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A Note on a Class of Noncoercive Functionals

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Abstract—Our goal here is to prove the existence of a nontrivial critical point to the following functional:

$$I(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega |u|^p \ln\left(1+u^2\right).$$

by using the well-known Mountain-Pass theorem with the Cerami Palais-Smale condition.

Keywords—Mountain-Pass theorem, Nontrivial solution, Cerami PS condition, Noncoercive functionals, "Superlinear" elliptic problems, Critical point.

1. INTRODUCTION

In this paper, we study the following functional:

$$I(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega |u|^p h(u)\,dx,$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, with a sufficient smooth boundary $\partial\Omega$ and $h : \mathbb{R} \to \mathbb{R}$ is a C^1 function. We are going to use the Mountain-Pass theorem in order to prove the existence of a nontrivial critical point $u \in W_o^{1,p}(\Omega)$. We suppose that $p > n$.

Let us state the assumptions on h.

(i) $h(\cdot)$ is increasing and $h(r) \to \infty$ as $r \to \pm\infty$;
(ii) $\lim_{u\to\pm\infty}(h(u)/u) = 0$;
(iii) there exists some $M > 0$ such that

$$|u| - |u|^p uh'(u) \le M,$$

for all $u \in \mathbb{R}$;
(iv) $h(0) \le \mu < \lambda_1/p$;
(v) for every $\mu > 0$, $h(\mu) > 0$.

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REMARK. It is easy to check that $h(u) = \ln(1+u^2)$ or $h(u) = ((\ln(1+u^2))^2$ verifies all the above conditions.

Let us denote by $F(u) = |u|^p h(u)$, $f(u) = |u|^{p-2}uh(u) + |u|^p h'(u)$. It is well known that I is well defined and C^1.

Let us introduce the (PS) that we are going to use.

CERAMI (PS) CONDITION. For every $\{u_n\} \subseteq W_o^{1,p}(\Omega)$ with $|I(u_n)| \leq M$ and $(1 + \|u_n\|_{1,p}) < I'(u_n), \phi > \to 0$ for every $\phi \in W_o^{1,p}(\Omega)$, there exists a strongly convergent subsequent. This condition has introduced by Cerami (see [1,2]).

Finally, we are going to use the first eigenvalue and eigenfunction of the p-Laplacian and for more details we refer to [3].

2. BASIC RESULTS

We are going to use the Mountain-Pass theorem, so our first lemma is that I satisfies the (PS) condition due to Cerami.

LEMMA 1. *I satisfies the Cerami (PS) condition.*

PROOF. Suppose that there exists a sequence $\{u_n\} \subseteq X$ such that $|I(u_n)| \leq M$ and $(1 + \|u_n\|_{1,p}) < I'(u_n), \phi > \to 0$ for every $\phi \in X$.

Then we have

$$-M \leq -\|Du_n\|_p^p + \int_\Omega pF(u_n)\,dx \leq M, \tag{1}$$

and, choosing $\phi = u_n$,

$$-\varepsilon_n \frac{\|u_n\|_{1,p}}{1+\|u_n\|_{1,p}} \leq \|Du_n\|_p^p - \int_\Omega f(u_n)\,u_n\,dx \leq \varepsilon_n \frac{\|u_n\|_{1,p}}{1+\|u_n\|_{1,p}}. \tag{2}$$

Now, consider the sequence $a_n = 1/p\|u_n\|_\infty^{p-1}h(\|u_n\|_\infty)$. Then multiplying inequality (1) with $a_n + 1$ and substituting with (2), we arrive at

$$\begin{aligned}\frac{\|Du_n\|_p^p}{cp\,\|Du_n\|_p^{p-1}\,h\left(\|Du_n\|_p\right)} &\leq a_n\,\|Du_n\|_p^p \\ &\leq \int_\Omega (a_n+1)\,pF(u_n) - f(u_n)\,u_n\,dx \\ &\quad + (a_n+1)\,M + \varepsilon_n\frac{\|u_n\|_{1,p}}{1+\|u_n\|_{1,p}}.\end{aligned} \tag{3}$$

Then we can say

$$\begin{aligned}\int_\Omega (a_n+1)\,pF(u_n) - f(u_n)\,u_n\,dx &\leq \int_\Omega \frac{p\,|u_n(x)|^p\,h(u_n(x))}{p\,|u_n(x)|^{p-1}\,h(u_n(x))} - |u_n(x)|^p\,u_n(x)\,h'(u_n(x))\,dx \\ &= \int_\Omega |u_n(x)| - |u_n(x)|^p\,u_n(x)\,h'(u_n(x))\,dx \leq M.\end{aligned}$$

Suppose now that $\|Du_n\|_p \to \infty$. Next, we will show that there exists some $c > 0$ such that $\|u_n\|_\infty > c$ for big enough n. Suppose not. Then, $\|u_n\|_\infty \to 0$. But we have supposed that $I(u_n) \leq M$ and it is easy to see that $\int_\Omega |u_n(x)|^p h(u_n(x))\,dx \to 0$. That is, we have a contradiction, because we have supposed that $\|Du_n\|_{1,p} \to \infty$.

Going back to (3), we obtain a contradiction to the hypothesis that u_n is not bounded. Using well-known arguments, we can prove that in fact $\{u_n\}$ have a convergent subsequence. ∎

LEMMA 2. *There exists some $e \in W_o^{1,p}(\Omega)$ such that $I(e) \leq 0$.*

PROOF. Choose $\phi \in W_o^{1,p}(\Omega)$ such that $\phi(x) \geq \mu > 0$ on some ball $B(x_o, r) \subseteq \Omega$ and zero elsewhere. Then we claim that there exists big enough $\xi \in \mathbb{R}$ such that $I(\xi\phi) \leq 0$.

Suppose not. Then there exists a sequence $\xi_n \to \infty$ such that $I(\xi_n \phi) \geq c > 0$. That means

$$\frac{\xi_n^p}{p} \|D\phi\|_p^p - \int_\Omega |\xi_n \phi(x)|^p h(\xi_n \phi(x))\, dx \geq c > 0.$$

Then we can say that

$$\begin{gathered}\int_\Omega |\phi(x)|^p h(\xi_n \phi(x))\, dx \leq M \Rightarrow \\ \int_{B(x_o,r)} |\phi(x)|^p h(\xi_n \mu)\, dx \leq M \Rightarrow \\ h(\xi_n \mu) \int_{B(x_o,r)} |\phi(x)|^p\, dx \leq M.\end{gathered}$$

That is, we have a contradiction. ∎

LEMMA 3. *There exists some $\rho > 0$ small enough and $a > 0$ such that $I(u) \geq a$ for all $\|u\|_{1,p} = \rho$.*

PROOF. Suppose not. Then there exists some sequence $\{u_n\} \subseteq W_o^{1,p}(\Omega)$ with $\|u_n\|_{1,p} = \rho_n \to 0$, such that $I(u_n) \leq 0$. That is,

$$\frac{1}{p} \|Du_n\|_p^p \leq \int_\Omega |u_n(x)|^p h(u_n(x))\, dx.$$

Let $y_n(x) = u_n(x)/\|u_n\|_{1,p}$.

Divide this inequality by $\|u_n\|_{1,p}^p$. Then we arrive at

$$\frac{\lambda_1}{p} \|y_n\|_p^p \leq \frac{1}{p} \|Dy_n\|_p^p \leq \int_\Omega |y_n(x)|^p h(u_n(x))\, dx. \tag{4}$$

Note that $\|y_n\|_{1,p} = 1$, so $y_n \to y$ weakly in $W_o^{1,p}(\Omega)$ and $y_n \to y$ strongly in $L^p(\Omega)$. Also, because $\|u_n\|_{1,p} \to 0$ and $p > n$ we have $\|u_n\|_\infty \to 0$. So, from the above, we deduce that $\|Dy_n\|_p \to \lambda_1 \|y\|_p$. Also, from the weak lower semicontinuity of the norm functional, we have that $(1/p)\|Dy\|_{1,p} \leq (\lambda_1/p)\|y\|_p^p$. Now, using the variational characterization of the first eigenvalue, we arrive at the fact that $\|Dy\|_p^p = \lambda_1 \|y\|_p^p$ and $\|Dy_n\|_p \to \|Dy\|_p$. From the uniform convexity of $W_o^{1,p}(\Omega)$, we deduce that $y_n \to y$ strongly in $W_o^{1,p}(\Omega)$ and, because $\|y_n\|_{1,p} = 1$, we obtain that $y \neq 0$ and in fact $y = u_1$, i.e., the first eigenfunction.

So, going back to (4), we obtain

$$\frac{1}{p} \|Du_1\|_p^p \leq \int_\Omega |u_1(x)|^p h(0)\, dx < \frac{\lambda_1}{p} \|u_1\|_p^p.$$

That is, we have a contradiction. ∎

Then we can use the Mountain-Pass theorem to obtain a nontrivial critical point.

3. APPLICATIONS TO DIFFERENTIAL EQUATIONS

Consider the following elliptic equation:

$$\begin{aligned} -\Delta_p(u) &= |u|^{p-2} u \ln(1+u^2) + |u|^p \frac{2u}{1+u^2}, && \text{a.e. on } \Omega, \\ u &= 0, && \text{a.e. on } \partial\Omega, \quad 2 \leq p < \infty. \end{aligned} \tag{5}$$

Here, as before, $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth enough boundary $\partial\Omega$ and $p > n$.

Let us denote by $f(u) = |u|^{p-2}u\ln(1+u^2) + |u|^p(2u/(1+u^2))$. Then it is easy to see that $\lim_{u\to\infty}(f(u)/(|u|^{p-2}u)) \to \infty$.

It is well known that for such kind of problems Ambrosetti-Rabinowitz [4] had introduced a hypothesis which states as follows.

There exists some $\theta > p$ such that

$$0 < \theta F(u) \le f(u)\, u,$$

for all $|u| > M$ for big enough M with $F(u) = \int_o^u f(r)\, dr$. From this condition, we can easily prove that $F(u) \ge |u|^\theta$. So, there is not such a $\theta > p$ for $f(u) = |u|^{p-2}u\ln(1+u^2) + |u|^p(2u/(1+u^2))$ to satisfy the above condition. But it is easy to see that we can apply the results of the previous section, to the corresponding energy functional, and then derive a nontrivial critical point which in fact is a solution to problem (5).

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