Decomposition of Hessenberg DAE Systems to State Space Form

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ABSTRACT

An algorithm is given for symbolically decoupling the solutions to a linear, time dependent differential-algebraic equation $Ez' = A(t)z + f(t)$, $z(t) \in R^s$, in Hessenberg form into state and algebraic components. The state variables are the solutions to an ordinary differential equation with initial conditions restricted to a subspace of R^s, while the algebraic components are linear functions of the state variables involving derivatives of the coefficients and input functions up to order $r - 1$, where r is the index of the system. This decomposition provides closed form solutions to linear Hessenberg DAEs in terms of the fundamental solutions of the state variable system. The implications of the algorithm for computing consistent initial conditions, for certain singular optimal control problems, and for numerical solutions are briefly discussed.

1. INTRODUCTION

In this paper we formally describe an algorithm for decomposing the solutions of a higher index, time dependent *differential-algebraic equation* (DAE)

$$E(t)z'(t) = A(t)z(t) + f(t), \qquad t \in \Omega = [t_0, t_1], \tag{1.1}$$

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in *Hessenberg* form into state and algebraic components. The Hessenberg form is characterized by $E(t) = E$ being a semiexplicit projector, $A(t)$ being block upper Hessenberg, and an invertibility condition on the subdiagonal blocks of $A(t)$. We explicitly define the Hessenberg form and review some of its most relevant properties in the next section. In addition to the algorithm description, the purpose of this paper is to show that the state variables for these problems are the solutions to an ordinary differential equation with initial conditions constrained to a proper subspace of R^s which is invariant for the flow of this ODE; to show that the algebraic variables are completely determined by the state variables, coefficients, and inputs, and possibly derivatives of these functions; and to explore some of the consequences of these results for computing consistent initial conditions, for certain classes of singular optimal control problems, and also for the numerical solution of boundary value problems involving Hessenberg forms.

Hessenberg forms involving a single matrix (or *free* Hessenberg forms [15]) commonly occur in control and numerical linear algebra applications due to the information which they convey about the eigenstructure of a system and to the existence of numerically stable algorithms for computing them. Nonlinear DAE *system* Hessenberg forms $F(z', z, t) = 0$, $F_{z'}$ singular, arise naturally as models of constrained mechanical systems, in trajectory-prescribed-path control problems, as necessary conditions in optimal control, and as finite difference approximations to the incompressible Navier-Stokes equations. Typically these systems have index 2 or 3, where the index is a measure of the number of differentiations of the problem data involved in the solutions. Systems with index 5 or higher can arise in robotics problems, but arbitrarily high index problems can be constructed by linking lower index problems in cascade. Thus it is of interest to study the generic properties of large index problems which exhibit the structural characteristics applicable at the lower index level. Also, whereas many of the relevant applications involve nonlinearities, many of the difficulties of analyzing these systems either numerically or analytically are already present in the linear, time dependent problem, so that it is of interest to understand this problem as a step towards understanding the nonlinear case.

The numerical solution of the index 2 and index 3 problems was first investigated in [2] and simultaneously in [16]. The methods used in these investigations were fixed stepsize backward differentiation formulas (BDF) due to the similarity of constrained systems to reduced order models of singularly perturbed equations and also to the stiff stability of the BDF. Subsequently this form was generalized to arbitrary index in [7] and studied in some detail theoretically. Variable step BDF for Euler-Lagrange equations (index 2 and 3) have been investigated in [9], and a class of generalized BDF methods for problems with index 2 through 4 (which include variable step BDFs as a special case) has been studied in [14]. Investigations of implicit Runge-Kutta methods for Hessenberg DAEs are described in [3, 13]. In these papers the errors are analyzed by decomposition into projected components, where the projectors can be computed directly from the explicit description of the system.

It is the main purpose of this paper to show that an analogous decomposition can be carried out for the exact solution of a general Hessenberg form and that it yields significant information about the system, such as explicit expressions for consistent initial conditions, complementary conditions for the well-posing of DAE boundary value problems, and even explicit solutions in terms of the fundamental solutions of the state space system. To our knowledge this is the first description of such a decomposition tailored specifically for general Hessenberg form DAE systems. For discussions regarding the reduction of general linear DAEs to other state space forms, we refer the reader to the work of März et al. [12] using matrix chains and to results due to Campbell [5] for computing completions based on derivative arrays. These latter two approaches produce underlying ODE systems for Hessenberg forms which are different from the approach advocated in this paper.

In the next section we review some of the necessary background and give some basic results which will be used in the remainder of the paper. In Section 3 we present and analyze the reduction algorithm, and in Section 4 we discuss some applications of these results to differential-algebraic boundary value problems, in particular BVPs arising in singular optimal control problems. Finally, in Section 5 we make some closing remarks on applications of these results and techniques to other questions related to Hessenberg DAEs.

2. BACKGROUND AND TERMINOLOGY

In this section we briefly review some definitions and develop the necessary background which will be used throughout the remainder of the paper. By a solution of (1.1) we mean a C^1 function $z(t)$ which satisfies the DAE on an open subinterval $\mathcal{O} \subset \Omega$. Then (1.1) is *solvable* on Ω if for every sufficiently smooth input function $f(t)$ there exists at least one solution which can be extended to Ω, all solutions are $C^1(\Omega)$, and in addition every solution is uniquely determined by its value $z(t)$ for any $t \in \Omega$. In particular, solutions neither bifurcate nor escape to infinity in Ω. This definition is slightly stronger than is necessary, since different components of z will in general have varying degrees of smoothness (cf. Corollary 4). If $z \equiv 0$ is the unique solution to the homogeneous problem corresponding to $f \equiv 0$, then (1.1) is *degenerate* [8]. Otherwise, it is nondegenerate and there exists an integer $p > 0$ and a smooth basis of functions $\{\phi_1(t), \dots, \phi_p(t)\}$ for the solution space of the homogeneous problem. The number p is called the dimension of the solution manifold for (1.1).

The system (1.1) is in *Hessenberg form of size r* (denoted by $\mathscr{H}_r$) if there is an integer $r > 1$ such that $E(t) = E$ is a semiexplicit projector

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{2.1}$$

and such that $A(t)$ is block upper Hessenberg of the form

$$A(t) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,r-1} & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2,r-1} & 0 \\ & A_{32} & \cdots & A_{3,r-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & A_{r,r-1} & 0 \end{pmatrix}, \tag{2.2}$$

where $A_{ij} = A_{ij}(t)$ is a sufficiently smooth $n_i \times n_j$ matrix, and where the matrix Π defined by

$$\Pi = A_{r,r-1} A_{r-1,r-2} \cdots A_{21} A_{1r} \tag{2.3}$$

is $n_r \times n_r$ and invertible for all $t \in \Omega$. The parameter r, the *index* of the system [1], is a measure of the sensitivity of a DAE to smooth perturbations of its inputs and coefficients. The size of the system is $s = \sum_{i=1}^{r} n_i$, and the identity block in (2.1) is $(s - n_r) \times (s - n_r)$.

In linear systems theory system Hessenberg forms are typically defined either in terms of the controllability pair (A, B) or the observability pair (A^T, C^T) for the system

$$\begin{aligned} x' &= Ax + Bu + f(t), \\ y &= Cx + Du + g(t) \end{aligned} \tag{2.4}$$

(see [15]). What distinguishes our definition (2.1), (2.2) from the usual definitions in control theory is the invertibility condition on Π and the fact that $\mathscr{H}_r$ involves the structure of the triple (A, B, C). In fact one can easily show that $\mathscr{H}_r$ is structurally equivalent to the case where (A, B) and (A, C^T) are simultaneously in the system Hessenberg form described in [15]. In this paper we are not concerned with the computation of the DAE Hessenberg form from a more general system description. However, it is important to note that system Hessenberg forms for a pair (A, B) can be highly ill conditioned even though the corresponding free Hessenberg form (in terms of A) may be well conditioned.

Partition the variable $z \in R^s$ in (1.1) into

$$\left(z_1, \dots, z_{r-1}, u\right)^t \equiv \left(z_1^T, \dots, z_{r-1}^T, u^T\right)^T$$

where $z_i \in R^{n_i}$ and $u \in R^{n_r}$. Define the matrices $P_i(t)$ (for $i = 1, \dots, r-1$) by

$$P_i(t) = \begin{cases} A_{1r}\Pi^{-1}A_{r,r-1}A_{r-1,r-2} \cdots A_{21}, & i = 1, \\ A_{i,i-1}A_{i-1,i-2} \cdots A_{21}A_{1r}\Pi^{-1}A_{r,r-1} \cdots A_{i+1,i} & i \neq 1. \end{cases}$$

Associated with P_i are the subspaces $\mathscr{R}_i$ and $\mathscr{N}_i$ given by

$$\mathscr{R}_i = \mathscr{R}\left(A_{i,i-1} \cdots A_{21}A_{1r}\right),$$

$$\mathscr{N}_i = \mathscr{N}\left(A_{r,r-1} \cdots A_{i+2,i+1}A_{i+1,i}\right), \qquad i \neq 1,$$

with $\mathscr{R}_1 = \mathscr{R}(A_{1r})$ and $\mathscr{N}_1 = \mathscr{N}(A_{r,r-1}A_{r-1,r-2} \cdots A_{21})$, where $\mathscr{R}(\cdot)$ and $\mathscr{N}(\cdot)$ denote range and nullspace, respectively. In particular, $P_i(t)$ is a rank n_r projector onto $\mathscr{R}_i$ along $\mathscr{N}_i$, so that $R^{n_i} = \mathscr{R}_i \oplus \mathscr{N}_i$. Furthermore, the following partial commutativity relationships hold:

$$\begin{aligned} P_1A_{1r} &= A_{1r}, \\ P_iA_{i,i-1} &= A_{i,i-1}P_{i-1}, \qquad i = 2, \dots, r-1, \\ A_{r,r-1}P_{r-1} &= A_{r,r-1}. \end{aligned} \tag{2.5}$$

With P_i defined above, let $P = \operatorname{diag}(P_1, \dots, P_{r-1})$, so that

$$\mathscr{R}(P) = \prod_{j=1}^{r-1} \mathscr{R}(P_i) \subset R^{s-n_r}.$$

The conditions in (2.5) imply the invertibility of the linear transformation $\hat{A}_i : \mathscr{R}_{i-1} \to \mathscr{R}_i$ defined by $\hat{A}_i v = A_{i,i-1}v$. This follows from the invertibility of Π and the facts that rank $A_{i,i-1} \geqslant n_r$ and also rank P_{i-1} = rank $P_i = n_r$. Coupled with (2.5) and interpreted geometrically, this says that the path from

u to Πu through the intermediate subspaces $\mathscr{R}_i$ can be traversed uniquely in the reverse direction via the $\hat{A}_i^{-1}$.

The following simple results will be useful in the next section.

PROPOSITION 1. *Let $\mathscr{O} \subset R$ be an open interval with $P : \mathscr{O} \mapsto R^{s \times s}$ $(s \geq 1)$ any smooth projector-valued mapping. Then*

(i) $PP'P \equiv 0$;
(ii) $(I - P)P'P = (I - P)P'$.

Proof. Differentiate $P^2 \equiv P$. Premultiplying by P proves (i); premultiplying by $I - P$ proves (ii). ■

COROLLARY 1. *Let $\mathscr{O} \subset R$ be an open interval with $P : \mathscr{O} \mapsto R^{s \times s}$ $(s \geq 1)$ any smooth projector-valued mapping. For any smooth function $z : \mathscr{O} \mapsto R^s$, define $v = (I - P)z$. Then $Pv' = -PP'v$.*

Proof. A straightforward calculation using Proposition 1 shows that $Pv' = P\{(I - P)z' - P'z\} = -PP'z = -PP'\{v + Pz\} = -PP'v$. ■

3. A DECOMPOSITION ALGORITHM FOR HESSENBERG SYSTEMS

For the purposes of presentation, in this section we write $\mathscr{H}_r$ $(r \geq 3)$ by

$$z_1' = A_{11} z_1 + \sum_{j=2}^{r-1} A_{1j} z_j + A_{1r} u + f_1, \tag{3.1}$$

$$z_i' = A_{i,i-1} z_{i-1} + A_{ii} z_i + \sum_{j=i+1}^{r-1} A_{ij} z_j + f_i \qquad (i = 2, \ldots, r-1), \tag{3.2}$$

$$0 = A_{r,r-1} z_{r-1} + f_r, \tag{3.3}$$

where the summation in (3.2) is not present if $r = 3$. The index 2 case is treated separately. For each z_i define $v_i = (I - P_i) z_i$ and $w_i = P_i z_i$. Then $z_i = v_i \oplus w_i$ according to the splitting $R^{n_i} = \mathscr{R}(I - P_i) \oplus \mathscr{N}(I - P_i)$. Multiplying (3.3) by $A_{r-1,r-2} \cdots A_{21} A_{1r} \Pi^{-1}$ gives an exact expression for w_{r-1}:

$$w_{r-1} = -A_{r-1,r-2} \cdots A_{21} A_{1r} \Pi^{-1} f_r. \tag{3.4}$$

Furthermore, if w_i and z_j $(j \geqslant i+1)$ are known, we can split z_j and premultiply (3.2) by $I - P_i$ to get

$$(I - P_i)v_i' = (I - P_i)A_{i,i-1}z_{i-1} + (I - P_i)A_{ii}v_i + \sum_{j=i+1}^{r-1}(I - P_i)A_{ij}v_j$$

$$+ (I - P_i)\left(\sum_{j=i}^{r-1}A_{ij}w_j + f_i - w'_i\right). \tag{3.5}$$

Then using (2.5) and adding $P_i v_i'$ to both sides of (3.5) according to Corollary 1 gives

$$v_i' = A_{i,i-1}v_{i-1} + \left[(I - P_i)A_{ii} - P_iP_i'\right]v_i + \sum_{j=i+1}^{r-1}(I - P_i)A_{ij}v_j$$

$$+ (I - P_i)\left(\sum_{j=i}^{r-1}A_{ij}w_j + f_i - w_i'\right). \tag{3.6}$$

We can solve for w_{i-1} by premultiplying (3.2) by

$$A_{i-1,i-2} \cdots A_{21}A_{1r}\Pi^{-1}A_{r,r-1} \cdots A_{i+1,i}$$

and rearranging to get

$$w_{i-1} = A_{i-1,i-2} \cdots A_{21}A_{1r}\Pi^{-1}A_{r,r-1} \cdots A_{i+1,i}$$

$$\times\left[\left(v_i' - \sum_{j=i}^{r-1}A_{ij}v_j\right) + \left(w_i' - \sum_{j=i}^{r-1}A_{ij}w_j - f_i\right)\right]. \tag{3.7}$$

Continuing in this fashion, we obtain a similar expression to (3.6) for v_1' excluding the first term on the right-hand side. Finally, we solve (3.1) for u as

$$u = \Pi^{-1}A_{r,r-1}A_{r-1,r-2} \cdots A_{21}$$

$$\times\left[\left(v_1' - \sum_{j=1}^{r-1}A_{1j}v_j\right) + \left(w_1' - \sum_{j=1}^{r-1}A_{1j}w_j - f_1\right)\right]. \tag{3.8}$$

The preceding discussion motivates the formal description of Algorithm 1 given below and the results which follow.

ALGORITHM 1. {Hessenberg state space decomposition}
begin
  if $(r = 2)$ **then begin**
    $w_1 = -A_{12}\Pi^{-1}f_2;$
    $v_1' = [(I - P_1)A_{11} - P_1P_1']v_1 + (I - P_1)[f_1 + A_{11}w_1 - w_1'];$
    $u = \Pi^{-1}A_{21}[(v_1' - A_{11}v_1) + (w_1' - A_{11}w_1 - f_1)]$
    end
  else begin
    $w_{r-1} = -A_{r-1,r-2} \cdots A_{21}A_{1r}\Pi^{-1}f_r;$
    for $i = r - 1$ **downto** 2 **do begin**
      $v_i' = A_{i,i-1}v_{i-1} + [(I - P_i)A_{ii} - P_iP_i']v_i$
        $+\sum_{j=i+1}^{r-1}(I - P_i)A_{ij}v_j + (I - P_i)[\sum_{j=i}^{r-1} A_{ij}w_j + f_i - w_i'];$
      $Q_i = A_{i-1,i-2} \cdots A_{21}A_{1r}\Pi^{-1}A_{r,r-1} \cdots A_{i+1,i};$
      $w_{i-1} = Q_i[(v_i' - \sum_{j=i}^{r-1} A_{ij}v_j) + (w_i' - \sum_{j=i}^{r-1} A_{ij}w_j - f_i)]$
      end;
    $v_1' = [(I - P_1)A_{11} - P_1P_1']v_1$
      $+(I - P_1)[\sum_{j=2}^{r-1} A_{1j}v_j + (\sum_{j=1}^{r-1} A_{1j}w_j + f_1 - w_1')];$
    $u = \Pi^{-1}A_{r,r-1}A_{r-1,r-2} \cdots A_{21}$
      $\times[(v_1' - \sum_{j=1}^{r-1} A_{1j}v_j) + (w_1' - \sum_{j=1}^{r-1} A_{1j}w_j - f_1)];$
    for $i = 1$ **to** $r - 1$ **do**
      $z_i = v_i + w_i$
    end
end. {Hessenberg decomposition}

THEOREM 1. *Assume that $A(t)$ and $f(t)$ are $C^r(\Omega)$. If $z(t)$ is a solution of $\mathscr{H}_r$ for $r \geqslant 2$, then Algorithm 1 produces the direct sum decomposition $z_i = v_i \oplus w_i$ where $v_i = (I - P_i)z_i$ and $w_i = P_iz_i$. The variable $v = (v_1, \ldots, v_{r-1})^t$ is a solution of the linear ordinary differential equation*

$$v' = \hat{A}(t)v + g(t), \tag{3.9}$$

where $\hat{A}(t)$ is block upper Hessenberg of the form

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1,r-1} \\ A_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2,r-1} \\ & A_{32} & \cdots & \vdots \\ & & \ddots & \vdots \\ & & A_{r-1,r-2} & \hat{A}_{r-1,r-1} \end{pmatrix}$$

such that

$$\hat{A}_{ii} = (I - P_i) A_{ii} - P_i P_i', \tag{3.10}$$

$\mathscr{R}(\hat{A}_{ij}(t)) \subseteq \mathscr{R}(I - P_i(t)) = \mathscr{N}_i$ *for* $i < j \leqslant r - 1$, *and* $g_i(t) \in \mathscr{N}_i$ *for* $i = 1, \dots, r - 1$. *The functions* $\hat{A}_{ij}(\cdot)$ *and* $g(\cdot)$ *are completely determined by* $A^{(k)}$ *and* $f^{(l)}$ *for* $0 \leqslant k \leqslant r - 1$ *and* $0 \leqslant l \leqslant r - 2$.

Furthermore, the variable $(w, u)^t \equiv (w_1, \dots, w_{r-1}, u)^t$ *can be written as*

$$(w, u)^t = \hat{B}(t)v + q(t) \tag{3.11}$$

where $\hat{B}(t)$ *is an* $s \times (s - n_r)$ *matrix of the form*

$$\hat{B}(t) = \begin{pmatrix} 0 & B_{12} & B_{13} & \cdots & B_{1,r-1} \\ & 0 & B_{23} & \cdots & B_{2,r-1} \\ & & 0 & \ddots & \vdots \\ & & & \ddots & B_{r-2,r-1} \\ & & & & 0 \\ B_{u1} & B_{u2} & B_{u3} & \cdots & B_{u,r-1} \end{pmatrix} \tag{3.12}$$

and where the B_{uj}, B_{ij}, *and* $q(t) \equiv (q_1, \dots, q_{r-1}, q_u)^t$ *are also completely determined by* $A^{(k)}$ *and* $f^{(k)}$ *for* $0 \leqslant k \leqslant r - 1$.

Proof. The proof is by induction on i for $1 \leqslant i \leqslant r - 1$ for $r \geqslant 3$. First, the index two case is treated separately. From (3.1)–(3.3) and Algorithm 1 we have

$$\begin{aligned} v_1' &= [(I - P_1) A_{11} - P_1 P_1'] v_1 + (I - P_1)[f_1 + A_{11} w_1 - w_1'], \\ w_1 &= -A_{12} \Pi^{-1} f_2, \\ u &= \Pi^{-1} A_{21}[(v_1' - A_{11} v_1) + (w_1' - A_{11} w_1 - f_1)]. \end{aligned} \tag{3.13}$$

Equations (2.5), (3.13) imply that $(I - P_1)w_1' = -\ (I - P_1) A_{12}' \Pi^{-1} f_2$. Also, Corollary 1 implies $A_{21} v_1' = A_{21}[(I - P_1)v_1' + P_1 v_1'] = A_{21}(-P_1 P_1' v_1)$

$= -A_{21}P_1'v_1$. Thus (3.13) is equivalent to

$$v_1' = \left[\left(I - P_1\right)A_{11} - P_1P_1'\right]v_1 + \left(I - P_1\right)\left[f_1 + \left(A_{12}' - A_{11}A_{12}\right)\Pi^{-1}f_2\right],$$

$$w_1 = -A_{12}\Pi^{-1}f_2, \tag{3.14}$$

$$u = \Pi^{-1}A_{21}\Big[-\left(P_1' + A_{11}\right)v_1 + \left(A_{11}A_{12}\Pi^{-1} - \left\{A_{12}\Pi^{-1}\right\}'\right)f_2 - A_{12}\Pi^{-1}f_2' - f_1\Big], \tag{3.15}$$

which clearly satisfies the statement of the theorem when $r = 2$.

To establish the theorem for the case $r \geqslant 3$ we will simultaneously show that:

(1) $w_i = (\sum_{j=i+1}^{r-1} B_{ij}v_j) + q_i$, where B_{ij} and q_i involve $r - i - 1$ derivatives of A and f, respectively;

(2) $v_i' = A_{i,i-1}v_{i-1} + [(I - P_i)A_{ii} - P_iP_i']v_i + \sum_{j=i+1}^{r-1} \hat{A}_{ij}v_j + (I - P_i)g_i$, where $\hat{A}_{ij}$ and g_i involve $r - i$ derivatives of f.

For $i = r - 1$, Equations (3.4), (3.6) imply

$$\begin{aligned} v_{r-1}' &= A_{r-1,r-2}v_{r-2} + \left[\left(I - P_{r-1}\right)A_{r-1,r-1} - P_{r-1}P_{r-1}'\right]v_{r-1} + \left(I - P_{r-1}\right) \\ &\quad \times\left\{\left[\left(A_{r-1,r-2} \cdots A_{21}A_{1r}\right)' - A_{r-1,r-1}A_{r-1,r-2} \times \cdots A_{21}A_{1r}\right]\Pi^{-1}f_r + f_{r-1}\right\}, \end{aligned} \tag{3.16}$$

$$w_{r-1} = -A_{r-1,r-2} \cdots A_{21}A_{1r}\Pi^{-1}f_r.$$

Thus (1) and (2) hold for $i = r - 1$. If they hold for $i \geqslant k$, then for $i = k - 1$ we have from (2.5), (3.7) and the substitution $v_k' = (I - P_k)v_k' - P_kP_k'v_k$

$$\begin{aligned} w_{k-1} &= A_{k-1,k-2} \cdots A_{21}A_{1r}\Pi^{-1}A_{r,r-1} \cdots A_{k+1,k} \\ &\quad \times\left[-\left(P_k' + A_{kk}\right)v_k + \sum_{j=k+1}^{r-1} A_{kj}v_j + \left(\sum_{j=k+1}^{r-1} B_{kj}v_j\right)' + q_k' - \sum_{j=k}^{r-1} A_{kj}w_j - f_k\right] \\ &= \left(\sum_{j=k}^{r-1} B_{k-1,j}v_j\right) + q_{k-1}, \end{aligned} \tag{3.17}$$

where $B_{k-1,j}$ and q_{k-1} involve one more derivative of A and f than B_{kj} and q_k. Thus (1) holds for every $i = 1, \ldots, r-1$. The form for (2) for $i = k-1$ follows immediately by substituting (3.17) into (3.6). Finally, the expression $u = (\sum_{j=1}^{r-1} B_{uj}v_j) + q_u$ follows directly from (3.8) in analogy with (3.17). This completes the proof of the theorem. ■

As an illustration of Theorem 1 and to get an idea of the rapid growth both in number [which is $O(r^2)$ for index $= r$] and in complexity of the formulas, we include the results of Algorithm 1 for the index three and index four Hessenberg forms in the appendix.

THEOREM 2. *The subspace* $\mathscr{R}(I-P) \subseteq R^{s-n_r}$ *is invariant for the flow of the system* (3.9). *That is, if* $v(t)$ *is a solution of* (3.9) *such that for some* $\hat{t} \in \Omega$ *one has* $P(\hat{t})v(\hat{t}) = 0$, *then* $P(t)v(t) \equiv 0$ *in* Ω.

Proof. Since P is block diagonal, it suffices to show that $P_i(\hat{t})v_i(\hat{t}) = 0$ implies $P_i(t)v_i(t) \equiv 0$ for $i = 1, \ldots, r-1$. When $i = 1$, multiplying (3.6) by P_1 yields $P_1 v_1' = -P_1 P_1' v_1$. But then $(P_1 v_1)' = (I - P_1)P_1' v_1 = (I - P_1)P_1' P_1 v_1$ by Proposition 1. Hence $P_1(\hat{t})v_1(\hat{t}) = 0$ implies $P_1(t)v_1(t) \equiv 0$. For $i > 1$, multiplying (3.6) by P_i yields $P_i v_i' = A_{i,i-1}P_{i-1}v_{i-1} - P_i P_i' v_i$. If $P_{i-1}v_{i-1} \equiv 0$, then Proposition 1 again applies to get $P_i(t)v_i(t) \equiv 0$. Hence by induction $P_i(t)v_i(t) \equiv 0$ for $i = 1, \ldots, r-1$. ■

Theorems 1 and 2 suggest the following definition.

DEFINITION 1. For Hessenberg systems (3.1)–(3.3), the subspace $\mathscr{R}(I - P_i)$ is the *state space* for the variable z_i, and $\mathscr{N}(I - P_i)$ is the corresponding *algebraic space* for z_i. Similarly, the projections $v_i = (I - P_i)z_i$ and $w_i = P_i z_i$ are the *state* and *algebraic* components for z_i. Finally, the state component of the solution $z(t)$ is $v(t)$ and the algebraic component is $(w(t), u(t))^t$.

Let $\Psi(t, t_0)$ be the $(s - n_r) \times (s - n_r)$ fundamental matrix solution of (3.9) such that $\Psi(t_0, t_0) = I$. For arbitrary continuous g and initial condition v_0, the general solution with starting at $t = t_0$ is given by

$$v(t) = \Psi(t, t_0)v_0 + \int_{t_0}^{t} \Psi(t, s)g(s)\, ds. \tag{3.18}$$

On the other hand, from Theorem 2, if g is determined from f by Algorithm 1 and if $[I - P(t_0)]v_0 = v_0$, then $[I - P(t)]v(t) = v(t)$ for every t. If $f \equiv 0$, it follows directly from the proof of Theorem 1 that $g \equiv 0$. Consequently,

$$\int_{t_0}^{t} \Psi(t, s) g(s)\, ds \in \mathscr{R}\big(I - P(t)\big)$$

independent of v_0, so that in general the complementary subspace $\mathscr{R}(P(t))$ is not invariant for (3.9).

THEOREM 3. *Let A and f be $C^r(\Omega)$. Then the converse of Theorem* 1 *holds. That is, suppose v, w, and u are differentiable solutions of* (3.9), (3.11) *such that $P(\hat{t})v(\hat{t}) = 0$ for some $\hat{t} \in \Omega$, where $\hat{A}$, $\hat{B}$, g, and q are determined by Algorithm* 1. *Define $z_i = v_i + u_i$ for $i = 1, \ldots, r-1$. Then $z = (z_1, \ldots, z_{r-1}, u)^t$ is a solution to* (3.1), (3.3).

Proof. By definition of z and the hypothesis of the theorem, we have that $(I - P_i)z_i = v_i$ and $P_i z_i = u_i$. Multiplying (3.4) by $A_{r,r-1}$ shows that z_{r-1} solves the last equation in (3.3). Rearranging (3.6) for $i = 1$ using Corollary (1) (excluding the first term on the right hand side) yields

$$(I - P_1)z_1' = \sum_{j=1}^{r-1} (I - P_1) A_{1j} z_j + (I - P_1) f_1. \tag{3.19}$$

Multiplying (3.8) by A_{1r} and adding to both sides of (3.19) shows that z_1 solves the first equation in (3.3). To show that z solves the second equation in (3.3), multiply (3.7) by $A_{i,i-1}$ and add to both sides of (3.6). This completes the proof of the theorem. ■

COROLLARY 2. *Let A and f be $C^r(\Omega)$. Then* (3.3) *is a solvable DAE on* Ω. *The general solution starting at t_0 is given by*

$$z(t) = \big[\hat{N} + \hat{B}(t)\big]\Psi(t, t_0)\big[I - P(t_0)\big]v_0$$

$$+ \big[\hat{N} + \hat{B}(t)\big] \int_{t_0}^{t} \Psi(t, s) g(s)\, ds + q(t), \tag{3.20}$$

where $\hat{B}$, g, and q are given in Theorem 1, *$v_0 \in R^{s-n_r}$ is arbitrary, $\Psi(t, t_0)$ is* *the fundamental solution for* (3.9), *and the* $s \times (s - n_r)$ *matrix*

$$\hat{N} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

with the identity block of size $s - n_r$.

Proof. The proof follows immediately by combining (3.18) with the previous results and the existence theory for linear ODEs with continuous inputs and coefficients. ∎

COROLLARY 3. *Let the assumptions of Corollary* 2 *hold, and define the* $s \times (s - n_r)$ *matrix* $G(t)$ *by*

$$G(t) = \left[\hat{N} + \hat{B}(t)\right]\left[I - P(t)\right].$$

Then z_0 *is a consistent initial condition for* (3.1)–(3.3) *at* $t = \hat{t} \in \Omega$ *if and only if* $z_0 = G(\hat{t})v_0 + q(\hat{t})$ *for some* v_0. *Consequently, the dimension* p *of the solution manifold for* (3.3) *is* $\sum_{i=1}^{r-1}(n_i - n_r)$, rank $G(t) = p$, *and* $\mathcal{R}(G(t))$ *is the solution manifold for* (3.3) *when* $f \equiv 0$.

Proof. That $p = \sum_{i=1}^{r-1}(n_i - n_r)$ follows directly from (3.20) and the fact that $\operatorname{rank}(I - P_i) = n_i - n_r$. Also, rank $G(t) = p$, since $\hat{N} + \hat{B}(t)$ has full column rank. To show that $\mathcal{R}(G(t))$ is the solution manifold for the homogeneous system (3.1)–(3.3) it suffices to prove that $f \equiv 0$ implies $q \equiv 0$ and then apply (3.20). But this easily follows from (3.16), (3.17) in the proof of Theorem 1 in the same way that $f \equiv 0$ implies $g \equiv 0$. ∎

The definition of solvability given in Section 2 required differentiability in every component of the solution $z(t)$. However, from (3.9), (3.11) the variables v and w are independent of u. Differentiability of u requires one more derivative of the coefficients and inputs than v and w. This leads to the following slight generalization of the previous results.

COROLLARY 4. *If* A *and* f *are in* $C^{r-1}(\Omega)$, *then there exists a* $\sum_{i=1}^{r-1}(n_i - n_r)$*-dimensional family of functions* $z = (z_1, \dots, z_{r-1}, u)^t$ *which satisfy* (3.3) *such that* $z_i \in C^1(\Omega)$ *and* $u \in C^0(\Omega)$.

4. APPLICATION TO BOUNDARY VALUE PROBLEMS

In this section we briefly discuss application of the results of the previous section to the formulation and numerical solution of differential-algebraic two-point boundary value problems (BVP) and also to the analysis of a simple class of singular optimal control problems. We consider systems of the form

$$\begin{aligned} \mathscr{L}[z(t)] &:= E(t)z'(t) - A(t)z(t) = f(t), \qquad z(t) \in R^s, \\ \mathscr{B}[z(t)] &:= B_0 z(t_0) + B_1 z(t_1) = \beta \in R^\mu, \end{aligned} \tag{4.1}$$

where (E, A) is a sufficiently smooth (at least C^r) $\mathscr{H}_r$ pair and $\mu = p$ is the dimension of the solution manifold for (4.1). We refer the interested reader to [8] for a detailed analysis of general linear DAE two-point BVPs. We assume that the boundary matrices B_0 and B_1 are $p \times s$ such that $\operatorname{rank}[B_0 \;\; B_1] = p$. The *shooting matrix* for (4.1) is

$$S = B_0\Phi(t_0) + B_1\Phi(t_1).$$

From [8] and the results of the previous section, we know that $p = \sum_{i=1}^{r-1}(n_i - n_r)$ and $\Phi(t) = [\phi_1(t) \;\; \phi_2(t) \;\; \cdots \;\; \phi_p(t)]$ is an $s \times p$ fundamental solution matrix whose columns span $\mathscr{R}(G(t))$, where $G(t)$ is defined in Corollary 3. More specifically, if Q is an $(s - n_r) \times p$ matrix consisting of p linearly independent columns of $I - P(t_0)$, then from Equation (3.16) we have that

$$\Phi(t) = [\hat{N} + \hat{B}(T_1)]\Psi(t, t_0)Q,$$

which implies

$$S = \{B_0[\hat{N} + \hat{B}(t_1)] + B_1[\hat{N} + \hat{B}(t_1)]\Psi(t_1, t_0)\}Q.$$

Then the system (4.1) is a *solvable* BVP if and only if S is nonsingular; that is, the DAE $\mathscr{L}[z(t)] = f(t)$ is solvable and for each $\beta \in R^p$ there exists a unique solution $z(t)$ to the DAE which satisfies the boundary conditions $\mathscr{B}[z(t)] = \beta$.

4.1. *Numerical Solution of DAE BVPs*

We consider the numerical solution of systems (4.1) in Hessenberg form by finite difference discretizations of the form

$$\mathscr{L}_h z_j \equiv \sum_{k=0}^{N} C_{jk}(h) z_k = F_j(h, f), \qquad j = 1, \ldots, N \tag{4.2}$$

$$\mathscr{B}_h \mathbf{z}_h \equiv B_0 z_0 + B_1 z_N = \beta, \tag{4.3}$$

where $\mathbf{z}_h = \{z_j\}_{j=0}^N$ is the numerical approximation to the solution $\{z(t_j)\}_{j=0}^N$, $h = \lfloor(t_1 - t_0)/N\rfloor$ is the stepsize, and $t_j = t_0 + jh$. The assumption that h is constant is not relevant to the discussion which follows but does influence the local stability properties of the method. In matrix form, (4.2), (4.3) is an underdetermined linear system of $Ns + p$ equations in $(N+1)s$ unknowns. In [8] it was proved that if (4.2) is convergent and stable for the DAE initial value problem, then complementing (4.2), (4.3) with $s - p$ linearly independent consistency conditions on z_0 yields a convergent and stable method for the BVP. Thus we seek conditions of the form

$$M_0 z_0 = b_0 \tag{4.4}$$

where M_0 is $(s-p) \times s$ with full row rank, where both M_0 and b_0 are determined by the data $A(t), f(t)$ and their derivatives at $t = t_0$, and where (4.4) is equivalent to the consistency of z_0.

From Corollaries 2 and 3, z_0 is a consistent initial condition for (3.1)–(3.3) if and only if

$$z_0 = G(t_0)v_0 + q(t_0), \tag{4.5}$$

where G is given as in Corollary 3 and B, q are determined by Algorithm 1. That rank $G(t_0) = p$ implies dim $\mathcal{N}(G(t_0)^T) = s - p$. Let V be a $s \times (s - p)$ full column rank matrix whose columns span $\mathcal{N}(G(t_0)^T)$. Then $S = V(V^TV)^{-1}V^T$ is an orthogonal projector onto $\mathcal{N}(G(t_0)^T)$, and $I - S$ projects onto the orthogonal complement $\mathcal{N}(G(t_0)^T)^\perp = \mathcal{R}(G(t_0))$. Clearly (4.5) implies

$$V^Tz_0 = V^Tq(t_0). \tag{4.6}$$

But (4.6) implies $z_0 - q(t_0) \in \mathcal{N}(S) = \mathcal{R}(I - S) = \mathcal{R}(G(t_0))$. Setting $M_0 = V^T$ and $b_0 = V^Tq(t_0)$ gives the desired $s - p$ consistency conditions. Combining these with (4.2), (4.3) yields the $(N+1)s \times (N+1)s$ nonsingular linear system

$$\mathcal{A}_h \mathbf{z}_h = F(h, f),$$

where

$$\mathcal{A}_h = \begin{pmatrix} B_0 & 0 & \cdots & B_1 \\ M_0 & 0 & \cdots & 0 \\ C_{10} & C_{11} & \cdots & C_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N0} & C_{N1} & \cdots & C_{NN} \end{pmatrix}.$$

4.2. DAE BVPs in Optimal Control

A well-known example of a DAE two-point BVP arises from the Euler-Lagrange equations for singular optimal control problems

$$\min_{u\in\mathscr{U}} J[x,u] = \frac{1}{2}\left\{x_1^TCx_1 + \int_{t_0}^{t_1}[x^TH(t)x + u^TR(t)u]\,dt\right\} \tag{4.7}$$

subject to the homogeneous *state space* system (in control terminology)

$$x'(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0. \tag{4.8}$$

Here the state variable x and the control u are of size n_x and n_u $(n_x > n_u)$, respectively, $x_1 = x(t_1)$ is a free endpoint value, and $\mathscr{U}$ is a set of admissible controls (for simplicity we assume at least continuous). The cost matrices C, $H(t)$, and $R(t)$ are assume to be real, symmetric, and positive semidefinite on $[t_0, t_1]$, while $R(t)$ may be singular but has constant rank on $[t_0, t_1]$. We ignore nonhomogeneities in (4.8) in this discussion, although these can be handled in a straightforward manner. In the control theory literature, (4.7), (4.8) is known as a free endpoint linear-quadratic regulator problem.

The first order necessary conditions for $(x, u)^t$ to be an optimal state and control pair are given by the homogeneous DAE boundary value problem

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)u(t), \\ \lambda'(t) &= -H(t)x(t) - A^T(t)\lambda(t), \\ 0 &= B^T(t)\lambda(t) + R(t)u(t) \end{aligned} \tag{4.9}$$

subject to the boundary conditions

$$x(t_0) = x_0, \tag{4.10}$$

$$\lambda(t_1) = -Cx(t_1), \tag{4.11}$$

where λ is the *costate* or Lagrange multiplier variable of size n_x. Other types of boundary conditions may exist. For example, if $x(t_1)$ is specified, then the *natural* terminal condition $\lambda(t_1) + Cx(t_1) = 0$ will not be present.

By replicating the argument in [4] it is straightforward to show that the semidefinite assumptions on C, H, and R imply that (4.9)–(4.11) is also sufficient for $(x, u)^t$ to be optimal. The approach involves showing that if $(x^*, \lambda^*, u^*)^t$ solves (4.9)–(4.11) then the positive semidefinite quadratic functional $\psi(\omega) = J[\omega x^* + (1-\omega)x, \omega u^* + (1-\omega)u]$ is minimized at $\omega = 1$ over all admissible pairs $(x, u)^t$ such that $x(t_0) = x_0$. The corresponding optimal cost is $\frac{1}{2}\lambda_0^{*T} x_0^*$, where $\lambda_0^* = \lambda^*(t_0)$. We leave the details of this exercise to the interested reader.

Consider the special case of (4.7), (4.8) where $R(t)$ is singular and $B^T HB$ is nonsingular. Then (4.9) can be reduced to a totally singular problem ($R \equiv 0$) with a first order singular arc, which is of the form $\mathscr{H}_3$; henceforth we assume that $R \equiv 0$. In this case the index 3 state space decomposition given in the appendix gives the separation of the true state components in x and λ from the corresponding algebraic components and the control u. The projectors for (4.9) are $P_1 = B(B^T HB)^{-1} B^T H$ and $P_2 = HB(B^T HB)^{-1} B^T$, respectively. Since H is symmetric, $P_1^T = P_2$; hence $\mathscr{R}(P_1^T) = \mathscr{N}(P_1)^{\perp} = \mathscr{R}(I - P_1)^{\perp} = \mathscr{R}(P_2)$. But $\mathscr{R}(P_2)$ is the algebraic space for λ, while $\mathscr{R}(I - P_1)$ is the state space for x. Thus with a slight abuse of the terminology we have established the following interesting result.

THEOREM 4. *For the singular optimal control problem* (4.7), (4.8) *under the assumption $B^T HB$ invertible, the state space for the costate variable λ is orthogonal to the algebraic space for the state variable x. Similarly, the algebraic space for the costate variable is orthogonal to the state space for the state variable. If $H = \pm I$ the state spaces for the state and costate variables are the same and are orthogonal to their respective algebraic counterparts.*

A system of the form (4.9) with $H = -I$ can arise if λ and x are position and velocity variables, respectively (with x' depending on λ), and (4.9) is the linearization of a constrained mechanical system with normalized masses [16].

The comments preceding Theorem 4 assumed the existence of solutions. Note that (4.9)–(4.11) is an overdetermined BVP, since $p = 2(n_x - n_u)$ and (4.10), (4.11) consists of $2n_x$ conditions. Consequently this BVP is not solvable in the sense defined above. However, the DAE initial value problem (4.9) is solvable. Therefore, in principle we can apply the results of Section 3 to determine consistency of the boundary conditions with existence and uniqueness of solutions. Using (3.20) and the formulas in the appendix, we have

$$\begin{bmatrix} x(t) \\ \lambda(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} I & B_{12}(t) \\ 0 & I \\ B_{u1}(t) & B_{u2}(t) \end{bmatrix} \Psi(t, t_0)\left[I - P(t_0)\right] v_0, \tag{4.12}$$

where

$$B_{12}(t) = -B(B^THB)^{-1}B^T(P_2' - A^T),$$

$$B_{u1}(t) = (B^THB)^{-1}B^TH(B_{12}H + P_1' + A) \tag{4.13}$$

$$B_{u2}(t) = -(B^THB)^{-1}B^TH\{B_{12}' - AB_{12} - B_{12}[(I - P_2)A^T + P_2P_2']\}.$$

Applying the boundary conditions yields

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = W[I - P(t_0)]v_0, \tag{4.14}$$

where

$$W = \begin{bmatrix} I & B_{12}(t_0) \\ [C \quad CB_{12}(t_1) + I] & \Psi(t_1, t_0) \end{bmatrix}. \tag{4.15}$$

THEOREM 5. *Let W be the matrix in* (4.15). *Then solutions to* (4.9)–(4.11) *exist if and only if* $(x_0, 0)^t \in \mathscr{R}(W[I - P(t_0)])$. *If* (4.14) *is consistent, a solution to* (4.9)–(4.11) *is unique if and only if W is nonsingular.*

Proof. Clearly any solution to (4.9)–(4.11) must satisfy (4.14). Conversely, if $(x_0, 0)^t \in \mathscr{R}(W[I - P(t_0)])$ and v_0 is any solution to (4.14), then the solution $(x, \lambda, u)^t$ in (4.12) corresponding to the initial condition $[I - P(t_0)]v_0$ will satisfy the boundary conditions (4.11). This takes care of the question of existence in the first part of the theorem. To address the question of uniqueness, note that by linearity, a solution to any solvable BVP of the form (4.1) is unique if and only if $z \equiv 0$ is the unique solution to the corresponding homogeneous BVP ($\beta = 0$, $f \equiv 0$). Thus suppose that W is nonsingular. Then $0 = W[I - P(t_0)]v_0$ if and only if $0 = [I - P(t_0)]v_0$ if and only if $(x, \lambda, u)^t \equiv 0$ is the unique solution to (4.9)–(4.11). Conversely, suppose that $(x, \lambda, u)^t \equiv 0$ is the unique solution to (4.9)–(4.11) but W is singular. Then there exists $\phi \neq 0$ such that $W[I - P(t_0)]\phi = 0$ but $[I - P(t_0)]\phi \neq 0$. Using the initial condition $[I - P(t_0)]\phi$ in (4.12) gives a nonzero solution to the homogeneous BVP, contradicting the assumption of solvability. Hence W must be nonsingular. ∎

COROLLARY 5. *Solutions to* (4.9)–(4.11) *are unique if and only if* $\mathscr{N}(W[I - P(t_0)]) = \mathscr{N}(I - P(t_0)) = \mathscr{R}(P(t_0))$ *if and only if* $\text{rank}\{W[I - P(t_0)]\} = \text{rank}[I - P(t_0)]$.

COROLLARY 6. *Solutions to* (4.9)–(4.11) *are unique for every* t_1 *in a sufficiently small interval containing* t_0.

Proof. Viewing $W(t_1) = W(t)$ for $t = t_1$ in (4.15), block row reduction shows that $W(t_0)$ is nonsingular. ∎

Note that Corollary 5 is slightly more general than the uniqueness result in Theorem 5, since it can be applied to Hessenberg DAE systems with more general boundary conditions than (4.10), (4.11). To summarize the results of this section, we state the following

THEOREM 6. *If* t_1 *is sufficiently close to* t_0, *an optimal control for* (4.7), (4.8) (*if one exists*) *is unique and is given in either closed or open loop form by*

$$u(t) = \begin{cases} \{B_{u1}(t)[B_{u2}(t) - B_{12}(t)B_{u1}(t)]\}(x(t), \lambda(t))^t & (\textit{closed loop}), \\ [B_{u1}(t) \quad B_{u2}(t)]\Psi(t, t_0)W^{-1}(x_0, 0)^t & (\textit{open loop}). \end{cases}$$

The corresponding optimal cost is given by $\frac{1}{2}x_0^T W_{21} x_0$, *where* W_{21} *is the* (2, 1) *block of* $W^{-1}(t_1)$.

5. DISCUSSION

In this paper we have presented and analyzed an algorithm for algebraically resolving the solutions of a linear, time dependent Hessenberg form DAE into its state and algebraic components. We have also discussed several applications of this procedure to boundary value problems arising in optimal control and to the numerical solution of these systems. In contrast to the reduction algorithm in [7], which produced a chain of generalized integrators and differentiators, the technique described here does not involve time variable coordinate changes. We have shown that in principle the Hessenberg reduction procedure can be used to compute exact initial conditions for smooth solutions, although the formulas are quite complicated and will probably require the use of symbolic packages in order to be usable for problems with index $\geqslant 5$. For boundary value problems, the ability to express the conditions for consistency of initial conditions is critical to well-posing of the BVP and for the formulation of a nonsingular linear system for finite difference methods. We have shown that with some minor modifications, Corollary 2 provides these for linear Hessenberg DAEs.

At this point the implications of the algorithm for nonlinear DAEs and its relationship to the various state space forms which can be produced by derivative array computations (see [1]) by different pivoting strategies is not known but is under investigation. Note that the existence of an invariant subspace for the differential part of the solution suggests that the same error control strategies which are used to govern the variation of stepsize in ODE solvers can be applied to the projected components in $\mathscr{R}(I - P)$ rather than in z; the known results for these methods should be directly applicable to Hessenberg DAE problems.

APPENDIX

In this appendix we give the decomposition formulas for components of the state variable ODE and the expressions for the algebraic variables for the index 3 and index 4 Hessenberg forms.

Index 3 state space decomposition:

$$\begin{aligned} v_1' &= \left[(I - P_1) A_{11} - P_1 P_1' \right] v_1 + \hat{A}_{12} v_2 + g_1, \\ v_2' &= A_{21} v_1 + \left[(I - P_2) A_{22} - P_2 P_2' \right] v_2 + g_2, \\ w_1 &= B_{12} v_2 + q_1, \\ w_2 &= q_2, \\ u &= B_{u1} v_1 + B_{u2} v_2 + q_u, \end{aligned} \tag{6.1}$$

where

$$q_2 = -A_{21} A_{13} \Pi^{-1} f_3,$$

$$g_2 = (I - P_2)\left\{ f_2 + \left([A_{21} A_{13}]' - A_{22} A_{21} A_{13} \right) \Pi^{-1} f_3 \right\},$$

$$B_{12} = -A_{13} \Pi^{-1} A_{32} (P_2' + A_{22}),$$

$$\begin{aligned} q_1 = A_{13} \Pi^{-1} A_{32} \Big\{ \Big(A_{22} A_{21} A_{13} \Pi^{-1} - \left[A_{21} A_{13} \Pi^{-1} \right]' \Big) f_3 \\ - A_{21} A_{13} \Pi^{-1} f_3' - f_2 \Big\}, \end{aligned}$$

$$\hat{A}_{21} = (I - P_1)\left[A_{12} + (A_{13}' - A_{11} A_{13}) \Pi^{-1} A_{32} (P_2' + A_{22}) \right],$$

$$g_1 = (I - P_1)\Big\{f_1 - (A'_{13} - A_{11}A_{13})\Pi^{-1}A_{32}\Big[\Big(A_{22}A_{21}A_{13}\Pi^{-1} - [A_{21}A_{13}\Pi^{-1}]'\Big)f_3 - A_{21}A_{13}\Pi^{-1}f'_3 - f_2\Big] + A_{12}q_2\Big\},$$

$$B_{u1} = \Pi^{-1}A_{32}A_{21}(B_{12}A_{21} - P'_1 - A_{11}),$$

$$B_{u2} = \Pi^{-1}A_{32}A_{21}\{B'_{12} - A_{12} - A_{11}B_{12} + B_{12}[(I - P_2)A_{22} - P_2P'_2]\},$$

$$q_u = \Pi^{-1}A_{32}A_{21}(B_{12}g_2 - q'_1 - A_{11}q_1 - A_{12}q_2 - f_2). \tag{6.2}$$

Index 4 state space decomposition:

$$\begin{aligned}
v'_1 &= [(I - P_1)A_{11} - P_1P'_1]v_1 + \hat{A}_{12}v_2 + \hat{A}_{13}v_3 + g_1,\\
v'_2 &= A_{21}v_1 + [(I - P_2)A_{22} - P_2P'_2]v_2 + \hat{A}_{23}v_3 + g_2,\\
v'_3 &= A_{32}v_2 + [(I - P_3)A_{33} - P_3P'_3]v_3 + g_3,\\
w_1 &= B_{12}v_2 + B_{13}v_3 + q_1,\\
w_2 &= B_{23}v_3 + q_2,\\
w_3 &= q_3,\\
u &= B_{u1}v_1 + B_{u2}v_2 + B_{u3} + q_u,
\end{aligned} \tag{6.3}$$

where

$$q_3 = -A_{32}A_{21}A_{14}\Pi^{-1}f_4,$$

$$g_3 = (I - P_3)(f_3 + A_{33}q_3 + A'_{32}A_{21}A_{14}\Pi^{-1}f_4),$$

$$B_{23} = -A_{21}A_{14}\Pi^{-1}A_{43}(P'_3 + A_{33}),$$

$$q_2 = A_{21}A_{14}\Pi^{-1}A_{43}(q'_3 - A_{33}q_3 - f_3),$$

$$\hat{A}_{23} = (I - P_2)[A_{23} + A_{22}B_{23} + A'_{21}A_{14}\Pi^{-1}A_{43}(P'_3 + A_{33})],$$

$$g_2 = (I - P_2)[f_2 + A_{22}q_2 + A_{23}q_3 - A'_{21}A_{14}\Pi^{-1}A_{43} \times (q'_3 - A_{33}q_3 - f_3)],$$

$$B_{12} = A_{14}\Pi^{-1}A_{43}A_{32}\left(B_{23}A_{32} - P_2' - A_{22}\right),$$

$$B_{13} = A_{14}\Pi^{-1}A_{43}A_{32}$$
$$\times\left\{B_{23}' - B_{23}\left[\left(I - P_3\right)A_{33} - P_3P_3'\right] - A_{23} - A_{22}B_{23}\right\},$$

$$q_1 = A_{14}\Pi^{-1}A_{43}A_{32}\left\{q_2' + B_{23}\left(I - P_3\right)\right.$$
$$\left.\times\left[f_3 + A_{33}q_3 + A_{32}'A_{21}A_{14}\Pi^{-1}f_4\right] - A_{22}q_2 - A_{23}q_3 - f_2\right\},$$

$$\hat{A}_{12} = \left(I - P_1\right)\left(A_{12} + A_{11}B_{12} - B_{12}'\right),$$

$$\hat{A}_{13} = \left(I - P_1\right)\left(A_{13} + A_{11}B_{13} + A_{12}B_{23} - B_{13}'\right),$$

$$g_1 = \left(I - P_1\right)\left\{f_1 + A_{11}q_1 + A_{12}q_2 + A_{13}q_3 - A_{14}'\Pi^{-1}A_{43}A_{32}\right.$$
$$\times\left[q_2' + B_{23}\left(I - P_3\right)\left(f_3 + A_{33}q_3 + A_{32}'A_{21}A_{14}\Pi^{-1}f_4\right)\right.$$
$$\left.\left. - A_{22}q_2 - A_{23}q_3 - f_2\right]\right\},$$

$$B_{u1} = \Pi^{-1}A_{43}A_{32}A_{21}\left(B_{12}A_{21} - P_1' - A_{11}\right),$$

$$B_{u2} = \Pi^{-1}A_{43}A_{32}A_{21}$$
$$\times\left\{B_{12}' + B_{12}\left[\left(I - P_2\right)A_{22} - P_2P_2'\right] - A_{12} - A_{11}B_{12} + B_{13}A_{32}\right\},$$

$$B_{u3} = \Pi^{-1}A_{43}A_{32}A_{21}$$
$$\times\left\{B_{12}\hat{A}_{23} + B_{13}'\left[\left(I - P_3\right)A_{33} - P_3P_3'\right] - A_{13} - A_{11}B_{13} - A_{12}B_{23}\right\},$$

$$q_u = \Pi^{-1}A_{43}A_{32}A_{21}$$
$$\times\left(B_{12}g_2 + B_{13}g_3 + q_1' - A_{11}q_1 - A_{12}q_2 - A_{13}q_3 - f_3\right). \qquad (6.4)$$

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