10, 27] and references there), second order ODEs (see [5, 19] and references there), as well as certain elliptic and hyperbolic problems (see [2, 8, 20, 21, 28]). In this section we recall basic definitions and facts which will be used in our considerations.

2.1. Isolating Neighbourhoods

Let \( H = (H, \langle \cdot, \cdot \rangle) \) be a real Hilbert space and \( L : H \to H \) be a linear bounded operator with spectrum \( \sigma(L) \) such that:

- \((H.1)\) \( H = \bigoplus_{k=0}^{\infty} H_k \) with all subspaces \( H_k \) being mutually orthogonal and of finite dimension;
- \((H.2)\) \( L(H_0) \subset H_0 \), \( H_0 \) is the invariant subspace of \( L \) corresponding to the part of spectrum \( \sigma_0(L) := \sigma(0) \cap \sigma(L) \) lying on the imaginary axis and \( L(H_k) = H_k \) for all \( k > 0 \);
- \((H.3)\) \( \sigma_0(L) \) is isolated in \( \sigma(L) \) i.e., \( \sigma_0(L) \cap ch(\sigma(L) \setminus \sigma_0(L)) = \emptyset \).

We say that a continuous map \( \eta : \mathbb{R} \times H \to H \) is a flow on \( H \) if \( \eta(0, x) = x \) and \( \eta(t, \eta(s, x)) = \eta(t + s, x) \) for all \( s, t \in \mathbb{R}, x \in H \).

**Definition 2.1.** If \( \eta \) is a flow and \( X \subset H \) then

\[ \text{Inv}(X, \eta) := \{ x \in X : \eta(t, x) \in X \text{ for all } t \in \mathbb{R} \} \] .

\( \text{Inv}(X, \eta) \) is the maximal \( \eta \)-invariant subset of \( X \).
Let $A$ denote a compact metric space. A continuous map $\eta: \mathbb{R} \times H \times A \to H$ is a family of flows indexed by $A$ if $\eta_\lambda$ (where $\eta_\lambda(t, x) := \eta(t, x, \lambda)$) is a flow of $H$ for all $\lambda \in A$. If $\eta: \mathbb{R} \times H \times A \to H$ is a family of flows and $X$ is a subset of $H$ then we let

$$\text{Inv}(X \times A, \eta) := \{ (x, \lambda) \in X \times A ; \ \eta(t, x, \lambda) \in X \text{ for all } t \in \mathbb{R} \}$$

Let $A$ be a metric space. We say that $f: H \times A \to H$ is a completely continuous map if $f$ is continuous and for any bounded subset $A \subset H \times A$ the closure of $f(A)$ is a compact subset of $H$.

**Definition 2.2.** Let $A$ be a compact metric space. We say that a family of flows $\eta: \mathbb{R} \times H \times A \to H$ is a family of $\mathcal{L} \mathcal{F}$-flows if

$$\eta(t, x, \lambda) = e^{\lambda t}x + U(t, x, \lambda),$$

where $U: \mathbb{R} \times H \times A \to H$ is completely continuous.

If $A$ consists of one point we drop the parameter space out from notation and we call $\eta$ an $\mathcal{L} \mathcal{F}$-flow. The class of $\mathcal{L} \mathcal{F}$-flows has a crucial compactness property which is precisely formulated in the following.

**Proposition 2.3.** Let $A$ be a compact metric space and let $\eta: \mathbb{R} \times H \times A \to H$ be a family of $\mathcal{L} \mathcal{F}$-flows. If $X \subset H$ is closed and bounded then $S := \text{Inv}(X \times A, \eta)$ is a compact subset of $X \times A$.

As we see in the next part of this section the above result plays an essential role in the construction of the $\mathcal{L} \mathcal{F}$-homotopy Conley index.

**Definition 2.4.** Let $A$ be a compact metric space. We say that $f: H \times A \to H$ is a family of $\mathcal{L} \mathcal{F}$-vector fields if there exists a completely continuous and locally Lipschitz continuous map $K: H \times A \to H$ such that

$$f(x, \lambda) = Lx + K(x, \lambda) \quad \text{for all } (x, \lambda) \in H \times A.$$

In the case when $A$ is a one pointed space we will call $f$ an $\mathcal{L} \mathcal{F}$-vector field.

If $f: H \to H$ is an $\mathcal{L} \mathcal{F}$-vector field and $x \in H$ then it is well known (e.g., see [11]) that there exists the maximal $C^1$-curve

$$(\alpha(x), \omega(x)) \ni t \mapsto \eta(t, x) \in H$$
satisfying
\[
\begin{align*}
\frac{d\eta}{dt} &= -f \cdot \eta \\
\eta(0, x) &= x
\end{align*}
\]
Moreover, if we set
\[
D(\eta) := \{(t, x) \in \mathbb{R} \times H : \sigma(x) < t < \omega(x)\},
\]
then \(D(\eta) \subset \mathbb{R} \times H\) is open and \(\eta : D(\eta) \ni (t, x) \mapsto \eta(t, x) \in H\) is continuous. In what follows we call \(\eta\) the local flow generated by \(f\).

Let \(f, f(x) = Lx + K(x)\) be an \(\mathcal{L}S\)-vector field. Then the local flow generated by \(f\) is of the form
\[
\eta(t, x) = e^{-Lt}x + U(t, x),
\]
where \(U : D(\eta) \to H\) is completely continuous (see \([20]\)).

We say that \(f\) is subquadratic if there exist \(a, b > 0\) such that
\[
|\langle K(x), x \rangle| \leq a \|x\|^2 + b \quad \forall x \in H. \tag{2.1}
\]

The following result is a direct consequence of Theorem 2.1 in \([29]\).

**Proposition 2.5.** Let \(f, f(x) = Lx + K(x)\) denote an \(\mathcal{L}S\)-vector field and let \(\eta\) be the local flow generated by \(f\). Then

(a) if \(x \in H\) and \((t_n)\) is a sequence in \((\sigma(x), \omega(x))\) convergent either to \(\sigma(x)\) or to \(\omega(x)\) then \((t_n, \eta(t_n, x))\) is an unbounded sequence (in \(\mathbb{R} \times H\));

(b) if \(f\) satisfies (2.1) then \(f\) generates an \(\mathcal{L}S\)-flow.

As it was pointed out in \([18]\) by Mischaikow, invariant sets are notoriously fickle with respect to perturbations. In particular, they can disappear, change their topological type or change their stability. Therefore instead of invariant sets it is sometimes more convenient to work with isolating neighbourhoods for flows. The reason why we have chosen that approach will be explained below.

**Definition 2.6.** We say that a bounded and closed subset \(X \subset H\) is an isolating neighbourhood for a flow \(\eta\) iff \(\text{Inv}(X) \subset \text{int}(X)\), the maximal \(\eta\)-invariant subset of \(X\) is included in the interior of \(X\).

In contrast to invariant sets, isolating neighbourhoods are robust in the sense given by the following.
**Theorem 2.7.** Let $A$ be a compact metric space and let $\eta: \mathbb{R} \times H \times A \to H$ be a family of $\mathcal{P}$-flows. Assume that $X \subset H$ is an isolating neighbourhood for a flow $\eta_\lambda$ for some $\lambda_0 \in A$. Then there is an open neighbourhood of $\lambda_0$, $V \subset A$ such that $X$ is an isolating neighbourhood for any flow $\eta_\lambda$ whenever $\lambda \in V$.

Actually, the above theorem can be reformulated in the following way.

**Corollary 2.8.** For any closed and bounded set $X \subset H$ the set

$$A(X) = \{ \lambda \in A; X \text{ is an isolating neighbourhood for } \eta_\lambda \text{ in } H \}$$

is open in $A$.

Thus an isolating neighbourhood is a nice object; one expects to be able to find it and once found it has a tendency to stay put. However the object of ultimate interest is really the invariant set. Fortunately as far as the $\mathcal{P}$-homotopy Conley index theory is concerned, information carried by an isolating neighbourhood $X$ for a flow $\eta$ can be translated into information about the dynamics of $\text{Inv}(X, \eta)$. Let us recall that we consider flows defined on an infinite dimensional Hilbert space and therefore the condition for the space to be locally compact is not satisfied which is in contrast to Conley [9], Salamon [26], and Mischaikow [18].

**2.2. Spectra**

Let $\mathcal{M}_0$ be the category of compact metrizable spaces with a base point. If $(X, x_0)$, $(Y, y_0)$ are objects in $\mathcal{M}_0$ then the set of morphisms $\text{Mor}(X, Y)$ consists of all continuous maps $f: X \to Y$ preserving base points. Recall that the closed inclusion $i \in \text{Mor}(A, X)$ is a cofibration if for any topological pointed space $(Z, z_0)$ and any continuous map

$$G: X \times \{0\} \cup A \times [0, 1] \to Z$$

satisfying $G(x_0, t) = z_0$ for all $t \in [0, 1]$, there is an extension of $G$ to the space $X \times [0, 1]$.

A pair of spaces $(X, A)$ in $\mathcal{M}_0$ is a pair of objects from the category $\mathcal{M}_0$ such that $A$ is a closed subset of $X$ and base points of $A$ and $X$ coincide. If in addition the inclusion map $i: (A, x_0) \to (X, x_0)$ is a cofibration then $(X, A)$ is called a $c$-pair in $\mathcal{M}_0$. Clearly, if $A$ is a base point in $X$ then $(X, A)$ is a $c$-pair in $\mathcal{M}_0$. A map of pairs $f: (X, A) \to (Y, B)$ is any continuous map from $X$ into $Y$ preserving base points and such that $f(A) \subset B$. Let $(X, A)$ be a pair in $\mathcal{M}_0$. Then the quotient space $X/A$ is obtained from $X$ by collapsing $A$ to a point, the base point of $X/A$. $X/A$ is an object of $\mathcal{M}_0$. If $X, Y$ are objects in $\mathcal{M}_0$ (with base points $x_0, y_0$ resp.) then the Cartesian product $X \times Y$ is also an object in $\mathcal{M}_0$ (with base point $(x_0, y_0)$). Moreover, their
wedge $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ is a closed subspace of $X \times Y$ and $(X \times Y, X \vee Y)$ is a pair in $\mathcal{M}_0$. Hence, the smash product $X \wedge Y$ defined as a quotient space $(X \times Y)/(X \vee Y)$ is an object in $\mathcal{M}_0$. Additionally to that, if $f : X \to Y$ and $g : X' \to Y'$ are morphisms in the category $\mathcal{M}_0$ then the induced map $f \wedge g : X \wedge X' \to Y \wedge Y'$ is a morphism of $\mathcal{M}_0$ as well.

The following properties of $\vee$ and $\wedge$ will be used in our considerations:

1. the smash product is commutative and associative up to natural homeomorphism;
2. the smash product is distributive over the wedge up to natural homeomorphism.

Denote by $I$ the unit interval with base point $\{0\}$, $\partial I = S^0$ the subspace $\{0, 1\}$ of $I$, $S = S^1 = I/\partial I$. In fact, the smash product is a functor in $\mathcal{M}_0$ and therefore it induces the suspension functor defined by $SX := S^1 \wedge X$. For any $m \in \mathbb{N}$ we define $S^mX = S(S^{m-1}X)$, (note, that $S^0 \wedge X$ is naturally homeomorphic with $X$). By property (1) we get immediately that $SX \wedge Y$ and $X \wedge SY$ are naturally homeomorphic. Finally, let us recall that $f \in \text{Mor}(X, Y)$ is a homotopy equivalence if there is $g \in \text{Mor}(Y, X)$ such that $g \circ f$ is homotopic with $\text{id}_X$ and $f \circ g$ is homotopic with $\text{id}_Y$, both homotopies are relative base points. If $f : X \to Y$ is a homotopy equivalence then we say that spaces $X$ and $Y$ are homotopy equivalent or they have the same homotopy type. Standard references for this section is Whitehead [30].

Let $\nu : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, be a fixed map. Suppose that $(E_n)^{\nu(n)}_{n=m(E)}$ is a sequence of objects in $\mathcal{M}_0$. Let $(\varepsilon_n : S^{\nu(n)}E_n \to E_{n+1})^\infty_{n=m(E)}$ be a sequence of morphisms.

**Definition 2.9.** We say that a pair $E := ((E_n)^{\nu(n)}_{n=m(E)}) \cdot (\varepsilon_n)^{\nu(n)}_{n=m(E)}$ is a spectrum if there exists $n_0 \geq n(E)$ such that $\varepsilon_n : S^{\nu(n)}E_n \to E_{n+1}$ is a homotopy equivalence for all $n \geq n_0$.

**Definition 2.10.** A map of spectra $f : E \to E'$ is a sequence of maps $(f_n)^{\nu(n)}_{n=m(E)} \in \text{Mor}(E_n, E'_n) n_0 \geq \max\{n(E), n(E')\}$ such that diagrams

$$
\begin{array}{ccc}
S^{\nu(n)}E_n & \xrightarrow{\varepsilon_n} & S^{\nu(n)}E'_n \\
\downarrow & & \downarrow \\
E_{n+1} & \xrightarrow{f_{n+1}} & E'_{n+1}
\end{array}
$$

are homotopy commutative for all $n \geq n_0$.

The category of spectra with a function $\nu$ is denoted $\mathcal{E}(\nu)$.
**Definition 2.11.** Two maps of spectra $f, f': E \to E'$ are homotopic if there is $n_1 \in \mathbb{N} \cup \{0\}$ such that $f_n \cong f'_n$ whenever $n \geq n_1$.

**Definition 2.12.** We say that $f: E \to E'$ is a homotopy equivalence of spectra $E$ and $E'$ if there exists $g: E' \to E$ such that $g \circ f: E \to E$ is homotopic with the identity map $id_E$ and $f \circ g: E' \to E'$ is homotopic with $id_{E'}$.

**Definition 2.13.** Two spectra $E, E'$ are said to be homotopy equivalent or they have the same homotopy type if there is a homotopy equivalence $f: E \to E'$. A homotopy type of spectrum is denoted by $[E]$.

Given two spectra $E$ and $E'$ their wedge $E^w = E \lor E'$ is defined as follows. For any $n \geq \max\{n(E), n(E')\}$ we put $E^w_n = E_n \lor E'_n$. By property (2) of $\lor$ there is the natural homeomorphism

$$z_n: S^{v(n)}(E_n \lor E'_n) \to S^{v(n)}E^w_n \lor S^{v(n)}E'^w_n$$

and therefore we can define a map $\varepsilon^w_n: S^{v(n)}E^w_n \to E^w_{n+1}$ as the composition

$$S^{v(n)}E^w_n = S^{v(n)}(E_n \lor E'_n) \xrightarrow{z_n} S^{v(n)}E_n \lor S^{v(n)}E'_n \xrightarrow{\varepsilon_n \lor \varepsilon'_n} E^w_{n+1} \lor E^w_{n+1} = E^w_{n+1}.$$  

Clearly, $\varepsilon^w_n$ is a homotopy equivalence for $n$ sufficiently large. Thus

$$E^w = ((E^w_n)_{n=\max\{n(E), n(E')\}}, (\varepsilon^w_n)_{n=\max\{n(E), n(E')\}})$$

is an object in $\mathcal{S}(v)$.

The category $\mathcal{S}(v)$ is not closed with respect to the smash product defined below. Let $E$ be an object of $\mathcal{S}(\mu)$ and $E'$ be an object of $\mathcal{S}(\nu)$. Define $E^\vee = E \wedge E'$ as follows. For each $n \geq \max\{n(E), n(E')\}$ we put $E^\vee_n = E_n \wedge E'_n$. By property (1) of $\wedge$ there is a natural homeomorphism

$$\psi_n: S^{\mu(n) + \nu(n)}(E_n \wedge E'_n) \to S^{\mu(n)}E_n \wedge S^{\nu(n)}E'_n$$

and thus we are able to define a homotopy equivalence map $\varepsilon^\vee_n: S^{\mu(n) + \nu(n)}E^\vee_n \to E^\vee_{n+1}$ as the composition

$$S^{\mu(n) + \nu(n)}E^\vee_n = S^{\mu(n) + \nu(n)}(E_n \wedge E'_n) \xrightarrow{\psi_n} S^{\mu(n)}E_n \wedge S^{\nu(n)}E'_n \xrightarrow{\varepsilon_n \lor \varepsilon'_n} E^\vee_{n+1} \wedge E^\vee_{n+1} = E^\vee_{n+1}.$$
so that the resulting spectrum $E^v = ((E^v_n)_{n = m(E)}), (e^v_n)_{n = m(E)})$ is an object in the category $\mathcal{P}(u + v)$. Let $S$ be a spectrum such that for each $n = 0, 1, 2, \ldots$, $E_n = S^1$, the unit sphere, $e_n = \text{id}_{S^1}$ and $\mu(n) = 0$. Then $SE := S \wedge E$ is the suspension functor in the category of spectra. Note that if $E$ is an object in \( \mathcal{P}(v) \) then $SE$ is an object of \( \mathcal{P}(v) \) as well. For any $m \in \mathbb{N}$ we define $S^mE = S^{m-1}(SE)$.

It is clear that both operations $\vee$ and $\wedge$ preserve homotopy type of spectra. This leads us to a conclusion that one can define “wedge” and “smash” of homotopy types of spectra putting $[E] \vee [E'] := [E \vee E']$ and $[E] \wedge [E'] := [E \wedge E']$, respectively. In particular, the suspension functor is defined, we put $S[E] := [SE]$.

Remark 2.14. For a given spectrum $E = ((E_n)_{n = m(E)}), (e_n)_{n = m(E)})$ its homotopy type is uniquely determined by the homotopy type of a pointed space $E_n$ with $n$ sufficiently large. In particular, if in the spectrum $E$ the sequence $((E_n)_{n = m(E)})$ is replaced by another sequence of homotopy equivalences $((E_n)_{n = m(E)})$ then the resulting spectrum has the same homotopy type as the original one. Therefore, in order to define the homotopy type $[E]$ one only needs a sequence of spaces $E = (E_n)_{n = m(E)}$ such that $S^mE_n$ is homotopy equivalent to $E_{n+1}$ for $n$ sufficiently large.

Denote by $0$ a spectrum such that for each $n \geq 0$, space $E_n$ consists only of a base point, $e_n$ maps the point in $E_n$ into the point in $E_{n+1}$. The suspension functor acts trivially on such spaces, i.e., $SE_n = E_n$ and therefore $0$ is an object of $\mathcal{P}(v)$ for arbitrary $v$.

**Definition 2.15.** We say that the homotopy type of spectrum $E$ is trivial if $E$ is homotopy equivalent with $0$.

**Definition 2.16.** Let $A$ and $E$ be objects of the category $\mathcal{P}(v)$. We say that $A$ is a subspectrum of $E$ if there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has:

(a) $(E_n, A_n)$ is a c-pair in $\mathcal{A}_0$;

(b) the inclusion map $i = (i_n: A_n \hookrightarrow E_n)_{n = n_0}$ is a map of spectra.

We say that $(E, A)$ is a pair of spectra if $A$ is a subspectrum of $E$.

2.3. The $\mathcal{L}^\infty$-Index

Referring for all technical details to [12] we briefly recall the construction of the $\mathcal{L}^\infty$-homotopy Conley index and its basic properties. Let $P_n: H \twoheadrightarrow H$ be the orthogonal projection onto $H^+ := \oplus_{i=0}^n H_i$. Let $H^-_n$ (resp. $H^+_n$), $n \geq 1$, denote the $L$-invariant subspace of $H_n$ corresponding to the part of spectrum of $L$ with the negative (resp. positive) real part. Define $v: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by $v(n) := \dim H^+_{n+1}$.
Let $S$ and $Y$ homotopy equivalent with $X \subset H$ be an isolating neighbourhood for the flow $\eta$.

Define $f_n: H^n \to H^n$ and $F_n: H^{n+1} \times [0, 1] \to H^{n+1}$ by

$$f_n(x) := Lx + P_n(K(x))$$

and

$$F_n(x, t) := Lx + (1 - t) P_n(K(x)) + tP_{n+1}(K(x)).$$

Let $\eta_n: \mathbb{R} \times H^n \to H^n$, $\xi_n: \mathbb{R} \times H^{n+1} \times [0, 1] \to H^{n+1}$ denote the family of flows induced by $f_n$ and $F_n$ as consequences of Proposition (2.3) we obtain that $X_n := X \cap H^n$ is an isolating neighbourhood for the flow $\eta_n$ and for the flow of flows $\xi_n$ for $n$ sufficiently large, say $n \geq n_0$. Choose $n \geq n_0$ and set $S_n := \text{Inv}(X_n, \eta_n)$. Thus $S_n$ admits an index pair $(Y_n, Z_n)$ and the Conley index of $S_n$ is the homotopy type of the pointed space $Y_n/Z_n$.

Let

$$D^+_n := \{x \in H^+_n ; \|x\| \leq 1\}, \quad D^-_n := \{x \in H^-_n ; \|x\| \leq 1\},$$

$$\partial D^-_n := \{x \in H^-_n ; \|x\| = 1\}.$$

Let $S_{n+1, n} := \text{Inv}(X_{n+1} \times [0, 1], \xi_n)$, $S_{n+1, n}(t) := \{x \in X_{n+1} ; (x, t) \in S_{n+1, n}(1)\}$.

Consider a family of flows $\theta_n: \mathbb{R} \times H^{n+1} \times [0, 1] \to H^{n+1}$ generated by

$$h_n: H^{n+1} \times [0, 1] \to H^{n+1}, h_n(x, s) = Lx + P_n(K(x + s(x - P_n(x)))).$$

Clearly

$$(Y_n \times D^+_n) \times D^-_{n+1}, Z_n \times D^+_n \times D^-_{n+1} \cup Y_n \times D^+_n \times \partial D^-_{n+1}$$

is an index pair for the isolated invariant set $\text{Inv}(X_{n+1}, \theta_n(\cdot, \cdot, 0)) = S_n$. Thus, the Conley index of $S_n = S_{n+1, n}(0)$ with respect to $\theta_n(\cdot, \cdot, 0)$ equals the homotopy type of

$$(Y_n \times D^+_n \times D^-_{n+1}, (Z_n \times D^+_n \times D^-_{n+1} \cup Y_n \times D^+_n \times \partial D^-_{n+1}))$$

which in turn is equal to the homotopy type of $S^{(n)}(Y_n, Z_n)$. Moreover, $X_{n+1}$ is an isolating neighbourhood for both families $\theta_n(\cdot, \cdot, s)$ and $\xi_n(\cdot, \cdot, s), s \in [0, 1]$ and $\theta_n(\cdot, \cdot, 1) = \xi_n(\cdot, \cdot, 0)$. Therefore, by the continuation property of the Conley index (see [9], [26]) $S^{(n)}(Y_n, Z_n)$ is homotopy equivalent with $Y_{n+1}/Z_{n+1}$. Thus, in view of Remark 2.2, the sequence

$$(E_n)_{n=n_0}^{\infty} := (Y_n/Z_n)_{n=n_0}^{\infty}$$

determines uniquely the homotopy type $[E]$. 

COHOMOLOGICAL CONLEY INDEX 31
Definition 2.17. Let \( \eta \) be an \( \mathcal{LF} \)-flow generated by a subquadratic \( \mathcal{LF} \)-vector field and let \( X \) be an isolating neighbourhood for \( \eta \). Define

\[
h_{\mathcal{LF}}(X, \eta) := [E].
\]

We call \( h_{\mathcal{LF}}(X, \eta) \) the \( \mathcal{LF} \)-homotopy Conley index of \( X \) with respect to \( \eta \) or simply the \( \mathcal{LF} \)-index.

Turning to the general case assume that \( f: H \to H \), \( f(x) = Lx + K(x) \) is an \( \mathcal{LF} \)-vector field and \( X / H \) is an isolating neighbourhood for the local flow \( \eta \) generated by \( f \). Choose \( s \in \mathbb{R} \) such that \( X \subset B(0, s) \) and define maps \( \mu: \mathbb{R} \to \mathbb{R} \),

\[
\mu(t) = \begin{cases} 
1 & \text{if } t \leq s \\
1 + s - t & \text{if } s < t \leq s + 1 \\
0 & \text{if } t \geq s + 1
\end{cases}
\]

and \( d: H \to [0, 1], \ d(x) = \mu(\|x\|) \).

Clearly, the map \( K_1: H \to H, \ K_1(x) := d(x) \cdot K(x) \) is completely continuous, loc. Lipschitz continuous and the closure of \( K_1(H) \) is compact. Hence \( f_1: H \to H, \ f_1(x) = Lx + K_1(x) \) is a subquadratic \( \mathcal{LF} \)-vector field. Moreover, the flow \( \eta_1 \) generated by \( f_1 \) and the local flow \( \eta \) coincide on \( X \). That is, \( \eta(t, x) \in X \) for \( t \in [0, a] \) implies \( \eta_1(t, x) = \eta(t, x) \) for \( t \in [0, a] \). In particular, \( X \) is an isolating neighbourhood for \( \eta_1 \). Note also that if \( \eta_2 \) is another \( \mathcal{LF} \)-flow generated by a subquadratic \( \mathcal{LF} \)-vector field \( f_2 \) such that \( f_2(x) = f(x) \) for all \( x \in X \) then

\[
h_{\mathcal{LF}}(X, \eta_1) = h_{\mathcal{LF}}(X, \eta_2).
\]

Definition 2.18. Let \( f \) be an \( \mathcal{LF} \)-vector field, \( \eta \) the local flow generated by \( f \) and let \( X \) be an isolating neighbourhood for \( \eta \). Define

\[
h_{\mathcal{LF}}(X, \eta) := h_{\mathcal{LF}}(X, \eta_1).
\]

We call \( h_{\mathcal{LF}}(X, \eta) \) the \( \mathcal{LF} \)-homotopy Conley index of \( X \) with respect to \( \eta \) or the \( \mathcal{LF} \)-index.

The following propositions give the basic properties of the \( \mathcal{LF} \)-homotopy Conley index.

Proposition 2.19 (Nontriviality). Let \( \eta: D(\eta) \to H \) be a local flow generated by an \( \mathcal{LF} \)-vector field and let \( X \subset H \) be an isolating neighbourhood for \( \eta \). If the \( \mathcal{LF} \)-homotopy Conley index \( h_{\mathcal{LF}}(X, \eta) \neq 0 \) then \( \text{Inv}(X, \eta) \neq \emptyset \).
Proposition 2.20 (Continuation). Let \( A \) be a compact, connected and locally contractible metric space. Assume that \( \eta: \mathbb{D}(\eta) \to H \) is a family of local flows generated by a family of \( \mathcal{L} \)-\( \mathcal{F} \)-vector fields \( f: H \times A \to H \). Let \( X \) be an isolating neighbourhood for a flow \( \eta \) for some \( \lambda \in A \). Then, there is a compact neighbourhood \( C \subset A \) of \( \lambda \) (\( \lambda \in \text{Int}(C) \)) such that
\[
h_{\mathcal{L} \mathcal{F}}(X, \eta_\mu) = h_{\mathcal{L} \mathcal{F}}(X, \eta_v)
\]
for all \( \mu, v \in C \).

3. COHOMOLOGY OF SPECTRA

In what follows, \( \hat{\mathbb{H}}^* \) denotes the reduced Čech cohomology theory with coefficients in some fixed ring \( \mathcal{F} \). This particular cohomology is chosen because it is defined for compact spaces and has the continuity property, i.e.,
\[
\hat{\mathbb{H}}^*(X) = \lim_{\mu \to X} \hat{\mathbb{H}}^*(X_\mu)
\]
if \( X = \bigcap X_\mu \).

Let \( A = (A_n, \omega_n) \) be a subspectrum of \( E = (E_n, \varepsilon_n) \). We define its quotient \( C = E/A \) (which is not necessarily a spectrum) as follows. Choose \( n_0 \) as in Definition (2.16). For each \( n \geq n_0 \) we put \( C_n := E_n/A_n \). Since \( \varepsilon_n: A_n \to E_n \) is a cofibration and \( i: A \to E \) is a map of spectra there is a map of pairs
\[
c_n: \left( S^{(\varepsilon_n)}E_n, S^{(\varepsilon_n)}A_n \right) \to \left( E_{n+1}, A_{n+1} \right)
\]
such that \( c_n: S^{(\varepsilon_n)}E_n \to E_{n+1} \) is homotopic with \( \varepsilon_n \) and the restriction of \( c_n \) to \( S^{(\varepsilon_n)}A_n \) considered as a map into \( A_{n+1} \) is equal to \( \omega_n \). Thus, the induced homomorphisms
\[
c_n^*: \hat{\mathbb{H}}^*(E_{n+1}) \to \hat{\mathbb{H}}^*(S^{(\varepsilon_n)}E_n)
\]
and
\[
c_n^*: \hat{\mathbb{H}}^*(A_{n+1}) \to \hat{\mathbb{H}}^*(S^{(\varepsilon_n)}A_n)
\]
are isomorphisms. This implies that
\[
c_n^*: \hat{\mathbb{H}}^*(E_{n+1}, A_{n+1}) \to \hat{\mathbb{H}}^*(S^{(\varepsilon_n)}E_n, S^{(\varepsilon_n)}A_n)
\]
is an isomorphism as well.

On the other hand the map \( c_n \) induces a map \( \gamma_n: S^{(\varepsilon_n)}C_n \to C_{n+1} \). If \((X, A)\) is a pair in \( \mathcal{D}_0 \) then \( \hat{\mathbb{H}}^*(X, A) \simeq \hat{\mathbb{H}}^*(X/A) \) and therefore \( \gamma_n^*: \hat{\mathbb{H}}^*(C_{n+1}) \to \hat{\mathbb{H}}^*(S^{(\varepsilon_n)}C_n) \) is an isomorphism. Now, by the quotient \( E/A \) of a pair \((E, A)\) we understand the pair \( C = (C_n)_{n} \), \( (\gamma_n)_{n} \). It may happen that maps \( \gamma_n \) are not homotopy equivalences and therefore \( C \) may not be a spectrum. However, as far as the \( \mathcal{L} \mathcal{F} \)-index is concerned we will always be able to arrange things so that without any extra assumptions the quotient of a pair of spectra is a spectrum itself.
Define a map $\rho: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, $\rho(0) = 0$ and

$$\rho(n) = \sum_{i=0}^{n-1} v(i), \quad n \geq 1.$$  

Let $E = (E_n)_{n=m(E)}, (E_n)_{n=m(E)}$ be a spectrum. For a fixed $q \in \mathbb{Z}$ consider a sequence of cohomology groups

$$\tilde{H}^{q+\rho(n)}(E_n), \quad n \geq n(E)$$

and define a sequence of homomorphisms

$$h_n: \tilde{H}^{q+\rho(n+1)}(E_{n+1}) \to \tilde{H}^{q+\rho(n+1)}(S\rho(n)E_n),$$

where $S\rho$ denotes the suspension isomorphism.

**Definition 3.1.** The $q$th cohomology group of a spectrum $E$ is the inverse limit group

$$H^q(E) = \lim_{\rightarrow} \{\tilde{H}^{q+\rho(n)}(E_n), h_n\}.$$  

Since $E$ is a spectrum there is $n_0 \in \mathbb{N}$ such that $\varepsilon_n$ is a homotopy equivalence whenever $n \geq n_0$. Thus

$$h_n: \tilde{H}^{q+\rho(n+1)}(E_{n+1}) \to \tilde{H}^{q+\rho(n)}(E_n)$$

is an isomorphism if $n \geq n_0$ and the sequence of groups $\tilde{H}^{q+\rho(n)}(E_n)$ stabilizes. Consequently we have the following

**Remark 3.2.** For a fixed spectrum $E$ there is $n_0 \in \mathbb{N}^*$ such that $H^q(E) \cong \tilde{H}^{q+\rho(n)}(E_n)$ for all $n \geq n_0$. Here are some consequences of that observation.

1. the graded group $H^*(E)$ is finitely generated if $\tilde{H}^*(E_n)$ is finitely generated;
2. the spectrum $E$ is of finite type (i.e., $H^*(E)$ is finitely generated and almost all groups are zero) if the space $E_n$ is of finite type.

**Remark 3.3.** Cohomology groups of $E$ can be nontrivial both for positive and negative $q \in \mathbb{Z}$.

Indeed, put $v(n) = 2$ for $n \in \mathbb{N} \cup \{0\}$ and let $E_n := S^{2n-1} \vee S^{2n+1}$, the wedge product of two spheres, $n \geq 1$. Then $\rho(n) = 2n$ and

$$H^q(E) \cong \tilde{H}^{q+\rho(n)}(E_n) = \begin{cases} \mathbb{Z} & \text{for } q = -1 \text{ or } 1 \\ 0 & \text{else.} \end{cases}$$
Let $f: E \to E'$ be a map of spectra and let $\{\tilde{H}^{q + \rho(n)}(E_n), h_n\}$, $\{\tilde{H}^{q + \rho(n)}(E'_n), h'_n\}$ be inverse systems of groups constructed for $E$ and $E'$, respectively. By functoriality of the suspension isomorphism the sequence of group homomorphisms $f_n^{q + \rho(n)}: \tilde{H}^{q + \rho(n)}(E'_n) \to \tilde{H}^{q + \rho(n)}(E_n)$ induced by $f$ satisfies

$$h_n : f^{q + \rho(n + 1)}_n = f^{q + \rho(n)}_n : h'_n$$

for all $n \geq \max\{n(E), n(E')\}$ and thus it defines a group homomorphism

$$f^*: H^q(E') \to H^q(E)$$

on inverse limits. Clearly, two homotopic maps of spectra $f$ and $g$ induce the same homomorphism on cohomology groups.

Let $(E, A)$ be a pair of spectra. For each $q \in \mathbb{Z}$ the quotient $E/A = ((C_n)_{n=0}^{\infty}, (\gamma_n)_{n=0}^{\infty})$ defines an inverse system of cohomology groups

$$\{\tilde{H}^{q + \rho(n)}(C_n), (S^*)^{(n-1)}; \gamma_n^{q + \rho(n)}\}.$$  

We let

$$H^q(E/A) := \lim_{\to} \{\tilde{H}^{q + \rho(n)}(C_n), (S^*)^{(n-1)}; \gamma_n^{q + \rho(n)}\}.$$  

Since $(S^*)^{(n-1)}; \gamma_n^{q + \rho(n)}: \tilde{H}^{q + \rho(n)}(C_n) \to \tilde{H}^{q + \rho(n)}(C_{n-1})$ is an isomorphism for $n > n_0$, the sequence of groups $\tilde{H}^{q + \rho(n)}(C_n)$ stabilizes and therefore $H^q(E/A) \cong \tilde{H}^{q + \rho(n)}(C_n)$ for $n$ sufficiently large. Define the relative cohomology groups as follows.

**Definition 3.4.** For a pair of spectra $(E, A)$ and $q \in \mathbb{Z}$ we define the $q$th cohomology group of $(E, A)$

$$H^q(E, A) := H^q(E/A).$$

Let $(E, A)$ be a pair of spectra. For every pair $(E_n, A_n)$ there is a long exact sequence

$$\cdots \overset{i^*}{\longrightarrow} \tilde{H}^{q-1}(A_n) \overset{\delta^*}{\longrightarrow} \tilde{H}^q(E_n, A_n) \overset{j^*}{\longrightarrow} \tilde{H}^q(E_n)$$

$$\overset{i^*}{\longrightarrow} \tilde{H}^q(E_n) \overset{\delta^*}{\longrightarrow} \cdots$$
with \( n \) sufficiently large. Passing to the inverse limits we obtain the long exact sequence

\[
\ldots \xrightarrow{f^{*-1}} \tilde{H}^{*+1}(A) \xrightarrow{\delta^{*-1}} \tilde{H}^{*}(E, A) \xrightarrow{f^*} \tilde{H}^{*}(E) \xrightarrow{f^*} \tilde{H}^{*}(A) \xrightarrow{\delta^*} \ldots
\]

(3.1)

which is functorial.

Denote by \( r^q(E) \) the rank (dimension if the coefficient ring is the field) of the \( q \)th cohomology group \( H^q(E) \) and by \( d^q(E, A) \) the rank of the image of

\[ \delta^q: H^q(A) \to H^{q+1}(E, A). \]

Assuming all groups in the sequence (3.1) are of finite rank we define the following generalized formal power series:

\[
\mathcal{P}(t, E) = \sum_{q \in \mathbb{Z}} r^q(E) \cdot t^q,
\]

\[
\mathcal{Q}(t, E, A) = \sum_{q \in \mathbb{Z}} d^q(E, A) \cdot t^q.
\]

If \( r^q(E) \) and \( d^q(E, A) \) are 0 for all \( q \) less than some fixed \( q_0 \in \mathbb{Z} \) then \( \mathcal{P}(t, E) \) and \( \mathcal{Q}(t, E, A) \) are called the generalized Poincaré series.

**Lemma 3.5.** Let \((E, A)\) be a pair of spectra such that \( \mathcal{P}(t, E), \mathcal{P}(t, A), \mathcal{P}(t, E/A) \) and \( \mathcal{Q}(t, E, A) \) are generalized Poincaré series. Then

\[
\mathcal{P}(t, E/A) + \mathcal{P}(t, A) = \mathcal{P}(t, E) + (1 + t) \cdot \mathcal{Q}(t, E, A).
\]

**Proof.** Choose \( q_0 \in \mathbb{Z} \) such that ranks of all groups appearing in the sequence (3.1) are zero whenever \( q < q_0 \). From the exactness of (3.1) we conclude that for every \( m \geq 0 \) one has

\[
\begin{align*}
&\quad r^q(E/A) - r^q(E) + r^q(A) - r^{q+1}(E/A) + \ldots \\
&+ (-1)^m r^{q+m}(E/A) - (-1)^m r^{q+m}(E) + (-1)^m r^{q+m}(A) \\
&- (-1)^m d^{q+m}(E, A) = 0.
\end{align*}
\]

From this we deduce that

\[
\begin{align*}
&\quad (-1)^m d^{q+m}(E, A) \\
&= (-1)^{m-1} d^{q+m-1}(E, A) \\
&+ (-1)^m r^{q+m}(E/A) - (-1)^m r^{q+m}(E) + (-1)^m r^{q+m}(A).
\end{align*}
\]
Multiplication of this equation by \((-1)^m\eta_0^m\) and addition over \(m\) yields
\[
\mathcal{P}(t, E, A) = -t \cdot \mathcal{P}(t, E, A) + \mathcal{P}(t, E/A) - \mathcal{P}(t, E) + \mathcal{P}(t, A)
\]
or equivalently
\[
\mathcal{P}(t, E/A) + \mathcal{P}(t, A) = \mathcal{P}(t, E) + (1 + t) \cdot \mathcal{P}(t, E, A)
\]
which proves the lemma. 

Remark 3.6. In fact, formal power series \(\mathcal{P}\) and \(\mathcal{P}\) are well defined for the homotopy type of spectra. This observation will be used in the next section.

4. ATTRACTORS, REPELLERS, AND MORSE DECOMPOSITIONS

In the classical Conley index theory it is well known that a Morse decomposition of an isolated invariant set is “robust” with regards to perturbations (cf. [18]). This is because of the local compactness of the phase space. As we show below the same result holds true for \(\mathcal{L}S\)-flows.

Recall that for a flow \(\eta: \mathbb{R} \times H \to H\) and every \(x \in H\) one can define its \(\omega\) and \(\alpha\) limits:
\[
\omega(x) := \bigcap_{t} \overline{c(\eta([t, \infty), x)} , \quad \alpha(x) := \bigcap_{t} c(\eta([-\infty, t)), x).
\]

Let \(\eta: \mathbb{R} \times H \times A \to H\) be a family of \(\mathcal{L} S\)-flows and assume that \(X \subset H\) is an isolating neighbourhood for a flow \(\eta_{A}\), where \(A \in A\) is fixed. Suppose that \((A, A^*)\) is an attractor–repeller pair in the invariant set \(S = \text{Inv}(X, \eta_{A})\).

Both sets \(A\) and \(A^*\) are isolated invariant sets themselves and therefore there are isolating neighbourhoods \(X_A, X_{A^*} \subset H\) for \(\eta_{A}\) such that \(A = \text{Inv}(X_A, \eta_{A})\) and \(A^* = \text{Inv}(X_{A^*}, \eta_{A})\), respectively. Obviously, we may suppose that \(X_A \cup X_{A^*} \subset X\) and \(X_A \cap X_{A^*} = \emptyset\).

**Theorem 4.1.** There is an open neighbourhood \(V\) of \(\lambda\) in \(A\) such that for each \(\mu \in V\) one has:

1. \(X, X_A, X_{A^*}\) are isolating neighbourhoods for \(\eta_{\mu}\),
2. if \(x \in \text{Inv}(X, \eta_{\mu}) \setminus (\text{Inv}(X_A, \eta_{\mu}) \cup \text{Inv}(X_{A^*}, \eta_{\mu}))\) then
   \[
   \omega(x) \subset \text{Inv}(X_A, \eta_{\mu}) \text{ and } \alpha(x) \subset \text{Inv}(X_{A^*}, \eta_{\mu}).
   \]

**Proof.** The first statement of our theorem is a direct consequence of Theorem 2.1 in [12]. In fact, one can choose a closed neighbourhood \(W \subset A\) such that (1) holds for all \(\mu \in W\). By Proposition 2.1 in [12] we
conclude that $S_W = \text{Inv}(X \times W, \eta_W)$ is compact, where $\eta_W$ is the restriction of the family $\eta$ to the set of indices $W$. Note, that $X_C = \text{cl}(X \setminus (X_D \cup X^*_D))$ is an isolating neighbourhood for $\eta_D$ and $\text{Inv}(X_C, \eta_D) = \emptyset$. Since $S_W$ is compact there is an open neighbourhood of $\lambda$, $V_1 \subset W \subset A$ such that $X_C$ is an isolating neighbourhood for $\eta_D$ and $\text{Inv}(X_C, \eta_D) = \emptyset$ whenever $\mu \in V_1$. The set $X_0 = X_D \cap S$ is a (closed) neighbourhood of $A$ in $S$ and for each $x \in X_0 \setminus D$ the set $\eta_1(( -\infty, 0], x)$ is not included in $X_0$.

Assume that there is a sequence $(\lambda_n) \subset V_1$ converging to $\lambda$ such that

$$A_n = \text{Inv}(X_D \times \{\lambda_n\}, \eta_D)$$

is not an attractor in $S_n = \text{Inv}(X \times \{\lambda_n\}, \eta_D)$. If $X_n = (X_D \times \{\lambda_n\}) \cap S_n$ then by Lemma 3.1 in [26] there is $x_n \in X_n \setminus A_n$ such that $\eta_D(( -\infty, 0], x_n) \subset X_n$. Consequently $\eta_D([0, \infty), x_n)$ is not included in $X_n$ and thus one can define

$$t_n = \sup \{t > 0 : \eta_D([0, t], x_n) \subset X_n\}$$

which is a positive, finite number for each $n \in \mathbb{N}$. Put $y_n = \eta_D(t_n, x_n) \in X_n$ \cap cl$(S \setminus X_0)$ which is easily seen from the construction of the sequence $(y_n)$. Moreover, since for each $n \in \mathbb{N}$, $\eta_D((-\infty, 0], y_n) \subset X_n \subset X_D$ we claim that $\eta_D(( -\infty, 0], y) \subset X_0$ which is a contradiction. Thus there is an open neighbourhood $V \subset V_1$ of $\lambda$ such that $A_n = \text{Inv}(X_D, \eta_D)$ is an attractor in $S_n = \text{Inv}(X, \eta_D)$, $\mu \in V$. Additionally to that, the set $A_n^* = \text{Inv}(X_n^*, \eta_n)$ is the complementary repellor to $A_n$ in $S_n$. Obviously, it may happen that $A_n^* \cap S_n'$ is empty.

Let $\eta$, $X \subset H$ be as above and let $S = \text{Inv}(X, \eta_D)$. Then the finite collection $\{M(\pi) : \pi \in \mathcal{D}\}$ of compact invariant sets in $S$ is said to be a Morse decomposition of $S$ if there exists an ordering $\pi_1, ..., \pi_n$ of $\mathcal{D}$ such that for every $x \in S \setminus \bigcup_{\pi \in \mathcal{D}} M(\pi)$ there exist indices $i, j \in \{1, 2, \ldots, n\}$ such that $i < j$ and $\omega(x) \subset M(\pi_i)$, $\alpha(x) \subset M(\pi_j)$. Every ordering of $\mathcal{D}$ with this property is said to be admissible. The sets $M(\pi)$ are called Morse sets.

**Theorem 4.2.** Let $\{M(\pi) : \pi \in \mathcal{D}\}$ be a Morse decomposition of $S = \text{Inv}(X, \eta_D)$ with $\mathcal{D}$ having $n$ elements. There are closed subsets $X_1, ..., X_n = X = X_0^* \setminus X_n^*$ of $X$ and an open neighbourhood $V$ of $\lambda \in A$ such that for each $\mu \in V$ the following conditions are satisfied:

1. $X_i, X^*_j \ i \in \{1, ..., n\}, j \in \{0, ..., n-1\}$ are isolating neighbourhoods for a flow $\eta_\mu$.
2. $\text{Inv}(X_i \cap X^*_j, \eta_\mu) = M(\pi_i), i = 1, ..., n$. 

Theorem 4.3. Under the above assumptions there exist spectra $E_A$, $E_A^*$, and $E_A^*$ representing $\mathcal{L} \mathcal{F}$-homotopy Conley indices of $X$, $X_A$, and $X_A^*$, respectively, such that the sequence

$$
\cdots \xrightarrow{\delta^{n-1}} H^{n-1}(E_A) \xrightarrow{\delta^{n-1}} H^n(E_A) \xrightarrow{\delta^n} H^n(E_A^*) \xrightarrow{\delta^n} \cdots
$$

(4.1)

is exact.

Proof. We use notation as in the construction of the $\mathcal{L} \mathcal{F}$-homotopy Conley index. Choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $X \cap H^n$, $X_A \cap H^n$, and $X_A^* \cap H^n$ are isolating neighbourhoods for the flow $\eta_n$ and the family of flows $\tilde{\eta}_n$.

Denote by $A_n = \text{Inv}(X \cap H^n, \eta_n)$, $A_n^* = \text{Inv}(X_A \cap H^n, \eta_n)$, and $S_n = \text{Inv}(X \cap H^n, \eta_n)$. By Theorem 4.1 the pair $(A_n, A_n^*)$ is an attractor-repeller pair in $S_n$. In [10] the existence of a filtration of index pairs for
Morse decomposition is proved (see Theorem 3.1). This means in particular, that there is a filtration \((N^0_n \subset N^1_n \subset N^2_n)\) of compact sets in \(X_n\), such that \((N^0_n, N^1_n), (N^1_n, N^2_n)\), and \((N^2_n, N^0_n)\) are index pairs for \(S_n, A_n\), and \(A^*_n\), respectively. Using the same arguments as in the construction of the \(L^\mathcal{P}\)-homotopy Conley index we show that spaces \((E_S, [E_A])\) is a pair of homotopy type of spectra, \([E_A^*] = [E_S/E_A]\) is the homotopy type of a quotient spectrum and all those spectra are of finite type we have the following

**Proposition 4.4.** Under the assumptions of Theorem 4.3 one has

\[
\mathcal{P}(t, [E_A^*]) + \mathcal{P}(t, [E_A]) = \mathcal{P}(t, [E_S]) + (1 + t) \cdot \mathcal{P}(t),
\]

where all coefficients of the generalized power series \(\mathcal{P}(t)\) are non-negative integers.

**Proof.** It is a direct consequence of Lemma 3.5 and Remark 3.6.

We keep all the assumptions and notations as above.

**Proposition 4.5.** Let \((A, A^*)\) be an attractor-repeller pair for the isolated invariant set \(S\) and suppose that \(S = A \cup A^*\). Then

\[
[E_S] = [E_A] \vee [E_A^*] = [E_A \vee E_A^*].
\]

**Proof.** This is a direct consequence of Proposition 2.3 in [12] and the classical version of the above proposition; see, e.g., Theorem 5.8 in [26].

**Corollary 4.6.** Assume that for some \(q \in \mathbb{Z}\) the boundary homomorphism \(\delta^q\) from the long exact sequence (4.1) is nonzero. Then there is a connecting orbit in \(S\) joining \(A\) and \(A^*\).

**Proof.** Suppose that \(S = A \cup A^*\). Then by Proposition 4.5, \([E_S] = [E_A] \vee [E_A^*]\) and

\[
H^q(E_S) = H^q(E_A) \oplus H^q(E_A^*), \quad q \in \mathbb{Z}.
\]

Now, the exactness of (4.1) implies that \(\delta^q = 0\) for each \(q \in \mathbb{Z}\).

In the sequel we use notation as in Theorem 4.2. Let \(\{M(\pi); \mathcal{D}\}\) be a Morse decomposition of \(S = \text{Inv}(X, \eta)\) with \(\mathcal{D}\) having \(n\) elements. Denote
by \([E_{M(n)}]\) the \(\mathcal{L}P\)-homotopy Conley index of \(X_i \cap X_{i+1}\) and by \([E_S]\) the \(\mathcal{L}P\)-index of \(X\).

**Theorem 4.7 (Morse Inequalities).** Under the above assumptions one has

\[
\sum_{i=1}^{n} \mathcal{P}(t, [E_{M(n)}]) = \mathcal{P}(t, [E_S]) + (1 + t) \cdot \mathcal{I}(t),
\]

where the coefficients of the generalized power series \(\mathcal{I}(t)\) are non-negative integers.

**Proof.** Choose any admissible ordering of \(\mathcal{D}, \pi_1, ..., \pi_n\). For each \(i = 1, ..., n\) one can define an attractor-repeller pair

\((\text{Inv}(X_i \cap X_{i+1}, \eta), \text{Inv}(X_{i-1}, \eta))\)

in the set \(\text{Inv}(X_i, \eta)\). Denote by \([E_{X_i}]\) the \(\mathcal{L}P\)-homotopy Conley index of \(X_i\). Note that \([E_{X_1}] = [E_{M(n)}]\) and \([E_{X_n}] = [E_S]\). By Proposition 4.4 we have for \(i = 2, ..., n\) the equalities

\[
\mathcal{P}(t, [E_{M(n)}]) + \mathcal{P}(t, [E_{X_{i-1}}]) = \mathcal{P}(t, [E_{X_i}]) + (1 + t) \cdot \mathcal{I}_{i}(t).
\]

Adding these equations over \(i \geq 2\) and setting

\[
\mathcal{I}(t) = \sum_{i=2}^{n} \mathcal{I}_{i}(t)
\]

one finds

\[
\sum_{i=1}^{n} \mathcal{P}(t, [E_{M(n)}]) = \mathcal{P}(t, [E_S]) + (1 + t) \cdot \mathcal{I}(t).
\]

\[
\Box
\]

### 5. APPLICATIONS TO HAMILTONIAN SYSTEMS

In this section we discuss some examples in which our theory is applied. Our aim is to show that the theory presented here can sometimes give better results than other Morse type theories recently developed. In each example we consider a Hamiltonian function, \(2\pi\)-periodic in \(t\) such that the corresponding Hamiltonian system is asymptotically linear at trivial (constant) solutions and at the infinity. In all cases we have resonance at some trivial solutions. Using Morse inequalities (4.2) we are able to prove the existence of at least one periodic solution of a given problem in addition to trivial ones.
Let us first recall a general setting in which our examples will be discussed. Given a Hamiltonian $G \in C^1(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})$ which is $2\pi$-periodic in $t$ consider the Hamiltonian system of differential equations

$$\dot{z} = J\nabla G(z, t)$$

(5.1)

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is the standard symplectic matrix and $\nabla$ denotes the gradient with respect to $z \in \mathbb{R}^{2n}$.

We will be concerned with the existence of $2\pi$-periodic solutions of (5.1). Let us denote by $H = H^{1,2}(S^1, \mathbb{R}^{2n})$ the Hilbert space of $2\pi$-periodic, $\mathbb{R}^{2n}$-valued functions

$$z(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(kt) + b_k \sin(kt)),$$

where $a_0, a_k, b_k \in \mathbb{R}^{2n}$

with the inner product given by

$$\langle z, z' \rangle_H = 2\pi \langle a_0, a'_0 \rangle + \pi \sum_{k=1}^{\infty} k (\langle a_k, a'_k \rangle + \langle b_k, b'_k \rangle),$$

(5.2)

where $\langle a, b \rangle$ denotes the standard inner product in $\mathbb{R}^{2n}$.

If $|\nabla G(z, t)| \leq c_1 + c_2 \cdot |z|^{s}$ for every $(z, t) \in \mathbb{R}^{2n} \times \mathbb{R}$ and some positive $s$ then $z(t)$ is a $2\pi$-periodic solution of (5.1) if and only if it is a critical point of the functional $\Phi \in C^1(H, \mathbb{R})$ defined by

$$\Phi(z) = \frac{1}{2} \langle L z, z \rangle_H + \phi(z),$$

(5.3)

where

$$\langle L z, z \rangle_H = \int_0^{2\pi} \langle -J\dot{z}, z \rangle dt\quad \text{and} \quad \phi(z) = -\int_0^{2\pi} G(z, t) dt$$

(5.4)

(cf. [20]). It is also shown in [20] that the mapping $\nabla \Phi$ is compact and therefore $\nabla \Phi: H \to H$ is a vector field which can be written in the form

$$\nabla \Phi(z) = Lz + K(z),$$

where $K: H \to H$ is completely continuous.

Choose $e_1, \ldots, e_{2n}$ the standard basis in $\mathbb{R}^{2n}$ and denote

$$H(0) = \text{span}\{e_1, \ldots, e_{2n}\}$$

$$H(k) = \text{span}\{(\cos(kt))\ e_i + (\sin(kt))\ J e_i : i = 1, \ldots, 2n\}, \quad k \in \mathbb{Z} - \{0\}.$$
It is seen from (5.2), (5.3) that $L$ is a differential operator in $H$ which is explicitly given by

$$Lz = \sum_{k=1}^{\infty} -Jb_k \cos(kt) + Ja_k \sin(kt).$$

So, $Lz = 0$ if $z$ is a constant function and $Lz = \pm z$ for every $z \in H(\pm k)$, $k > 0$. Put $H_0 = H(0)$, $H_k = H(k) \oplus H(-k)$, $k = 1, 2, \ldots$. Obviously, $H = \bigoplus_{k=0}^{\infty} H_k$, spaces $H_k$ are mutually orthogonal and $H_0 = \ker L$. For more details we refer the reader to [27].

Thus we conclude that $V\Phi$ is a $\mathcal{C}^\infty$-vector field provided it is locally Lipschitz continuous. Moreover, a map $v : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, which is needed to define spectrum is constant and for each $k \in \mathbb{N} \cup \{0\}$ one has $v(k) = 2n$.

Finally, let us recall that if $A$ is a symmetric $2n \times 2n$-matrix and

$$\dot{z} = JAz$$

is a linear Hamiltonian system then the vector field $V\Phi : H \to H$ corresponding to that system preserves all spaces $H_k$ and the restriction of $V\Phi$ to $H_k$, $k \geq 1$, may be identified with the linear map on $\mathbb{R}^{4n}$ whose matrix is

$$T_k(A) = \begin{bmatrix} \frac{1}{k} A & -J \\ J & \frac{1}{k} A \end{bmatrix} \quad (5.5)$$

and with $-A$ on $\mathbb{R}^{2n}$ if $k = 0$ (see [27]).

The following generalized Morse index and the nullity have been defined by Amann and Zehnder (see [2]).

$$i^- (A) := M^- (-A) + \sum_{k=1}^{\infty} (M^- (T_k(A)) - 2n)$$

$$i^0 (A) := M^0 (-A) + \sum_{k=1}^{\infty} M^0 (T_k(A)),$$

where $M^- (B)$ is the number (with multiplicity) of negative eigenvalues of a symmetric matrix $B$ and $M^0 (B)$ is the dimension of its kernel.

If $i^0 (A) = 0$ then $V\Phi$ is a linear isomorphism. In such a case $S := \{0\}$ is an isolated invariant set for the flow $\eta$ induced by $V\Phi$. Moreover, if $r > 0$ then $D(r) := \{z \in H : \|z\| \leq r\}$ is an isolating neighbourhood and

$$h_{\mathcal{N}\eta} (D(r), \eta) = [E],$$
where $E$ is a spectrum such that, $E_k = S^{n(k)}$ with $p(k) = i^-(A) + k \cdot 2n$ for sufficiently large $k$. Thus, using elementary properties of $h_{x,y}$ we obtain the following two observations:

**Remark 5.1.** Assume that $G(z, t) = \frac{1}{2} \langle A_z(z - z_0), z - z_0 \rangle + g(z, t)$ where $z_0 \in \mathbb{R}^m \subset H$, $A_z$ is linear symmetric and $\tilde{V}g(z, t) = o(|z|)$ uniformly in $t$ as $z \to z_0$. If $i^{n(A_z)} = 0$ then for $r$ sufficiently small and positive $D_{z_0}(r):= \{ z \in H : ||z - z_0|| \leq r \}$ is an isolating neighbourhood of $S = \{ z_0 \}$ and

$$h_{x,y}(D_{z_0}(r), \eta) = [E],$$

where $E_k = S^{n(k)}$ for $k$ sufficiently large.

**Remark 5.2.** Assume that $G(z, t) = \frac{1}{2} \langle A_z, z, z \rangle + g(z, t)$, where $A_z$ is linear symmetric and $\tilde{V}g(z, t)$ is bounded. If $i^{n(A_z)} = 0$ then for $r$ sufficiently large $D(r)$ is an isolating neighbourhood and

$$h_{x,y}(D(r), \eta) = [E],$$

where $E_k = S^{n(k)}$ for $k$ sufficiently large.

**Example 5.1.** Let $n = 1$ and $G: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be such that:

- \((c1)\) $G(x, y, t) = \frac{1}{2}(x^2 + y^2) + (x^2 + y^2)^2 + h(x, y, t)$ if $x^2 + y^2 \leq a_1$, $a_1 > 0$, with $h(x, y, t)$ being a smooth perturbation of order higher than 4 (with resp. to $x$ and $y$ variables);

- \((c2)\) $G(x, y, t) = \frac{1}{2}(x - x_0)^2 + \frac{1}{2}(y - y_0)^2 + ((x - x_0)^2 - 3(x - x_0)(y - y_0)^2) \cdot \cos(3t)$ if $(x - x_0)^2 + (y - y_0)^2 \leq a_2$, for some $(x_0, y_0) \neq (0, 0), a_2 > 0$;

- \((c3)\) $G(x, y, t) = \frac{1}{2}d(x^2 + y^2) + g(x, y, t)$, if $x^2 + y^2 \geq a_3$, $a_3 > 0$, $d > 0$ is not an integer and the derivative of $g(x, y, t)$ is bounded.

Clearly $z(t) = 0$ and $z(t) = (x_0, y_0)$ are trivial solutions of (5.1). It follows from our assumptions that an $x,y$-vector field $\Phi: H \to H$ has derivatives $\mathcal{A}_0$ at 0, $\mathcal{A}_z$ at $z_0 = (x_0, y_0)$ and $\mathcal{A}_\infty$ at the infinity which are selfadjoint operators. Additionally, $\mathcal{A}_\infty$ is an isomorphism and $\ker \mathcal{A}_0 = \ker \mathcal{A}_z$. Thus, $\mathcal{A}_\infty$ is the space of solutions of the linearizations of system (5.1)

$$\dot{z} = J\nabla^2 G(0, t) z = J\nabla^2 G(z_0, t) z = Jz, \quad z = (x, y) \in \mathbb{R}^2$$

(5.6)

at trivial solutions. This means in particular that 0 and $z_0$ are degenerate critical points for $\Phi: H \to \mathbb{R}$. Put $V_2 = \ker \mathcal{A}_0 = \ker \mathcal{A}_z$. Thus, $V_2$ is a subspace of dimension 2 spanned by $u_1(t) = \cos(t) \cdot e_1 + \sin(t) \cdot e_2$ and $u_2(t) = -\sin(t) \cdot e_1 + \cos(t) \cdot e_2$.
and this is in fact the space $H(1) = H_1$. Subspaces $V_1 = im \delta_0 = im \delta_0$ and $V_2$ are orthogonal in $H$ and $H = V_1 \oplus V_2$. Let $\pi: H \to H$ be the orthogonal projection onto $V_2$. Define $F: V_1 \oplus V_2 \to V_2$, $F(v_1, v_2) = \pi \circ \nabla \Phi(v_1, v_2)$. Choosing the basis $\{u_1, u_2\}$ we introduce a coordinate system in $V_2$. For each $v_2 \in V_2$ one has

$$v_2(t) = a \cdot u_1(t) + b \cdot u_2(t) = (a \cos(t) - b \sin(t)) e_1 + (a \sin(t) + b \cos(t)) e_2, \quad a, b \in \mathbb{R}.$$ 

For every constant map $z \in H$ one has $z = (z, 0) \in V_1 \oplus V_2$. It has been shown in [12] that up to a positive multiplier

$$F(z_0, (a, b)) = (a^2 - b^2, -2ab)$$

in a neighbourhood $U_0$ of $0 \in V_2$ if $z_0 = 0$. Obviously, the same arguments work if $0$ is replaced by an arbitrary $z_0 \in \mathbb{R}^{2m}$ as far as $c_2$ is satisfied. Moreover, Corollary 5.5 from [12] implies that $z_0 \in H$ is isolated in $\nabla \Phi^{-1}(0)$. Similarly, one shows that (up to a positive multiplier)

$$F(0, (a, b)) = (a^3 + ab^2, a^3b + b^3)$$

in a neighbourhood $U_0$ of $0 \in V_2$ and again by Corollary 5.5 from [12] $0 \in H$ is an isolated in $\nabla \Phi^{-1}(0)$. Since we deal with a gradient vector field there are isolating neighbourhoods $X_0$ and $X_2$ for the $\partial' F$-flow $\eta$ generated by such that $\text{Inv}(X_0, \eta) = \{0\}$ and $\text{Inv}(X_0, \eta) = \{z_0\}$.

Now, if the flow on $V_2$ is generated by $F(z_0, \cdot): V_2 \to V_2$ then we easily compute the Conley index of $[z_0]$ which is equal to $[S^1 \vee S^1, \ast]$, the homotopy type of the join of 2 copies of 1-dimensional pointed spheres. Using (5.5) (for $A_{z_0} = Id$: $\mathbb{R}^2 \to \mathbb{R}^2$) we then find that the $\partial' F$-index of $X_{z_0}$ with respect to $\eta$, $h_{\partial' F}(X_{z_0}, \eta)$ is equal to the homotopy type of spectrum $E$ for which one has $E_k = S^{2k+1} \vee S^{2k+1}$, the wedge of two pointed spheres of dimension $2k+3$, $k = 1, 2, \ldots$. Similarly, if the flow on $V_2$ is generated by $F(0, \cdot): V_2 \to V_2$ then the Conley index of $\{0\}$ is equal to $[S^3]$, the homotopy type of 2-dimensional sphere. Thus, by (5.5) (for $A_0 = Id$: $\mathbb{R}^2 \to \mathbb{R}^2$) the Conley index of $X_0$, $h_{\partial' F}(X_0, \eta)$ is equal to the homotopy type of spectrum $E$ such that $E_k = S^{2k+2}$, a pointed sphere of dimension $2k+2$, $k = 1, 2, \ldots$. Since the derivative of $\nabla \Phi$ at the infinity is an isomorphism there is an isolating neighbourhood $X_0$ for $\eta$ such that $S = \text{Inv}(X_0, \eta)$ is the maximal bounded isolated invariant set for $\eta$ in $H$.

Applying Remark 5.2 we find that the Conley index $h_{\partial' F}(X_0, \eta)$ is the homotopy type of spectrum $E^r$ such that $E_k^r = S^{2k+2m}$, a pointed sphere of dimension $2k+2m$ if $d \in (m-1, m)$ and $k$ is sufficiently large.
From the inequality \([E \setminus E'] \neq [E']\) we obtain \(S \neq \{0, z_0\}\). It might happen that \(S\) consists of two stationary points and a set of connecting orbits. We show that except \(\{0\}\) and \(\{z_0\}\) there is another rest point of \(\eta\) in \(S\). Suppose, on the contrary, that \(\{0\}\) and \(\{z_0\}\) are the only stationary points in \(S\). Since \(\eta\) is generated by the gradient vector field both points are Morse sets in \(S\) and the collection of sets \(\{M_1 = \{0\}, M_2 = \{z_0\}\}\) defines a Morse decomposition of \(S\). We easily compute the cohomology groups of spectra \(E, E', \) and \(E''\):

\[
H^q(E) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } q = 3, \\ 0 & \text{for } q \neq 3, \end{cases} \quad H^q(E') = \begin{cases} \mathbb{Z} & \text{for } q = 4 \\ 0 & \text{for } q \neq 4 \end{cases}
\]

\[
H^q(E'') = \begin{cases} \mathbb{Z} & \text{for } q = 2m \\ 0 & \text{for } q \neq 2m. \end{cases}
\]

Applying Theorem 4.7 we obtain the equality

\[2t^3 + t^4 = t^{2m} + (1 + t) \varphi(t)\] (5.7)

which cannot be true due to the fact that all coefficients of \(\varphi\) are non-negative. In fact, this proves that except two trivial solutions \(z_0(t) = 0\) and \(z_1(t) = (x_0, y_0)\) the Hamiltonian system (5.1) satisfying (c1), (c2), and (c3) has a third periodic solution. However, if in (5.7) \(t\) is replaced by \(-1\) we obtain \(-1 = 1\) which obviously says that (5.7) does not hold even if we know nothing about the polynomial \(\varphi(t)\). This may suggest that we are able to prove the existence of a third critical point of \(\Phi\) without using the gradient structure of the vector field \(\nabla \Phi\) and consequently without Morse theory methods.

Indeed, consider a linear isomorphism \(\tilde{L} : H \to H\)

\[
\tilde{L} = \begin{cases} Lz & \text{if } z \in \bigoplus_{k=1}^{\infty} H_k \\ z & \text{if } z \in H_0. \end{cases}
\]

Let \(U \subset H\) be an open and bounded set and let \(f: \tilde{U} \to H\) be a map of the form \(f(z) = \tilde{L}(z) + K(z)\), where \(K: \tilde{U} \to H\) is completely continuous and \(f\) has no zeroes on the boundary of \(\tilde{U}\). Then one defines the Leray–Schauder degree with respect to \(\tilde{L}\)

\[\deg(f, U, 0) := \deg_{LS}(\text{Id} + \tilde{L}^{-1} \circ K, U, 0).\]

Now, an easy computation shows that in our example the local degree of

\[\nabla \Phi(z) = Lz + K(z) = \tilde{L}z + \tilde{K}(z)\]
at 0 is equal to 1, at \( z_0 \) is equal to \(-2\) and if \( U \) is a disc centered at 0 with radius sufficiently large then \( \deg(\nabla \Phi, U, 0) = 1 \). By additivity of the Leray–Schauder degree we conclude that additionally to \( \{0\} \) and \( \{z_0\} \) there is at least one point \( \bar{z} \in H \) such that \( \nabla \Phi(\bar{z}) = 0 \).

In the next example this kind of argument cannot be used.

**Example 5.2.** Assume that additionally to (c1), (c2) and (c3) a map \( G: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies:

(c4) there are points \( z_1, z_2 \in \mathbb{R}^2 \), \( z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_1 \neq z_2 \) such that

\[
G(x, y, t) = \frac{1}{2}d_i((x-x_i)^2 + (y-y_i)^2) + h_i(x, y, t) \quad \text{if} \quad (x-x_i)^2 + (y-y_i)^2 < a_i, \quad a_i > 0 \quad i = 1, 2,
\]

where

- \( h_i(x, y, t) \) is a smooth perturbation of order higher than 2 at \((x_i, y_i)\) (with respect to \( x \) and \( y \) variables), \( i = 1, 2 \);
- \( d_1, d_2 > 0 \) are not integers.

This time the \( \mathbb{S}^2 \times \mathbb{R} \)-vector field \( \nabla \Phi: H \rightarrow H \) has derivatives at four points: \( 0, z_0, z_1, \) and \( z_2 \) which are critical points for \( \Phi \) and has the derivative at the infinity. One easily checks that the derivatives \( \partial_{x_i}, \partial_{y_i} \) of \( \nabla \Phi \) at \( z_1 \) and \( z_2 \) are isomorphisms. As \( z_1 \) and \( z_2 \) are nondegenerate critical points for \( \Phi \) there are isolating neighborhoods \( X_i \) for \( \eta \) such that \( \Inv(X_i, \eta) = \{ z_i \} \) for \( i = 1, 2 \). By Remark 5.1 we find the Conley index \( h_{\mathbb{S}^2 \times \mathbb{R}}(X_i, \eta) = [E^i] \) with

\[
E_k = S^{2k+2m}, \quad \text{a pointed sphere of dimension } 2k+2m, \text{ if } d_i \in (m_i - 1, m_i)
\]

for \( i = 1, 2 \), and \( k = 1, 2, ... \).

Since \([ E \vee E' \vee E^1 \vee E^2 ] \neq [ E^\ast ]\) we conclude that \( S \neq \{ 0, z_0, z_1, z_2 \} \). It turns out that one can give sufficient conditions for the existence of at least 5 rest points of \( \eta \) in \( S \) in terms of \( m_1, m_2 \), and \( m \). Indeed, assume that \( \eta \) has exactly four rest points in \( S \). Thus a Morse decomposition of \( S \) is defined by the collection of sets \( \{ M_1 = \{ 0 \}, M_i = \{ z_i \} : i = 0, 1, 2 \} \). We compute the cohomology groups of spectra \( E' \):

\[
H^q(E') = \begin{cases} 
\mathbb{Z} & \text{for } q = 2m_i, \\
0 & \text{for } q \neq 2m_i, 
\end{cases} \quad i = 1, 2.
\]

Applying Theorem 4.7 we obtain the equality

\[
2t^3 + t^4 + t^{2m_1} + t^{2m_2} = t^{2m} + (1 + t) \delta(t). \tag{5.8}
\]

Now, Eq. (5.8) does not hold in the following cases:

(T1) \( m = 1 \) and \( (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \setminus \{(1, 1), (2, 1), (1, 2)\} \);

(T2) \( m > 1 \) and \( (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \setminus \{(1, m), (2, m), (m, 1), (m, 2)\} \).
which is a consequence of the fact that \( \mathcal{A}(t) \) has nonnegative coefficients. Thus, if either (T1) or (T2) is satisfied then \( \eta \) has at least 5 rest points in \( S \). Notice, that both sides of (5.8) coincide at \( t = -1 \) so the polynomial \( \mathcal{A}(t) \) has been used in an essential way. The Leray-Schauder degree gives us no extra information as well. One easily computes local degrees at \( z_1 \) and \( z_2 \) which are equal to 1 so that the sum of local degrees is equal to the global degree.

In the third example all critical point of \( \Phi \) we know about are degenerate.

**Example 5.3.** Assume that additionally to (c1) and (c3) a map \( G: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) satisfies:

\[
\text{(c5) } G(x, y, t) = \frac{1}{2}(x - x_1)^2 + \frac{1}{2}(y - y_2)^2 - (x - x_1)^4 + h_1(x, y, t) \quad \text{if } (x - x_2)^2 + (y - y_2)^2 \leq \varepsilon_2, \text{ for some } (x_1, y_1), \varepsilon_2 > 0;
\]

\[
\text{(c6) } G(x, y, t) = \frac{1}{2}(x - x_2)^2 + \frac{1}{2}(y - y_2)^2 + (x - x_2)^4 \cdot \cos(2t) + h_2(x, y, t) \quad \text{if } (x - x_2)^2 + (y - y_2)^2 \leq \varepsilon_3, \text{ for some } (x_2, y_2), \varepsilon_3 > 0;
\]

where \( h_i(x, y, t) \) are smooth perturbations of order higher than 4 at \( z_1 = (x_1, y_1) \) and \( z_2 = (x_2, y_2), i = 1, 2 \) (with respect to \( x \) and \( y \) variables). Similarly to the above examples we check that \( \nabla \Phi \) is a \( \mathcal{G}/\mathcal{F} \)-vector field having derivatives \( \mathcal{A}_0, \mathcal{A}_z, \text{ and } \mathcal{A}_{z_2} \) at 0, \( z_1 \), and \( z_2 \), respectively, and the derivative \( \mathcal{A}_\infty \) at the infinity. Since

\[\ker \mathcal{A}_0 = \ker \mathcal{A}_z = \ker \mathcal{A}_{z_2} = H(1)\]

points 0, \( z_1 \) and \( z_2 \) are degenerate critical points for \( \Phi \). In the sequel, we use the same notation as in Example 5.1. We already know Conley indices for \( X_0 \) and \( X_z \). Using the same methods as in [12] we find that up to a positive multiplier

\[F(z_1, (a, b)) = -1 \cdot (a^1 \cdot a^2 + a^3 b + b^1)\]

and

\[F(z_2, (a, b)) = (a^1, -b^3)\]

in a neighbourhood \( U \) of \( 0 \in V_2 \). This implies that \( z_1 \) and \( z_2 \) are isolated in \( \mathcal{V} \Phi^{-1}(0) \). As we deal with the gradient vector field there are isolating neighbourhoods \( X_1, X_2 \) for \( \eta \) such that \( \text{Inv}(X_1, \eta) = \{z_1\} \) and \( \text{Inv}(X_2, \eta) = \{z_2\} \). Proceeding the same way as in Example 5.1 we find that \( h_{x,\mathcal{F}}(X_1, \eta) \) is the homotopy type of a spectrum \( E^1 \) such that \( E^1_k = S^{2k+4} \), the homotopy type of a pointed sphere if dimension \( 2k+4 \) and \( h_{x,\mathcal{F}}(X_2, \eta) \) is the homotopy type of a spectrum \( E^2 \) with \( E^2_k = S^{2k+3} \), the homotopy type of a pointed
sphere of dimension $2k + 3$. The inequality $[E \vee E^1 \vee E^2] \neq [E^*]$ implies $S \neq \{0, z_1, z_2\}$. Similarly to the above we compute the cohomology groups of spectra,

$$H^s(E^1) = \begin{cases} \mathbb{Z} & \text{for } q = 4, \\ 0 & \text{for } q \neq 4, \end{cases} \quad H^s(E^2) = \begin{cases} \mathbb{Z} & \text{for } q = 3 \\ 0 & \text{for } q \neq 3 \end{cases}$$

and assuming that $0, z_1$ and $z_2$ are the only stationary points for $\eta$ in $S$ we derive

$$t^3 + 2t^4 = t^{2m} + (1 + t) \beta(t). \quad (5.9)$$

Since $\beta(t)$ has positive coefficients (5.9) does not hold whenever $m \neq 2$. Thus, if $m \neq 2$ there are at least 4 rest point of $\eta$ in $S$. The Leray–Schauder degree gives us no extra information. The local degrees of $V\phi$ at $z_1$ and $z_2$ are equal to 1 and $-1$, respectively, so that the sum of local degrees is equal to the global one.

Note added in proof. The author thanks Jacek Tabor from Jagellonian University for pointing out that in the construction of the $\mathcal{L}_\theta$-index in [12] the statement: “Clearly $(Y_n \times D_{n+1}^* \times D_{n+1}^*, Z_n \times D_{n+1}^* \times D_{n+1}^*)$ is an index pair for the isolated invariant set $S_n$ with respect to the flow $\xi_t(\cdot, 0)$” is incorrect. The correct description of that part of our construction is given in this paper.

REFERENCES


