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PERFECT IMAGES OF SPACES WITH A $\delta\theta$-BASE AND WEAKLY θ-REFINABLE SPACES

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The main result shows that the class of spaces with a $\delta\theta$-base is invariant under perfect mappings. By using related techniques it is also shown that the class of weakly θ-refinable spaces is preserved by perfect images.

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1. Introduction

Since Moore spaces need not have a point-countable base the $\delta\theta$-base defined by Aull [1] provides a natural generalization of Moore spaces and spaces with a point-countable base. The $\delta\theta$-base concept has helped unify much of the theory involving certain generalized metric spaces. The reader may wish to consult [2], [9] and [11] for a survey of some of the interesting results concerning spaces with a $\delta\theta$-base.

The main result of this paper proves that the class of spaces with a $\delta\theta$-base is invariant under perfect images, answering a question asked in [2] and [6]. Using techniques suggested by this proof, we also show that the class of weakly θ-refinable spaces is preserved by perfect mappings. We finish this section with a few preliminary definitions and results. Other concepts will be reviewed as needed.

A base $\mathcal{B}$ for X is said to be a *$\delta\theta$-base* if $\mathcal{B}$ can be expressed as $\mathcal{B}=\bigcup_{n=1}^{\infty}\mathcal{H}_n$ such that if $x\in U\subset X$, where U is open, there is some $m\in N$ where $\operatorname{ord}(x,\mathcal{H}_m)\leq\omega$ and some $H\in\mathcal{H}_m$ with $x\in H\subset U$. (Recall that $\operatorname{ord}(x,\mathcal{H}_m)=|\{G\colon x\in G\in\mathcal{H}_m\}|$.) We assume the convention that whenever $\bigcup_{n=1}^{\infty}\mathcal{H}_n$ is called a $\delta\theta$-base for X that the collections $\mathcal{H}_i$ are arranged so that the above is true.

A space X is said to have *countable tightness* if whenever $A\subset X$ and $x\in\bar{A}$ there exists some countable $M\subset A$ such that $x\in\bar{M}$. Certainly, first countable spaces have countable tightness. The following lemma is well known and easy to prove.

Lemma 1.1. *Suppose* $\mathscr{H}$ *is a collection of open subsets of* X *and for* $n \leq \omega$, $F_n = \{x \in X: \operatorname{ord}(x, \mathscr{H}) \leq n\}$. *If* $n < \omega$ *then* F_n *is closed in* X, *and if* X *has countable tightness then* F_ω *is closed in* X.

In particular, we see that if $\bigcup_{n=1}^{\infty} \mathscr{H}_n$ is a $\delta\theta$-base for X and if $B_m = \{x \in X: \operatorname{ord}(x, \mathscr{H}_m) \leq \omega\}$ then B_m is closed in X for every $m \in N$.

We will need the following lemma by A. Miscenko [12].

Lemma 1.2. *If* $\mathscr{P}$ *is a point-countable collection of subsets of* X *then every* $Y \subset X$ *has at most countably many minimal finite covers by elements of* $\mathscr{P}$.

All mappings are continuous and onto, and a *perfect mapping* $f: X \to Y$ is a closed mapping such that $f^{-1}(y)$ is compact in X for every $y \in Y$.

2. Recipe for preserving base axioms

A 'perfect image theorem' has been given for spaces with a development [13], point-countable base [10], [8], σ-point-finite base [10], [4], σ-locally countable base [5], Primitive base [7] and θ-base [7]. Starting from scratch these proofs are all difficult and use a variety of techniques. Although we cannot give one proof which covers the $\delta\theta$-base case (Theorem 3.5) and all of the above, there is a common approach which can be used in all of these results. In an attempt to unify the results in this area we give below an outline of this general procedure. The reader may wish to check how each of these steps is realized in the proof of Theorem 3.5.

2.1. Given: A base $\mathscr{B}$ for X satisfying property P and a perfect mapping $f: X \to Y$.

2.2. Find appropriate definition for 'P-minimal cover' of a subset $E \subset X$.

2.3. For $C \subset U \subset X$, where C is compact and U is open, show there is a finite collection $\mathscr{U} \subset \mathscr{B}$ such that $C \subset \bigcup \mathscr{U} \subset U$ and $\mathscr{U}$ is a P-minimal cover of C.

2.4. 'Count' the number of P-minimal covers of a given subset E of X.

2.5. For $\mathscr{F} \subset \mathscr{B}$, $|\mathscr{F}| < \omega$, $\mathscr{G} \subset \mathscr{B}$, let

$$\mathscr{M}(\mathscr{F}, \mathscr{G}) = \{B \in \mathscr{G}: \mathscr{F} \text{ is a } P\text{-minimal cover of } f^{-1}(f(B))\}.$$

2.6. Use the open collections $\mathscr{M}(\mathscr{F}, \mathscr{G})$ in X to build, via the mapping f, a base (for Y) satisfying P.

3. Preserving a $\delta\theta$-base

Our main result is Theorem 3.5; we begin with a series of lemmas.

Lemma 3.1. *If X has a $\delta\theta$-base there is a $\delta\theta$-base $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ for X such that whenever $C \subset U \subset X$, where C is a nonempty compact set and U is open, then there exists $\mathcal{G}_m$, covering C, and a subcollection $\mathcal{W} \subset \mathcal{G}_m$ where $A_m \cap C \subset \bigcup \mathcal{W} \subset U$ and $A_m \cap C \neq \emptyset$. ($A_m = \{x \in X: \operatorname{ord}(x, \mathcal{G}_m) \leq \omega\}$.)*

Proof. Suppose $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ is a $\delta\theta$-base for X and for $n \in N$, $B_n = \{x \in X: \operatorname{ord}(x, \mathcal{H}_n) \leq \omega\}$. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots$ be an enumeration of the collections obtained by taking unions of a finite number of collections from $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$. We show $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ satisfies the desired conditions. Suppose $C \subset U \subset X$ with C a nonempty compact set and U open. There is a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers chosen in the following way:

(i) n_1 is the first integer such that for some $H_1 \in \mathcal{H}_{n_1}$, $H_1 \subset U$ and $H_1 \cap B_{n_1} \cap C \neq \emptyset$.

(ii) For $1 < i \leq k$, n_i is the first integer larger than n_{i-1} such that for some $H_i \in \mathcal{H}_{n_i}$, $H_i \subset U$ and $H_i \cap B_{n_i} \cap (C - \bigcup_{j=1}^{i-1} (\bigcup \mathcal{H}_{n_j})) \neq \emptyset$.

(iii) $\bigcup_{j=1}^{k} \mathcal{H}_{n_j}$ covers C.

There is some $r \in N$ such that $\mathcal{G}_r = \bigcup_{j=1}^{k} \mathcal{H}_{n_j}$. Let $A_r = \{x: 1 \leq \operatorname{ord}(x, \mathcal{G}_r) \leq \omega\}$; then $\mathcal{G}_r$ satisfies the following:

(a) $\mathcal{G}_r$ covers C.

(b) There is some $G \in \mathcal{G}_r$ ($G = H_k$ will work) such that $G \subset U$ and $G \cap A_r \cap C \neq \emptyset$.

Now, applying the above, we can find a finite sequence $r_1, r_2, \ldots, r_s$ satisfying the following:

(1) r_1 is the first integer such that $\mathcal{G}_{r_1}$ covers C and there is some $G \in \mathcal{G}_{r_1}$ such that $G \subset U$ and $G \cap A_{r_1} \cap C \neq \emptyset$.

(2) For $1 < i \leq s$, r_i is the first integer such that $\mathcal{G}_{r_i}$ covers

$$C_i = C \cap A_{r_1} \cap \cdots \cap A_{r_{i-1}} - \bigcup \{G \in \mathcal{G}_{r_j}: 1 \leq j \leq i-1,\ G \subset U,\ G \cap A_{r_1} \cap \cdots \cap A_{r_j} \neq \emptyset\}$$

and for some $G_i \in \mathcal{G}_{r_i}$, $G_i \subset U$ and $G_i \cap A_{r_i} \cap C_i \neq \emptyset$.

(3) $\{G \in \mathcal{G}_s: G \subset U \text{ and } G \cap C_s \neq \emptyset\}$ covers C_s.

(The above sequence can be found by using (1) and (2) to describe the process for choosing the r_i, and using the compactness of each C_i to insure the process stops to give (3).) Then $\bigcup_{i=1}^{s} \mathcal{G}_{r_i} = \mathcal{G}_m$ for some m where $A_m = A_{r_1} \cap A_{r_2} \cap \cdots \cap A_{r_s}$; it follows that $\mathcal{G}_m$ satisfies the desired conditions.

In the remainder of this section we assume $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ is a $\delta\theta$-base for X satisfying the conditions of Lemma 3.1.

Definition 3.2. If $E \subset X$ and $s = (s_1, \ldots, s_m) \in N^m$, a cover $\mathcal{U}$ of E is said to be an *s-minimal* cover (with respect to $\mathcal{G}$) if $\mathcal{U}$ can be expressed as $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_m$

where for $1 \leqslant i \leqslant m$:

(1) $\mathcal{U}_i \subset \mathcal{G}_{s_i}$

(2) $\mathcal{G}_{s_i}$ covers $E - \bigcup_{j<i} (\bigcup \mathcal{U}_j)$

(3) $\mathcal{U}_i$ is a minimal cover of $E \cap A_{s_i} - \bigcup_{j<i} (\bigcup \mathcal{U}_j)$.

Lemma 3.3. *If $C \subset U \subset X$, where C is a nonempty compact set and U is open, there is a finite s-minimal cover $\mathcal{U}$ of C (for some $s \in \bigcup_{r=1}^{\infty} N^r$) such that $\bigcup \mathcal{U} \subset U$.*

Proof. Let $C_1 = C$ and s_1 be the first integer such that $\mathcal{G}_{s_1}$ covers C_1 and there is a (finite) subcollection $\mathcal{U}_1 \subset \mathcal{G}_{s_1}$ where $\mathcal{U}_1$ is a minimal cover of $A_{s_1} \cap C_1$, $\bigcup \mathcal{U}_1 \subset U$, and $A_{s_1} \cap C_1 \neq \emptyset$ (use Lemma 3.1). Continue by induction so that when $k > 1$ we have $C_k = C_{k-1} - \bigcup \mathcal{U}_{k-1}$ and if $C_k \neq \emptyset$, s_k is the smallest integer such that $\mathcal{G}_{s_k}$ covers C_k and there is a finite $\mathcal{U}_k \subset \mathcal{G}_{s_k}$ where $\mathcal{U}_k$ is a minimal cover of $A_{s_k} \cap C_k$, $\bigcup \mathcal{U}_k \subset U$, and $A_{s_k} \cap C_k \neq \emptyset$.

If each $C_k \neq \emptyset$, let $K = \bigcap_{i=1}^{\infty} C_i$; then K is a nonempty compact set contained in U, so there is some $q \in N$ such that $\mathcal{G}_q$ covers K and there is a finite subcollection $\mathcal{H} \subset \mathcal{G}_q$ such that $\mathcal{H}$ is a minimal cover of $A_q \cap K$, $\bigcup \mathcal{H} \subset U$, and $A_q \cap K \neq \emptyset$. Since $C_i \downarrow K$ there is some r such that $C_r \subset \bigcup \mathcal{G}_q$, and for $j \geqslant r$, $C_j \cap A_q$ is compact and $C_j \cap A_q \downarrow K \cap A_q$ so there is some $p \geqslant r$ with $s_p > q$ such that $A_q \cap C_p \subset \bigcup \mathcal{H}$. (Note that the sequence $\{s_i\}$ may not be increasing but is unbounded.) Now, $\mathcal{G}_q$ covers C_p, $\mathcal{H}$ is a minimal cover of $A_q \cap C_p$, $\bigcup \mathcal{H} \subset U$ and $A_q \cap C_p \neq \emptyset$. This contradicts the minimal condition put on s_p since $s_p > q$. Hence there is some m with $C_{m+1} = \emptyset$ and $C_1, \ldots, C_m$ all nonempty. The collection $\mathcal{U} = \bigcup_{i=1}^{m} \mathcal{U}_i$ is the desired $(s_1, \ldots, s_m)$-minimal cover of C.

Lemma 3.4. *If $E \subset X$ and $s \in N^m$, some $m \in N$, there are at most a countable number of s-minimal covers of E (with respect to $\mathcal{G}$).*

Proof. Suppose a typical s-minimal cover is represented by $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_m$ where each $\mathcal{U}_i \subset \mathcal{G}_{s_i}$ and $\mathcal{U}_i$ is a minimal cover of $E \cap A_{s_i} - \bigcup_{j<i} (\bigcup \mathcal{U}_{n_j})$. Working inductively through the integers $1, 2, \ldots, m$ it follows from Lemma 1.2 that there are at most countably many choices for $\mathcal{U}_i$. This will give the desired result.

Theorem 3.5. *If $f: X \to Y$ is a perfect mapping and X has a $\delta\theta$-base then Y has a $\delta\theta$-base.*

Proof. Suppose $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ is a $\delta\theta$-base for X satisfying the conditions of Lemma 3.1. For any $r \in N$ and $s \in N^r$ let $\mathcal{C}(s) = \mathcal{C}(s_1, \ldots, s_r)$ denote the family of all finite subcollections of $\mathcal{G}_{s_1} \cup \cdots \cup \mathcal{G}_{s_r}$. For any $\mathcal{U} \in \mathcal{C}(s)$ and any $k \in N$ let

$$\mathcal{M}(\mathcal{U}, s, k) = \{W \in \mathcal{G}_k : \mathcal{U} \text{ is an } s\text{-minimal cover of } f^{-1}(f(W))\}.$$

Let $S(\mathcal{U}, s, k)$ be the saturated part (with respect to f) of

$$(\bigcup \mathcal{M}(\mathcal{U}, s, k)) \cup (\bigcup \mathcal{U} - A_k),$$

$$B(\mathcal{U}, s, k) = f(S(\mathcal{U}, s, k)),$$

$$\mathcal{B}(s, k) = \{B(\mathcal{U}, s, k) : \mathcal{U} \in \mathcal{C}(s)\},$$

and

$$\mathcal{B} = \bigcup \left\{ \mathcal{B}(s, k) : s \in \bigcup_{r=1}^{\infty} N^r, k \in N \right\}.$$

To see that $\mathcal{B}$ is a $\delta\theta$-base for Y suppose $y \in V \subset Y$ for V open in Y. By Lemma 3.3 there is some $r \in N$, $s \in N^r$, and an s-minimal cover $\mathcal{U}$ of $f^{-1}(y)$ such that $\bigcup \mathcal{U} \subset f^{-1}(V)$. Say $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_r$ where each $\mathcal{U}_i \subset \mathcal{G}_{s_i}$, $\mathcal{G}_{s_i}$ covers $f^{-1}(y) - \bigcup_{j<i} (\bigcup \mathcal{U}_j)$, and $\mathcal{U}_i$ is a minimal cover of $f^{-1}(y) \cap A_{s_i} - \bigcup_{j<i} (\bigcup \mathcal{U}_j)$. Let

$$H = \left[\bigcup_{i=1}^{r} \left(\bigcup \mathcal{U}_i - \left(\bigcup_{j<i} A_{s_j} \right)\right) \right] \cap \left[\bigcap_{i=1}^{r} \left((\bigcup \mathcal{G}_{s_i}) \cup \left(\bigcup_{j<i} (\bigcup \mathcal{U}_j) \right) \right) \right]$$

and let $S(H)$ denote the saturated part of H. Note that $f^{-1}(y) \subset S(H)$ so there exists some $\mathcal{G}_k$, covering $f^{-1}(y)$, and a finite nonempty subcollection $\mathcal{W} \subset \mathcal{G}_k$ where

$$A_k \cap f^{-1}(y) \subset \bigcup \mathcal{W} \subset S(H),$$

and $A_k \cap f^{-1}(y) \cap W \neq \emptyset$ for every $W \in \mathcal{W}$. If $W \in \mathcal{W}$ notice that $f^{-1}(y) \subset f^{-1}(f(W)) \subset H$. The condition $f^{-1}(f(W)) \subset H$ forces

$$f^{-1}(f(W)) \cap A_{s_i} - \bigcup_{j<i} (\bigcup \mathcal{U}_j) \subset \bigcup \mathcal{U}_i$$

and since $\mathcal{U}_i$ is a minimal cover of $f^{-1}(y) \cap A_{s_i} - \bigcup_{j<i} (\bigcup \mathcal{U}_j)$ we see that $\mathcal{U}_i$ is a minimal cover of $f^{-1}(f(W)) \cap A_{s_i} - \bigcup_{j<i} (\bigcup \mathcal{U}_j)$. Also $\mathcal{G}_{s_i}$ covers $f^{-1}(f(W)) - \bigcup_{j<i} (\bigcup \mathcal{U}_i)$ so it follows that $\mathcal{U}$ is an s-minimal cover of $f^{-1}(f(W))$. This says that $\mathcal{W} \subset \mathcal{M}(\mathcal{U}, s, k)$, hence

$$f^{-1}(y) \subset S(\mathcal{U}, s, k) \subset \bigcup \mathcal{U} \subset f^{-1}(V).$$

This gives $y \in B(\mathcal{U}, s, k) \subset V$, showing that $\mathcal{B}$ is a base for Y. To complete the proof that $\mathcal{B}$ is a $\delta\theta$-base, pick a fixed $x \in A_k \cap f^{-1}(y)$ (for the above choice of k). If $y \in B(\mathcal{K}, s, k)$ for some $\mathcal{K} \in \mathcal{C}(s)$ then $A_k \cap f^{-1}(y) \subset \bigcup \mathcal{M}(\mathcal{K}, s, k)$ so there must be some $D \in \mathcal{G}_k$ such that $x \in D$ and $\mathcal{K}$ is an s-minimal cover of $f^{-1}(f(D))$. Since $x \in D$ for only countably many $D \in \mathcal{G}_k$ and each $f^{-1}(f(D))$ has only a countable number of s-minimal covers (by Lemma 3.4) it follows that $\operatorname{ord}(y, \mathcal{B}(s, k)) \leq \omega$. That completes the proof of the theorem.

4. Weakly θ-refinable spaces

A space X is *weakly θ-refinable* [3] if for any open cover $\mathcal{W}$ of X there is an open refinement $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ such that if $x \in X$ there is some $n \in N$ where $1 \leq \operatorname{ord}(x, \mathcal{H}_n) < \omega$. In this section we show the class of weakly θ-refinable spaces is preserved by perfect images. We need the following characterization from [3].

Theorem 4.1. (Bennett and Lutzer). *A space X is weakly θ-refinable if and only if any open cover $\mathcal{W}$ of X has an open refinement $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ where for any $x \in X$ there is some $n \in N$ with* $\operatorname{ord}(x, \mathcal{H}_n) = 1$.

The proof of the next lemma is similar to the first part of the proof of Lemma 3.1.

Lemma 4.2. *Suppose $\{\mathcal{H}_n\}_{n=1}^{\infty}$ is a sequence of open collections in X such that for any $x \in X$ there exists $m \in N$ where* $\operatorname{ord}(x, \mathcal{H}_m) = 1$. *Let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ be an enumeration of all collections obtained by taking unions of a finite number of collections from $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$. If C is a compact subset of X there exists some $k \in N$ and finite $\mathcal{U} \subset \mathcal{G}_k$ such that $C \subset \bigcup \mathcal{G}_k$ and*

$$\emptyset \neq C \cap \{x : \operatorname{ord}(x, \mathcal{G}_m) = 1\} \subset \bigcup \mathcal{U}.$$

In the remainder of this section, $\{\mathcal{G}_n\}_{n=1}^{\infty}$ denotes a sequence of open collections in X satisfying the conditions of Lemma 4.2 and, for each $n \in N$, $E_n = \{x \in X : \operatorname{ord}(x, \mathcal{G}_n) \leq 1\}$. We continue with some additional notation.

If $\mathcal{F}_n = \{G \cap E_n : G \in \mathcal{G}_n\}$ then $\mathcal{F}_n$ is relatively closed and discrete in $\bigcup \mathcal{G}_n$. Hence, whenever C is compact and $C \subset \bigcup \mathcal{G}_n$ then $|\{G \in \mathcal{G}_n : G \cap C \cap E_n \neq \emptyset\}| < \omega$. For $r \in N$, $s, t \in N^r$, $\mathcal{U}_i \subset \mathcal{G}_{s_i}$, $1 \leq i \leq r$, let

$$H(s, \mathcal{U}_1, \ldots, \mathcal{U}_r) = \bigcup_{i=1}^{r} \left(\bigcup \mathcal{U}_i - \bigcup_{j<i} E_{s_j} \right)$$

and

$$\mathcal{H}(s, t) = \{H(s, \mathcal{U}_1, \ldots, \mathcal{U}_r) : \mathcal{U}_i \subset \mathcal{G}_{s_i}, \quad |\mathcal{U}_i| = t_i, 1 \leq i \leq r\}.$$

It should be clear that the next lemma will provide the main tool for the proof of Theorem 4.4.

Lemma 4.3. *If C is compact in X there exists some $r \in N$ and $s, t \in N^r$ such that C is contained in exactly one element of $\mathcal{H}(s, t)$.*

Proof. Let $C_1 = C$. By the conditions given in Lemma 4.2 there is a smallest integer s_1 such that $\mathcal{G}_{s_1}$ covers C_1 and there is a finite subcollection $\mathcal{U}_1 \subset \mathcal{G}_{s_1}$ where $\mathcal{U}_1$ is a minimal cover of $E_{s_1} \cap C_1$ and $E_{s_1} \cap C_1 \neq \emptyset$. If $t_1 = |\mathcal{U}_1|$ notice that $\mathcal{U}_1$ is the only subcollection of $\mathcal{G}_{s_1}$ having cardinality t_1 and covering $E_{s_1} \cap C_1$. Using induction and an argument similar to that used in the proof of Lemma 3.3, there is a finite sequence $s=(s_1,\ldots,s_r)$ of positive integers and finite $\mathcal{U}_i \subset \mathcal{G}_{s_i}$ for $1 \le i \le r$ such that if $C_i = C - \bigcup_{j<i} \mathcal{U}_i$ then $\mathcal{G}_{s_i}$ covers C_i, $\mathcal{U}_i$ is a minimal cover of $E_{s_i} \cap C_i$, $E_{s_i} \cap C_i \neq \emptyset$ and $C \subset \bigcup_{i=1}^r (\bigcup \mathcal{U}_i)$. Let $t_i = |\mathcal{U}_i|$; then $\mathcal{U}_i$ is the only subcollection of $\mathcal{G}_{s_i}$ having cardinality t_i and covering $E_{s_i} \cap C_i$. If $t=(t_1,\ldots,t_r)$ it follows that $C \subset H(s, \mathcal{U}_1, \ldots, \mathcal{U}_r)$ and $H(s, \mathcal{U}_1, \ldots, \mathcal{U}_r)$ is the only element of $\mathcal{H}(s,t)$ containing C.

Theorem 4.4. *If X is weakly θ-refinable and $f: X \to Y$ is a perfect mapping then Y is weakly θ-refinable.*

Proof. Suppose $\mathcal{V}$ is an open cover of Y and let $\mathcal{V}^F$ denote the collection of all unions of finite subcollections from $\mathcal{V}$. It suffices to show that $\mathcal{V}^F$ has an open refinement $\bigcup_{n=1}^\infty \mathcal{S}_n$ where for each $y \in Y$ there is some $m \in N$ with $\mathrm{ord}(y, \mathcal{S}_m) = 1$. Let $\mathcal{W} = \{f^{-1}(V)\colon V \in \mathcal{V}\}$. Since X is weakly θ-refinable we see that $\mathcal{W}$ has an open refinement $\bigcup_{n=1}^\infty \mathcal{G}_n$ satisfying the conditions of Lemma 4.2 and inducing the open collections $\mathcal{H}(s,t)$ for $s, t \in \bigcup_{r=1}^\infty N^r$. For any $B \in \mathcal{H}(s,t)$ let $D(B) = Y - f(X-B)$ and $\mathcal{D}(s,t) = \{D(B)\colon B \in \mathcal{H}(s,t)\}$. Since $\bigcup_{r=1}^\infty N^r$ is countable it follows from Lemma 4.3 that

$$\bigcup \left\{ \mathcal{D}(s,t)\colon s, t \in \bigcup_{r=1}^\infty N^r \right\}$$

is the desired open refinement of $\mathcal{V}^F$.

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