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A locally connected continuum without convergent sequences

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Abstract

We answer a question of Juhász by constructing under CH an example of a locally connected continuum without nontrivial convergent sequences.

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1. Introduction

During the ninth Prague Topological Symposium, Juhász asked whether there is a locally connected continuum without nontrivial convergent sequences. This question arose naturally in his investigation in [7] with Gerlits, Soukup, and Szentmiklóssy on characterizing continuity in terms of the preservation of compactness and connectedness. The aim of this note is to answer this question in the affirmative under the Continuum Hypothesis (abbreviated: CH).

The standard example of a continuum not containing nontrivial convergent sequences is $\beta\mathbb{H}\setminus\mathbb{H}$, the Čech–Stone remainder of $\mathbb{H}=[0,\infty)$. But this space is not locally connected.

Fedorchuk [6] constructed a consistent example of a compact space of cardinality $\mathfrak{c}$ containing no nontrivial convergent sequences. See also van Douwen and Fleissner [4] for a somewhat simpler construction under the Definable Forcing Axiom. These constructions yield zero-dimensional spaces. As a consequence, our construction has to be somewhat

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different. As in [6,4], we 'kill' all possible nontrivial convergent sequences in a transfinite process of length ω_1. However, our 'killing' is done in the Hilbert cube $Q = \prod_{n=1}^{\infty}[-1,1]_n$ instead of the Cantor set. Specifically, we will construct under CH an inverse ω_1-sequence of Hilbert cubes the inverse limit of which is the desired example. This yields an infinite-dimensional locally connected continuum without nontrivial convergent sequences. The construction works since the Hilbert cube is 'sufficiently' homogeneous. Similar constructions can be performed in other 'sufficiently' homogeneous continua as well.

For all undefined notions, see [5,10].

2. The construction

There basically seem to be two ways to obtain compact spaces without nontrivial convergent sequences. The first way is to prove that a certain known space does not have nontrivial convergent sequences. An example of such a space is the one mentioned in the introduction: $\beta\mathbb{H} \setminus \mathbb{H}$. This space surfaces at many places in the literature, and one can prove that it has no convergent sequences for example by observing that it is an F-space. The second way is to build in that the space one gets at the end of a certain process has no nontrivial convergent sequences. This is usually done by a transfinite inverse limit construction. An advantage of this procedure is that along the way one can try to build in additional desirable properties. However, it often turns out that for these benefits one has to pay a price. These constructions quite often demand complex bookkeeping and require additional set theoretic assumptions.

Since there is no natural locally connected continuum for which one can hope to be able to prove that it has no nontrivial convergent sequences, we are forced to try to use the second method. Let us start with the Hilbert cube Q. There are many convergent sequences to be killed, so let us first think of the question how to kill a single sequence. To this end, let S be convergent sequence in Q and let x be its limit. We assume that $x \notin S$. Consider a space M which admits a continuous surjection $f : M \to Q$. We think of Q as a step in our inverse limit procedure, and M as its successor step. We want M to 'kill' the sequence $S \cup \{x\}$ in such a way that it cannot be resurrected in the steps to come. It is clear what is needed for that. Let us split S into two disjoint infinite subsets S_0 and S_1. If we can construct M and f in such a way that the sets $f^{-1}[S_0]$ and $f^{-1}[S_1]$ have disjoint closures in M, then we clearly achieved our goal. This can be done by 'blowing up' the point $x \in Q$ to an interval, say J, and to let S_0 and S_1 converge to two different points of this interval. The map $f : M \to Q$ simply shrinks the interval J to a single point (the point x). Since we want a locally connected continuum at the end of our process, M should be a locally connected continuum as well. If we can choose M to be homeomorphic to Q then the construction can be continued by dealing with M in the same way.

That this can indeed be done rather easily, follows from the following considerations. Write $S = \langle x_n \rangle_n$ and consider, in the product $Q \times \mathbb{I}$, the sequence

$$y_n = \begin{cases} \langle x_n, 0\rangle & (n \text{ even}), \\ \langle x_n, 1\rangle & (n \text{ odd}). \end{cases}$$

In $Q \times \mathbb{I}$ shrink the interval $J = \{x\} \times \mathbb{I}$ to a single point. Bing's Shrinking Criterion yields that $(Q \times \mathbb{I})/J$ is homeomorphic to Q. Since Q is homogeneous with respect to convergent sequences, there is a homeomorphism $\alpha : Q \to (Q \times \mathbb{I})/J$ sending $\{x\}$ onto $\{J\}$ and for every n, x_n to y_n. This means that the spaces $Q \times \mathbb{I}$ and $(Q \times \mathbb{I})/J$ and the natural decomposition map $Q \times \mathbb{I} \to (Q \times \mathbb{I})/J$ demonstrate that what we want can be done.

This is unfortunately not the whole story since we want to kill *all* convergent sequences in Q, and also all sequences that surface in the spaces that we will create in later steps of the construction. We could try to kill all sequences at the same time, but then the resulting space is out of control. So it is inevitable to aim for killing the sequences one by one. So we have to ensure that at the end of our inductive process all sequences will be dealt with. At step α of the construction many sequences from steps $\beta < \alpha$ will still be 'alive'. So it is unavoidable that sequences from the previous steps have to be pulled back. However, a pulled back sequence from step $\beta < \alpha$ does not need to be a sequence anymore since in the intermediate steps between β and α it could have been changed considerably. So this tells us that we should try to understand what the pulled back sets look like for otherwise control is impossible. We will have to dig a little deeper for achieving that. Details can be found in Section 3.

We will construct below for every $\alpha < \omega_1$ a space M_α and for every $\beta \leqslant \alpha$ a continuous function $f^\alpha_\beta : M_\alpha \to M_\beta$ such that, among other things, $M_\alpha \approx Q$ and each f^α_β is a so-called cell-like Z^*-map. (Cell-like maps are monotone maps with certain additional properties.) The inverse sequence will be *continuous*, which means that if α is a limit ordinal then M_α will be the inverse limit of the previous M_β's. The fact that the functions are Z^*-maps will ensure that at successor stages we are able to do our splitting in such a way that the new space that we are creating is homeomorphic to the Hilbert cube.

We will now perform the construction under CH. It is modulo the results in Section 3 very similar to known constructions in the literature (see, e.g., Kunen [9]).

We work in the cube Q^{ω_1}; for every $\alpha < \omega_1$, we identify Q^α with

$$\{x \in Q^{\omega_1} :\ \beta > \alpha \to x_\beta = 0\}.$$

For every $1 \leqslant \alpha < \omega_1$ let $\{S^\alpha_\xi :\ \xi < \omega_1\}$ list all nontrivial convergent sequences in Q^α that do not contain their limits. For all $\alpha, \xi < \omega_1$ pick disjoint complementary infinite subsets A^α_ξ and B^α_ξ of S^α_ξ.

We shall construct for $1 \leqslant \alpha \leqslant \omega_1$ a closed subspace $M_\alpha \subseteq Q^\alpha$. The space we are after will be M_{ω_1}.

Let $\tau : \omega_1 \to \omega_1 \times \omega_1$ be a surjection such that $\tau(\beta) = \langle \alpha, \xi \rangle$ implies $\alpha \leqslant \beta$.

For $\alpha \leqslant \beta \leqslant \omega_1$ let π^β_α be the natural projection from Q^β onto Q^α. The following conditions will be satisfied:

(A) $M_\alpha \approx Q$ for every $1 \leqslant \alpha < \omega_1$, and if $\alpha \leqslant \beta$ then $\pi^\beta_\alpha[M_\beta] = M_\alpha$. We put $\rho^\beta_\alpha = \pi^\beta_\alpha \restriction M_\beta : M_\beta \to M_\alpha$.

(B) If $\alpha \leqslant \beta$ then $\rho^\beta_\alpha : M_\beta \to M_\alpha$ is a cell-like Z^*-map.

(C) If $\beta < \omega_1$, $\tau(\beta) = \langle \alpha, \xi \rangle$, and $S^\alpha_\xi \subseteq M_\alpha$ then $(\rho^{\beta+1}_\alpha)^{-1}[A^\alpha_\xi]$ and $(\rho^{\beta+1}_\alpha)^{-1}[B^\alpha_\xi]$ have disjoint closures in $M^{\beta+1}$.

Observe that the construction is determined at all limit ordinals γ. By compactness and (A) we must have

$$M_\gamma = \{x \in Q^\gamma \colon (\forall \alpha < \gamma)(\pi^\gamma_\alpha(x) \in M_\alpha)\}.$$

Also, if $(\gamma_n)_n$ is any strictly increasing sequence of ordinals with $\gamma_n \nearrow \gamma$ then M_γ is canonically homeomorphic to

$$\varprojlim (M_{\gamma_n}, \rho^{\gamma_{n+1}}_{\gamma_n})_n.$$

By Theorem 3.2 below this implies that $M_\gamma \approx Q$ and also that $\rho^\gamma_{\gamma_n}$ is a cell-like Z^*-map for every n. Since γ_1 can be *any* ordinal smaller than γ, the same argument yields that ρ^γ_α is a cell-like Z^*-map for *every* $\alpha < \gamma$. So in our construction we need only worry about successor steps.

Put $M_1 = Q^{\{0\}}$, and let $1 \leqslant \beta < \omega_1$ be arbitrary. We shall construct $M_{\beta+1}$ assuming that M_β has been constructed. To this end, let $\tau(\beta) = \langle \alpha, \xi \rangle$. We make the obvious identification of $Q^{\beta+1}$ with $Q^\beta \times Q$. If $S^\alpha_\xi \nsubseteq M_\alpha$ then there is nothing to do. We then fix any element $q \in Q$, and put

$$M_{\beta+1} = M_\beta \times \{q\}.$$

So assume that $S^\alpha_\xi \subseteq M_\alpha$. By Theorem 3.3 there exists a cell-like Z^*-map $f : Q \to M_\beta$ such that

$$f^{-1}[(\rho^\beta_\alpha)^{-1}[A^\alpha_\xi]], \qquad f^{-1}[(\rho^\beta_\alpha)^{-1}[B^\alpha_\xi]]$$

have disjoint closures in Q. Put

$$M_{\beta+1} = \{\langle f(x), x\rangle \in Q^\beta \times Q \colon x \in Q\}.$$

So $M_{\beta+1}$ is nothing but the graph of f. It is clear that $M_{\beta+1}$ is as required.

Now put $M = M_{\omega_1}$. Observe that M is a locally connected continuum, being the inverse limit of an inverse system of locally continua with monotone surjective bonding maps (see, e.g., [5, 6.3.16 and 6.1.28]). Assume that T is a nontrivial convergent sequence with its limit x in M. Since $T \cup \{x\}$ is countable, there exists $\alpha < \omega_1$ such that $\rho^{\omega_1}_\beta \restriction (T \cup \{x\})$ is one-to-one and hence a homeomorphism for every $\beta \geqslant \alpha$. Pick $\xi < \omega_1$ such that $S^\alpha_\xi = \rho^{\omega_1}_\alpha[T]$, and $\beta \geqslant \alpha$ such that $\tau(\beta) = \langle \alpha, \xi \rangle$. Then $\rho^{\omega_1}_{\beta+1}[T \cup \{x\}]$ is a nontrivial convergent sequence with its limit in $M_{\beta+1}$ which is mapped by $\rho^{\beta+1}_\alpha$ onto S^α_ξ with its limit. But this is clearly in conflict with (C).

3. The Hilbert cube

A *Hilbert cube* is a space homeomorphic to Q. Let M^Q denote an arbitrary Hilbert cube. A closed subset A of M^Q is a *Z-set* if for every $\varepsilon > 0$ there is a continuous function $f : M^Q \to M^Q \setminus A$ which moves the points less than ε. It is clear that a closed subset of a Z-set is a Z-set. We list some other important properties of Z-sets.

(1) Every singleton subset of M^Q is a Z-set.
(2) A countable union of Z-sets is a Z-set provided it is closed.
(3) A homeomorphism between Z-sets can be extended to a homeomorphism of M^Q.
(4) If X is compact and $f : X \to M^Q$ is continuous then f can be approximated arbitrarily closely by an imbedding whose range is a Z-set.

See [10, Chapter 6] for details.

Observe that by (1) and (2), every nontrivial convergent sequence with its limit is a Z-set in M^Q.

A *near homeomorphism* between compacta X and Y is a continuous surjection $f : X \to Y$ which can be approximated arbitrarily closely by homeomorphisms. This means that for every $\varepsilon > 0$ there is a homeomorphism $g : X \to Y$ such that for every $x \in X$ the distance between $f(x)$ and $g(x)$ is less than ε.

A closed subset $A \subseteq M^Q$ has *trivial shape* if it is contractible in any of its neighborhoods. A continuous surjection f between Hilbert cubes M^Q and N^Q is *cell-like* provided that $f^{-1}(q)$ has trivial shape for every $q \in N^Q$. The following fundamental result is due to Chapman [3] (see also [10, Theorem 7.5.7]).

(5) Let $f : M^Q \to N^Q$ be cell-like, where M^Q and N^Q are Hilbert cubes. Then f is a near homeomorphism.

It is easy to see that if $f : M^Q \to N^Q$ is a near homeomorphism between Hilbert cubes then f is cell-like. So within the framework of Hilbert cubes the notions 'near homeomorphism' and 'cell-like' are equivalent.

A continuous surjection f between Hilbert cubes M^Q and N^Q is called a Z^*-map provided that for every Z-set $A \subseteq N^Q$ the preimage $f^{-1}[A]$ is a Z-set in M^Q.

Lemma 3.1. *Let M^Q and N^Q be Hilbert cubes, and let $f : M^Q \to N^Q$ be a continuous surjection for which there is a Z-set $A \subseteq M^Q$ which contains all nondegenerate fibers of f. Then f is a Z^*-map.*

Proof. Let $B \subseteq N^Q$ be an arbitrary Z-set, and put $B_0 = B \setminus f[A]$. Write B_0 as $\bigcup_{n=1}^{\infty} E_n$, where each E_n is compact. It follows from [10, Theorem 7.2.5] that for every n the set $f^{-1}[E_n]$ is a Z-set in M^Q. As a consequence,

$$f^{-1}[B] \subseteq A \cup \bigcup_{n=1}^{\infty} f^{-1}[E_n]$$

is a countable union of Z-sets and hence a Z-set by (2). □

Theorem 3.2. *Let $(Q_n, f_n)_n$ be an inverse sequences of Hilbert cubes such that every f_n is cell-like as well as a Z^*-map. Then*

(A) *$\varprojlim (Q_n, f_n)_n$ is a Hilbert cube.*
(B) *The projection $f_n^{\infty} : \varprojlim (Q_n, f_n)_n \to Q_n$ is a cell-like Z^*-map for every n.*

Proof. It will be convenient to let Q_∞ denote $\varprojlim(Q_n, f_n)_n$.

By (5), every f_n is a near homeomorphism. Hence we get (A) from Brown's Approximation Theorem for inverse limits in [2]. It follows from [10, Theorem 6.7.4] that every projection $f_n^\infty : Q_\infty \to Q_n$ is a near homeomorphism, hence is cell-like.

For every n let ϱ_n be an admissible metric for Q_n which is bounded by 1. The formula

$$\varrho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \varrho_n(x_n, y_n)$$

defines an admissible metric for Q_∞. With respect to this metric f_n^∞ is a $2^{-(n-1)}$-mapping [10, Lemma 6.7.3].

For (B) it suffices to prove that f_1^∞ is a Z^*-map. To this end, let $A \subseteq Q_1$ be a Z-set, and let $\varepsilon > 0$. Pick $n \in \mathbb{N}$ so large that $2^{-(n-1)} < \varepsilon$. It follows that for every $x \in Q_n$ the diameter of the fiber $(f_n^\infty)^{-1}(x)$ is less than ε. An easy compactness argument gives us an open cover $\mathcal{U}$ of Q_n such that for every $U \in \mathcal{U}$ we have that

$$\operatorname{diam}\big(f_n^\infty\big)^{-1}[U] < \varepsilon. \tag{$*$}$$

Let $\gamma > 0$ be a Lebesgue number for this cover [10, Lemma 1.1.1]. Since f_n^∞ is a near homeomorphism, there is a homeomorphism $\varphi : Q_\infty \to Q_n$ such that for every $x \in Q_\infty$ we have

$$\varrho_n\big(f_n^\infty(x), \varphi(x)\big) < \tfrac{1}{2}\gamma.$$

Observe that $A_n = (f_1^n)^{-1}[A]$ is a Z-set in Q_n. There consequently is a continuous function $\xi : Q_n \to Q_n \setminus A_n$ which moves the points less than $\frac{1}{2}\gamma$. Now define $\eta : Q_\infty \to Q_\infty$ by

$$\eta = \varphi^{-1} \circ \xi \circ f_n^\infty.$$

It is clear that $\eta[Q_\infty]$ misses $(f_1^\infty)^{-1}[A]$. In order to check that η is a 'small' move, pick an arbitrary element $x \in Q_\infty$. By construction, $\varrho_n(x_n, \xi(x_n)) < \frac{1}{2}\gamma$. Since $\eta(x) = \varphi^{-1}(\xi(x_n))$, clearly $\varrho_n(\eta(x)_n, \xi(x_n)) < \frac{1}{2}\gamma$. We conclude that $\varrho_n(\eta(x)_n, x_n) < \gamma$. Pick an element $U \in \mathcal{U}$ which contains both $\eta(x)_n$ and x_n. By $(*)$ it consequently follows that $\varrho(\eta(x), x) < \varepsilon$, which is as required. □

Theorem 3.3. *If $(A_n)_n$ is a relatively discrete sequence of closed subsets of Q such that $\overline{\bigcup_{n=1}^\infty A_n}$ is a Z-set then there are a Hilbert cube M and a continuous surjection $f : M \to Q$ such that*

(A) *f is a cell-like Z^*-map.*
(B) *The closures of the sets $\bigcup_{n=1}^\infty f^{-1}[A_{2n}]$ and $\bigcup_{n=0}^\infty f^{-1}[A_{2n+1}]$ are disjoint.*

Proof. Consider the subspace $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ of Q, and the 'remainder' $R = A \setminus \bigcup_{n=1}^{\infty} A_n$. Observe that R is compact since the sequence $(A_n)_n$ is relatively discrete. Let T denote the product $A \times \mathbb{I}$; put

$$S = (R \times \mathbb{I}) \cup \left(\bigcup_{n=1}^{\infty} A_{2n} \times \{0\} \right) \cup \left(\bigcup_{n=0}^{\infty} A_{2n+1} \times \{1\} \right).$$

Then S is evidently a closed subspace of T. Let $\pi : R \times \mathbb{I} \to R$ denote the projection. It is clear that the adjunction space $S \cup_\pi R$ is homeomorphic to A (cf., [11, p. 507]). By (4), any constant function $S \to Q$ can be approximated by an imbedding whose range is a Z-set. So we may assume without loss of generality that S is a Z-subset of some Hilbert cube M^Q. Now consider the space $N = M^Q \cup_\pi R$ with natural decomposition map f. It is clear that f is cell-like, each nondegenerate fiber of f being an arc [10, Corollary 7.1.2]. We will prove below that $N \approx Q$. Once we know that, we also get by Lemma 3.1 that f is a Z^*-map. Observe that the projection $\pi : R \times \mathbb{I} \to R$ is a hereditary shape equivalence. So by a result of Kozlowski [8] (see also [1]), it follows that N is an AR. Since S is a Z-set in M^Q it consequently follows from [10, Proposition 7.2.12] that $f[S] \approx A$ is a Z-set in N. But $N \setminus f[S]$ is obviously a Q-manifold, and consequently has the disjoint-cells property. But this implies that N has the disjoint-cells property, i.e., $N \approx Q$ by Toruńczyk's topological characterization of Q in [12] (see also [10, Corollary 7.8.4]). So we conclude that $f[S] \approx A$ is a Z-set in the Hilbert cube N. By (3) there is a homeomorphism of pairs $(Q, A) \approx (N, f[S])$. (There are several ways to arrive at the same conclusion.) This homeomorphism may be chosen to be the 'identity' on A. This shows that we are done by Lemma 3.1 and the obvious fact that the sets

$$\bigcup_{n=1}^{\infty} A_{2n} \times \{0\}, \qquad \bigcup_{n=0}^{\infty} A_{2n+1} \times \{1\}$$

have disjoint closures in M^Q. □

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