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Almost sure convergence for the maxima of strongly dependent stationary Gaussian vector sequences ☆

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A B S T R A C T

In this paper, we prove the almost sure limit theorem of the maxima for a kind of strongly dependent stationary Gaussian vector sequences.

1. Introduction

The almost sure limit theorem (ASLT) for the maximum of independent identically distributed (i.i.d.) random variables has been studied by Fahrner and Stadtmüller [6] and Cheng et al. [4] respectively. For more related works on ASLT, see [1,10–13]. For the weakly dependent stationary Gaussian sequence $\{X_n, n \geqslant 1\}$ with $\mathrm{E} X_1 = 0$, $\mathrm{Var}\, X_1 = 1$, Csáki and Gonchigdanzan [5] obtained the ASLT for the maxima if their correlation $r_n = \mathrm{E} X_1 X_{n+1}$ satisfies $r_n \log n (\log\log n)^{1+\varepsilon} = O(1)$ for some $\varepsilon > 0$. For some extensions see [2] and [9]. Chen et al. [3] studied the ASLT of extremes for weakly dependent stationary Gaussian vector sequences. For some potential applications of ASLT, see [10]. Lin [8] extended this principle to a kind of strongly dependent Gaussian sequence. He proved

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}\left(\frac{M_k - b_k}{a_k} \leqslant x\right) = \int_{-\infty}^{\infty} \exp\left(-e^{-x-\rho+\sqrt{2\rho}z}\right)\phi(z)\,dz \quad \text{a.s.} \tag{1.1}$$

if

$$|r_n \log n - \rho| (\log\log n)^{1+\varepsilon} = O(1), \tag{1.2}$$

where the normalizing constants are

$$a_n = (2\log n)^{-1/2}, \qquad b_n = a_n^{-1} - a_n(\log\log n + \log 4\pi)/2. \tag{1.3}$$

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For the multivariate setting, let $\{\mathbf{X}_s, s \geqslant 1\}$ be a standard stationary d-dimensional Gaussian sequence, i.e., $\mathbf{X}_s = (X_{s1}, X_{s2}, \ldots, X_{sd})$ with $\mathrm{E}X_{sj} = 0, \mathrm{Var}\, X_{sj} = 1$ and correlations $r_{ij}(|t-s|) = \mathrm{Cov}(X_{si}, X_{tj})$ for $1 \leqslant i, j \leqslant d,\ s, t \geqslant 1$. Suppose that $r_{ij}(n)$ satisfies

$$r_{ij}(n)\log n \to \rho_{ij} \in (0, \infty), \quad 1 \leqslant i, j \leqslant d, \tag{1.4}$$

as $n \to \infty$ and

$$\sup_{\substack{1\leqslant i,j\leqslant d\\ n\geqslant 1}} \left|r_{ij}(n)\right| < 1. \tag{1.5}$$

Let $\mathbf{M}_n^{(k)}$ stand for the d-dimensional vector of kth extreme-order statistic of the sequence $\{\mathbf{X}_s, 1 \leqslant s \leqslant n\}$, where $1 \leqslant k \leqslant n$, i.e., $\mathbf{M}_n^{(k)} = (M_{n1}^{(k)}, M_{n2}^{(k)}, \ldots, M_{nd}^{(k)})$, where $M_{nj}^{(k)}$ denotes the kth order statistic of $\{X_{sj}, 1 \leqslant s \leqslant n\}$, $j = 1, 2, \ldots, d$. Thus we have $\mathbf{M}_n^{(1)}$ as the vector maxima and $\mathbf{M}_n^{(n)}$ as the vector minima. We also define the normalized constants

$$\mathbf{a}_n = (a_n, \ldots, a_n), \qquad \mathbf{b}_n = (b_n, \ldots, b_n)$$

where a_n and b_n are defined by (1.3). Wiśniewski [14] investigated the limit distribution of $\mathbf{M}_n^{(k)}$ for a kind of Gaussian vector sequence with equal correlation. Under conditions (1.4) and (1.5), Wiśniewski [15] studied the limiting distribution of the d-variate point processes of exceedances formed by $\{\mathbf{X}_s, 1 \leqslant s \leqslant n\}$ and obtained:

Theorem A. *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be a standardized stationary Gaussian vector sequence satisfying* (1.4) *and* (1.5), *then for fixed k,*

$$\frac{\mathbf{M}_n^{(k)} - \mathbf{b}_n}{\mathbf{a}_n} \xrightarrow{d} \mathbf{M}_{\boldsymbol{\rho}}^{(k)} + \mathbf{R}_{\boldsymbol{\rho}}\mathbf{Z}, \tag{1.6}$$

where $\mathbf{R}_{\boldsymbol{\rho}} = (\sqrt{2\rho_{ii}})_{1\leqslant i\leqslant d}$ and $\mathbf{Z}$ is the standard Gaussian vector with the covariance coefficients $\mathrm{Cov}(Z_i, Z_j) = \rho_{ij}/\sqrt{\rho_{ii}\rho_{jj}}$. $\mathbf{Z}$ and $\mathbf{M}_{\boldsymbol{\rho}}^{(k)}$ are independent and

$$P\big(\mathbf{M}_{\boldsymbol{\rho}}^{(k)} \leqslant \mathbf{x}\big) = \prod_{i=1}^{d} \exp\big(-e^{-x_i-\rho_{ii}}\big) \sum_{m=0}^{k-1} \frac{(e^{-x_i-\rho_{ii}})^m}{m!}.$$

This paper is devoted to the study of the ASLT for the maxima of $\{\mathbf{X}_s, 1 \leqslant s \leqslant n\}$ under conditions similar to (1.2). This paper is organized as follows: In Section 2, we give the main result. Related proofs are provided in Section 3.

2. Main result

In this section we state our main result, i.e., for the ASLT of the maxima of $\{\mathbf{X}_s, 1 \leqslant s \leqslant n\}$. The main result is:

Theorem 2.1. *Let $\{\mathbf{X}_s, s \geqslant 1\}$ be a sequence of d-dimensional stationary standard Gaussian random vectors with correlation $r_{ij}(s)$ satisfying conditions* (1.4) *and* (1.5), *and additionally for $1 \leqslant i, j \leqslant d$ let the following condition hold*:

$$\left|r_{ij}(n) - t_{ij}(n)\right|(\log n)(\log\log n)^{1+\varepsilon} = O(1). \tag{2.1}$$

Then we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{s=1}^{n} \frac{1}{s} \mathbb{I}\left(\frac{\mathbf{M}_s - \mathbf{b}_s}{\mathbf{a}_s} \leqslant \mathbf{x}\right) = (\Lambda_{\boldsymbol{\rho}} * \Phi_{\boldsymbol{\rho}})(\mathbf{x}), \quad a.s.$$

for $\mathbf{x} \in R^d$, where $t_{ij}(n) = \rho_{ij}/\log n$, $\mathbf{M}_s = \mathbf{M}_s^{(1)}$, and $\boldsymbol{\rho} = (\rho_{11}, \rho_{22}, \ldots, \rho_{dd})$. Operator $*$ denotes convolution, and

$$\Lambda_{\boldsymbol{\rho}}(\mathbf{x}) = \Lambda(\mathbf{x} + \boldsymbol{\rho}), \qquad \Lambda(\mathbf{x}) = \prod_{i=1}^{d} \exp\big(-e^{-x_i}\big),$$

$$\Phi_{\boldsymbol{\rho}}(\mathbf{x}) = \Phi\big(2^{-1/2}\mathbf{x}\mathbb{A}^{-1}(\boldsymbol{\rho})\big)$$

and Φ represents the joint distribution function of a Gaussian vector $\mathbf{Y}_0$ with $\mathrm{Cov}(\mathbf{Y}_0) = (\rho_{ij}/\sqrt{\rho_{ii}\rho_{jj}})_{1\leqslant i,j\leqslant d}$ and $\mathrm{E}\mathbf{Y}_0 = \mathbf{0}$. $\mathbb{A}^{-1}(\mathbf{t})$ is the inverse of $\mathbb{A}(\mathbf{t})$, and the latter is defined by (3.2).

3. Proofs of the main results

We first define a triangular array of d-dimensional random Gaussian vectors $\{\mathbf{Y}_m^{(n)}\colon 1 \leqslant m \leqslant n,\ n > 1\}$, the rows of $\{\mathbf{Y}_m^{(n)}\colon 1 \leqslant m \leqslant n\}$ are standard Gaussian equally correlated sequences with the covariance coefficients

$$\mathrm{Cov}\big(Y_{mi}^{(n)}, Y_{mj}^{(n)}\big) = r_{ij}(0), \qquad \mathrm{Cov}\big(Y_{mi}^{(n)}, Y_{lj}^{(n)}\big) = t_{ij}(n), \tag{3.1}$$

where $1 \leqslant m \neq l \leqslant n$ and $n > 1$. And suppose that $\{\mathbf{X}_s,\ s \geqslant 1\}$ and $\{\mathbf{Y}_m^{(n)}\colon 1 \leqslant m \leqslant n,\ n > 1\}$ are independent. Write $\mathbf{M}_n^*$ for the vector maxima in $\{\mathbf{Y}_m^{(n)}\colon 1 \leqslant m \leqslant n\}$. For each $\mathbf{s} \in (0,\infty)^d$ and $\mathbf{v} \in (0,\infty)^d$ denote

$$\mathbb{A}(\mathbf{s}) = \begin{bmatrix} s_1^{1/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_d^{1/2} \end{bmatrix}, \qquad \mathbb{B}(\mathbf{v}) = \begin{bmatrix} (1-v_1)^{1/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (1-v_d)^{1/2} \end{bmatrix}. \tag{3.2}$$

The following representation of the standard Gaussian array $\{\mathbf{Y}_m^{(n)}\colon 1 \leqslant m \leqslant n,\ n > 1\}$ is due to Wiśniewski [14]:

Lemma 3.1. *Assume that the standard Gaussian array $\{\mathbf{Y}_m^{(n)}\colon 1 \leqslant m \leqslant n,\ n > 1\}$ satisfies condition (3.1). Then the rows of the array can be represented by means of sums of independent vectors in the following way:*

$$\big(\mathbf{Y}_1^{(n)}, \ldots, \mathbf{Y}_n^{(n)}\big) \overset{\text{a.s.}}{=} \big(\mathbf{Z}_0^{(n)}\mathbb{A}\big(\mathbf{t}(n)\big) + \mathbf{Z}_1^{(n)}\mathbb{B}\big(\mathbf{t}(n)\big), \ldots, \mathbf{Z}_0^{(n)}\mathbb{A}\big(\mathbf{t}(n)\big) + \mathbf{Z}_n^{(n)}\mathbb{B}\big(\mathbf{t}(n)\big)\big),$$

where $\mathbf{t}(n) = (t_{11}(n), \ldots, t_{dd}(n))$ and $\{\mathbf{Z}_m^{(n)}\colon m \in \{0\} \cup N\}$ is an independent Gaussian sequence with covariance matrices

$$\mathrm{Cov}\big(\mathbf{Z}_0^{(n)}\big) = \left(\frac{t_{ij}(n)}{\sqrt{t_{ii}(n)t_{jj}(n)}}\right)_{1 \leqslant i,j \leqslant d},$$

$$\mathrm{Cov}\big(\mathbf{Z}_m^{(n)}\big) = \left(\frac{r_{ij}(0) - t_{ij}(n)}{\sqrt{(1-t_{ii}(n))(1-t_{jj}(n))}}\right)_{1 \leqslant i,j \leqslant d}$$

and with vectors of mean values

$$\mathrm{E}\mathbf{Z}_0^{(n)} = \mathrm{E}\mathbf{Z}_m^{(n)} = \mathbf{0}.$$

Proof. See Proposition 1 of [14]. □

The following bound is from the Normal Comparison Lemma (cf. [7]):

Lemma 3.2. *Suppose that the Gaussian vector sequence $\{\mathbf{X}_n,\ n \geqslant 1\}$ satisfies the conditions (1.4) and (1.5) and the triangular Gaussian array $\{\mathbf{Y}_k^{(n)},\ 1 \leqslant k \leqslant n\}$ satisfies (3.1). Set $\mathbf{u}_n = \mathbf{a}_n\mathbf{x} + \mathbf{b}_n$, then we have*

$$\big|P(\mathbf{M}_n \leqslant \mathbf{u}_k) - P\big(\mathbf{M}_n^* \leqslant \mathbf{u}_n\big)\big| \leqslant \mathfrak{C} \sum_{i,j=1}^{d} \sum_{1 \leqslant s,t \leqslant n} \big|r_{ij}\big(|t-s|\big) - t_{ij}(n)\big| \exp\left\{-\frac{u_{ki}^2 + u_{nj}^2}{2(1+\omega_{ij}(|t-s|,n))}\right\},$$

where $\omega_{ij}(s,n) = \max\{|r_{ij}(s)|, t_{ij}(n)\}$ and $\mathfrak{C}$ is an absolute positive constant which may change from line to line.

Proof. Using Theorem 4.2.1 in [7], we get the desired result. □

Lemma 3.3. *Suppose that the condition (2.1) holds and set $\mathbf{u}_n = \mathbf{a}_n\mathbf{x} + \mathbf{b}_n$. Then for large n we have*

$$\sup_{1 \leqslant k \leqslant n} k \sum_{i,j=1}^{d} \sum_{s=1}^{n} \big|r_{ij}(s) - t_{ij}(n)\big| \exp\left\{-\frac{u_{ki}^2 + u_{nj}^2}{2(1+\omega_{ij}(s,n))}\right\} \leqslant \mathfrak{C}(\log\log n)^{-(1+\varepsilon)}.$$

Proof. As $t_{ij}(n) = \rho_{ij}/\log n$, for large n we have

$$\sigma(1) := \sup_{\substack{1 \leqslant m,l \leqslant n \\ 1 \leqslant i,j \leqslant d}} \big|\omega_{ij}(m,l)\big| < 1$$

by (1.5). According to Leadbetter et al. [7], for large n we have

$$\exp\left\{-\frac{u_{ni}^2}{2}\right\} \sim \frac{\mathfrak{C}u_{ni}}{n}, \qquad u_{ni} \sim (2\log n)^{\frac{1}{2}}, \quad \text{for } 1 \leqslant i \leqslant d. \tag{3.3}$$

Notice

$$\begin{aligned}
&k\sum_{i,j=1}^{d}\sum_{s=1}^{n}\left|r_{ij}(s)-t_{ij}(n)\right|\exp\left\{-\frac{u_{ki}^2+u_{nj}^2}{2(1+w_{ij}(s,n))}\right\}\\
&\quad=k\sum_{i,j=1}^{d}\sum_{s=1}^{[n^\alpha]}\left|r_{ij}(s)-t_{ij}(n)\right|\exp\left\{-\frac{u_{ki}^2+u_{nj}^2}{2(1+w_{ij}(s,n))}\right\}\\
&\qquad+k\sum_{i,j=1}^{d}\sum_{s=[n^\alpha]+1}^{n}\left|r_{ij}(s)-t_{ij}(n)\right|\exp\left\{-\frac{u_{ki}^2+u_{nj}^2}{2(1+w_{ij}(s,n))}\right\}\\
&\quad=T_1+T_2,
\end{aligned}$$

where $0<\alpha<(1-\sigma(1))/(1+\sigma(1))$. For T_1, by using (3.3), we have

$$\begin{aligned}
T_1&\leqslant k\sum_{i,j=1}^{d}n^\alpha\exp\left\{-\frac{u_{ki}^2+u_{nj}^2}{2(1+\sigma(1))}\right\}\\
&\leqslant\mathfrak{C}\sum_{i,j=1}^{d}kn^\alpha\left(\frac{u_{ki}u_{nj}}{n^2}\right)^{1/(1+\sigma(1))}\\
&\leqslant\mathfrak{C}\sum_{i,j=1}^{d}n^{1+\alpha-2/(1+\sigma(1))}(\log n)^{1/(1+\sigma(1))}.
\end{aligned}$$

As $1+\alpha-2/(1+\sigma(1))<0$, we get $T_1\leqslant\mathfrak{C}n^{-\delta}$ for some $\delta>0$. The remainder is to estimate the bound of T_2. Letting $p=[n^\alpha]$, we have

$$\begin{aligned}
T_2&\leqslant k\sum_{i,j=1}^{d}\sum_{p<s\leqslant n}\left|r_{ij}(s)-t_{ij}(n)\right|\exp\left\{-\frac{u_{ki}^2+u_{nj}^2}{2(1+\sigma(p))}\right\}\\
&=\sum_{i,j=1}^{d}k\exp\left\{-\frac{u_{ki}^2+u_{nj}^2}{2(1+\sigma(p))}\right\}\sum_{p<s\leqslant n}\left|r_{ij}(s)-t_{ij}(n)\right|\\
&=\sum_{i,j=1}^{d}\frac{kn}{\log n}\exp\left\{-\frac{u_{ki}^2+u_{nj}^2}{2(1+\sigma(p))}\right\}\frac{\log n}{n}\sum_{p<s\leqslant n}\left|r_{ij}(s)-t_{ij}(n)\right|\\
&=\sum_{i,j=1}^{d}D_{ij}.
\end{aligned}$$

By the arguments similar to those of Lemma 2.1 in [8], we get

$$D_{ij}\leqslant\mathfrak{C}(\log\log n)^{-(1+\varepsilon)}$$

uniformly for $1\leqslant i,j\leqslant d$. The proof is complete. □

Lemma 3.4. *Under the conditions of Lemma* 3.2*, we have*

$$\operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k\leqslant\mathbf{u}_k\}-\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\}\right)\leqslant\mathfrak{C}(\log n)^2(\log\log n)^{-(1+\varepsilon)}.$$

Proof. Note that

$$\begin{aligned}
&\operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\left(\mathbb{I}\{\mathbf{M}_k\leqslant\mathbf{u}_k\}-\mathbb{I}\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\}\right)\right)\\
&\quad=\sum_{k=1}^{n}\frac{1}{k^2}\operatorname{Var}\left(\mathbb{I}\{\mathbf{M}_k\leqslant\mathbf{u}_k\}-\mathbb{I}\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\}\right)
\end{aligned}$$

$$+2\sum_{1\leqslant i<j\leqslant n}\frac{1}{ij}\operatorname{Cov}\big(\mathbb{I}\{\mathbf{M}_i\leqslant \mathbf{u}_i\}-\mathbb{I}\{\mathbf{M}_i^*\leqslant \mathbf{u}_i\},\mathbb{I}\{\mathbf{M}_j\leqslant \mathbf{u}_j\}-\mathbb{I}\{\mathbf{M}_j^*\leqslant \mathbf{u}_j\}\big)$$
$$=J_1+J_2.$$

Obviously $J_1<\infty$. To estimate J_2, for $i<j$ define $\mathbf{M}_{i,j}=\max\{\mathbf{X}_s\colon\ i+1\leqslant s\leqslant j\}$. Similarly define $\mathbf{M}^*_{i,j}$ and $\widetilde{\mathbf{M}}_{i,j}$. Notice for $i<j$

$$\begin{aligned}
&\big|\operatorname{Cov}\big(\mathbb{I}\{\mathbf{M}_i\leqslant \mathbf{u}_i\}-\mathbb{I}\{\mathbf{M}_i^*\leqslant \mathbf{u}_i\},\mathbb{I}\{\mathbf{M}_j\leqslant \mathbf{u}_j\}-\mathbb{I}\{\mathbf{M}_j^*\leqslant \mathbf{u}_j\}\big)\big|\\
&\quad\leqslant \big|\operatorname{Cov}\big(\mathbb{I}\{\mathbf{M}_i\leqslant \mathbf{u}_i\}-\mathbb{I}\{\mathbf{M}_i^*\leqslant \mathbf{u}_i\},\mathbb{I}\{\mathbf{M}_j\leqslant \mathbf{u}_j\}-\mathbb{I}\{\mathbf{M}_j^*\leqslant \mathbf{u}_j\}-\big(\mathbb{I}\{\mathbf{M}_{i,j}\leqslant \mathbf{u}_j\}-\mathbb{I}\{\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\}\big)\big)\big|\\
&\qquad+\big|\operatorname{Cov}\big(\mathbb{I}\{\mathbf{M}_i\leqslant \mathbf{u}_i\}-\mathbb{I}\{\mathbf{M}_i^*\leqslant \mathbf{u}_i\},\mathbb{I}\{\mathbf{M}_{i,j}\leqslant \mathbf{u}_j\}-\mathbb{I}\{\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\}\big)\big|\\
&\quad=:P_{i,j}^{(1)}+P_{i,j}^{(2)}.
\end{aligned}$$

For $P_{i,j}^{(1)}$ we get

$$\begin{aligned}
P_{i,j}^{(1)}&\leqslant 2\,\mathrm{E}\big|\mathbb{I}\{\mathbf{M}_j\leqslant \mathbf{u}_j\}-\mathbb{I}\{\mathbf{M}_{i,j}\leqslant \mathbf{u}_j\}\big|+2\,\mathrm{E}\big|\mathbb{I}\{\mathbf{M}_j^*\leqslant \mathbf{u}_j\}-\mathbb{I}\{\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\}\big|\\
&=2\big(P(\mathbf{M}_{i,j}\leqslant \mathbf{u}_j)-P(\mathbf{M}_j\leqslant \mathbf{u}_j)\big)+2\big(P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)-P\big(\mathbf{M}_j^*\leqslant \mathbf{u}_j\big)\big)\\
&=2\big[P(\mathbf{M}_{i,j}\leqslant \mathbf{u}_j)-P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)\big]-2\big[P(\mathbf{M}_j\leqslant \mathbf{u}_j)-P\big(\mathbf{M}_j^*\leqslant \mathbf{u}_j\big)\big]+4\big(P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)-P\big(\mathbf{M}_j^*\leqslant \mathbf{u}_j\big)\big)\\
&\leqslant 2\big|P(\mathbf{M}_{i,j}\leqslant \mathbf{u}_j)-P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)\big|+2\big|P(\mathbf{M}_j\leqslant \mathbf{u}_j)-P\big(\mathbf{M}_j^*\leqslant \mathbf{u}_j\big)\big|+4\big(P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)-P\big(\mathbf{M}_j^*\leqslant \mathbf{u}_j\big)\big)\\
&\leqslant \mathfrak{C}(\log\log j)^{-(1+\varepsilon)}+4P(\widetilde{\mathbf{M}}_i>\widetilde{\mathbf{M}}_{i,j})\\
&\leqslant \mathfrak{C}(\log\log j)^{-(1+\varepsilon)}+4d\frac{i}{j}.
\end{aligned}\tag{3.4}$$

The third inequality follows by Lemmas 3.2 and 3.3. The last inequality comes from Lemma 3.1 and the arguments in [1].

For $P_{i,j}^{(2)}$, noting that the $\{\mathbf{X}_s,\ s\geqslant 1\}$ and $\{\mathbf{Y}_m^{(n)}\}$ are independent, for $i<j\leqslant n$ we can get

$$\begin{aligned}
P_{i,j}^{(2)}&=\big|P(\mathbf{M}_i\leqslant \mathbf{u}_i,\mathbf{M}_{i,j}\leqslant \mathbf{u}_j)+P\big(\mathbf{M}_i^*\leqslant \mathbf{u}_i,\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)\\
&\qquad-P(\mathbf{M}_i\leqslant \mathbf{u}_i)P(\mathbf{M}_{i,j}\leqslant \mathbf{u}_j)-P\big(\mathbf{M}_i^*\leqslant \mathbf{u}_i\big)P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)\big|\\
&\leqslant \big|P(\mathbf{M}_i\leqslant \mathbf{u}_i,\mathbf{M}_{i,j}\leqslant \mathbf{u}_j)-P\big(\mathbf{M}_i^*\leqslant \mathbf{u}_i,\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)\big|\\
&\qquad+\big|P(\mathbf{M}_i\leqslant \mathbf{u}_i)-P\big(\mathbf{M}_i^*\leqslant \mathbf{u}_i\big)\big|+\big|P(\mathbf{M}_{i,j}\leqslant \mathbf{u}_j)-P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)\big|\\
&\qquad+2\big|P\big(\mathbf{M}_i^*\leqslant \mathbf{u}_i,\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)-P\big(\mathbf{M}_i^*\leqslant \mathbf{u}_i\big)P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)\big|.
\end{aligned}$$

By the Normal Comparison Lemma, we can check that

$$\big|P\big(\mathbf{M}_i^*\leqslant \mathbf{u}_i,\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)-P\big(\mathbf{M}_i^*\leqslant \mathbf{u}_i\big)P\big(\mathbf{M}_{i,j}^*\leqslant \mathbf{u}_j\big)\big|\leqslant \sum_{1\leqslant l,m\leqslant d}(ij)^{\frac{\rho_{lm}}{\log n+\rho_{lm}}}\big[(\log i)\log j\big]^{\frac{\log n}{2(\log n+\rho_{lm})}}(\log n)^{-1}.\tag{3.5}$$

Hence by Lemmas 3.2 and 3.3 and (3.5),

$$P_{i,j}^{(2)}\leqslant \mathfrak{C}\big((\log\log i)^{-(1+\varepsilon)}+(\log\log j)^{-(1+\varepsilon)}\big)+\sum_{1\leqslant l,m\leqslant d}(ij)^{\frac{\rho_{lm}}{\log n+\rho_{lm}}}\big[(\log i)\log j\big]^{\frac{\log n}{2(\log n+\rho_{lm})}}(\log n)^{-1}\tag{3.6}$$

for $i<j\leqslant n$. By (3.4) and (3.6), we have

$$\begin{aligned}
J_2&=2\sum_{1\leqslant i<j\leqslant n}\frac{1}{ij}\big(P_{i,j}^{(1)}+P_{i,j}^{(2)}\big)\\
&\leqslant 4d\sum_{1\leqslant i<j\leqslant n}\frac{1}{j^2}+\mathfrak{C}\sum_{1\leqslant i<j\leqslant n}\frac{1}{ij}\big((\log\log i)^{-(1+\varepsilon)}+(\log\log j)^{-(1+\varepsilon)}\big)\\
&\qquad+\sum_{1\leqslant l,m\leqslant d}\sum_{1\leqslant i<j\leqslant n}\frac{1}{ij}(ij)^{\frac{\rho_{lm}}{\log n+\rho_{lm}}}\big[(\log i)\log j\big]^{\frac{\log n}{2(\log n+\rho_{lm})}}(\log n)^{-1}\\
&=:F_1+F_2+F_3.
\end{aligned}$$

Clearly,

$$F_1\leqslant \mathfrak{C}\log n\leqslant \mathfrak{C}(\log n)^2(\log\log n)^{-(1+\varepsilon)}$$

and

$$\begin{aligned}
F_3 &\leqslant \sum_{1\leqslant l,m\leqslant d} (\log n)^{\frac{-\rho_{lm}}{\log n+\rho_{lm}}} \sum_{j=2}^{n} j^{\frac{-\log n}{\log n+\rho_{lm}}} \sum_{i=1}^{j-1} i^{\frac{-\log n}{\log n+\rho_{lm}}} \\
&\leqslant \sum_{1\leqslant l,m\leqslant d} \frac{\rho_{lm}}{\log n+\rho_{lm}} (\log n)^{\frac{-\rho_{lm}}{\log n+\rho_{lm}}} \sum_{j=1}^{n} j^{\frac{-2\log n}{\log n+\rho_{lm}}+1} \\
&\leqslant \sum_{1\leqslant l,m\leqslant d} \frac{2\rho_{lm}^2}{(\log n+\rho_{lm})^2} (\log n)^{\frac{-\rho_{lm}}{\log n+\rho_{lm}}} n^{\frac{2\rho_{lm}}{\log n+\rho_{lm}}} \\
&\leqslant \sum_{1\leqslant l,m\leqslant d} 2\rho_{lm}^2 \exp\left(\frac{-(3\rho_{lm}+2\log n)\log\log n+2\rho_{lm}\log n}{\log n+\rho_{lm}}\right) \\
&\leqslant \mathfrak{C}\exp\left(-\frac{\log\log n}{2}\right) = \mathfrak{C}(\log n)^{-1/2}
\end{aligned}$$

for large n. We also need to estimate F_2. For large n, let A be an integer such that $\log A \sim (\log n)^{\delta}$ for some $0<\delta<1$. Hence

$$\begin{aligned}
F_2 &= \mathfrak{C}\Bigg(\sum_{\substack{1\leqslant i<j\leqslant n \\ i\leqslant A}} + \sum_{\substack{1\leqslant i<j\leqslant n \\ i>A}}\Bigg) \frac{1}{ij}\big((\log\log i)^{-(1+\varepsilon)} + (\log\log j)^{-(1+\varepsilon)}\big) \\
&\leqslant \mathfrak{C}(\log n)^2(\log\log n)^{-(1+\varepsilon)}.
\end{aligned}$$

The proof is complete. □

Lemma 3.5. *Let (η_n) be a sequence of bounded random variables. If*

$$\operatorname{Var}\left(\sum_{k=1}^{n} \frac{1}{k}\eta_k\right) \leqslant \mathfrak{C}(\log n)^2(\log\log n)^{-(1+\varepsilon)} \quad \text{for some } \varepsilon>0,$$

then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}(\eta_k - \mathrm{E}\,\eta_k) = 0 \quad \text{a.s.}$$

Proof. See Lemma 3.1 of [5]. □

Proof of Theorem 2.1. By Lemma 3.5 and Theorem A, we only need to check that Lemma 3.5 holds for $(\mathbb{I}\{\mathbf{M}_k \leqslant \mathbf{u}_k\}, k\geqslant 1)$. By the well-known C_2-inequality, we have

$$\begin{aligned}
\operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k\leqslant\mathbf{u}_k\}\right) &= \operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k\leqslant\mathbf{u}_k\} - \sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\} + \sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\}\right) \\
&\leqslant 2\left[\operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\}\right) + \operatorname{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k\leqslant\mathbf{u}_k\} - \sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\}\right)\right] \\
&=: 2(L_1+L_2).
\end{aligned}$$

By Lemma 3.4, we get $L_2 \leqslant \mathfrak{C}(\log n)^2(\log\log n)^{-(1+\varepsilon)}$. The remainder is to estimate L_1. Using Lemma 3.1, write L_1 as

$$\begin{aligned}
&\mathrm{E}\left(\sum_{k=1}^{n}\frac{1}{k}\big(\mathbb{I}\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\} - P\{\mathbf{M}_k^*\leqslant\mathbf{u}_k\}\big)\right)^2 \\
&\quad= \mathrm{E}\left(\sum_{k=1}^{n}\frac{1}{k}\big(\mathbb{I}\{\widetilde{\mathbf{M}}_k\leqslant(\mathbf{u}_k-\mathbf{Z}_0^{(n)}\mathbb{A}(\mathbf{t}(k)))\mathbb{B}^{-1}(\mathbf{t}(k))\} - P\{\widetilde{\mathbf{M}}_k\leqslant(\mathbf{u}_k-\mathbf{Z}_0^{(n)}\mathbb{A}(\mathbf{t}(k)))\mathbb{B}^{-1}(\mathbf{t}(k))\}\big)\right)^2 \\
&\quad= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \mathrm{E}\left(\sum_{k=1}^{n}\frac{1}{k}\zeta_k\right)^2 d\Phi(\mathbf{z}), \qquad (3.7)
\end{aligned}$$

where

$$\widetilde{\mathbf{M}}_k = \left(\max\left\{Z_{1j}^{(k)}, Z_{2j}^{(k)}, \ldots, Z_{kj}^{(k)}\right\}:\ 1 \leqslant j \leqslant d\right),$$

and

$$\zeta_k = \mathbb{I}\left\{\widetilde{\mathbf{M}}_k \leqslant \left(\mathbf{u}_k - \mathbf{z}\mathbb{A}\left(\mathbf{t}(k)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(k)\right)\right\} - P\left\{\widetilde{\mathbf{M}}_k \leqslant \left(\mathbf{u}_k - \mathbf{z}\mathbb{A}\left(\mathbf{t}(k)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(k)\right)\right\}.$$

Notice

$$\mathrm{E}\left(\sum_{k=1}^{n}\frac{1}{k}\zeta_k\right)^2 = \sum_{k=1}^{n}\frac{1}{k^2}\mathrm{E}\,|\zeta_k|^2 + 2\sum_{1\leqslant k<l\leqslant n}\frac{|\mathrm{E}(\zeta_k\zeta_l)|}{kl} =: H_1 + H_2. \tag{3.8}$$

Clearly, $H_1 \leqslant \sum_{k=1}^{n}\frac{1}{k^2} < \infty$ as $|\zeta_k| \leqslant 1$. To estimate the bound of H_2, for $k<l$ we get

$$\begin{aligned}
\left|\mathrm{E}(\zeta_k\zeta_l)\right| &\leqslant \Big|\mathrm{Cov}\Big(\mathbb{I}\left\{\widetilde{\mathbf{M}}_k \leqslant \left(\mathbf{u}_k - \mathbf{z}\mathbb{A}\left(\mathbf{t}(k)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(k)\right)\right\},\\
&\qquad \mathbb{I}\left\{\widetilde{\mathbf{M}}_l \leqslant \left(\mathbf{u}_l - \mathbf{z}\mathbb{A}\left(\mathbf{t}(l)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(l)\right)\right\} - \mathbb{I}\left\{\widetilde{\mathbf{M}}_{k,l} \leqslant \left(\mathbf{u}_l - \mathbf{z}\mathbb{A}\left(\mathbf{t}(l)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(l)\right)\right\}\Big)\Big|\\
&\leqslant 2\left[\mathrm{E}\left|\mathbb{I}\left\{\widetilde{\mathbf{M}}_l \leqslant \left(\mathbf{u}_l - \mathbf{z}\mathbb{A}\left(\mathbf{t}(l)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(l)\right)\right\} - \mathbb{I}\left\{\widetilde{\mathbf{M}}_{k,l} \leqslant \left(\mathbf{u}_l - \mathbf{z}\mathbb{A}\left(\mathbf{t}(l)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(l)\right)\right\}\right|\right]\\
&= 2\left[P\left\{\widetilde{\mathbf{M}}_{k,l} \leqslant \left(\mathbf{u}_l - \mathbf{z}\mathbb{A}\left(\mathbf{t}(l)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(l)\right)\right\} - P\left\{\widetilde{\mathbf{M}}_l \leqslant \left(\mathbf{u}_l - \mathbf{z}\mathbb{A}\left(\mathbf{t}(l)\right)\right)\mathbb{B}^{-1}\left(\mathbf{t}(l)\right)\right\}\right]\\
&= 2\left[\left(\prod_{i=1}^{d}\Phi\left(\left(u_{li} - z_i\sqrt{t_{ii}(l)}\right)\left(1-t_{ii}(l)\right)^{-\frac{1}{2}}\right)\right)^{l-k} - \left(\prod_{i=1}^{d}\Phi\left(\left(u_{li} - z_i\sqrt{t_{ii}(l)}\right)\left(1-t_{ii}(l)\right)^{-\frac{1}{2}}\right)\right)^{l}\right]\\
&\leqslant \mathfrak{C}\frac{k}{l}.
\end{aligned}$$

Thus

$$H_2 \leqslant \mathfrak{C}\sum_{1\leqslant k<l\leqslant n}\frac{1}{kl}\left(\frac{k}{l}\right) \leqslant \mathfrak{C}\log n \leqslant \mathfrak{C}(\log n)^2(\log\log n)^{-(1+\varepsilon)}. \tag{3.9}$$

Combining with (3.7), (3.8) and (3.9), we can get $L_1 \leqslant \mathfrak{C}(\log n)^2(\log\log n)^{-(1+\varepsilon)}$. The proof is complete. □

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