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A Weil–Barsotti formula for Drinfeld modules

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Abstract

We study the group of extensions in the category of Drinfeld modules and Anderson's t-modules, and we show in certain cases that this group can itself be given the structure of a t-module. Our main result is a Drinfeld module analogue of the Weil–Barsotti formula for abelian varieties. Extensions of general t-modules are also considered, in particular extensions of tensor powers of the Carlitz module. We motivate these results from various directions and compare to the situation of elliptic curves.

MSC: 11G09

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1. Introduction and statement of results

In this paper, we investigate extensions of Drinfeld modules using the well-known analogy between abelian varieties and Drinfeld modules. We prove analogues for Drinfeld modules of the classical Weil–Barsotti formula and the Cartier–Nishi biduality theorem for abelian varieties.

Let A be an abelian variety over a field k; we denote the dual abelian variety by $A^\vee$. The Weil–Barsotti formula states that for any k-algebra R, there is a natural, functorial isomorphism $\mathrm{Ext}^1_R(A, \mathbb{G}_m) \cong A^\vee(R)$ where the first group is calculated in the category of group schemes over $\mathrm{Spec}\, R$ [15]. In other words, the functor

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$R \mapsto \mathrm{Ext}^1_R(A, \mathbb{G}_m)$ (on k-algebras) is represented by the dual abelian variety $A^\vee$. The *biduality* theorem of Cartier–Nishi states that there is a canonical isomorphism $\mathrm{Ext}^1_R(A^\vee, \mathbb{G}_m) \cong A(R)$; this can also be restated: there is a canonical isomorphism of abelian varieties $(A^\vee)^\vee \cong A$. Theorem 1.1 provides a Drinfeld module analogue of these results.

There is also an important relationship between the de Rham cohomology (in characteristic zero) of A and the universal additive (or vectorial) extension $A^\natural$ of A [14]. A de Rham theory for Drinfeld modules based on additive extensions has already been developed by Anderson, Deligne, Gekeler, and Yu [11]. We pursue generalizations for t-modules.

We remark that extensions by $\mathbb{G}_m$ and $\mathbb{G}_a$ have been used by Deligne [9, Section 10] to define Cartier duality and the de Rham theory for 1-motives.

Notation: Let K be a perfect field of characteristic $p > 0$, and let $\mathbb{F}_q[t]$ be the polynomial ring in one variable over the finite field $\mathbb{F}_q$ where $q = p^m$. Fix an $\mathbb{F}_q$-linear homomorphism $\iota : \mathbb{F}_q[t] \to K$ with $\theta := \iota(t)$. Throughout, all Drinfeld modules and t-modules are defined with respect to the map ι, and in particular all Drinfeld modules are $\mathbb{F}_q[t]$-modules.

The ring $K\{\tau\}$ is the ring of twisted polynomials in K such that for $x \in K$, $\tau x = x^q \tau$. A d-dimensional t-module over K is at first an $\mathbb{F}_q$-linear ring homomorphism

$$\Phi : \mathbb{F}_q[t] \to \mathrm{Mat}_d(K\{\tau\}),$$

such that, as a polynomial in τ with coefficients in $\mathrm{Mat}_d(K)$,

$$\Phi(t) = (\theta I_d + N)\tau^0 + M_1 \tau^1 + \cdots,$$

where I_d is the identity matrix and N is nilpotent. In general, a t-module over K is an algebraic group E defined over K, which is isomorphic over K to $\mathbb{G}_a^d$, together with a choice of $\mathbb{F}_q$-linear endomorphism $t : E \to E$ such that $d(t - \theta)^n \mathrm{Lie}(E) = 0$ for all n sufficiently large. By choosing an isomorphism $E \cong \mathbb{G}_a^d$, one can specify a homomorphism $\Phi : \mathbb{F}_q[t] \to \mathrm{Mat}_d(K\{\tau\})$ as above. To denote this choice of coordinates, we write $E = (\mathbb{G}_a^d, \Phi)$.

Let C denote the Carlitz module, $C : \mathbb{F}_q[t] \to K\{\tau\}$, defined by $C(t) = \theta + \tau$.

We take $\mathrm{Ext}^1(\cdot, \cdot)$ to be the bifunctor Ext^1 from the additive category of t-modules to the category of abelian groups. In Section 2 we see that, for two t-modules E and F, those extensions which induce trivial t-module extensions of their respective tangent spaces comprise a canonical subgroup $\mathrm{Ext}^1_0(E, F) \subseteq \mathrm{Ext}^1(E, F)$. For a t-module E, we let $E^\vee := \mathrm{Ext}^1_0(E, C)$.

Our analogue of the classical Weil–Barsotti formula and the Cartier–Nishi biduality theorem is the following

Theorem 1.1. *Let E be a Drinfeld module of rank $r \geqslant 2$.*

(a) *The group* $\mathrm{Ext}^1(E, C)$ *is naturally a t-module of dimension r and sits in an exact sequence of t-modules*

$$0 \to E^\vee \to \mathrm{Ext}^1(E, C) \to \mathbb{G}_a \to 0.$$

Furthermore, $E^\vee$ *is the Cartier–Taguchi dual t-module associated to E* [21], *and in particular,* $E^\vee$ *is isomorphic to the* $(r-1)$*th exterior power* $\bigwedge^{r-1} E$ *of E.*

(b) *The group* $\mathrm{Ext}^1(E^\vee, C)$ *is also naturally a t-module of dimension r and sits in an exact sequence*

$$0 \to E \to \mathrm{Ext}^1(E^\vee, C) \to \mathbb{G}_a^{r-1} \to 0.$$

Moreover, we have a biduality: $(E^\vee)^\vee \cong E$.

(c) *Any morphism* $\beta : E \to F$ *of Drinfeld modules* (*of rank* $\geqslant 2$) *induces a morphism of dual t-modules* $\beta^\vee : F^\vee \to E^\vee$.

The proof of Theorem 1.1 also shows that the t-module structure on $\mathrm{Ext}^1(E, C)$ is compatible with base change of the field K; see Section 5.

Parts (a) and (b) of Theorem 1.1 for Drinfeld modules of rank 2 have been proven by Woo [22]. Taguchi [21] has constructed a Weil pairing (compatible with the Galois action) on the torsion points of E and $E^\vee$. Taguchi remarks in [21] that his definition of the Cartier dual $E^\vee$ of a Drinfeld module E does generalize to some (but not all) t-modules.

Theorem 1.1 requires us to work outside the category of Drinfeld modules, and one may ask for general t-modules E and F over K whether $\mathrm{Ext}^1(E, F)$ has the structure of a t-module. In this vein, we have the following result. Let $C^{\otimes n}$ denote the nth tensor power of the Carlitz module [2].

Theorem 1.2. *If* $n > m$, *then* $\mathrm{Ext}^1(C^{\otimes m}, C^{\otimes n})$ *has the structure of a t-module, and there is an exact sequence of t-modules*

$$0 \to C^{\otimes(n-m)} \to \mathrm{Ext}^1(C^{\otimes m}, C^{\otimes n}) \to L \to 0,$$

where L is an m-dimensional iterated extension of $\mathbb{G}_a$. *Moreover,* $\mathrm{Ext}^1_0(C^{\otimes m}, C^{\otimes n}) \cong C^{\otimes(n-m)}$.

Since the (tractable) period $\tilde{\pi}^{n-m}$ of $C^{\otimes(n-m)}$ is a power of the period $\tilde{\pi}$ of the Carlitz module (see Goss [12, Chapter 3]), one should compare Theorem 1.2 with the isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(m), \mathbb{Z}(n)) \cong \mathbb{C}/(2\pi i)^{n-m}\mathbb{Z}, \quad n > m,$$

from the theory of mixed Hodge structures [8].

The question of whether an analogue of the Weil–Barsotti formula holds for general t-modules is also interesting. Experimental evidence suggests that the general situation is subtle and that such formulas are not always valid for pure t-modules, e.g. in the form of Theorem 1.1, a Weil–Barsotti formula does not hold for $C^{\otimes m}$ because $\mathrm{Ext}^1(C^{\otimes m}, C^{\otimes n})$ is not well behaved for $n \leqslant m$. See Section 4 for more details.

As pointed out by the referee, it would be worth investigating the extent to which Theorems 1.1 and 1.2 are true for Drinfeld modules over rings more general than $\mathbb{F}_q[t]$. This raises some technical issues, which we discuss in Section 5.

The outline of this paper is as follows. In Section 2, we present definitions and fundamental results on extensions of t-modules. We prove Theorems 1.1 and 1.2 in Sections 3 and 4. In Section 5, we consider extensions of t-modules from an analytic viewpoint, so as to motivate the expectation that $\mathrm{Ext}^1(E, F)$ can be represented by a t-module for certain t-modules E and F. We consider the situation of elliptic curves in Section 6 and compare our results to an unpublished theorem of S. Lichtenbaum about extensions of elliptic curves over $\mathbb{C}$. We conclude in Section 7 with some remarks about extensions of t-motives.

2. Extensions of t-modules and biderivations

In this section, we establish definitions and results about extensions of t-modules. For general definitions of t-modules, we follow the terminology in [12, Chapter 5]. So as not to lead to confusion, we adhere to the following convention: a "t-module" refers to the object of the same name defined in Section 1, whereas an "$\mathbb{F}_q[t]$-module" is simply a module over the ring $\mathbb{F}_q[t]$.

We point out that the results in this section remain valid in the case that K is not perfect, though we do not make use of this fact later on.

Let E and F be t-modules over K. An extension of E by F is a t-module X fitting into an exact sequence of t-modules

$$0 \to F \to X \to E \to 0. \tag{1}$$

Then $\mathrm{Ext}^1(E, F)$ is defined to be the group (under Baer sum) of t-module extensions of E by F up to Yoneda equivalence.

The main tool which enables us to compute this group is that of biderivations. The following definitions run parallel to those of Brownawell and the first author [6] and Gekeler [11], where extensions of t-modules by $\mathbb{G}_a$ were investigated.

Let $\Phi : \mathbb{F}_q[t] \to \mathrm{Mat}_d(K)\{\tau\}$ and $\Psi : \mathbb{F}_q[t] \to \mathrm{Mat}_e(K)\{\tau\}$ be choices of coordinates for E and F, respectively, where $\mathrm{Mat}_d(K)\{\tau\}$ is the ring of twisted polynomials with matrix coefficients. A (Φ, Ψ)-*biderivation* is an $\mathbb{F}_q$-linear map

$$\delta : \mathbb{F}_q[t] \to \mathrm{Mat}_{e \times d}(K)\{\tau\},$$

which satisfies

$$\delta(ab) = \Psi(a)\delta(b) + \delta(a)\Phi(b) \quad \forall a, b \in \mathbb{F}_q[t]. \tag{2}$$

The $\mathbb{F}_q$-vector space of all (Φ, Ψ)-biderivations is denoted $\mathrm{Der}(\Phi, \Psi)$. It is straightforward to check that a biderivation δ is uniquely determined by the single value $\delta(t)$, and so if $V \in \mathrm{Mat}_{e\times d}(K)\{\tau\}$, we define $\delta_V \in \mathrm{Der}(\Phi, \Psi)$ to be that biderivation such that $\delta_V(t) = V$. In this way, we have an isomorphism of $\mathbb{F}_q$-vector spaces

$$V \mapsto \delta_V : \ \mathrm{Mat}_{e\times d}(K)\{\tau\} \xrightarrow{\sim} \mathrm{Der}(\Phi, \Psi). \tag{3}$$

A biderivation δ is called *inner* if for some $U \in \mathrm{Mat}_{e\times d}(K\{\tau\})$ we have

$$\delta(a) = \delta^{(U)}(a) := U\Phi(a) - \Psi(a)U \quad \forall a \in \mathbb{F}_q[t]. \tag{4}$$

The subspace of $\mathrm{Der}(\Phi, \Psi)$ of inner biderivations is denoted $\mathrm{Der}_{\mathrm{in}}(\Phi, \Psi)$.

Every (Φ, Ψ)-biderivation δ gives rise to an extension $X = (\mathbb{G}_a^{d+e}, \Upsilon)$ of E by F by defining

$$\Upsilon(a) := \begin{pmatrix} \Phi(a) & 0 \\ \delta(a) & \Psi(a) \end{pmatrix} \quad \forall a \in \mathbb{F}_q[t].$$

Again it is straightforward, using (2), to check that Υ is well defined. Moreover, every extension of E by F defines a unique biderivation.

We note that if $\delta^{(U)}$ is an inner biderivation then in fact X is split. In this case the matrix $\Theta := \begin{pmatrix} I_d & 0 \\ U & I_e \end{pmatrix}$ provides the splitting, where I_d, I_e are identity matrices:

$$\Theta^{-1}\Upsilon(a)\Theta = \begin{pmatrix} \Phi(a) & 0 \\ 0 & \Psi(a) \end{pmatrix} \quad \forall a \in \mathbb{F}_q[t].$$

Furthermore, it follows from the above discussion that every split extension arises in this way.

Suppose we are given two extensions of E by F which are Yoneda equivalent. It follows easily from the definition of Yoneda equivalence that the corresponding biderivations differ by an inner biderivation. It is straightforward to check that the (Baer) sum on $\mathrm{Ext}^1(E, F)$ corresponds to usual addition on the level of biderivations.

Now the endomorphisms of E and F induce (identical) $\mathbb{F}_q[t]$-module structures on $\mathrm{Ext}^1(E, F)$. That is, if X represents a class in $\mathrm{Ext}^1(E, F)$ and $b \in \mathbb{F}_q[t]$, we can define two t-modules $X * b$ and $b * X$, which ultimately represent the same class in $\mathrm{Ext}^1(E, F)$. Explicitly, suppose δ is the (Φ, Ψ)-biderivation corresponding to X and $\pi : X \to E$ is the natural map in (1). Let

$$X * b := \ker((e, x) \mapsto \Phi(b)(e) - \pi(x) : E \oplus X \to E).$$

Then $X * b$ is itself a t-module extension of E by F, and the operation of $\mathbb{F}_q[t]$ on it is given by

$$(\Upsilon * b)(a) = \begin{pmatrix} \Phi(a) & 0 \\ \delta(a)\Phi(b) & \Psi(a) \end{pmatrix} \quad \forall a \in \mathbb{F}_q[t].$$

On the other hand, we can similarly use endomorphisms of F to define an extension $b * X$ whose t-module structure is given by

$$(b * \Upsilon)(a) = \begin{pmatrix} \Phi(a) & 0 \\ \Psi(b)\delta(a) & \Psi(a) \end{pmatrix} \quad \forall a \in \mathbb{F}_q[t].$$

To see that $X * b$ and $b * X$ are equivalent extensions, we note that $\varepsilon : \mathbb{F}_q[t] \to \mathrm{Mat}_{e \times d}(K)\{\tau\}$ defined by

$$\varepsilon(a) := \delta(a)\Phi(b) - \Psi(b)\delta(a) \quad \forall a \in \mathbb{F}_q[t]$$

is in fact the inner biderivation $\delta^{(U)}$, with $U = \delta(t)$ in (4). That is,

$$b : \delta(\cdot) \mapsto \delta(\cdot)\Phi(b) \quad \text{and} \quad b : \delta(\cdot) \mapsto \Psi(b)\delta(\cdot) \tag{5}$$

define $\mathbb{F}_q[t]$-module structures on $\mathrm{Der}(\Phi, \Psi)$ which are the same modulo $\mathrm{Der}_{\mathrm{in}}(\Phi, \Psi)$. We record the results from the preceding paragraphs in the following lemma.

Lemma 2.1. *Let $E = (\mathbb{G}_a^d, \Phi)$ and $F = (\mathbb{G}_a^e, \Psi)$ be t-modules. Then*

$$\mathrm{Ext}^1(E, F) \cong \mathrm{Der}(\Phi, \Psi)/\mathrm{Der}_{\mathrm{in}}(\Phi, \Psi)$$

as $\mathbb{F}_q[t]$-modules.

For $U \in \mathrm{Mat}_{d_1 \times d_2}(K)\{\tau\}$, we let $dU \in \mathrm{Mat}_{d_1 \times d_2}(K)$ be the constant term of U as a polynomial in τ, and we define the following subspaces of $\mathrm{Der}(\Phi, \Psi)$:

$$\mathrm{Der}_0(\Phi, \Psi) := \{\delta \in \mathrm{Der}(\Phi, \Psi) : d\delta(t) = 0\},$$

$$\mathrm{Der}_{\mathrm{si}}(\Phi, \Psi) := \{\delta^{(U)} \in \mathrm{Der}_{\mathrm{in}}(\Phi, \Psi) : dU = 0\}.$$

The utility of $\mathrm{Der}_0(\Phi, \Psi)$ is derived from the following lemma, whose immediate corollary follows from Lemma 2.1. Note that Der_0 represents a different object here than in [6,11]. Biderivations in $\mathrm{Der}_{\mathrm{si}}(\Phi, \Psi)$ are called *strictly inner*, and clearly $\mathrm{Der}_{\mathrm{si}}(\Phi, \Psi) \subseteq \mathrm{Der}_0(\Phi, \Psi)$. We will study $\mathrm{Der}_{\mathrm{si}}(\Phi, \Psi)$ in more detail in Section 5.

The map $d\Phi : \mathbb{F}_q[t] \to \mathrm{Mat}_d(K)$ defines a non-abelian t-module whose underlying space is the tangent space $\mathrm{Lie}(E) \cong K^d$. Furthermore, the map

$$\delta \mapsto d\delta : \ \mathrm{Der}(\Phi, \Psi) \to \mathrm{Der}(d\Phi, d\Psi)$$

is $\mathbb{F}_q$-linear, and it is $\mathbb{F}_q[t]$-linear modulo inner biderivations.

Lemma 2.2. *Let $E = (\mathbb{G}_a^d, \Phi)$ and $F = (\mathbb{G}_a^e, \Psi)$ be t-modules. The following is an exact sequence of $\mathbb{F}_q[t]$-modules*:

$$0 \to \frac{\mathrm{Der}_0(\Phi, \Psi)}{\mathrm{Der}_0(\Phi, \Psi) \cap \mathrm{Der}_{\mathrm{in}}(\Phi, \Psi)} \to \frac{\mathrm{Der}(\Phi, \Psi)}{\mathrm{Der}_{\mathrm{in}}(\Phi, \Psi)} \to \frac{\mathrm{Der}(d\Phi, d\Psi)}{\mathrm{Der}_{\mathrm{in}}(d\Phi, d\Psi)}. \tag{6}$$

If $\theta \in K$ is transcendental over $\mathbb{F}_q$, then the final map in this sequence is surjective.

We define $\mathrm{Ext}_0^1(E, F)$ to be the $\mathbb{F}_q[t]$-submodule of $\mathrm{Ext}^1(E, F)$ corresponding to $\mathrm{Der}_0(\Phi, \Psi)/(\mathrm{Der}_0(\Phi, \Psi) \cap \mathrm{Der}_{\mathrm{in}}(\Phi, \Psi))$.

Corollary 2.3. *The sequence in Lemma 2.2 corresponds to an exact sequence of $\mathbb{F}_q[t]$-modules,*

$$0 \to \mathrm{Ext}_0^1(E, F) \to \mathrm{Ext}^1(E, F) \to \mathrm{Ext}^1(\mathrm{Lie}(E), \mathrm{Lie}(F)),$$

where the final map is surjective if $\theta \in K$ is transcendental over $\mathbb{F}_q$.

Proof of Lemma 2.2. Injectivity on the left of (6) is clear. To show exactness in the center, first any $\delta \in \mathrm{Der}_0(\Phi, \Psi)$ maps to 0 in $\mathrm{Der}(d\Phi, d\Psi)$. On the other hand, suppose $\delta \in \mathrm{Der}(\Phi, \Psi)$ and $d\delta \in \mathrm{Der}(d\Phi, d\Psi)$ is inner, say $d\delta(t) = Ud\Phi(t) - d\Psi(t)U$ with $U \in \mathrm{Mat}_{e\times d}(K)\{\tau\}$. Then $\delta - \delta_{\Phi,\Psi}^{(U)} \in \mathrm{Der}_0(\Phi, \Psi)$ represents the same class as δ in $\mathrm{Der}(\Phi, \Psi)/\mathrm{Der}_{\mathrm{in}}(\Phi, \Psi)$.

In the case that $\theta \in K$ is transcendental over $\mathbb{F}_q$, we show surjectivity on the right. We suppose $d\Phi(t) = I_d\theta + M$ and $d\Psi(t) = I_e\theta + N$, where M and N are nilpotent. Without loss of generality, we can assume M and N are both upper triangular. The spaces of biderivations $\mathrm{Der}(d\Phi, d\Psi)$ and $\mathrm{Der}_{\mathrm{in}}(d\Phi, d\Psi)$ are both naturally K-linear, and the map defining elements of $\mathrm{Der}_{\mathrm{in}}(d\Phi, d\Psi)$,

$$U \mapsto \delta_{d\Phi,d\Psi}^{(U)}(t) \colon \mathrm{Mat}_{e\times d}(K)\{\tau\} \to \mathrm{Mat}_{e\times d}(K)\{\tau\} \tag{7}$$

is K-linear and respects the grading by degrees in τ. Now

$$\delta_{d\Phi,d\Psi}^{(U)}(t) = U\theta - \theta U + UM - NU. \tag{8}$$

If U_{ij} is the matrix in $\mathrm{Mat}_{e\times d}(K)$ with a 1 in the ijth entry and zeros elsewhere, then $\{U_{m1}\tau^r, U_{m-1,1}\tau^r, \ldots\} = \{U_{ij}\tau^r\}_{i=m,\ldots,e}^{j=1,\ldots,d}$ is an ordered K-basis for $\mathrm{Mat}_{e\times d}(K)\tau^r$. Using this basis, it follows from (8) and the fact that M and N are both upper triangular that the map in (7), restricted to $\mathrm{Mat}_{e\times d}(K)\tau^r$, is lower triangular with $\theta^{q^r} - \theta$ along the diagonal. Since $\theta^{q^r} - \theta$ is non-zero by our assumption on θ,

$$U \mapsto \delta_{d\Phi,d\Psi}^{(U)}(t) \colon \; \mathrm{Mat}_{e\times d}(K)\{\tau\}\tau \xrightarrow{\sim} \mathrm{Mat}_{e\times d}(K)\{\tau\}\tau$$

is an isomorphism of K-vector spaces. Therefore, for every $\delta \in \mathrm{Der}(d\Phi, d\Psi)$, if $d\delta(t) = 0$, then δ is inner, and so the right-hand side of (6) is surjective. $\square$

3. A Weil–Barsotti formula

In this section, we prove Theorem 1.1. Let $E = (\mathbb{G}_a, \Phi)$ be a Drinfeld module of rank $r \geqslant 2$, where $\Phi(t) = \theta + a_1\tau + \cdots + a_r\tau^r$. For an extension X of E by C, we let $\delta_X \in \mathrm{Der}(\Phi, C)$ be its associated biderivation.

Proof of Theorem 1.1. (a) An extension X of E by C splits if and only if $\delta_X \in \mathrm{Der}_{\mathrm{in}}(\Phi, C)$, i.e. if there exists $u \in K\{\tau\}$ such that $\delta_X(t) = u\Phi(t) - (\theta + \tau)u$. If $u = c\tau^m$, then we obtain an inner biderivation $\delta^{(u)}$ such that

$$\begin{aligned}\delta^{(u)}(t) &= u\Phi(t) - (\theta + \tau)u \\ &= c\theta^{q^m}\tau^m + ca_1^{q^m}\tau^{m+1} + \cdots + ca_r^{q^m}\tau^{m+r} - \theta c\tau^m - c^q\tau^{m+1}.\end{aligned} \tag{9}$$

If $n := \deg_\tau(\delta_X(t))$ is greater than $r-1$, we can repeatedly subtract (9) from $\delta_X(t)$, with $m = n - r,\ n - r - 1, \ldots, 0$, to reduce the τ-degree of $\delta_X(t)$; eventually this degree will be $< r$. Namely, any extension X is equivalent to an extension X' with $\deg_\tau(\delta_{X'}(t)) \leqslant r - 1$, which we call the reduced representative of X.

According to (9), the non-zero inner biderivations of least degree have degree r, so two extensions Z and Y satisfying $\deg_\tau(\delta_Z(t)) \leqslant r-1$, $\deg_\tau(\delta_Y(t)) \leqslant r-1$, and $\delta_Z(t) \neq \delta_Y(t)$, are inequivalent. Therefore the map

$$X \mapsto \delta_{X'}(t) \colon \mathrm{Ext}^1(E, C) \xrightarrow{\sim} V := \{b_0 + b_1\tau + \cdots + b_{r-1}\tau^{r-1} \colon b_i \in K\}, \tag{10}$$

where X' is the reduced representative of X, induces an isomorphism of $\mathbb{F}_q[t]$-modules.

We now turn to the t-module structure on $\mathrm{Ext}^1(E, C)$. Recall from (5) that multiplication-by-t on $\mathrm{Ext}^1(E, C)$ is defined by $t * \alpha = (\theta + \tau)\alpha = \alpha\Phi(t)$. Here, we think of α as being an element of V. In order to see the action explicitly, it is enough to consider $\alpha = b_i\tau^i$:

$$\begin{aligned} t*(b_0) &= (\theta+\tau)b_0 = \theta b_0 + b_0^q\tau, \\ t*(b_1\tau) &= (\theta+\tau)b_1\tau = \theta b_1\tau + b_1^q\tau^2, \\ &\ \ \vdots \qquad\qquad \vdots \\ t*(b_{r-1}\tau^{r-1}) &= (\theta+\tau)b_{r-1}\tau^{r-1} = \theta b_{r-1}\tau^{r-1} + b_{r-1}^q\tau^r. \end{aligned} \tag{11}$$

Using (9) with $m = 0$, $c = b_{r-1}^q/a_r$, we can rewrite the last identity of (11) as

$$\begin{aligned} t*(b_{r-1}\tau^{r-1}) = {} & \left(\frac{b_{r-1}^{q^2}}{a_r^q} - \frac{b_{r-1}^q a_1}{a_r}\right)\tau - \frac{b_{r-1}^q}{a_r}(a_2\tau^2 + \cdots + a_{r-2}\tau^{r-2}) \\ & + \left(\theta b_{r-1} - \frac{b_{r-1}^q a_{r-1}}{a_r}\right)\tau^{r-1}. \end{aligned}$$

Thus in terms of the elements $e_i = \tau^i$ $(i = 0, \dots, r-1)$ of V, the t-module structure on $\mathrm{Ext}^1(E, C)$ can be expressed by the map $\Pi : \mathbb{F}_q[t] \to \mathrm{Mat}_r(K)\{\tau\}$, defined by

$$\Pi(t) := \begin{pmatrix} \theta & 0 & \cdots & \cdots & 0 & 0 \\ \tau & \theta & 0 & & \vdots & -\dfrac{a_1}{a_r}\tau + \dfrac{1}{a_r^q}\tau^2 \\ 0 & \tau & \theta & 0 & \vdots & -\dfrac{a_2}{a_r}\tau \\ \vdots & 0 & \tau & \theta & 0 & \vdots \\ \vdots & & 0 & \tau & \theta & -\dfrac{a_{r-2}}{a_r}\tau \\ 0 & \cdots & \cdots & 0 & \tau & \theta - \dfrac{a_{r-1}}{a_r}\tau. \end{pmatrix}.$$

Comparing with Taguchi [21, Section 5], it is clear that the t-module $\mathrm{Ext}^1(E, C)$ is an extension of $\mathbb{G}_a$ by the t-module denoted there $\check{E}$. By Corollary 2.3 and the characterization of V in (10), it is clear $\mathrm{Ext}^1(E, C)$ is an extension of $\mathbb{G}_a$ by $E^\vee$. Thus $E^\vee$ is the same as Taguchi's t-module. Moreover, Taguchi shows [21, Theorem 5.1] that $E^\vee$ is a pure t-module isomorphic to $\bigwedge^{r-1} E$. $\square$

We turn to part (b), and for simplicity we assume that $a_r = 1$; the general case follows similarly. The t-module structure on $E^\vee$ is then defined by

$$\Psi(t) := \begin{pmatrix} \theta & 0 & \cdots & 0 & -a_1\tau + \tau^2 \\ \tau & \theta & 0 & \vdots & -a_2\tau \\ 0 & \tau & \theta & 0 & \vdots \\ \vdots & 0 & \tau & \theta & -a_{r-2}\tau \\ 0 & \cdots & 0 & \tau & \theta - a_{r-1}\tau. \end{pmatrix}.$$

Consider the biderivation $\delta_X : \mathbb{F}_q[t] \to \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$ of any extension X of $E^\vee$ by C. The inner biderivations are of the form $\delta^{(U)}(t) = U\Psi(t) - (\theta + \tau)U$ for $U = (u_i) \in \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$. Explicitly, $v = (v_i) \in \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$ defines an inner biderivation, $\delta^{(U)} = \delta_v$, if it has the form

$$v_i = u_i\theta - \theta u_i + u_{i+1}\tau - \tau u_i \quad (1 \leqslant i \leqslant r-2),$$

$$v_{r-1} = u_{r-1}\theta - \theta u_{r-1} + u_1\tau^2 - \tau u_{r-1} - \sum_{j=1}^{r-1} u_j a_j \tau.$$

An inner biderivation δ_v is said to be basic if $u_i = 0$ for all $i \neq s$ and $u_s = c\tau^m$ with $c \in K$; we write $v = v(s, c, m) = (v_1, \dots, v_{r-1})$. Explicitly written, these are

$$v(1, c, m) = (c\theta^{q^m}\tau^m - c\theta\tau^m - c^q\tau^{m+1}, 0, \dots, 0, c\tau^{m+2} - ca_1^{q^m}\tau^{m+1}),$$

$$v(2,c,m) = (c\tau^{m+1}, c\theta^{q^m}\tau^m - c\theta\tau^m - c^q\tau^{m+1}, 0, \ldots, 0, -ca_2^{q^m}\tau^{m+1}),$$

$$v(z,c,m) = (0, \ldots, c\tau^{m+1}, c\theta^{q^m}\tau^m - c\theta\tau^m - c^q\tau^{m+1}, 0, \ldots, 0, -ca_z^{q^m}\tau^{m+1}),$$

$$v(r-1,c,m) = (0, \ldots, 0, c\tau^{m+1}, c\theta^{q^m}\tau^m - c\theta\tau^m - c^q\tau^{m+1} - ca_{r-1}^{q^m}\tau^{m+1}), \qquad (12)$$

where for $2 \leqslant z \leqslant r-2$ the possible non-zero coordinates of $v(z,c,m)$ are v_{z-1}, v_z, and v_{r-1}. Every inner biderivation arises from an additive combination of basic δ_v.

Consider $G \subseteq \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$ consisting of elements $u := (u_i)$ with τ-degrees of $u_1, \ldots, u_{r-2}$ zero and the τ-degree of u_{r-1} less than or equal one. In other words, $u_1, \ldots, u_{r-2} \in K$ and $u_{r-1} = c + d\tau$ with $c, d \in K$. Elements of G give rise via (3) to biderivations which we will call reduced.

Lemma 3.1. *The map* $u \mapsto \delta_u : G \to \mathrm{Der}(\Psi, C)$ *induces an isomorphism*

$$G \cong \frac{\mathrm{Der}(\Psi, C)}{\mathrm{Der}_{\mathrm{in}}(\Psi, C)} \cong \mathrm{Ext}^1(E^\vee, C)$$

of $\mathbb{F}_q$*-vector spaces.*

Proof. We need only to prove the first isomorphism by Lemma 2.1. Let X be an extension. We want to subtract appropriate v's in (12) from $\delta_X(t) = u = (u_1, u_2, \ldots, u_{r-1})$ so that the resulting biderivation is reduced. In this process, we need to keep track of the τ-degrees of the u_j's. We define the τ-degree of u to be the vector $d_u := (d_1(u), \ldots, d_{r-1}(u))$ with $d_j(u) = \tau$-degree of u_j. Given two vectors d and d' with integer coefficients, we shall say $d \leqslant d'$ if $d_i \leqslant d'_i$ for all i. So our claim is that u can be reduced to a biderivation $\tilde{u}$ such that $d_{\tilde{u}} \leqslant (0, \ldots, 0, 1)$.

Let n be the maximum of the integers $d_j(u)$; one has $d_j(u) \leqslant n$ for all j. We can modify u by $v(2, c, d_1 - 1)$ for an appropriate $c \in K$ to obtain u' such that $d_1(u') < d_1(u)$ and $d_g(u') \leqslant n$ for $g \geqslant 2$. Subtracting an appropriate $v(3, c, d_2(u'))$, we obtain u'' such that $d_2(u'') < d_2(u')$ and $d_g(u'') \leqslant n$ for $g \geqslant 3$. Repeating this for $z = 4, \ldots, r-2$ using appropriate $v(z, c, m)$, we obtain a $w \in \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$ whose degree vector is less than or equal to $(n-1, n-1, \ldots, n-1, n)$.

If $n \leqslant 1$, we are done. If not ($n \geqslant 2$), we can subtract an appropriate $v(1, c, n-2)$ to obtain a vector w whose degree vector is less than or equal to $(n-1, n-1, \ldots, n-1, n-1)$. We repeat the procedure in the two paragraphs above until we arrive at a vector w whose degree vector is less than or equal to $(0, \ldots, 0, 1)$. $\square$

We can now determine the t-module structure on $\mathrm{Ext}^1(E^\vee, C)$. By the previous lemma, it suffices to see this structure on G. Consider the elements $e_i \in G \subseteq \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$, $1 \leqslant i \leqslant r$, defined as follows. For $1 \leqslant i \leqslant r-1$, we take e_i to be the vector with 1 in the ith coordinate and zeros elsewhere. We take e_r to be the vector with τ in the last coordinate and zeros elsewhere. The structure of a

$\mathbb{F}_q[t]$-module on G is completely described by the action of t on elements of the form be_i, $b \in K$, since additive combinations of such elements give all of G.

Consider $t*(be_i) = (\theta + \tau)be_i = \theta e_i + b^q \tau e_i$. The last is no longer an element of G, and we need its equivalent vector in G. Using the reduction procedure of Lemma 3.1, it is easily computed that

$$(\theta + \tau)(be_i) = \theta b e_n + \left(\sum_{f=0}^{r-i-1} b^{q^{r-i-f}} a_{r-f} \right) e_r, \quad 1 \leqslant i \leqslant r-1,$$

$$(\theta + \tau)(be_{r-1}) = \theta b e_{r-1} + b^q e_r,$$

$$(\theta + \tau)(be_{r-2}) = \theta b e_{r-2} + (b^{q^2} + b^q a_{r-1}) e_r,$$

$$(\theta + \tau)(be_r) = \left(\theta + \sum_{f=1}^{r} b^{q^f} a_f \right) e_r.$$

Thus there is a t-module structure $\Xi : \mathbb{F}_q[t] \to \mathrm{Mat}_r(K)\{\tau\}$ on $\mathrm{Ext}^1(E^\vee, C)$, which is completely described by

$$\Xi(t) := \begin{pmatrix} \theta & 0 & \cdots & \cdots & 0 \\ 0 & \theta & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \theta & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{r-1} & \alpha_r \end{pmatrix}.$$

Here $\alpha_r = \theta + a_1\tau + a_2\tau^2 + \cdots + a_{r-1}\tau^{r-1} + \tau^r$, and the others are given by $\alpha_n = \sum_{f=0}^{r-n-1} \tau^{r-n-f} a_{r-f}$, for $1 \leqslant n \leqslant r-1$. Moreover, $\mathrm{Ext}^1(E^\vee, C)$ is an extension of $\mathbb{G}_a^{r-1}$ by E. By Lemma 3.1, $(E^\vee)^\vee$ is one-dimensional, which completes the proof. $\square$

Proof of Theorem 1.1. (c) If $F = (\mathbb{G}_a, \Psi)$ is a Drinfeld module, then a morphism $\beta : E \to F$ is represented by $\beta = u_0 + \cdots + u_d\tau^d \in K\{\tau\}$ such that $\beta\Phi(a) = \Psi(a)\beta$ for all $a \in \mathbb{F}_q[t]$. Then β induces an $\mathbb{F}_q[t]$-module homomorphism $\beta^\vee : \mathrm{Ext}^1(F, C) \to \mathrm{Ext}^1(E, C)$, and on the level of biderivations,

$$(\beta^\vee(\delta))(a) = \delta(a)\beta \quad \forall a \in \mathbb{F}_q[t].$$

Also $\beta^\vee$ takes $\mathrm{Ext}_0^1(F, C) = F^\vee$ into $\mathrm{Ext}_0^1(E, C) = E^\vee$. Since Drinfeld modules of different ranks have no non-zero morphisms between them, we can assume that the rank of E is the same as the rank of F.

We continue with the considerations of the proof of part (a). As in (11), we need to measure the effect of $\beta^\vee$ on biderivations in $\mathrm{Der}(\Psi, C)$ represented by $\alpha = b_i\tau^i$

in (10). We see that for $0 \leqslant i \leqslant r-1$,

$$\beta^\vee(\delta_{b_i\tau^i})(t) = b_i\tau^i\beta = b_i u_0^{q^i}\tau^i + \cdots + b_i u_d^{q^i}\tau^{i+d}.$$

Using (9) we can subtract suitable inner biderivations in $\mathrm{Der}_{\mathrm{in}}(\Phi, C)$ and find that in $\mathrm{Ext}^1(E, C)$,

$$\beta^\vee(\delta_{b_i\tau^i})(t) = c_{i,0}(b_i)\tau^0 + \cdots + c_{i,r-1}(b_i)\tau^{r-1},$$

where each $c_{i,j}(x)$ is an $\mathbb{F}_q$-linear polynomial in $K[x]$ whose coefficients depend only on β. Thus $\beta^\vee : \mathrm{Ext}^1(F, C) \to \mathrm{Ext}^1(E, C)$ is represented by a matrix in $\mathrm{Mat}_{r\times r}(K\{\tau\})$, and $\beta^\vee$ restricts to a t-module morphism $\beta^\vee : F^\vee \to E^\vee$. $\square$

4. Extensions of tensor powers of the Carlitz module

In this section, we prove Theorem 1.2. Recall that the nth tensor power of the Carlitz module is the n-dimensional pure t-module $C^{\otimes n} : \mathbb{F}_q[t] \to \mathrm{Mat}_n(K)\{\tau\}$ defined by

$$C^{\otimes n}(t) := \begin{pmatrix} \theta & 1 & 0 \\ \vdots & \ddots & 1 \\ \tau & \cdots & \theta \end{pmatrix}.$$

That is, $C^{\otimes n}(t) = \theta I_n + N_n + E_n\tau$, where

$$N_n := \begin{pmatrix} 0 & 1 & 0 \\ \vdots & \ddots & 1 \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad E_n := \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

See [2] for more details.

Fix $m<n$. The following lemma determines representatives for elements of $\mathrm{Der}(C^{\otimes m}, C^{\otimes n})/\mathrm{Der}_{\mathrm{in}}(C^{\otimes m}, C^{\otimes n})$. For $V \in \mathrm{Mat}_{n\times m}(K\{\tau\})$, recall the definition of δ_V from (3).

Lemma 4.1. *Let*

$$G := \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_{n\times m}(K) \right\}.$$

Then $V \mapsto \delta_V : G \xrightarrow{\sim} \mathrm{Der}(C^{\otimes m}, C^{\otimes n})/\mathrm{Der}_{\mathrm{in}}(C^{\otimes m}, C^{\otimes n})$ *is an isomorphism of* $\mathbb{F}_q$*-vector spaces.*

Proof. Let Q_{ij} be the $n \times m$ matrix with a 1 in the ijth entry and zeros elsewhere. For $c \in K$ and $k \geqslant 0$, by taking $cQ_{ij}\tau^k$ for U in $\delta^{(U)}(t)$, we define

$$\begin{aligned}\delta_{ijk}^{[c]} &:= \delta^{(cQ_{ij}\tau^k)}(t) = (cQ_{ij}\tau^k)C^{\otimes m}(t) - C^{\otimes n}(t)(cQ_{ij}\tau^k)\\ &= c(\theta^{q^k} - \theta)Q_{ij}\tau^k + c(Q_{ij}N_m - N_n Q_{ij})\tau^k\\ &\quad + (cQ_{ij}E_m - c^q E_n Q_{ij})\tau^{k+1}. \end{aligned} \tag{13}$$

Since every $U \in \mathrm{Mat}_{n\times m}(K\{\tau\})$ is an $\mathbb{F}_q$-linear combination of matrices of the form $cQ_{ij}\tau^k$, biderivations arising from (13) generate $\mathrm{Der}_{\mathrm{in}}(C^{\otimes m}, C^{\otimes n})$ as an $\mathbb{F}_q$-vector space.

Suppose that $V = (v_{ij}) \in \mathrm{Mat}_{n\times m}(K\{\tau\})$ is arbitrary and that $\deg_\tau(v_{ij}) \leqslant r$ for some $r \geqslant 1$. We will show that by subtracting matrices in (13) from V in various ways we can replace V by a matrix V' which has each entry of τ-degree $\leqslant r-1$ and also $\delta_{V'}$ equivalent to δ_V modulo $\mathrm{Der}_{\mathrm{in}}(C^{\otimes m}, C^{\otimes n})$.

We bootstrap our way through the entries of V in the following way. Let

$$D_{ij} := \{\delta_{ijk}^{[c]}\colon\ k \geqslant 0,\ \ c \in K\}.$$

Define a function F from the set of subsets of $I := \{(i,j)\colon\ 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ to itself. We set $F(S)$ to be those entries of V whose degrees in τ can be decreased by subtracting an element of $\sum_{(i,j)\in S} D_{ij}$, without increasing the degrees of the other entries. Our claim then is that $F(I) = I$. The following containments can be easily checked:

$$F(\{2 \leqslant i \leqslant n; j = m\}) \supseteq \{2 \leqslant i \leqslant n; j = 1\},$$

$$F(\{2 \leqslant i \leqslant n; j = m\} \cup \{i - j \geqslant 2\}) \supseteq \{i - j \geqslant 1\},$$

$$F(\{j = m\} \cup \{i - j \geqslant 2\}) \supseteq \{i - j \geqslant 1\} \cup \{(1,1),(1,m)\},$$

$$F(\{j = m\} \cup \{i - j \geqslant 1\}) \supseteq \{i - j \geqslant 0\} \cup \{(1,m)\},$$

$$F(\{j = m\} \cup \{i - j \geqslant -\ell\}) \supseteq \{i - j \geqslant -\ell - 1\} \cup \{(1,m)\}, \tag{14}$$

where the last containment holds for all $\ell = 0, \ldots, m-1$.

Therefore, we can assume that every entry of V is a constant from K. Now that the τ-degree of each entry of V is 0, one checks the containments in (14) still hold with the exception that all sets on the right-hand side must have $\{j = 1\}$ removed. That is, we can adjust V so that it can be replaced by a matrix in G, but elements of G can be reduced no further. $\square$

Proof of Theorem 1.2. Let G be given as in Lemma 4.1 so that by Lemma 2.1

$$G \cong \mathrm{Ext}^1(C^{\otimes m}, C^{\otimes n}).$$

Recalling the definition of Q_{ij} from the proof above, let

$$e_1 := Q_{11}, \ldots, e_n := Q_{n1}$$

be basis vectors for G over K. Combining Lemmas 2.1 and 4.1, G has a natural $\mathbb{F}_q[t]$-module structure which we now make explicit. For $c \in K$,

$$t * (ce_1) = C^{\otimes n}(t)(ce_1) = \theta ce_1 + c^q \tau e_n \tag{15}$$

and

$$t * (ce_i) = C^{\otimes n}(t)(ce_i) = \theta ce_i + ce_{i-1}, \quad 2 \leqslant i \leqslant n. \tag{16}$$

We note that as defined in (13),

$$\sum_{i=0}^{m-1} \delta_{n-i,m-i,0}^{[c^q]} = c^q Q_{n1}\tau - c^q Q_{n-m,1} = c^q \tau e_n - c^q e_{n-m},$$

and since this sum defines an inner $(C^{\otimes m}, C^{\otimes n})$-biderivation, we subtract it from (15) and find that

$$t * (ce_1) = \theta ce_1 + c^q e_{n-m}.$$

Therefore, combining this with (16), we see that the $\mathbb{F}_q[t]$-module structure on G can be expressed as a t-module by the map $\Pi : \mathbb{F}_q[t] \to \mathrm{Mat}_n(K)\{\tau\}$ defined by

$$\Pi(t) := \left(\begin{array}{c|ccc} & 0 & \cdots & 0 \\ C^{\otimes(n-m)}(t) & \vdots & & \vdots \\ & 1 & \cdots & 0 \\ \hline 0 & & I_m\theta + N_m & \end{array} \right).$$

Thus there is an exact sequence of t-modules,

$$0 \to C^{\otimes(n-m)} \to G \to L \to 0,$$

where L is an m-dimensional iterated extension of $\mathbb{G}_a$, and one checks that moreover $C^{\otimes(n-m)} = \mathrm{Ext}^1_0(C^{\otimes m}, C^{\otimes n})$. $\square$

Remark 4.2. The $\mathbb{F}_q[t]$-module structure on $\mathrm{Ext}^1(C^{\otimes m}, C^{\otimes n})$ when $n \leqslant m$ provides a different picture. We consider the following examples.

Using similar methods as in the proof of Theorem 1.2 it is possible to show that $\mathrm{Ext}^1(C^{\otimes 2}, C) \cong K^2$ as $\mathbb{F}_q$-vector spaces and that the $\mathbb{F}_q[t]$-module structure on

$\mathrm{Ext}^1(C^{\otimes 2}, C)$ is given by

$$t * \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 1 & \theta + \tau^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad a, b \in K.$$

Thus $\mathrm{Ext}^1(C^{\otimes 2}, C)$ is an extension of $\mathbb{G}_a$ by the adjoint of the Carlitz module (see Goss [12, Section 3.6]), but not a t-module.

As for an example with $n = m$, the first case to consider is $\mathrm{Ext}^1(C, C)$, which itself is quite subtle. For an inner biderivation $\delta^{(u)} \in \mathrm{Der}_{\mathrm{in}}(C, C)$ arising from $u = c\tau^k \in K\{\tau\}$,

$$\delta^{(u)}(t) = c\tau^k C(t) - C(t)c\tau^k = c(\theta^{q^k} - \theta)\tau^k + (c - c^q)\tau^{k+1}.$$

Unless K is algebraically closed as well as perfect, it is not possible to systematically decrease the degree in τ of $\delta(t)$ for an arbitrary $\delta \in \mathrm{Der}(C, C)$ (one needs to be able to solve equations of the form $c - c^q = \alpha$ for $\alpha \in K$). If K is algebraically closed, the situation improves, and it is possible to show that then $\mathrm{Ext}^1(C, C) \cong K$ and that in fact $\mathrm{Ext}^1(C, C) \cong \mathbb{G}_a$ as $\mathbb{F}_q[t]$-modules.

5. Periods of t-modules and extensions

Here, we would like to motivate the expectation that $\mathrm{Ext}^1(E, F)$, for certain t-modules E and F, can be given the structure of a t-module. Overall, we have made the following observations. If $E = (\mathbb{G}_a^d, \Phi)$ is a pure t-module of rank r, then the weight of E is defined to be

$$\mathrm{wt}(E) := d/r.$$

What Theorems 1.1 and 1.2 have in common is that, for certain pure t-modules E and F, the $\mathbb{F}_q[t]$-module $\mathrm{Ext}^1(E, F)$ has the structure of a t-module provided $\mathrm{wt}(E) < \mathrm{wt}(F)$ and that the submodule $\mathrm{Ext}^1_0(E, F)$ is itself pure and uniformizable.

Also in the situations of Theorems 1.1 and 1.2, $\mathrm{Ext}^1_0(E, F)$ behaves well under base-extension. That is, if $L \supseteq K$ is a perfect field, then $\mathrm{Ext}^1_{0/K}(E, F) \cong \mathrm{Ext}^1_{0/L}(E, F)(K)$, where here $\mathrm{Ext}^1_{0/K}(E, F)$ is the usual $\mathrm{Ext}^1_0(E, F)$ over K and $\mathrm{Ext}^1_{0/L}(E, F)(K)$ is the set of K-valued points on $\mathrm{Ext}^1_{0/L}(E, F)$.

These results, along with other experimental evidence, suggest that in general $\mathrm{Ext}^1_0(E, F)$ is pure, uniformizable, and functorial in K, for arbitrary pure uniformizable t-modules E and F with $\mathrm{wt}(E) < \mathrm{wt}(F)$. However, the examples in Remark 4.2 show that the situation when $\mathrm{wt}(E) \geqslant \mathrm{wt}(F)$ is somewhat different.

In this section, we investigate the structure of $\mathrm{Ext}^1(E, F)$ from an analytic point of view so as to support the veracity of the claims above. Our main tool will be generalizations of quasi-periodic functions defined in [6,11] for extensions by $\mathbb{G}_a$. Moreover, we study an analogue of the de Rham map of Gekeler [11]. For more details on the general analytic theory of t-modules see [12, Chapter 5].

Let $\mathbb{K}$ be the completion of the algebraic closure of the Laurent series field $\mathbb{F}_q((1/\theta))$, where θ is an independent variable. Let $\iota : \mathbb{F}_q[t] \to \mathbb{K}$ be defined by $t \mapsto \theta$.

Let $E = (\mathbb{G}_a^d, \Phi)$ and $F = (\mathbb{G}_a^e, \Psi)$ be uniformizable t-modules over $\mathbb{K}$. We take Exp_E: $\mathrm{Lie}(E)(\mathbb{K}) \to E(\mathbb{K})$ to be the exponential map of E. Because E is uniformizable, there is an exact sequence of $\mathbb{F}_q[t]$-modules

$$0 \to \Lambda_\Phi \to \mathbb{K}^d \overset{\mathrm{Exp}_\Phi}{\to} \mathbb{K}^d \to 0,$$

where $\mathbb{F}_q[t]$ operates by $d\Phi$ on the central $\mathbb{K}^d$ and by Φ on the right one. Also, Λ_Φ is the period lattice of Φ and is a discrete $\mathbb{F}_q[t]$-submodule of $\mathbb{K}^d$. Similarly, we define Exp_Ψ and Λ_Ψ for F.

Lemma 5.1. *For each $\delta \in \mathrm{Der}_0(\Phi, \Psi)$ there is a unique $\mathbb{F}_q$-linear entire function $F_\delta : \mathbb{K}^d \to \mathbb{K}^e$ such that for $z \in \mathbb{K}^d$,*

$$F_\delta(z) \equiv 0 \ (\mathrm{mod}\ \deg q),$$

$$F_\delta(d\Phi(a)z) = \Psi(a)F_\delta(z) + \delta(a)\mathrm{Exp}_\Phi(z) \quad \forall a \in \mathbb{F}_q[t].$$

Proof. The proof here is essentially the same as the that of the existence of the exponential function, and in particular we can easily adapt the proof of Proposition 2.1.4 in [1] to this situation. □

Lemma 5.2. *Suppose $\delta^{(U)} \in \mathrm{Der}_0(\Phi, \Psi) \cap \mathrm{Der}_{\mathrm{in}}(\Phi, \Psi)$. Then*

$$F_\delta(z) = U\mathrm{Exp}_\Phi(z) - \mathrm{Exp}_\Psi(dU \cdot z).$$

Proof. This follows directly from Lemma 5.1, using the fact that $\delta^{(U)} \in \mathrm{Der}_0(\Phi, \Psi)$ if and only if $dU\, d\Phi(a) = d\Psi(a)\, dU$ for all $a \in \mathbb{F}_q[t]$. □

If $X = (\mathbb{G}_a^{d+e}, \Upsilon)$ is an extension of E by F defined by a biderivation $\delta \in \mathrm{Der}_0(\Phi, \Psi)$, then Lemma 5.1 and the uniqueness of the exponential function imply that

$$\mathrm{Exp}_\Upsilon \begin{pmatrix} z \\ u \end{pmatrix} = \begin{pmatrix} \mathrm{Exp}_\Phi(z) \\ \mathrm{Exp}_\Psi(u) + F_\delta(z) \end{pmatrix}$$

is the exponential function for X. Also, since E and F are both uniformizable, so is X. Moreover, the period lattice of X is

$$\Lambda_\Upsilon = \left\{ \begin{pmatrix} \lambda \\ \eta \end{pmatrix} : \lambda \in \Lambda_\Phi, \mathrm{Exp}_\Psi(\eta) + F_\delta(\lambda) = 0 \right\}.$$

For each $(\lambda, \eta) \in \Lambda_\Upsilon$, it follows that $(\lambda, \eta + d\Psi(a)\mu) \in \Lambda_\Upsilon$ for all $a \in \mathbb{F}_q[t]$ and $\mu \in \Lambda_\Psi$.

Proposition 5.3. *The following map is a homomorphism of* $\mathbb{F}_q[t]$*-modules*:

$$\mathrm{DR} := \mathrm{DR}_{(\Phi,\Psi)} : \mathrm{Der}_0(\Phi,\Psi)/\mathrm{Der}_{\mathrm{si}}(\Phi,\Psi) \to \mathrm{Hom}_{\mathbb{F}_q[t]}(\Lambda_\Phi, \Psi(\mathbb{K})),$$

$$\delta \mapsto (\lambda \mapsto F_\delta(\lambda)).$$

Proof. The map $\mathrm{DR}_0 : \delta \mapsto (\lambda \mapsto F_\delta(\lambda))$ on $\mathrm{Der}_0(\Phi,\Psi) \to \mathrm{Hom}_{\mathbb{F}_q[t]}(\Lambda_\Phi, \Psi(\mathbb{K}))$ is clearly well defined and $\mathbb{F}_q$-linear by Lemma 5.1. If $\delta^{(U)} \in \mathrm{Der}_{\mathrm{si}}(\Phi,\Psi)$, then by definition $dU = 0$, and so by Lemma 5.2, $\mathrm{DR}_0(\delta^{(U)}) = 0$. Therefore DR is well defined. Furthermore, from (5) and Lemma 5.1, it follows that

$$F_{a*\delta}(z) = F_{\Psi(a)\delta(\cdot)}(z) = \Psi(a)F_\delta(z) \quad \forall a \in \mathbb{F}_q[t],$$

and so $\mathrm{DR}(a*\delta)(\lambda) = \Psi(a)(\mathrm{DR}(\delta)(\lambda))$, for all $a \in \mathbb{F}_q[t]$. $\square$

Remark 5.4. In the case of extensions of Drinfeld modules by $\mathbb{G}_a$, the map DR specializes to the de Rham homomorphism of Gekeler [11]. Gekeler shows in this case that the de Rham map is in fact an isomorphism. In general, determining the kernel and cokernel of DR is a delicate matter and will require further study. However, Theorems 1.1 and 1.2 can be used to imply that the de Rham maps in their respective situations are indeed isomorphisms. A straightforward modification of the proof of [11, Theorem 3.1] yields the following partial result.

Proposition 5.5. *Suppose* $E = (\mathbb{G}_a, \Phi)$ *and* $F = (\mathbb{G}_a, \Psi)$ *are Drinfeld modules of ranks* r *and* s, *respectively. If* $r > s$, *then* $\mathrm{DR}_{(\Phi,\Psi)}$ *is injective.*

Remark 5.6. If A is the ring of functions on a smooth curve $X/\mathbb{F}_q$ which are regular away from a fixed point ∞, then one can also consider Drinfeld A-modules (see [12, Chapter 4]) and their groups of extensions. In [11], Gekeler considers $\mathrm{Ext}^1(E, \mathbb{G}_a)$, where E is a Drinfeld A-module of rank r and shows that

$$\mathrm{Ext}^1_0(E, \mathbb{G}_a) \cong \mathbb{G}_a^{r-1} \tag{17}$$

as A-modules. To prove this, Gekeler uses the de Rham isomorphism to show that these two spaces match up exactly. In the special case that $A = \mathbb{F}_q[t]$, the isomorphism in (17) can be generalized and proven without the de Rham isomorphism (see [6, Section 3]). This is possible essentially because all t-modules and biderivations of t-modules are determined by values of homomorphisms on t alone.

For Drinfeld A-modules E and F over a general ring A, one can define biderivations and inner biderivations just as in Section 2. However, the identifications in (3) and thus in Lemma 2.1 are more complicated, and this difficulty makes the A-module structure of $\mathrm{Ext}^1(E,F)$ hard to characterize.

Although one would want to find generalizations of Theorems 1.1 and 1.2 in the setting of Drinfeld A-modules, it is not immediately clear what the precise form these analogues would take. Since there are in general several choices of non-isomorphic rank 1 Drinfeld A-modules to consider, the exact structure of extensions by these modules is a direction for future investigations.

For the remainder of this section, we will consider the implications of the assumption that DR is an isomorphism. In this way, we attempt to motivate the idea that $\mathrm{Ext}^1_0(E,F)$ can be given the structure of a t-module.

Suppose $\mathbb{K}^d$ is an $\mathbb{F}_q[t]$-module defined by an $\mathbb{F}_q$-linear representation $\phi:\mathbb{F}_q[t]\to \mathrm{Mat}_d(\mathbb{K})$ such that $\phi(t)=\theta I_d+N$ with N nilpotent. If Λ is any finitely generated discrete $\mathbb{F}_q[t]$-submodule of $\mathbb{K}^d$ of rank r, then following the language of Anderson [1, Section 4.4], we call $\mathbb{K}^d/\Lambda$ a *t-torus* of dimension d and rank r. Through its exponential function, every uniformizable t-module is isomorphic to a t-torus.

Let $\lambda_1,\dots,\lambda_r$ be an $\mathbb{F}_q[t]$-basis for Λ_Φ. Choosing this basis fixes isomorphisms

$$\mathrm{Hom}_{\mathbb{F}_q[t]}(\Lambda_\Phi,\Psi(\mathbb{K}))\cong\Psi(\mathbb{K})^r\cong(\mathbb{K}^e)^r/(\Lambda_\Psi)^r. \tag{18}$$

We let

$$W:=W_{(\Phi,\Psi)}:=\left\{(U\lambda_1,\dots,U\lambda_r)\in\mathrm{Mat}_{e\times r}(\mathbb{K}):\begin{array}{c}U\in\mathrm{Mat}_{e\times d}(\mathbb{K}),\\ Ud\Phi(t)=d\Psi(t)U\end{array}\right\}. \tag{19}$$

Suppose $\delta^{(U)}\in\mathrm{Der}_0(\Phi,\Psi)\cap\mathrm{Der}_{\mathrm{in}}(\Phi,\Psi)$ for $U\in\mathrm{Mat}_{e\times d}(\mathbb{K})$. By Lemma 5.2 and (18),

$$\mathrm{DR}(\delta^{(U)})=(\mathrm{Exp}_\Psi(U\lambda_1),\dots,\mathrm{Exp}_\Psi(U\lambda_r))\in\mathrm{Mat}_{e\times r}(\mathbb{K}),$$

and so $\mathrm{DR}(\mathrm{Der}_0(\Phi,\Psi)\cap\mathrm{Der}_{\mathrm{in}}(\Phi,\Psi))=\mathrm{Exp}_\Psi^{\oplus r}(W)$. Thus by Corollary 2.3, $\mathrm{DR}_{(\Phi,\Psi)}$ induces a homomorphism of $\mathbb{F}_q[t]$-modules,

$$\mathrm{DR}_{(\Phi,\Psi)}:\ \mathrm{Ext}^1_0(E,F)\to\mathrm{Mat}_{e\times r}(\mathbb{K})/(W+\Lambda^r_\Psi), \tag{20}$$

which is an isomorphism if the original $\mathrm{DR}_{(\Phi,\Psi)}$ is one. Depending on E and F, the right-hand side may or may not be isomorphic (rigid analytically) to a t-torus. We consider the following situations.

Example 5.7. Suppose $E = (\mathbb{G}_a, \Phi)$ and $F = (\mathbb{G}_a, \Psi)$ are Drinfeld modules of rank 2 and s, respectively. Let λ_1, λ_2 and $\mu_1, \ldots, \mu_s$ be generators for their period lattices over $\mathbb{F}_q[t]$. The exact sequence

$$0 \to W \to \mathbb{K}^2 \xrightarrow{f} \mathbb{K} \to 0,$$

where $f(x, y) = \lambda_1 y - \lambda_2 x$, provides a choice of coordinates on the right-hand side of (20), and $f(\Lambda_\Psi \oplus \Lambda_\Psi)$ is the $\mathbb{F}_q[t]$-submodule generated by $\{\lambda_i \mu_j\}_{i=1,2}^{j=1,\ldots,s}$.

If $s = 1$, then Proposition 5.5 implies that $\mathrm{Ext}_0^1(E, F) \hookrightarrow \mathbb{K}/(\mathbb{F}_q[t]\lambda_1\mu_1 + \mathbb{F}_q[t]\lambda_2\mu_1)$, and Theorem 1.1 (and also [22, Proposition 7]) confirms that this is in fact an isomorphism of t-tori. If $s \geqslant 2$, then it is easy to construct examples where $f(\Lambda_\Psi \oplus \Lambda_\Psi)$ is not discrete in $\mathbb{K}$, and so in such cases the right-hand side of (20) is not a t-torus (cf. Theorem 6.1). When we compare weights, these observations are consistent with the discussion at the beginning of the section, i.e. $\mathrm{wt}(E) \geqslant \mathrm{wt}(F)$ precisely when $s \geqslant 2$.

Proposition 5.8. *Suppose $E = (\Phi, \mathbb{G}_a)$ and $F = (\Psi, \mathbb{G}_a)$ are Drinfeld modules of rank r and s, respectively, with $r > s$. If $\mathrm{DR}_{(\Phi,\Psi)}$ is an isomorphism, then $\mathrm{Ext}_0^1(E, F)$ is isomorphic as an $\mathbb{F}_q[t]$-module to a t-torus of dimension $r-1$ and rank rs.*

Proof. Let $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_s$ be $\mathbb{F}_q[t]$-bases for Λ_Φ and Λ_Ψ, respectively. Because E and F are both one-dimensional, $\mathrm{Der}_{\mathrm{in}}(\Phi, \Psi) \subseteq \mathrm{Der}_0(\Phi, \Psi)$. As in (19),

$$W = W_{(\Phi,\Psi)} = \{(U\lambda_1, \ldots, U\lambda_r) \in \mathrm{Mat}_{1\times r}(\mathbb{K}) \colon\ U \in \mathbb{K}\}.$$

We claim that $W \cap (\Lambda_\Psi)^r = \{0\}$. Suppose $w \in W \cap (\Lambda_\Psi)^r$. Then

$$w = (U\lambda_1, \ldots, U\lambda_r) = (v_1, \ldots, v_r), \quad v_i \in \Lambda_\Psi.$$

Because $r > s$, there is a non-trivial $\mathbb{F}_q[t]$-linear dependency $\sum d\Psi(a_i)v_i = 0$, and thus $U \cdot \sum d\Phi(a_i)\lambda_i = 0$. Since $\lambda_1, \ldots, \lambda_r$ are linearly independent over $\mathbb{F}_q[t]$, it follows that $U = 0$.

By the following argument, the image of Λ_Ψ^r is discrete in $\mathrm{Mat}_{1\times r}(\mathbb{K})/W$. We proceed by induction on r, for which the base case ($r = 2$) is trivial. Furthermore, it suffices to continue with $r = s + 1$, which we will now assume; the cases where $r > s + 1$ follow as straightforward consequences. The $\mathbb{K}$-linear map

$$\begin{array}{rccc} f\colon & \mathrm{Mat}_{1\times r}(\mathbb{K}) & \to & \mathrm{Mat}_{1\times(r-1)}(\mathbb{K}), \\ & (x_1, \ldots, x_r) & \mapsto & (\lambda_2 x_1 - \lambda_1 x_2, \ldots, \lambda_r x_{r-1} - \lambda_{r-1} x_r) \end{array}$$

has kernel W. We need to show that $V := \mathbb{F}_q((1/\theta)) \cdot f(\Lambda_\Psi) \subseteq \mathrm{Mat}_{1\times(r-1)}(\mathbb{K})$ is an rs-dimensional vector space over $\mathbb{F}_q((1/\theta))$. The typical element of V has the form

$$v = (\mu_1, \ldots, \mu_s) A \begin{pmatrix} \lambda_2 & \cdots & 0 \\ -\lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \lambda_r \\ 0 & \cdots & -\lambda_{r-1} \end{pmatrix}, \tag{21}$$

where $A = (\alpha_{ij}) \in \mathrm{Mat}_{s\times r}(\mathbb{F}_q((1/\theta)))$. Let B denote the $r \times (r-1)$ matrix in the above formula. By our assumption that $s = r-1$, the matrix AB is square, and its determinant is

$$\det(AB) = \lambda_2 \cdots \lambda_{r-1} \sum_{j=1}^{r} (-1)^{r-j} \lambda_j \det(A_j),$$

where A_j is the $s \times s$ minor of A with the jth column removed. If $v = 0$, then this implies that $\det(AB) = 0$, and since $\lambda_1, \ldots, \lambda_r$ are $\mathbb{F}_q((1/\theta))$-linearly independent, $\det(A_j) = 0$ for each j. Thus the rank of the matrix A is less than s, and we can rewrite the last row of A as an $\mathbb{F}_q((1/\theta))$-linear combination of the other rows, say $\alpha_{sj} = \sum_{i=1}^{s-1} \beta_i \alpha_{ij}$ with $\beta_i \in \mathbb{F}_q((1/\theta))$. The formulation in (21) can be rewritten as

$$v = (\mu_1 + \beta_1 \mu_s, \ldots, \mu_{s-1} + \beta_{s-1}\mu_s)\tilde{A}B,$$

where $\tilde{A}$ is the $(s-1) \times r$ matrix obtained by removing the last row of A. Again $v = 0$, but then our induction hypothesis with r replaced by $r-1$ allows us to conclude that $\tilde{A} = 0$. $\square$

Remark 5.9. Letting E and F be Drinfeld modules of rank r and s, with $r > s$, Proposition 5.8 shows that $\mathrm{Ext}_0^1(E, F)$ is isomorphic to a t-torus as long as $\mathrm{DR}_{(\Phi,\Psi)}$ is an isomorphism. Since in this case $\mathrm{wt}(E) < \mathrm{wt}(F)$, our discussion at the beginning of this section leads us to speculate that $\mathrm{DR}_{(\Phi,\Psi)}$ is an isomorphism and that the t-torus isomorphic to $\mathrm{Ext}_0^1(E, F)$ is in fact a pure uniformizable t-module. In addition, using the techniques of Theorem 1.1, it is possible to show that $\mathrm{Ext}_0^1(E, F)$ is isomorphic to a t-module of dimension $r-1$, though we omit the details.

6. Elliptic curves

Let E_1 and E_2 be elliptic curves over $\mathbb{C}$. Let $G := \mathrm{Ext}_{\mathbb{C}}^1(E_1, E_2)$ be the extension group in the category of complex abelian varieties. By the Poincaré reducibility theorem, this is a torsion group. The set of complex points $E(\mathbb{C})$ of an elliptic curve E over $\mathbb{C}$ can be viewed as a complex torus written E^{an}; so one may also consider

$A := \mathrm{Ext}^1(E_1^{\mathrm{an}}, E_2^{\mathrm{an}})$, the extension group in the category of complex tori. There is a natural homomorphism $G \to A$; the image is the torsion subgroup of A (see [4, Remark 6.2, p. 23]).

We present next a theorem of Lichtenbaum (1960s, unpublished); however, our proof is different in that it is based on periods. By comparison to Example 5.7 and Proposition 5.8, the theorem below indicates that the situation for elliptic curves runs quite parallel to our own. Specifically, in light of in Example 5.7, it suggests that $\mathrm{Ext}^1(E, F)$ will rarely be representable as a t-module for non-isogenous Drinfeld modules E and F of rank 2. Moreover, it suggests that $\mathrm{Ext}^1(E, F)$, for general t-modules E and F, will not always have the structure of a t-module.

Theorem 6.1 (Lichtenbaum). *Let E_1 and E_2 be elliptic curves over $\mathbb{C}$, and let $G := \mathrm{Ext}^1_{\mathbb{C}}(E_1, E_2)$ and $A := \mathrm{Ext}^1(E_1^{\mathrm{an}}, E_2^{\mathrm{an}})$.*

(a) *If $E = E_1 = E_2$ has complex multiplication, then A is naturally isomorphic to $E(\mathbb{C})$ as abelian groups. Under this isomorphism, the group G is identified with the torsion points of $E(\mathbb{C})$.*
(b) *If E_1 and E_2 are isogenous and admit complex multiplication, then A is isogenous to $E_1(\mathbb{C})$ and $E_2(\mathbb{C})$.*
(c) *If at least one of E_1 and E_2 does not admit complex multiplication, then the natural topology on G is non-Hausdorff. Therefore, G is not the set of complex points of a complex algebraic variety with the classical topology.*

Proof. Given E_i $(i = 1, 2)$ as the quotient of $\mathbb{C}$ by lattices $\mathbb{Z} + \mathbb{Z}\tau_i$, where each τ_i may be taken to be have positive imaginary part, we see from [4, Proposition 5.7, p. 21] that A is naturally the quotient of $\mathbb{C}$ by the subgroup Λ generated by $1, \tau_1, \tau_2$ and $\tau_1\tau_2$.

Let us suppose that an elliptic curve X has complex multiplication; let us think of $X(\mathbb{C})$ as a quotient of $\mathbb{C}$ by $\mathbb{Z} + \mathbb{Z}\tau$. By the theory of complex multiplication [19], we have the following: (i) τ lies in an imaginary quadratic field; (ii) $\tau^2 = a\tau + b$ for some integers a and b; and (iii) if X' (with $X'(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau')$) is an elliptic curve isogenous to X, then τ' and τ lie in the same imaginary quadratic field and X' also has complex multiplication with the same CM-field.

If $X(\mathbb{C}) = E = E_1 = E_2$, then we have $\tau = \tau_1 = \tau_2$. By (i) and (ii), we get that Λ is the lattice $\mathbb{Z} + \mathbb{Z}\tau$. Thus we obtain that A is naturally identified with $E(\mathbb{C})$. The last statement in (a) follows from [4, Remark 6.2, p. 23].

If E_1 and E_2 are isogenous curves with complex multiplication, then τ_1 and τ_2 lie in the same imaginary quadratic field F; in this case, Λ is isomorphic to a fractional ideal of an order of K. This proves (b).

For (c), suppose at least one of E_1 and E_2 does not admit complex multiplication. By the fundamental theorem of complex multiplication, τ_1 and τ_2 are not both contained in one imaginary quadratic field. In other words, in this case Λ is a subgroup of $\mathbb{Z}$-rank greater than two, and so it is not a discrete subgroup of $\mathbb{C}$. $\square$

Remark 6.2 (Schoen). (a) If A and B are abelian varieties defined over $\bar{\mathbb{Q}}$, then the natural map $\mathrm{Ext}^1_{\bar{\mathbb{Q}}}(A,B) \to \mathrm{Ext}^1_{\mathbb{C}}(A,B)$ is an isomorphism.

(b) Any abelian surface corresponding to an element of $\mathrm{Ext}^1_{\mathbb{C}}(E_1,E_2)$ is isogenous to the product $E_1 \times E_2$. Any complex abelian variety, which is isogenous to a product of CM-elliptic curves, is itself a product of CM-elliptic curves [18].

7. Extensions of t-motives

In this section, we explore extensions from the standpoint of t-motives and examine avenues for further study. Given two t-modules E, F over K, we can consider the associated Anderson t-motives $M(E)$ and $M(F)$ [1]. Since the functor M which sends a t-module to its associated t-motive is contravariant, we obtain a map

$$M^*\colon \mathrm{Ext}^1(E,F) \to \mathrm{Ext}^1((M(F), M(E)),$$

because M gives an anti-equivalence of categories of t-modules and t-motives [1, Theorem 1], M^* is an isomorphism. If one is interested in computing just the group of extensions of t-modules, then it is relatively easy to compute extensions in the category of t-motives. We formulate this precisely in the next lemma.

The evident functor f from the category $\mathscr{T}$ of t-motives to the category $\mathscr{C}$ of left $K[t,\tau]$-modules is fully faithful [1, Section 1.2]. Here $K[t,\tau]$ is the non-commutative ring generated by t and τ with the relations, $t\tau = \tau t, xt = tx, \tau x = x^q\tau$, for all $x \in K$.

Lemma 7.1. *For any t-motives A and B over K, the natural map*

$$f^*\colon\ \mathrm{Ext}^1_{\mathscr{T}}(A,B) \to \mathrm{Ext}^1_{\mathscr{C}}(f(A), f(B))$$

is an isomorphism.

Proof. The fact that f is fully faithful implies that f^* is injective in the following way. Suppose the images of α and β under f^* coincide. Pick representatives of α and β, i.e. extensions X_1 and X_2 of A by B which satisfy $f(X_1) \cong f(X_2)$. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f(B) \arrow[r] \arrow[d, equal] & f(X_1) \arrow[r] \arrow[d, "\gamma"'] & f(A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & f(B) \arrow[r] & f(X_2) \arrow[r] & f(A) \arrow[r] & 0,
\end{tikzcd}$$

where γ is an isomorphism of $K[t,\tau]$-modules. Since f is fully faithful, γ is an isomorphism of t-motives. More precisely, γ is induced by an isomorphism $X_1 \xrightarrow{\sim} X_2$ of t-motives.

It remains to show the surjectivity of f^*. For this, we have to show the following: given any extension X of $f(A)$ by $f(B)$ in $\mathscr{C}$, the left $K[t,\tau]$-module X is a t-motive, i.e. (i) it is free and finitely generated as a $K[\tau]$-module and (ii) the associated primes of $X' := X/\tau X$, viewed as a module over the commutative ring $R := K[t]$, consist only of the principal ideal $I := (t-\theta)$. Geometrically, we want the coherent sheaf associated to X' on $\mathbb{A}^1$ to be supported only at the point $t = \theta$.

Condition (i) is clearly satisfied by X by general properties of modules over the ring $K[\tau]$ [12, Proposition 5.4.9]. For (ii), consider an extension Q of P by N where P and N are finitely generated R-modules. Every associated prime of Q is an associated prime of either P or N. Also, if H is a quotient module of N, then every associated prime of H is an associated prime of N (this is easy to see via the geometric interpretation).

Now, by assumption, the associated primes of the R-modules $A' := f(A)/\tau f(A)$ and $B' := f(B)/\tau f(B)$ consist of just the ideal I. The R-module X' is an extension of A' by B'' (= a quotient module of B'). So we may apply the comments in the previous paragraph to the extension X' to deduce that X' satisfies (ii). $\square$

Remark 7.2. Lemma 7.1 shows that extensions may be computed via resolutions of t-motives by free $K[t,\tau]$-modules. Furthermore, it implies the injectivity of the map

$$f^*\colon \mathrm{Ext}^2_{\mathscr{T}}(A,B) \to \mathrm{Ext}^2_{\mathscr{C}}(f(A), f(B)).$$

For M and N as in Remark 7.4 (and $K = \mathbb{F}_q$), this gives $\mathrm{Ext}^2_{\mathscr{T}}(M,N) = 0$.

Remark 7.3 (Analogy with $\mathscr{D}$-modules). A. Rosenberg points out that $K[t,\tau]$ and, more generally, skew polynomial algebras are analogous to Weyl algebras in $\mathscr{D}$-module theory (the first Weyl algebra is $\mathbb{C}[x,\partial]$ with the relation $\partial x - x\partial = 1$) in that they are all special cases of *hyperbolic* algebras [17, Chapter II]. Since Ext's of certain (but not all) $\mathscr{D}$-modules possess a nontrivial structure of a $\mathscr{D}$-module, one may expect the same to be true for t-modules.

The analogy with $\mathscr{D}$-modules is best viewed within the context of opers (see [3, Section 7.3.14]; a Drinfeld module is an example of a Frobenius oper) and non-commutative algebraic geometry (see [5, Remark 5.3.5, Section 6], [7, Section 0.6], [17,20]). The analogy between Drinfeld modules and non-commutative tori is explained in [13].

Remark 7.4. Geometrically interpreting the definition of a t-motive [1, Section 1.2], a t-motive M is a special sheaf $\mathscr{F}_M$ over a non-commutative surface S [17] given by the product of the non-commutative affine line $\mathbb{A}^1_{\mathrm{nc}}$ by the commutative affine line $\mathbb{A}^1$. We think of S as fibered over $\mathbb{A}^1$ (viewed horizontally) with (vertical) fibers $\mathbb{A}^1_{\mathrm{nc}}$. The sheaf $\mathscr{F}_M$ is the ideal sheaf of a curve $X_M \subseteq S$ for which the following hold.

(a) X_M is finite over the two axes, via the projections to the components; in other words, X_M is transversal to the horizontal and vertical fibers.

(b) The intersection of X_M with the horizontal axis (which corresponds to $\tau = 0$) is a nilpotent subscheme of $\mathbb{A}^1$ supported at the point $t = \theta$; (a) assures us that the intersection is a proper subscheme of $\mathbb{A}^1$.

If M and N are distinct t-motives, then the group $\mathrm{Ext}^1_{\mathscr{T}}(M, N)$ can be interpreted via the "intersection scheme" of the curves X_M and X_N in S. We can view the subgroup $\mathrm{Ext}^1_0(M, N)$ (cf. Corollary 2.3) of $\mathrm{Ext}^1_{\mathscr{T}}(M, N)$ as the non-trivial part of the intersection locus corresponding to points distinct from $t = \theta, \tau = 0$, as in (b).

The geometric situation is especially clear in the case $K = \mathbb{F}_q$: S is the usual commutative affine plane $\mathbb{A}^2$; if $g(t, \tau)$ and $h(t, \tau)$ (assumed to have no common factors) are defining equations for the curves X_M and X_N, then $\mathrm{Ext}^1_{\mathscr{T}}(M, N)$ is isomorphic to the quotient module $\mathbb{F}_q[t, \tau]/(g, h)$ of the commutative ring $\mathbb{F}_q[t, \tau]$.

Remark 7.5. The results and ideas in this paper are used in an ongoing project with Thakur, whose aim is to relate extension groups of t-modules to values of zeta functions in the spirit of [2,10,16].

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