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The influence of SS-quasinormality of some subgroups on the structure of finite groups ☆

Shirong Li, Zhencai Shen *, Jianjun Liu, Xiaochun Liu

Department of Mathematics, Guangxi University, Nanning, Guangxi 530004, PR China

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Abstract

The following concept is introduced: a subgroup H of the group G is said to be SS-quasinormal (Supplement-Sylow-quasinormal) in G if H possesses a supplement B such that H permutes with every Sylow subgroup of B. Groups with certain SS-quasinormal subgroups of prime power order are studied. For example, fix a prime divisor p of $|G|$ and a Sylow p-subgroup P of G, let d be the smallest generator number of P and $\mathcal{M}_d(P)$ denote a family of maximal subgroups $P_1, \ldots, P_d$ of P satisfying $\bigcap_{i=1}^{d}(P_i) = \Phi(P)$, the Frattini subgroup of P. Assume that the group G is p-solvable and every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-supersolvable.

Keywords: SS-quasinormal subgroups; Maximal subgroups; 2-maximal subgroups; p-nilpotent groups; p-supersolvable groups

1. Introduction

All groups considered in this paper will be finite, the notation and terminology used in this paper are standard, as in [7]. Given a finite group G, two subgroups H and K of G are said to permute if $HK = KH$, that is, HK is a subgroup of G. A subgroup H of G is said to be S-

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* Corresponding author.
E-mail addresses: shirong@gxu.edu.cn (S. Li), zhencai688@sina.com (Z. Shen).

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quasinormal in G if H permutes with every Sylow subgroup of G. This concept was introduced by O.H. Kegel in 1962 and was investigated by many authors, for example, see [1–5,9]. Recently, in [6], Ballester-Bolinches and Pedraza-Aguilera extended this concept to S-quasinormally embedded subgroups. A subgroup H of G is S-quasinormally embedded in G if, for every Sylow subgroup P of H, there is an S-quasinormal subgroup K in G such that P is also a Sylow subgroup of K.

In the present paper, we study another generalization of S-quasinormal subgroup in a new way. Recall that a supplement of H to G is a subgroup B such that $G = HB$. There is at least one such supplement for every subgroup, for instance, let $B = G$. Based on the above concepts, we give the following definition:

Definition 1.1. Let G be a finite group. A subgroup H of G is said to be an SS-quasinormal subgroup (Supplement-Sylow-quasinormal subgroup) of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B.

Obviously, every S-quasinormal subgroup of G is SS-quasinormal and S-quasinormally embedded in G. In general, an SS-quasinormal subgroup need not be S-quasinormally embedded. For instance, S_3 is an SS-quasinormal subgroup of the symmetric group S_4, but S_3 is not S-quasinormally embedded and so not S-quasinormal. The converse is also true, for example, a Sylow 3-subgroup of A_5 is S-quasinormally embedded but not SS-quasinormal. In fact, there is no inclusion-relationship between the two concepts. In Section 2, we give some properties of SS-quasinormal subgroups and the following comparisons.

Proposition 1.1.

(i) *If every S-quasinormally embedded subgroup of G is also SS-quasinormal in G, then G is solvable.*
(ii) *The group G in which every SS-quasinormal subgroup is S-quasinormally embedded need not be solvable.*

On the other hand, in 1980, Srinivasan established an interesting theorem on supersolvable groups. For convenience, let $\mathcal{M}(G)$ denote the family of all maximal subgroups of all Sylow subgroups of G. Srinivasan [12] proved that a finite group G is supersolvable if every member of $\mathcal{M}(G)$ is S-quasinormal in G. This led a famous topic on group theory, which was to study the influence of the members of $\mathcal{M}(G)$ on the structure of G. This topic had been investigated by many authors (see [2,3,6] and [15]). More recently, in [6], Ballester-Bolinches and Pedraza-Aguilera showed that if every member of $\mathcal{M}(G)$ is S-quasinormally embedded in G, then G is supersolvable. Asaad and Heliel [3] extended Ballester-Bolinches' result to saturated formations $\mathcal{F}$ containing the class $\mathcal{U}$ of all supersolvable groups. They showed that $G \in \mathcal{F}$ if and only if there is a normal subgroup H such that $G/H \in \mathcal{F}$ and every member of $\mathcal{M}(H)$ is S-quasinormally embedded in G. Recall that a formation is a class $\mathcal{F}$ of groups satisfying the following conditions: (i) if $G \in \mathcal{F}$ and $N \trianglelefteq G$, then $G/N \in \mathcal{F}$, and (ii) if $N_1, N_2 \trianglelefteq G$ such that $G/N_1, G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$.

In the present paper, we consider a subset $\mathcal{M}_d(P)$ of $\mathcal{M}(P)$ for a given Sylow p-subgroup P of G defined by the following:

Definition 1.2. Let d be the smallest generator number of a p-group P and $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$ be a set of maximal subgroups of P such that $\bigcap_{i=1}^d P_i = \Phi(P)$.

Such subset $\mathcal{M}_d(P)$ is not unique for a fixed P in general. We know that $|\mathcal{M}(P)| = (p^d - 1)/(p-1)$, $|\mathcal{M}_d(P)| = d$ and $\lim_{d\to\infty}((p^d-1)/(p-1))/d = \infty$, so $|\mathcal{M}(P)| \gg |\mathcal{M}_d(P)|$.

In Section 3, we study the influence of the members of some fixed $\mathcal{M}_d(G_p)$ on the structure of group G. The main results are as follows:

Theorem 1.1. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Theorem 1.2. *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Theorem 1.3. *Let G be a p-solvable group for a prime p and P a Sylow p-subgroup of G. Suppose that every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G. Then G is p-supersolvable.*

Theorem 1.4. *Let G be a group. If, for every prime p dividing the order of G and $P \in Syl_p(G)$, every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is supersolvable.*

Theorem 1.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then the following two statements are equivalent*:

(i) *$G \in \mathcal{F}$.*
(ii) *There exists a normal subgroup H of G such that $G/H \in \mathcal{F}$, for every prime p dividing the order of H, $P \in Syl_p(H)$ and every member of $\mathcal{M}(P)$ is SS-quasinormal in G.*

Theorem 1.5 would be false if $\mathcal{M}(P)$ is replaced by some fixed $\mathcal{M}_d(P)$ in general. We give the following example.

Example 1.6. There exist a saturated formation $\mathcal{F}$ containing $\mathcal{U}$ and a solvable group G with a normal p-subgroup P such that $G/P \in \mathcal{F}$, and every member of some fixed $\mathcal{M}_d(P)$ is S-quasinormal (hence SS-quasinormal) in G, but $G \notin \mathcal{F}$.

If M is a maximal subgroup of G and H is a maximal subgroup of M, then we call H a 2-maximal subgroup of G. We say the group G is A_4-free if there are no subgroups in G for which A_4 is an isomorphic image. In Section 4, we prove the following two theorems.

Theorem 1.7. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every 2-maximal subgroup of P is SS-quasinormal in G and G is A_4-free, then G is p-nilpotent.*

Theorem 1.8. *Let $\mathcal{F}$ be the class of groups with the Sylow tower property of supersolvable type and N a normal subgroup of a group G such that $G/N \in \mathcal{F}$. If, for every prime p dividing the* *order of N and $P \in Syl_p(N)$, every 2-maximal subgroup of P is SS-quasinormal in G and, in addition, G is A_4-free, then G belongs to $\mathcal{F}$.*

2. Preliminaries

Lemma 2.1. *Suppose that H is SS-quasinormal in a group G, $K \leqslant G$ and N a normal subgroup of G. We have*:

(i) *If $H \leqslant K$, then H is SS-quasinormal in K.*
(ii) *HN/N is SS-quasinormal in G/N.*
(iii) *If $N \leqslant K$ and K/N is SS-quasinormal in G/N, then K is SS-quasinormal in G.*
(iv) *If K is quasinormal in G, then HK is SS-quasinormal in G.*

Proof. By definition, H has a supplement B to G such that $G = HB$ and $HX = XH$, $\forall X \in Syl(B)$. By the Dedekind identity, we have

$$K = (HB) \cap K = H(B \cap K) = HB_1,$$

which shows that B_1 is a supplement of H in K. Now, for any $T \in Syl(B_1)$, there is a $Y \in Syl(B)$ such that $T \leqslant Y$. By definition, $HY = YH$. Thus

$$(HY) \cap K = H(Y \cap K) = HT,$$
$$K \cap (YH) = (K \cap Y)H = TH.$$

Hence

$$HT = TH, \quad \forall T \in Syl(B_1).$$

By definition, H is SS-quasinormal in K. Thus (i) holds.

Now, let us show (ii). It is clear that BN/N is a supplement of HN/N to G/N. For any prime p, any Sylow p-subgroup of BN/N has the form XN/N, where X is a Sylow p-subgroup of BN. Further, by [8, VI. 4.7], there exist Sylow p-subgroups B_p of B and N_p of N such that $Y = B_pN_p$ is a Sylow p-subgroup of BN. Now, both XN/N and YN/N are Sylow p-subgroups of BN/N, by Sylow's theorem, $XN/N = (YN/N)^{bN} = (B_pN/N)^{bN} = B_p^bN/N$ for some $b \in B$. By hypotheses, H permutes with every Sylow subgroup of B, so

$$HB_p^b = B_p^bH, \quad b \in B.$$

We thus get

$$\begin{aligned} HN/N \cdot XN/N &= H(XN)/N = H\big(B_p^bN\big)/N \\ &= B_p^bHN/N = XN/N \cdot HN/N, \end{aligned}$$

which shows that HN/N is SS-quasinormal in G/N and hence (ii) is proved.

We show (iii). By definition, there is a supplement of K/N to G/N, say B/N. Then obviously B is also a supplement of K to G. For any $X \in Syl(B)$, XN/N is a Sylow subgroup of B/N. Applying the condition, we have

$$K/N \cdot XN/N = XN/N \cdot K/N.$$

So $KXN = XNK$. Moreover, $N \leqslant K$, hence $KX = XK$ holds for all $X \in Syl(B)$. By definition, K is SS-quasinormal in G.

The proof of (iv). By definition, there is a subgroup B such that $G = HB$ and $HX = XH$ for all $X \in Syl(B)$. As K is quasinormal in G, it follows that HK is a subgroup of G and B is a supplement of HK to G. Thus $(HK)X = H(KX) = H(XK) = (HX)K = (XH)K = X(HK)$ and HK is SS-quasinormal. □

Lemma 2.2. *Let H be a nilpotent subgroup of G. Then the following statements are equivalent*:

(i) *H is S-quasinormal in G.*
(ii) *$H \leqslant F(G)$ and H is SS-quasinormal in G.*
(iii) *$H \leqslant F(G)$ and H is S-quasinormally embedded in G.*

Proof. (i) ⇒ (ii). If H is S-quasinormal in G, then H is subnormal in G by [9], it follows that $H \leqslant F(G)$. Also, an S-quasinormal subgroup is clearly SS-quasinormal. Thus (ii) holds.

(ii) ⇒ (iii). Suppose that G satisfies (ii). We need to prove that H is S-quasinormal in G and thus (iii) follows. Let p be a prime dividing $|H|$ and let H_p be a Sylow p-subgroup of H. By hypothesis, there is a subgroup $B \leqslant G$ such that $G = HB$ and $HX = XH$ for all $X \in Syl(B)$. In particular, let $X = Q \in Syl_q(B)$, $q \neq p$, then $HQ = QH$. Then the subgroup HQ contains a Sylow q-subgroup Q^* of G and H_p is a Sylow p-subgroup of HQ. As $H_p \leqslant O_p(G)$, it follows that $H_p = O_p(G) \cap (HQ) \trianglelefteq HQ$, and thus Q^* normalizes H_p. Because this holds for all primes $q \neq p$, we have $O^p(G) \leqslant N_G(H_p)$. By [11, Lemma 2.2], H_p is S-quasinormal in G. It follows from [11, Proposition B] that H is S-quasinormal in G, as desired.

(iii) ⇒ (i). By definition, for each prime p and Sylow p-subgroup H_p of H, there is an S-quasinormal subgroup $M(p)$ of G such that H_p is a Sylow p-subgroup of $M(p)$. As the intersection of two S-quasinormal subgroups is S-quasinormal [9], and notice that $H_p = O_p(G) \cap M(p)$, we conclude that H_p is S-quasinormal in G. Now all Sylow subgroups of H is S-quasinormal, consequently, H is S-quasinormal in G, as desired. □

Proof of Proposition 1.1. Because every Sylow subgroup of G is always normally embedded and, of course, S-quasinormally embedded, it follows by hypotheses that G_p is SS-quasinormal in G. By definition, there is a subgroup B such that $G = G_pB$ and $G_pX = XG_p$ for all $X \in Syl(B)$. Then all Sylow p-subgroups of B are contained in G_p, and so in $G_p \cap B$, consequently, B has a unique Sylow p-subgroup, namely $G_p \cap B$ and hence B is p-closed. By Schur–Zassenhaus's theorem [8, I, 18.1–18.2], B has a Hall p'-subgroup K. Of course, K is also a Hall subgroup of G. Thus every Sylow subgroup of K is S-quasinormally embedded in G and so is SS-quasinormal in G by hypotheses, and hence is SS-quasinormal in K by Lemma 2(i). It follows by induction that K is solvable. In the light of Hall's theorem [8, VI, 2.3], K has a Sylow system $\{G_{p_1}, \ldots, G_{p_r}\}$. Thus $\{G_p, G_{p_1}, \ldots, G_{p_r}\}$ is a Sylow system of G. Again applying the Hall's theorem, we conclude that G is solvable, which finishes the proof of (i).

For (ii), assume $G = PSL(2,7)$. By the list of subgroups of $PSL(2,7)$ [8, II, 8.27], we know that the maximal subgroups of G possess only two types: S_4 with index 7 and the non-abelian group L of order 21 with index 2^3. Because, in general, subgroups with prime power index are certainly SS-quasinormal, the subgroups of G that are conjugate to one of S_4 and L are all SS-quasinormal. These are the only SS-quasinormal subgroups of G. On the other hand, each Hall subgroup of a group is always normally embedded, so both S_4 and L are normally embedded in G, hence S-quasinormally embedded in G. Thus each SS-quasinormal subgroup of G is certainly S-quasinormally embedded in G. However, $PSL(2,7)$ is not solvable. □

In order to develop the SS-quasinormal concept, we give some introductions and statements of results.

Lemma 2.3. *Let H be a subgroup of G and H_G denote the normal core of H in G. Then the following statements are equivalent*:

(i) *H is S-quasinormal in G.*
(ii) *$H/H_G \leqslant F(G/H_G)$ and H/H_G is SS-quasinormal in G/H_G.*
(iii) *$H/H_G \leqslant F(G/H_G)$ and H/H_G is S-quasinormally embedded in G/H_G.*

Proof. (i) ⇒ (ii). As H is S-quasinormal in G, by [11, Proposition A], H^G/H_G is nilpotent. It follows that $H/H_G \leqslant F(G/H_G)$. Moreover, an S-quasinormal subgroup is certainly SS-quasinormal, so H is SS-quasinormal in G, hence H/H_G is SS-quasinormal in G/H_G by Lemma 2.1(ii). Conclusion (ii) follows.

(ii) ⇒ (iii). This is immediate from Lemma 2.2.

(iii) ⇒ (i). By Lemma 2.2, we know that H/H_G is S-quasinormal in G/H_G. It follows that H is S-quasinormal in G. □

Lemma 2.4. *Let P be a normal elementary abelian p-subgroup of G. If all maximal subgroups of P are SS-quasinormal in G, then each chief factor of G contained in P is cyclic.*

Proof. By Lemma 2.2, all maximal subgroups of P are S-quasinormal in G. Now, let N be a minimal normal subgroup of G which is contained in P. It suffices to show that N is of order p. Let G_p be a Sylow p-subgroup of G. Then $N \cap Z(G_p) > 1$, so we can find a subgroup X of order p such that $X \leqslant N \cap Z(G_p)$. Let $\{P_1, \ldots, P_m\}$ be a set of maximal subgroups of P satisfying $X \leqslant P_i$. If such P_i does not exist, then $P = X$, as desired. So let $m \geqslant 1$. Then we have

$$X = \bigcap_{i=1}^{m} P_i.$$

As the intersection of S-quasinormal subgroups is S-quasinormal in G (see [9]), it follows that X is S-quasinormal in G. Thus, by a lemma of [11], we know that $O^p(G) \subseteq N_G(X)$. Recall that G_p centralizes X and $G = O^p(G)G_p$. Consequently, X is normal in G. Now, as N is a minimal normal subgroup of G containing X, we conclude $N = X$, as desired. □

The following lemmas are needed to prove our theorem.

Lemma 2.5. *If a p-subgroup P of G is SS-quasinormal* (p *a prime*), *then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Proof. By definition, there exists a supplement B of P to G such that $G = BP$ and P permutes with every Sylow subgroup of B. Let Q be a Sylow q-subgroup of G with $q \neq p$. Then Q is conjugate to a Sylow q-subgroup of B. That is, there is an element $g = ax, a \in B, x \in P$ such that $Q^{ax} \subseteq B$. Thus $PQ^{ax} = Q^{ax}P$, which implies that $PQ = QP$, as desired. □

Lemma 2.6. *(See [16, Lemma 2.6].) Let N be a normal subgroup of a group G* ($N \neq 1$). *If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in $F(N)$.*

Lemma 2.7. *(See [16].) Let G be a finite group and p a prime dividing the order of G, G is A_4-free and* $(|G|, p-1) = 1$. *Assume that N is a normal subgroup of G such that G/N is p-nilpotent and the order of N is not divisible by p^3. Then G is p-nilpotent.*

Lemma 2.8. *(See [13].) If P is a Sylow p-subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leqslant \Phi(P)$, then N is p-nilpotent.*

3. Maximal subgroups

Proof of Theorem 1.1. Set the $\mathcal{M}_d(P) = \{H_1, \ldots, H_d\}$. Fix an $H \in \mathcal{M}_d(P)$. We have:

($*$) There exists a Hall p'-subgroup K of G such that HK is a subgroup of index p in G.

By condition, there is a subgroup $B \leqslant G$ such that $G = HB$ and $HX = XH$ for all $X \in Syl(B)$. From $G = HB$, we obtain $|B : H \cap B|_p = |G : H|_p = p$, and hence $H \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $H \cap B$. Thus $S \nsubseteq H$ for all $S \in Syl_p(B)$ and $HS = SH$ is a Sylow p-subgroup of G. In view of $|P : H| = p$ and by comparison of orders, $S \cap H = B \cap H$, for all $S \in Syl_p(B)$. So

$$B \cap H = \bigcap_{b \in B} \left(S^b \cap H\right) \leqslant \bigcap_{b \in B} S^b = O_p(B).$$

We claim that B has a Hall p'-subgroup. Because $|O_p(B) : B \cap H| = p$ or 1, it follows that $|B/O_p(B)|_p = p$ or 1. As p is the smallest prime dividing $|G|$, by a well-known theorem of Burnside, $B/O_p(B)$ is p-nilpotent, and hence B is p-solvable. So B has a Hall p'-subgroup by [8, VI, 1.7]. Thus the claim holds. Now, let K be a Hall p'-subgroup of B, $\pi(K) = \{p_2, \ldots, p_s\}$ and $P_i \in Syl_{p_i}(K)$. By condition, H permutes with every P_i and so H permutes with the subgroup $\langle P_2, \ldots, P_s\rangle = K$. Thus $HK \leqslant G$. Obviously, K is a Hall p'-subgroup of G and HK is a subgroup of index p in G, as desired.

Now, for each $H_i \in \mathcal{M}_d(P)$, by ($*$), G has a Hall p'-subgroup K_i such that the subgroup $M_i = H_iK_i$ has index p in G. As p is the smallest prime dividing $|G|$, it follows that $M_i \trianglelefteq G$ and hence G/M_i is p-group. Set

$$N = \bigcap_{i=1}^{d} M_i.$$

Then N is a normal subgroup of G such that G/N is p-group. Since H_i is a Sylow p-subgroup of M_i, it follows that $P \cap N = \bigcap_{i=1}^{d} H_i = \Phi(P)$. Thus N is p-nilpotent by Lemma 2.8. Hence G is p-nilpotent, contrary to the choice of G. □

Corollary 3.1. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of some fixed $\mathcal{M}_d(P)$ is S-quasinormal in G, then G is p-nilpotent.*

Proof of Theorem 1.2. It is easy to see that the theorem holds when $p = 2$ by Theorem 1.1, so it suffices to prove the theorem for the case when p is odd. Suppose that the theorem is not true and let G be a counterexample of the smallest order. We have the following claims:

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, we consider the quotient group $G/O_{p'}(G)$. By Lemma 2.1, $G/O_{p'}(G)$ satisfies the hypotheses of the theorem, it follows that $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Hence G is p-nilpotent, a contradiction.

(2) If $P \leqslant H < G$, then H is p-nilpotent.

Noting that $N_H(P) \leqslant N_G(P)$, we have $N_H(P)$ is p-nilpotent. By Lemma 2.1, H satisfies the hypotheses of the theorem. By the choice of G, H is p-nilpotent, as desired.

(3) $G = PQ$, where Q is a Sylow q-subgroup of G, $q \neq p$.

By the choice of G, G is not p-nilpotent. In the light of a result of Thompson [14, Corollary], there exists a non-trivial characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Choose T such that the order of T is as large as possible. Since $N_G(P)$ is p-nilpotent, we have $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P satisfying $T < K \leqslant P$. Now, T char $P \trianglelefteq N_G(P)$, which gives $T \trianglelefteq N_G(P)$. So $N_G(P) \leqslant N_G(T)$. By (2), we get $N_G(T) = G$ and hence $T = O_p(G)$. Now, applying the result of Thompson again, we have that $G/O_p(G)$ is p-nilpotent, therefore G is p-solvable. Thus, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G by [7, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by (2), contrary to the choice of G. Consequently, $PQ = G$, as desired.

(4) Final contradiction.

We now make use of the above claims to finish our proof. As $O_{p'}(G) = 1$, we have $O_p(G) > 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N \leqslant \Phi(P)$, then $N \leqslant \Phi(G)$ by [8, III, 3.3], and the quotient group G/N satisfies the hypotheses of the theorem, thus G/N is p-nilpotent by the choice of G. It follows that $G/\Phi(G)$ is p-nilpotent and hence G is p-nilpotent, a contradiction. Thus $N \leqslant \Phi(P)$ cannot happen, so $N \nleqslant \Phi(P)$. Because $\Phi(P) = \bigcap_{i-1}^{d} P_i$, where $P_i \in \mathcal{M}_d(P)$, without loss of generality, we may assume that $N \nleqslant P_1$. Put $N_1 = N \cap P_1$. Then $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$. Also, by hypotheses, P_1 is SS-quasinormal in G, which indicates that P_1Q is a subgroup of G. As $N \trianglelefteq G$, we have $N_1 = N \cap P_1Q \trianglelefteq P_1Q$, and it follows that $N_1 \trianglelefteq \langle P_1Q, N\rangle = G$. Moreover, since N is a minimal normal subgroup of G, we have $N_1 = 1$ and N is a cyclic subgroup of order p.

Now, $NP_1 = P$ and $N \cap P_1 = 1$. By W. Gaschutz's theorem [8, I, 17.4], there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Of course, $N \nleqslant \Phi(G)$. By Lemma 2.6, we have $O_p(G) = R_1 \times \cdots \times R_r$, where R_i $(i = 1, \ldots, r)$ is minimal normal subgroup of G of order p. We therefore get $P \leqslant \bigcap_{i=1}^{r} C_G(R_i) = C_G(O_p(G))$. Moreover, by [10, Theorem 9.31] and (3), $C_G(O_p(G)) \leqslant O_p(G)$, it follows that $P = O_p(G)$ and so $G = N_G(P)$. Now, we apply the hypotheses that $N_G(P)$ is p-nilpotent to conclude that G is p-nilpotent. This is a contradiction, which completes the proof. □

Proof of Theorem 1.3. Suppose that the theorem is false so that there exists a counterexample G of minimal order. Set $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$ with $\bigcap_{i=1}^{d}(P_i) = \Phi(P)$. We shall finish the proof by following claims:

(1) $O_{p'}(G) = 1$ and $\Phi(O_p(G)) = 1$.

This follows from the choice of G as in the proof of Theorem 1.2.

(2) Every G-chief factor contained in $O_p(G)$ is cyclic.

As G is p-solvable and $O_{p'}(G) = 1$, we have $O_p(G) > 1$. Thus we can find a minimal normal subgroup N of G contained in $O_p(G)$. If $N \leqslant \Phi(P)$, then $N \leqslant \Phi(G)$, and the quotient group G/N satisfies the hypotheses of the theorem, by the minimality of G, G/N is p-supersolvable. As the class of p-supersoluble groups is a saturated formation, we have that G is p-supersolvable, a contradiction. Thus $N \nleqslant \Phi(P)$. We may assume that $N \nleqslant P_1$. Let $N_1 = N \cap P_1$. Then $|N : N_1| = p$. By hypotheses, P_1 is SS-quasinormal in G, so there is a supplement B of P_1 to G such that P_1 permutes with every Sylow subgroup of B. For any prime q distinct from p and a Sylow q-subgroup B_q of B, we have $P_1 B_q$ is a subgroup. It follows that $N_1 = N \cap P_1 B_q$ is normal in $P_1 B_q$. Therefore N_1 is normal in the subgroup $\langle N, P_1 B_q \mid q \in \pi(G), q \neq p\rangle = G$. The minimality of N yields $N_1 = 1$. Consequently, N is a cyclic subgroup of order p and hence $N \cap P_1 = 1$. By W. Gaschutz's theorem [8, I, 17.4], there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Of course, $N \nleqslant \Phi(G)$. Now, we can apply Lemma 2.6 to conclude that $O_p(G)$ is a direct product of normal subgroups of G of order p, thus (2) follows.

(3) The final contradiction.

Since $G/C_G(R_i)$ is a cyclic group of order $p - 1$, of course,

$$G/\bigcap_{i=1}^{r} C_G(R_i) = G/C_G\big(O_p(G)\big)$$

is p-supersolvable. On the other hand, as G is p-solvable and $O_{p'}(G) = 1$, by [10, Theorem 9.3.1], $C_G(O_p(G)) \leqslant O_p(G)$. Thus $G/O_p(G)$ is p-supersolvable. Now, the claim (2) implies that G is p-supersolvable, completing the proof. □

Remark 3.1. In Theorem 1.3, we cannot remove the assumption that G is p-solvable in general. For example, let $G = A_5$. Clearly, every maximal subgroup of the Sylow 5-subgroups of A_5 is the identity subgroup and of course, it is SS-quasinormal in A_5. But A_5 is not 5-supersolvable.

Proof of Theorem 1.4. Applying Lemma 2.1(i), we easily see that the hypotheses are inherited by Hall subgroups of G. Applying Theorem 1.1 to know that G possesses Sylow tower property of supersolvable type. Let q be the largest prime dividing $|G|$ and Q a Sylow q-subgroup of G. Then $Q \trianglelefteq G$. Let H be a member of $\mathcal{M}_d(Q)$. By Lemma 2.2, H is S-quasinormal in G. By a lemma of [11], $O^q(G)$ normalizes H. Moreover, H is maximal in Q, so Q normalizes H as well. It follows that $G = QO^q(G)$ normalizes H, i.e., $H \trianglelefteq G$. Now, we apply Lemma 2.1(ii) to see that G/H satisfies the hypotheses condition, and hence G/H is supersolvable for all $H \in \mathcal{M}_d(Q)$. Because

$$\bigcap_{H \in \mathcal{M}_d(Q)} H = \Phi(Q)$$

and the class of all supersolvable groups is a saturated formation, we can see that $G/\Phi(Q)$ is supersolvable. Finally, by [8, III, 3.3], $\Phi(Q) \leqslant \Phi(G)$, it follows that $G/\Phi(G)$ is supersolvable and hence G is supersolvable. The proof is now completed. □

Corollary 3.2. *Let G be a group. If, for every prime p dividing the order of a group G and $P \in Syl_p(G)$, every member of some fixed $\mathcal{M}_d(P)$ is S-quasinormal in G, then G is supersolvable.*

Proof of Theorem 1.5. We suppose this is the beginning of the proof of (ii) implies (i) and that (i) implies (ii) obviously. By Lemma 2.1(i), every member of $\mathcal{M}(H)$ is SS-quasinormal in H. It follows by Theorem 1.4 that H is supersolvable, so the Sylow q-subgroup Q of H is normal in G where q is the largest prime dividing $|H|$. As $(G/Q)/(H/Q) \cong G/H \in \mathcal{F}$, and $(G/Q, H/Q)$ satisfies the hypotheses condition, it follows by induction that $G/Q \in \mathcal{F}$. As $\Phi(Q) \leqslant \Phi(G)$ by [8, III, 3.3], applying induction again, we can assume that $\Phi(Q) = 1$. Now, by hypotheses, each member of $\mathcal{M}(Q)$ is SS-quasinormal in G. It follows by Lemma 2.4 that each chief factor of G contained in Q is cyclic. Notice that the saturated formation $\mathcal{F}$ contains all supersolvable groups, we thus conclude $G \in \mathcal{F}$. □

The following example shows that Theorem 1.5 is false if $\mathcal{M}(P)$ is replaced by some fixed $\mathcal{M}_d(P)$ in general.

Proof of Example 1.6. Let f be a formation function defined by $f(p)$ being the class of p'-groups for any prime p and let $\mathcal{F}$ be the formation locally defined by $f(p)$. If Y is a supersolvable group, then any p-chief factor H/N of Y is a cyclic group of order p, it follows that $Y/C_Y(H/N)$ is a cyclic group of order dividing $p-1$ and hence $Y/C_Y(H/N) \in f(p)$. Therefore, $Y \in \mathcal{F}$ and so $\mathcal{F}$ contains $\mathcal{U}$. Clearly, A_4 belongs to $\mathcal{F}$.

Let $P = \langle a, b, c\rangle$ be an elementary abelian group of order 3^3, and let α, β be two automorphisms of P defined respectively by defining

$$\alpha = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \qquad \beta = \begin{pmatrix} a & b & c \\ b & c^{-1} & a^{-1} \end{pmatrix}.$$

Then $\alpha^3 = \beta^3 = (\alpha\beta)^2 = 1$, so $H = \langle \alpha, \beta\rangle \cong A_4$. Then H acts on P as a group of automorphisms. Let $G = PH$ be the corresponding semidirect product. In fact, P is an irreducible and faithful A_4-module on $GF(p)$ and so P is a minimal normal subgroup of G with $C_H(P) = 1$. Because $A_4 \in \mathcal{F}$ and $G/P \cong H \cong A_4$, we have $G/P \in \mathcal{F}$. Let $K = PS$, where S is a Sylow

2-subgroup of G. We have $O^3(G) \leqslant K \trianglelefteq G$. Since S is an elementary abelian group of order 4, it follows that a minimal normal subgroup of K contained in P is of order p. By Maschke's theorem [8, I, 17.7], P is a completely reducible S-module. Hence $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle$, where $\langle a_i \rangle$ $(i = 1, 2, 3)$ is S-invariant. Let $P_i = \langle a_j\colon\ j \neq i \rangle$. Then every P_i is S-quasinormal in G and $\mathcal{M}_d(P) = \{P_1, P_2, P_3\}$. On the other hand, P is a 3-chief factor of G and $G/C_G(P) = G/P \cong A_4$, which is not a $3'$-group. Hence $G \notin \mathcal{F}$. □

4. 2-maximal subgroups

Proof of Theorem 1.7. Assume that the theorem is false. We consider a counterexample G of minimal order. Let N be a minimal normal subgroup of G. Then we have:

(1) N is the unique minimal normal subgroup of G and G/N is p-nilpotent, in addition, $N \nleqslant \Phi(G)$.

By a routine check, we know that G/N satisfies the condition of the theorem. By the choice of G, G/N is p-nilpotent, hence N is a unique minimal normal subgroup of G and $N \nsubseteq \Phi(G)$.

(2) $O_{p'}(G) = 1$ and every proper subgroup of G that containing P is p-nilpotent.

If $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ satisfies the hypotheses condition. The minimality of G implies that $G/O_{p'}(G)$ is p-nilpotent and hence G is p-nilpotent, a contradiction. By Lemma 2.1(i) and induction, every proper subgroup of G that containing P is p-nilpotent. Thus claim (2) holds.

(3) $|P| \geqslant p^3$.

This can be seen from Lemma 2.7.

(4) $O_p(G) = 1$.

If $O_p(G) \neq 1$, then $N \leqslant O_p(G)$. By (1), G/N is p-nilpotent. Let T/N be the normal p-complement of G/N. By the Schur–Zassenhaus theorem [10, I, 18.1], there exists a Hall p'-subgroup H of T such that $T = [N]H$, a semidirect product. By the Frattini argument, we have $G = NN_G(H)$.

Set $M = N_G(H)$. Then $G = NM$. By (1), $N \cap M = 1$ and M is a maximal subgroup of G. In particular, the normal core $M_G = 1$. As p is the smallest prime dividing $|G|$, it follows that $|G : M| \neq p$. So we have $|G : M| = |N| = p^n, n \geqslant 2$. Choose a Sylow p-subgroup P_0 of M such that $P = NP_0$. Of course, we can find a subgroup P_1 of P of index p^2 such that P_1 contains P_0. Then P_1 is SS-quasinormal in G by hypotheses. By Lemma 2.5, P_1 permutes with every Sylow subgroup of H, thus P_1H is a subgroup. Moreover, $M = P_0H \leqslant P_1H < G$, so the maximality of M gives $M = P_1H$. Consequently, $|N| = p^2$. Now, Lemma 2.7 indicates the subgroup NH is p-nilpotent, hence G is p-nilpotent, a contradiction.

(5) G is a non-solvable group.

By (2) and (4), G is a non-solvable group.

(6) The final contradiction.

Define the family $\Sigma = \{Q \mid Q \in Syl_q(G),\ q \in \pi(G),\ q \neq p\}$ and

$$L = \bigcap_{Q \in \Sigma} N_G(Q).$$

Then $L \trianglelefteq G$. As L normalizes each Q, it follows that L normalizes $Q \cap L$. We thus deduce $Q \cap L \leqslant O_{p'}(L) \leqslant O_{p'}(G) = 1$ by (2). Consequently, L is a p-subgroup and hence $L = 1$ by (4).

Suppose that P is generated by its 2-maximal subgroups. By hypotheses, every 2-maximal subgroup of P is SS-quasinormal in G. By Lemma 2.5, every 2-maximal subgroup of P permutes with every member Q of Σ. Thus P permutes with each Q in Σ. Burnside's $p^a q^b$-theorem implies that PQ is a solvable group and hence $PQ < G$ by (5). By the choice of G and (2), PQ is p-nilpotent and so $P \leqslant N_G(Q)$ for all $Q \in \Sigma$, contrary to $L = 1$. We thus conclude that the subgroup of P generated by its 2-maximal subgroups is a proper subgroup. In this case, P has a cyclic maximal subgroup P_1. If not, P has at least two maximal subgroups P_3, P_4 and $P = \langle P_3, P_4\rangle$. Moreover, P_3 has at least two maximal subgroups P_{31} and P_{32}, P_4 has at least two maximal subgroups P_{41} and P_{42}. It follows that $P_3 = \langle P_{31}, P_{32}\rangle$ and $P_4 = \langle P_{41}, P_{42}\rangle$. Thus $P = \langle P_3, P_4\rangle = \langle P_{31}, P_{32}, P_{41}, P_{42}\rangle$, this is a contradiction. By (3), $|P| \geqslant p^3$, so $|P_1| \geqslant p^2$. Let P_2 be a maximal subgroup of P_1. Then P_2 is a 2-maximal subgroup of P and $1 < P_2 \trianglelefteq P$. By hypotheses, P_2 is SS-quasinormal in G and so P_2Q is a subgroup of G by Lemma 2.5. Since P_2 is a cyclic subgroup and p is the smallest prime dividing the order of G, the Burnside theorem asserts that P_2Q is also p-nilpotent. We deduce that $P_2 \leqslant N_G(Q)$ for all $Q \in \Sigma$, contrary to $L = 1$. This completes the proof. □

Remark 4.1. In Theorem 1.7, we cannot remove the assumption that G is A_4-free in general. For example, $G = A_4$. Clearly, every 2-maximal subgroup of the Sylow 2-subgroups of A_4 is the identity subgroup and of course, SS-quasinormal in A_4. But A_4 is not 2-nilpotent.

From Theorem 1.7, we can get the following corollary.

Corollary 4.1. *Let G be a group. If, for every prime p dividing the order of G and $P \in Syl_p(G)$, every 2-maximal subgroup of P is SS-quasinormal in G and G is A_4-free, then G is a Sylow tower group of supersolvable type.*

Proof of Theorem 1.8. By Lemma 2.1 and Corollary 4.1, we use induction on $|G|$ to see that N is a Sylow tower group of supersolvable type. Let r be the largest prime number in $\pi(N)$ and $R \in Syl_p(N)$. Then R is normal in G and $(G/R)/(N/R) \cong G/N$ is a Sylow tower group of supersolvable type. By induction, $G/R \in \mathcal{F}$. Let q be the largest prime divisor of $|G|$ and Q a Sylow q-subgroup of G. Then $RQ \trianglelefteq G$. If $q = r$, then G has the Sylow tower property, as desired. Hence we assume that $r < q$.

Case 1. $RQ < G$. In this case, RQ is a Sylow tower group of supersolvable type by induction, it follows that $Q \trianglelefteq RQ$ and so $Q \trianglelefteq G$. Consider a Hall q'-subgroup M. Then $R \leqslant M$ and $M \in \mathcal{F}$ by induction. Thus $G \in \mathcal{F}$, as desired again.

Case 2. $G = RQ$. Let L be a minimal normal subgroup of G with $L \leqslant R$. Then the quotient group G/L satisfies the hypotheses. By induction, we see that G/L is a Sylow tower group of supersolvable type. As the class of all Sylow tower groups is a saturated formation, we have that $L \nsubseteq \Phi(G)$ and L is the unique minimal normal subgroup of G which is contained in R. Therefore $L = F(R) = R$ by Lemma 2.6. In particular, R is an abelian group.

If R is a cyclic subgroup of order r, then $r < q$ implies that $G = R \times Q$. Of course, $G \in \mathcal{F}$, which completes the proof. Thereby we may assume that $|R| \geqslant r^2$. Let R_1 be a 2-maximal subgroup of R. By hypotheses, R_1 is SS-quasinormal in G. By Lemma 2.2, R_1 is S-quasinormal in G and $N_G(R_1) \geqslant O^r(G)$ by [11, Lemma 2.2]. Thus R_1 is normal in G, and $R_1 = 1$ by minimality of R. Hence R is an elementary abelian group of order r^2. Now, any element g of Q induces an automorphism φ of R. When $|R| = r^2$, we know that $|Aut(R)| = (r+1)r(r-1)^2$. If $r = 2$ and some $\varphi \neq 1$, then the order of φ must be 3 as $r < q$. Thus the subgroup $R\langle g\rangle$ is not A_4-free, contrary to the hypotheses. Hence all $\varphi = 1$, i.e., $G = R \times Q$, completing the proof. The remainder is to consider the case when $r > 2$. Noticing that $r + 1$ is not a prime, so we have that all $\varphi = 1$ and $G = R \times Q$, hence $G \in \mathcal{F}$. The proof is now completed. □

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